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ON MEASURING INFLUENCE IN NON-BINARY VOTING GAMES*

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Abstract

In this note, we demonstrate using two simple examples that generalization of the Banzhaf measure of voter influence to non-binary voting games that requires as starting position a voter's membership in a winning coalition is likely to incompletely reflect the influence a voter has on the outcome of a game. Generalization of the Banzhaf measure that takes into consideration all possible pivot moves of a voter including those moves originating from a losing coalition will, on the other hand, result in a measure that is proportional to the Penrose measure only in the ternary case.

JEL Classification: C6, D7

Keywords: Penrose measure, Banzhaf index, ternary games, multicandidate weighted voting games

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1. Introduction

In a seminal paper in 1946, LS Penrose posited a measure of voter influence in binary majority voting games in which a given voter's influence is proportional to the number of divisions of other voters in which the given voter can make a difference. This is directly related to the number of instances that the given voter is a member of the "winning side" given that all other voters vote in a random manner. It is well known that Penrose's measure is directly equivalent to the absolute Banzhaf index, one based on the ability of a voter in a "winning" coalition to affect the passage or defeat of an issue [see Banzhaf (1965)]. Thus, it is a matter of indifference when measuring influence whether one focuses attention on the number of divisions of other voters in which the given voter can make a difference or on the number of pivot (or outcome-changing) moves the given voter has given his membership in a winning coalition.

In his generalization of the Banzhaf measure to non-binary voting games, Bolger (1983, 1986, 1990, 1993, SCW2002) proceeded along the latter line by requiring as starting position membership of the given voter in a winning coalition and modifying the definition of a pivot move to take into consideration the destination of the move since there is now more than one destination for a voter who defects from the winning coalition. In enumerating the number of pivot moves of a voter, the requirement that the given voter must be initially a member of the winning coalition, however, results in the exclusion of outcome-changing moves relating to those moves of voter when he is originally a member of a losing coalition. In the binary case, defection of a member from the losing coalition cannot be pivotal. In the *r*-alternative case, this continues to be true only if the winning coalition wins *unconditionally*, that is, when the win is independent of how other voters outside of the winning subset is distributed across the other (r-1) alternatives. Otherwise, such pivotal moves become possible. We demonstrate this point by appealing to two simple examples: the UN Security Council game, a game discussed in Bolger's work including Bolger (2002) and the set of non-dictatorial *3-voter 3 –candidate* weighted plurality games.

2. The UN Security Council Game

The procedure for computing the Banzhaf values for voters in ternary games and the Banzhaf values for a permanent (and rotating) member in the UN Security Council game are already a part of the established literature [see Felsenthal and Machover (1997, 1998)]. It is thus not the purpose of this example to belabor these points further. Rather, the intention here is to highlight the problematic nature of the Bolger measure that enumerates pivot moves only when such moves emanates from a winning coalition.

On non-procedural issues, the practical implementation of Article 27c of the UN Charter suggests that securing 9 affirmative votes out of the possible 15 is sufficient to secure passage of a resolution *provided* no permanent member vetoes the resolution. Under such a voting structure, a rotating member is pivotal for a given partition of voters *if and only if* it is a member of a strictly minimal winning coalition. This occurs when exactly nine members of the Council, inclusive of the rotating member in question, vote in the affirmative and no permanent member vote against the proposal. Accordingly, the total number of pivot moves of such a member is 106,308 as reported in Bolger (2002, p.711).

If a permanent member is a member of a strictly minimal winning coalition, then any move by such a member will be pivotal. Otherwise, the member is pivotal only if the move involves a veto. Again as enumerated in Bolger (2002, p.711), such moves total 178,212. Any measure that relies solely on these values to infer the relative influence a permanent member has in this game, however, is likely to be inappropriate since it does not fully account for all the instances in which a permanent member may influence the outcome of a vote in the Security Council.

In addition to the 178,212 pivot moves enumerated above, there are altogether 38,460 other instances in which a permanent member may change the outcome of a vote. All such instances invariably involve the situation whereby the permanent member in question had abstained but nonetheless the motion will be carried because nine or more affirmative votes have been obtained and there is no veto vote. In such instances, a move by the permanent member in question from an abstention to a *nay* vote would veto the resolution. We list these scenarios below using the following notation. The letter P denotes a permanent member, the letter R denotes a rotating member and the permanent member in focus is denoted by the letter P*. The letters Y, N and A refer to Yes, No and Abstain respectively.

