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# The Elasticity of Substitution and Endogenous Growth

Hing-Man LEUNG
*Singapore Management University*, hmleung@smu.edu.sg

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**H. M. Leung**

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School of Economics and Social Sciences
Singapore Management University
Eu Tong Sen #02-28
469 Bukit Timah Road
Singapore 259756
Fax: 65-68220777
**E-mail: hmleung@smu.edu.sg**

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**ABSTRACT**

The endogenous growth literature focuses exclusively on Cobb-Douglas. Elasticities other than unity are ignored. A recent paper by Klump and Grandville (2000) examined other elasticities but assumed an exogenous saving rate. By contrast, this paper studies elasticity and endogenous growth. Endogeneity is important since elasticity preserves capital's productivity and encourages saving. Two models are presented. The first assumes exogenous technological change. We find elasticity to have a positive level effect on income. No rate of growth effect is found. The second model allows learning by doing from capital accumulation. In addition to the level effect, rate of growth effects are found.



Keywords: endogenous growth, elasticity of substitution, level effect, rate of growth effect

*JEL* classification: O0, O4

The elasticity of substitution between capital and labor is an important aspect of the production function. Suppose the capital stock of a country rises. The marginal productivity of capital falls more rapidly if the elasticity of substitution is small. This has a direct implication on the returns to investment, and the role of the elasticity of substitution in economic growth is thus hard to ignore. At present, our understanding of this role is scanty.

The recent literature on endogenous growth has not helped very much in this direction. The leading models, such as Romer (1985, 1990), Lucas (1988) and Aghion and Howitt (1992) have not devoted attention to substitution elasticities. We have learned a great deal about growth when the production function is Cobb-Douglas, which has unit elasticity of substitution. Little is known, however, when the latter is either less than or greater than 1. Filling this gap in our knowledge is important. There is little evidence to believe that the elasticity of substitution between capital and labor is, or is even close to, 1.[1]

Two important papers, one old and the other recent, have dealt with this important issue. The old one is Professor Robert Solow's celebrated 'Contribution' (1956). His example 3 (p.77) examined the case when the elasticity of substitution is 2. The system turned out to be highly productive. He then derived the 'threshold' – when the saving rate is sufficiently higher than the population growth rate – beyond which a balanced growth path will not exist. One gets a hint from this that a 'more elastic system' is more productive than a less elastic one. For that reason, elasticity of substitution might impact positively on growth. The recent paper is Rainer Klump and Olivier de La Grandville (2000), which showed that the elasticity of substitution has a positive level effect on per-capita income. Starting from the same income level, a country with

[1] Four decades ago, Arrow et al. (1961) introduced the constant elasticity of substitution CES form. They motivated that by a time-series analysis of all non-farm production in the United States. Their results showed "an over-all elasticity of substitution between capital and labor significantly less than unity" (p.226). Many studies since then, surveyed in Yuhn (1991, p.343), reported elasticities in the U.S. not exceeding 0.76.

higher elasticity will be richer than one with lower elasticity. In a related work, Olivier de La Grandville (1989) tried to show that elasticity also had a rate of growth effect. Neither this pair of recent papers, nor Solow's (ibid) earlier, allowed consumption and saving to be endogenously chosen by agents in the system.

The primary purpose of this paper is to study the role that the elasticity of substitution has in endogenous growth. As argued earlier, elasticity affects capital productivity and the incentive to save. Thus, treating saving as endogenous is crucial. In our first model (section I), we assume an exogenous rate of technological change. In this case, the elasticity of substitution is shown to have a positive level effect, but no rate-of-change effect. In the second model (section II), a link is inserted between capital accumulation and learning by doing. Here, the elasticity of substitution not only has a level effect, but also a rate-of-change effect on per-capita output growth.

A short concluding remark is provided in section III.

## I. A Model with Endogenous Saving and Exogenous Technological Progress

The purpose of this paper is to investigate the impact of the elasticity of substitution ($\sigma$ hereafter) on economic growth. Saving should be endogenous in this investigation. Some allowance for endogenous technological progress is important, too, but we leave that for section II. The ease with which capital is substituted for labor in production affects the marginal productivity of capital at each point in time, and the size of $\sigma$, therefore, influences the incentive to save. That, in turn, changes the rate of capital accumulation, and, perhaps, the rate of output growth. Our results confirm and enrich that of Klump and De La Grandville (2000), which was based on the assumption of an exogenous saving rate.