(1) Y(4Ps, 5Rs), A(P*), 5Rs (either A or N)	: $^{10}C_5 x 2^5$	= 8,064 instances
(2) Y(4Ps, 6Rs), A(P*), 4Rs (either A or N)	: $^{10}C_6 x 2^4$	= 3,360 instances
(3) Y(4Ps, 7Rs), A(P*), 3Rs (either A or N)	: $^{10}C_7 x 2^3$	= 960 instances
(4) Y(4Ps, 8Rs), A(P*), 2Rs (either A or N)	: $^{10}C_8 x 2^2$	= 180 instances
(5) Y(4Ps, 9Rs), A(P*), 1R (either A or N)	: $^{10}C_9 x 2^1$	= 20 instances
(6) Y(4Ps, 10Rs), A(P*)	: $^{10}C_{10} x 2^0$	= 1 instance
(7) Y(3Ps, 6Rs), A(P*, P), 4Rs (either A or N)	: $^4C_3 x ^{10}C_6 x 2^4$	= 13,440 instances
(8) Y(3Ps, 7Rs), A(P*, P), 3Rs (either A or N)	: $^4C_3 x ^{10}C_7 x 2^3$	= 3,840 instances
(9) Y(3Ps, 8Rs), A(P*, P), 2Rs (either A or N)	: $^4C_3 x ^{10}C_8 x 2^2$	= 720 instances
(10) Y(3Ps, 9Rs), A(P*, P), 1R (either A or N)	: $^4C_3 x ^{10}C_9 x 2^1$	= 80 instances
(11) Y(3Ps, 10Rs), A(P*, P)	: $^4C_3 x ^{10}C_{10} x 2^0$	= 4 instances
(12) Y(2Ps, 7Rs), A(P*, 2Ps), 3Rs (either A or N)	: $^4C_2 x ^{10}C_7 x 2^3$	= 5,760 instances
(13) Y(2Ps, 8Rs), A(P*, 2Ps), 2Rs (either A or N)	: $^4C_2 x ^{10}C_8 x 2^2$	= 1,080 instances
(14) Y(2Ps, 9Rs), A(P*, 2Ps), 1R (either A or N)	: $^4C_2 x ^{10}C_9 x 2^1$	= 120 instances
(15) Y(2Ps, 10Rs), A(P*, 2Ps)	: $^4C_2 x ^{10}C_{10} x 2^0$	= 6 instances
(16) Y(P, 8Rs), A(P*, 3Ps), 2Rs (either A or N)	: $^4C_1 x ^{10}C_8 x 2^2$	= 720 instances
(17) Y(P, 9Rs), A(P*, 3Ps), 1R (either A or N)	: $^4C_1 x ^{10}C_9 x 2^1$	= 80 instances
(18) Y(P, 10Rs), A(P*, 3Ps)	: $^4C_1 x ^{10}C_{10} x 2^0$	= 4 instances
(19) Y(9Rs), A(P*, 4Ps), 1R (either A or N)	: $^{10}C_9 x 2^1$	= 20 instances
(20) Y(10Rs), A(P*, 4Ps)	: $^{10}C_{10} x 2^0$	= 1 instance
		38,460 instances

Thus, focusing solely on pivot moves when the member is originally a member of the winning coalition will not result in a measure that fully reflects the ability a permanent member has in influencing the outcome of a vote in the Security Council. Under Bolger's definition, the influence a permanent member has relative to that of a rotating member is $\frac{178{,}212}{106{,}308}$ (or 1.676 times) whereas under the alternative definition, this ratio is $\frac{216{,}672}{106{,}308}$ (or 2.038 times). This latter number coincides with that obtained in Felsenthal and Machover (1997, p. 348). In this 1997 paper, the reported Banzhaf value for a permanent member is 0.1009 and that for a rotating member is 0.0495 and the ratio of these numbers yields the value 2.038. That the value obtained, based on a complete enumeration of all pivot moves, is consistent with the Penrose measure may be directly inferred from Felsenthal and Machover (1997) and thus requires no further elaboration.

At this point, it is important to highlight an alternative extension of the Banzhaf measure to the UN Security Council game, one that is probably more appropriate given the particular structure of this ternary game. This is the hybrid ternary-binary model proposed in Felsenthal and Machover (1997, 1998). As noted in these works, "since abstention by an ordinary member has exactly the same effect as a 'no' vote, these members have in effect just two voting options -- `no' and 'yes' -- whereas only for the permanent members is abstention a distinct *tertium quid*." Explicit recognition of this fact suggests that the total number of relevant and "distinct" divisions of the voter set for this game is $3^5 2^{10}$ or 248,832 divisions, not 3^{15} or 14,348,907 divisions. Thus, a member's ability to influence outcomes in these 248,832 divisions becomes the relevant indication of his influence in the UN Security Council game. This line of reasoning will lead to a measure that assigns approximately 0.1038 to a permanent member and 0.0482 to a rotating member, thus giving rise to the outcome that a permanent member is $\frac{0.1038}{0.0482}$ or 2.154 times more influential than a rotating member.

3. The set of non-dictatorial *3-voter 3-candidate* weighted plurality games

In this second example, we demonstrate once again that Bolger's measure fails to fully account for all the instances in which a voter is pivotal. In addition, we would like to also highlight that in the proper (n, r) multi-candidate setting, such as in this example, both the Bolger measure and the measure derived from a complete enumeration of all pivot moves do not coincide with the Penrose measure, one that enumerates, for a given voter, the number of *r*-partitions of the other (n-1) voters in which the given voter may make a difference.