Imagine an economy employing capital $K$ and labor $L$, both having homogenous quality, to produce a single output $Y$. To study $\sigma$, we would obviously want to avoid the Cobb-Douglas production function. The main assumption we require is the linear homogeneity of output in the two inputs used. Technological progress is crucial in this exercise, and for this we assume the Hicks-neutral variety. Technology at time $t$ is denoted $A(t)$. We may write the production function as

$$Y\left[A(t), K(t), L(t)\right] = A(t) \cdot f[K(t), L(t)]. \tag{1}$$

Capital accumulation is the amount of per-period output not consumed

$$\dot{K}(t) = A(t) \cdot f\left[K(t), L(t)\right] - c(t)L(t). \tag{2}$$

We will equate the labor force with the population size. Both are denoted $L$.

Preference over the consumption stream is given by

$$\int_0^\infty e^{-\rho t} \cdot \frac{c(t)^{1-\alpha} - 1}{1-\alpha} L(t)dt \tag{3}$$

where $\rho$ is time preference and $1/\alpha$ is the elasticity of consumption. Per period consumption is $c(t)$. The economy is competitive and closed, in which each identical rational agent determines his per-period consumption and saving to maximize the discounted utility stream (3).

Using (1) to (3), we formulate the current-value Hamiltonian $H$ with one state variable $K(t)$, one control variable $c(t)$, and one costate (shadow price of saving) variable $\theta(t)$

$$H(K, \theta, c, t) = \frac{c^{1-\alpha} - 1}{1-\alpha} \cdot L + \theta\left[A \cdot f(K, L) - cL\right]. \tag{4}$$

By the Maximum Principle, the first order condition from the choice of consumption $c$ is

$$c^{-\alpha} = \theta. \tag{5}$$

The marginal return from consumption on the left-hand side must be equal to $\theta$ at each $t$. $\theta$ is shadow price of the capital stock. It reflects, among other things, the marginal returns from capital investment. We expect $\sigma$, which measures the degree of substitutability between capital and labor, to play a role in growth by determining $\theta$.

Again, by the Maximum Principle, the time-path for $\theta$ is

$$\dot{\theta} = \rho\theta - \theta A \cdot f_K . \qquad (6)$$

where $A \cdot f_K = A \cdot \partial f(K,L)/\partial K$ is the marginal product of capital (MPK). The transversality condition is $\lim_{t\to\infty} e^{-\rho t}\theta(t)K(t) = 0$, implying that the shadow value of the accumulated capital stock must vanish eventually. It should be clear that if a 'balanced growth path' exists, then this will be satisfied.

If MPK is large at time *t*, the capital stock will be larger in subsequent periods, and (6) requires the shadow price of capital stock $\theta$ to *fall* more rapidly at *t*. Differentiating (5) with respect to time yields $\dot{\theta}/\theta = -\alpha\left(\dot{c}/c\right)$. We are interested in the balanced path of growth. By definition along this path, the growth rate of per-capita consumption *c* and the growth rate of per-capita capital accumulation must be constant. Let the constant growth rate of a variable *z* be denoted $\dot{z}/z = g_z$. Combining $\dot{\theta}/\theta = -\alpha g_c$ with (6), we get

$$A \cdot f_K = \rho + \alpha g_c . \qquad (7)$$

Condition (7) tells us a great deal about the balanced growth path, if it exists. Along this path, the right-hand side must remain constant and so, too, must the left-hand side. For whatever magnitude of $g_c$, the economic agent's problem can be seen as choosing an investment plan such that the time-profile of the capital stock keeps the MPK $A \cdot f_K$ constant. We will now examine how the economic system solves this problem, given $\sigma$ and any constant $g_c$. Then, we will return to $g_c$ and see how its magnitude relates to $\sigma$, if it does.

Since technology progresses at an exogenous constant pace $\dot{A}/A = \mu$, (7) implies that the $f_K$ must *fall* at the same constant rate

$$\frac{\dot{f}_K}{f_K} = -\mu . \qquad (8)$$

Further, because of the linear homogeneity of *f*, $f_K$ is a function of the capital-labor ratio, denoted $r \equiv K/L$. For $f_K$ to fall, *r* must rise – capital is more abundant, so its marginal product declines. How fast *r* needs to rise in order to satisfy (8) depends critically on the substitution relations between *K* and *L*, i.e. $\sigma$.

Differentiating the partial derivative $f_K(K,L)$ with respect to time, we have

$$\dot{f}_K = \frac{df_K}{dt} = \frac{d^2 f}{dK^2}\frac{dK}{dt} + \frac{d^2 f}{dKdL}\frac{dL}{dt} = f_{KK}\dot{K} + f_{KL}\dot{L} .$$

Dividing through by $f_K$ and using (8), we get

$$\frac{\dot{f}_K}{f_K} = \frac{f_{KK}}{f_K}\dot{K} + \frac{f_{KL}}{f_K}\dot{L} = -\mu .$$

Using the familiar property of the linear homogeneity of *f*, $f_{KK} = -\frac{L}{K} f_{KL}$. Rearranging this gives

$$-L\frac{f_{KL}}{f_K}\left(\frac{\dot{K}}{K} - \frac{\dot{L}}{L}\right) = -\mu .$$

Finally, introducing $\sigma = \frac{f_K f_L}{f\, f_{KL}}$ we obtain

$$\frac{\dot{r}}{r} \equiv \frac{\dot{K}}{K} - \frac{\dot{L}}{L} = \mu\sigma\frac{f}{Lf_L} . \qquad (9)$$

Using the consumption good as numeraire, the fraction on the right-hand side $f/Lf_L$ is the inverse of labor's relative share in national income. Since $\left(Lf_L/f\right) \in [0,1]$, $\left(f/Lf_L\right) \in [1,\infty)$.

Further, for $\mu, \sigma > 0$, (9) implies $\dot{r} > 0$. We know that as *K* rises (relative to *L*), $f_L$ rises relative to $f_K$, but the share of *L* in total output is determined by whether $\sigma$ is greater than, equal to, or less than unity. This important result is attributed to the late J.R. Hicks, who stated as his third proposition in the famous chapter six, "Distribution and Economic Progress", of his *Theory of Wages* in 1932,

> "An increase in the supply of any factor will increase its relative share (i.e. its proportion of the National Dividend) if its 'elasticity of substitution' is greater than unity." (Hicks, ibid. p.117; for proof see his appendix on pp.246-247)

More concretely, it follows from this proposition that

$$\left( \frac{d\dfrac{f}{Lf_L}}{d\dfrac{K}{L}} \right) \overset{>}{\underset{<}{=}} 0 \quad \text{as} \quad \sigma \overset{>}{\underset{<}{=}} 1. \tag{10}$$

Recall that $f/Lf_L \in [1, \infty)$, and $\dot{r} > 0$ indefinitely. It follows from (10) and $\left(f/Lf_L\right) \in [1, \infty)$ that asymptotically $f/Lf_L$ must reach one of its limits, thus proving Proposition 1 below. This proposition establishes the constancy of $\dot{r}/r$ along the steady-state balanced growth path, which only exists if $0 < \sigma < 1$.

PROPOSITION 1. Assuming a positive rate of technological progress $\mu > 0$, the asymptotic capital-labor ratio is given by

(i) If $0 < \sigma < 1$, then $\lim_{t \to \infty} \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) = \mu \cdot \sigma \cdot 1 = \mu \cdot \sigma$.

(ii) If $\sigma > 1$, then $\lim_{t \to \infty} \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) = \mu \cdot \sigma \cdot \infty = \infty$.

For the economy to possess a steady-state balanced growth path, the capital-labor ratio must be a finite constant along this path. We have to conclude from Proposition 1 that such a path exists only if $\sigma \leq 1$. We may recall when Solow (1956) examined the case of CES with $\sigma = 2$, assuming an exogenous saving rate (his 'third example', pp.77-78), he found that a balanced growth path might not exist unless the saving rate is *less* than the exogenous rate of population growth. But the endogenous saving rate must be bigger when $\sigma$ is bigger. This is because easier substitution increases the marginal productivity of capital and the incentive to save. Our first contribution here is to show, in Proposition 1, that the threshold for the balanced path to exist should be expressed as $\sigma \leq 1$ in the case of exogenous technological change, $\mu$.