Let N={1, 2, 3} be a set of voters and let the set of coalitions be denoted by ***N*** ={{123},{12},{13},{23},{1},{2},{3}, ∅}. Let the weight vector associated with the voter set be (w_1, w_2, w_3) where, without loss of generality, we let $w_1>w_2>w_3$ and $w_2+w_3>w_1$. Whether a coalition is winning or otherwise will depend on how the 3 voters are distributed across the 3 alternatives. The following list completely enumerates the *3*-partitions and the corresponding embedded winning coalitions (bold-faced) for this set of games up to a permutation of alternatives: {(**{1,2,3}**,∅,∅), (**{1,2}**,{3},∅), (**{1,3}**,{2},∅), (**{2,3}**,{1},∅), (**{1}**,{2},{3})}.

Using Bolger's definition of an ordered pivot move, we obtain the number of pivot moves for each voter corresponding to each of these partitions as follows. Voter 1 has no pivot move in partitions (**{1,2,3}**,∅,∅) and (**{2,3}**,{1},∅), two pivot moves each in partitions (**{1,2}**,{3},∅), (**{1,3}**,{2},∅) and (**{1}**,{2},{3}) making the total count 6. Voter 2 has one pivot move in partition (**{1,2}**,{3},∅) and two pivot moves in partition (**{2,3}**,{1},∅) while Voter 3 has one pivot move in partition (**{1,3}**,{2},∅) and two pivot moves in partition (**{2,3}**,{1},∅). There are no other pivot moves for Voter 2 and Voter 3. Thus, under Bolger's definition, the vector of pivot moves for the voters are (6, 3, 3) giving rise to a measure of relative influence for the voters denoted by the vector (0.5, 0.25, 0.25).

Recall the partition (**{1}**,{2},{3}) and note that, for this partition, although Voter 2 and Voter 3 are not members of the winning coalition {**1**}, both voters can influence the outcome of the game by forming the coalition {2,3} either by Voter 2 moving to the third alternative or by Voter 3 moving to the second. In each of these instances, the resulting outcome will differ from alternative 1. If these extra pivotal moves of Voter 2 and Voter 3 are taken into consideration when computing the measure of influence, the resulting vector of pivot moves for the voters would have been (6, 4, 4) giving rise to a normalized influence vector equal to (0.428, 0.286, 0.286) as opposed to Bolger's (0.5, 0.25, 0.25). Thus, for this set of games, by ignoring the pivot moves of the smaller voters originating from a losing coalition, Bolger has inadvertently overstated the influence of the largest voter. Under the Bolger measure, Voter 1 has twice as much influence as Voter 2 or Voter 3. Under complete enumeration, the influence of Voter 1 is only 1.5 times that of Voter 2 or Voter 3.

On the Penrose measure, it may be deduced from the following set of *3*-partitions (up to a permutation) {(**{2,3}**,∅,∅), (**{2}**,{3},∅)} that Voter 1 is influential in the partition (**{2}**,{3},∅). Again, up to a permutation of alternatives, the *3*-partitions in which Voter 2 and Voter 3 are, respectively, influential are (**{1}**,{3},∅) and (**{1}**,{2},∅). Thus, the number of *3*-partititons inclusive of permutations in which the different voters are influential are respectively (6, 6, 6) yielding a relative influence vector equal to (0.33, 0.33, 0.33). This is clearly different from the preceding measures. In this instance, Voter 2 and Voter 3 are just as influential. This outcome arises because in this truly multicandidate situation, when a given voter is influential for a particular *3*-partition of the other voters, the influence may be to a different degree. For instance, consider Voter 1 and the *3*-partition (**{2}**,{3},∅). Without Voter 1, the outcome is the first alternative. Here, Voter 1 is influential and may be able to effect the second alternative or the third alternative. This is because he is the largest voter and the other voters are divided. On the other hand, consider Voter 2 and the *3*-partition (**{1}**,{3},∅). Without Voter 2, the outcome is the first alternative. Here, Voter 2 is influential but is able to change the outcome only to the second alternative, not the third. For the above set of (3, 3) games, therefore, because the Penrose measure does not make the distinction between the two different degree of influence, the resulting measure is biased against the largest voter, making him only just as influential as each of the other two smaller voters.

4. Some concluding remarks

In both examples discussed above, failure to consider pivot moves by voters originally in a losing coalition will cause the resulting measure of influence to be inappropriate. Loosely speaking, the underlying reason for the inappropriateness has to do with the fact that, in some instances, the winning coalition "wins" only because of the particular way in which the other voters are distributed over the remaining alternatives -- a *conditional* win. In the UN Security Council game, this difficulty would not arise if all permanent members and at least four other rotating members vote in the affirmative because, in this instance, the win is unconditional and how the other voters are distributed over the remaining alternatives would be largely irrelevant to the outcome. Likewise, in the set of (3, 3) weighted plurality games, if the winning coalition has a simple majority of the votes, this difficulty will also not arise as the win would be unconditional. The difficulty with Bolger's generalization highlighted in this note is not confined to the games exposited above but applies generally across (n, r) games.

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