In addition, part (i) of Proposition 1 says that for $\sigma \leq 1$, the capital-labor ratio grows at a constant rate, which is directly proportional to $\sigma$. In other words, an economy with higher elasticity of substitution between capital and labor will experience faster growth in its capital stock. We will show shortly that this is not sufficient to guarantee a higher growth rate of per-capita output along the balanced path. Before examining this rate-of-change effect, we present a relatively straightforward result concerning the level effect of $\sigma$.

PROPOSITION 2. Assume $\sigma \leq 1$. For two economies starting from the same economic base at time $t = 0$ and both experiencing the same rate of technological change, the one with a larger elasticity of substitution between capital and labor will enjoy a higher *level* of per-capita income at every time $t > 0$ along the balanced path.

PROOF. By constant returns, the per-capita output function is $A\phi(r,1)$, $r = K/L$ and $\phi$ increases with *r*. Proposition 2 is directly implied by Proposition 1. Q.E.D.

The result of Proposition 2 differs from Klump and De La Grandville's (2000) Theorem 1 (p.285) in two respects. First, we do not assume an exogenous saving rate. This brings our results closer to the neo-classical growth literature. Second, we show that there must be technological progress if the elasticity of substitution is to make a difference to output levels along the balanced path. If $\mu = 0$, $\dot{r}/r = 0$ from Proposition 1 and $A\phi(r,1)$ is constant. Per-capita output must be constant no matter what elasticity we have. This, of course, is simply reiterating the result of Solow (ibid).

Even though the level of per-capita output is higher when substitution is more elastic, the same cannot be said about its rate of growth along the balanced path, at least not under an exogenous constant $\mu$. This is shown in the following proposition.

PROPOSITION 3. Assume $\sigma \leq 1$. Suppose an economy is described by a CES production function $Y = A\left[K^{-\lambda} + L^{-\lambda}\right]^{-\frac{1}{\lambda}}$ and it experiences a constant rate of technological change $\dot{A}/A = \mu$. Along its balanced growth path, per-capita income grows at the rate $\mu$ for all $\sigma \leq 1$.

PROOF. For the CES production function specified, $\sigma = \dfrac{1}{1+\lambda}$. We may write it in per-capita income terms $y \equiv Y/L = A\left[r^{-\lambda} + 1\right]^{-\frac{1}{\lambda}}$. Now, differentiate this with respect to time to get $\dot{y} = \dot{A}\left[r^{-\lambda} + 1\right]^{-\frac{1}{\lambda}} + A\left[r^{-\lambda} + 1\right]^{-\frac{1}{\lambda}-1} r^{-\lambda-1}\dot{r}$. Dividing through by $y$ and simplifying, we get

$$\frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \left[\frac{r^{-\lambda}}{r^{-\lambda}+1}\right] \cdot \frac{\dot{r}}{r}. \qquad (11)$$

We know from Proposition 1 that $\dot{r}/r > 0$ for all $\sigma > 0$. By L'Hôpital's rule we obtain

$$\lim_{r\to\infty}\left[\frac{r^{-\lambda}}{r^{-\lambda}+1}\right]=0 \text{ if } 0<\lambda<\infty \text{ (i.e. } 0<\sigma<1\text{)}.$$

Using this in (11) completes the proof. Q.E.D.

The two terms on the right-hand side of (11) show rather nicely the two effects acting on the per-capita income growth path. $\sigma$ changes the productivity of capital and the incentives to accumulate it, thus affecting $g_r$ in the second term. This influence ceases, however, when the balanced path is reached. That leaves $g_y$ to be determined entirely by the exogenous technological change. $\sigma$ loses its rate-of-change effect along the balanced path.

In short, the neo-classical conclusion that per-capita output growth rate is determined by the exogenous technological change rate $\mu$ carries over intact for all $\sigma \le 1$. Elasticity of substitution adds only level effects, but not rate of growth effects. A country with greater $\sigma(\le 1)$ will be richer everywhere along the balanced growth path, but she will not grow faster. This should not be very surprising. Capital that is more easily substitutable for labor is more productive. A country with such advantages will accumulate a larger capital stock, which produces larger output per-capita. However, along the balanced path this richer country will not accumulate at a faster rate, because the larger capital stock leads to a lower marginal product of capital that counters any additional incentive to invest. The absolute amount saved and invested at each period is larger for the richer country, but the rate of saving and growth will not be different between countries with different $\sigma$. Much of this, as we will show next, depends on the assumption of exogenous $\mu$.

## II. Learning by Doing

The purpose of the present section is to show that the conclusions of the last section are substantially altered when the rate of technological change is endogenously determined. In particular, the elasticity of factor substitution $\sigma$ will be shown not only to have level effects but also rate of growth effects. There is more than one way to introduce endogenous technological change, and probably just as many ways to introduce the rate of growth effects from $\sigma$. Our focus on the learning by doing model is largely a strategic one. The main purpose here is to show that $\sigma$ can make a difference to long-term growth rates. It turns out that the intuition may be exposed most clearly in a simple model of learning by doing.

Adopt the same production function as in section I. This is $A(t)f\left[K(t),L(t)\right]$ where $f[\cdot]$ is linearly homogenous in $K$ and $L$. Following Arrow (1962), we link the accumulation of experience, or learning, to the process of capital accumulation. The stock of experience is captured by $A(t)$. Suppose its rate of change $\dot{A}(t)/A(t)$ is linked to the rate of capital accumulation $\dot{K}(t)/K(t)$. For notation, write this as $g_A = \phi\left(g_K\right)$.

To begin with, we assume strict concavity of the function $\phi\left(g_K\right)$. There are two justifications for this assumption. The first is that learning is bounded [Young (1993)]. Second and more importantly, $\phi\left(g_K\right)$ links the speed of learning to the speed of information arrival, which is represented by the rate of capital accumulation. Learning takes time. When anyone performs a task, there is probably some optimal speed at which information arrives for it to be properly studied, digested and absorbed. Nelson and Phelps (1966) argued that on the one hand, the environment has to be changing to provide the stimulus for learning. On the other hand, Atkinson and Stiglitz (1969) and more recently Basu and Weil (1998) pointed out that an excessively rapid pace of technological change may make learning difficult. Assumption (12) below requires capital accumulation within an appropriate range of speed for learning to take place. This will be relaxed later in our investigation.

ASSUMPTION. $$g_A = \phi(g_K), \quad \phi(0) = \phi(\bar{g}_K) = 0 \quad \text{for} \quad 0 < \bar{g}_K < \infty; \\ \phi' > 0, \quad \phi'(0) = \infty, \quad \text{and} \quad \phi'' < 0. \tag{12}$$

Assumption (12) gives rise to a learning-investment curve with the shape of a parabolic arc as shown in Figure 1. Learning takes place when $0 < g_K < \bar{g}_K$. The condition $\phi'(0) = \infty$ means that the marginal benefit to learning is large as the company begins to learn. We include this to ensure the existence of equilibrium. The strict concavity of $\phi$ implies the existence of $\phi(\hat{g}_K) = \max \phi(g_K)$.

The economy described by equations (3) to (10) remains unchanged, except that the exogenous $\mu$ is now replaced by $\phi(g_K)$. The capital-labor ratio $r$ along the steady-state balanced path behaves quite differently compared to that of Proposition 1 above. Assume, for simplicity, $g_L = 0$. This assumption implies $g_r = g_K$, but obviously none of our results will depend on this. We focus on $0 < \sigma \le 1$. The case of $\sigma > 1$ is discussed later in this section.

PROPOSITION 4. Assuming (12), and $0 < \sigma \le 1$, capital-labor ratio along the balanced path (asterisked) is characterized by

$$\lim_{t \to \infty} \left( \frac{\dot{K}}{K} - \frac{\dot{L}}{L} \right) = g_K^*(\sigma), \quad 0 < g_K^*(\sigma) \le \bar{g}_K \quad \text{and} \quad \frac{dg_K^*(\sigma)}{d\sigma} > 0.$$

PROOF. Follow the argument we used in the proof of Proposition 1. The equilibrium path is given by (9). Using (10) , we know for $0 < \sigma \le 1$

$$\lim_{t\to\infty}\frac{\dot{K}}{K} \equiv g_K^* = \phi\left(g_K^*\right)\cdot\sigma .$$

Thus $\sigma = g_K^* \big/ \phi\left(g_K^*\right)$, which is the inverse of the slope of the rays from origin to $\phi$. At $\sigma = 1$, the slope of the ray is 1, and by the Inada condition (12) $K$ must grow at a rate $g_K^*\big|_{\sigma=1} < \overline{g}_K$ (see Figure 1). As $\sigma$ falls below 1 towards 0, the ray becomes steeper, tracing out a monotonic decline in $g_K^*$. Q.E.D.

Proposition 4(i) is in some ways quite similar to Proposition 1(i). Easier substitution between factors encourages capital to accumulate, which is accomplished by slowing down the fall of the marginal product of capital. Learning introduces a feedback loop that affects income and in turn regulates $g_K$. The shape of the learning curve assumed in (12) implies that if $\sigma = 1$, equilibrium occurs on the falling portion of this curve (see Figure 1). The Inada assumption $\phi'(0) = \infty$ implies that maximum learning speed is achieved at some $\hat{\sigma} < 1$.

We finally come to the rate-of-change effect of $\sigma$. Again, we deploy the CES production function.

PROPOSITION 5. Assume $0 < \sigma \le 1$ and (12). Suppose an economy is described by the CES production function $Y = A\left[K^{-\lambda} + L^{-\lambda}\right]^{-\frac{1}{\lambda}}$. Then

(i) a balanced growth path exists;

(ii) there exists a unique $0 < \hat{\sigma} < 1$ such that per-capita income grows at a maximum rate $g_Y^*(\hat{\sigma}) = \max\left[g_Y^*(\sigma)\right]$. The comparative statics between $\sigma$ and the per-capita income growth rate along the balanced path is given by

$$\begin{cases} \text{for } 0 < \sigma < \hat{\sigma}, & \partial g_Y^* / \partial \sigma > 0 \\ \text{for } \hat{\sigma} < \sigma < 1, & \partial g_Y^* / \partial \sigma < 0. \end{cases}$$

PROOF. Using the same procedure as in Proposition 3 yields equation (11): $g_y \equiv \frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \left[ \frac{r^{-\lambda}}{r^{-\lambda}+1} \right] \cdot \frac{\dot{r}}{r}$. By L'Hôpital's rule, $\lim_{r \to \infty} \left[ \frac{r^{-\lambda}}{r^{-\lambda}+1} \right] = 0$ for $0 \le \lambda < \infty \left( \text{i.e. } 0 < \sigma \le 1 \right)$.

Hence, $g_y^* = g_A^* > 0$. Using Proposition 4, we have $g_y^* = \phi(\sigma)$. This proves (i) of the proposition.

Since the maximum learning speed is given by $\phi(\hat{\sigma})$, $g_Y^*(\hat{\sigma}) = \max\left[ g_Y^*(\sigma) \right]$ of (ii) follows immediately. It follows from Proposition 4 that

$$\begin{cases} \text{for } 0 < \sigma < \hat{\sigma}, & \partial g_Y^* / \partial \sigma > 0 \\ \text{for } \hat{\sigma} < \sigma < 1, & \partial g_Y^* / \partial \sigma < 0. \end{cases} \qquad \text{Q.E.D.}$$

Again, the two terms on the right-hand side of (11) show the two effects of $\sigma$ on the growth rate. $\sigma$ directly affects the second effect $\left[ \frac{r^{-\lambda}}{r^{-\lambda}+1} \right] \cdot \frac{\dot{r}}{r}$ by changing the incentive to accumulate capital. As in the case of exogenous $\mu$ this effect vanishes along the balanced path. But now $\sigma$ retains an indirect effect, via $\phi(g_K)$, on $g_A$ along the balanced path in the long run.

Finally, we take a quick look at the remaining case $\sigma > 1$.

PROPOSITION 6. Assume (12), and $\sigma > 1$. Then

(i) $\lim_{t \to \infty} g_K = 0$;

(ii) $g_y^* = 0$.

PROOF. Follow the argument we used in the proof of Proposition 4. From (9) we have $g_K = \phi(g_K)\sigma \frac{f}{Lf_L}$. Using (10), noting $g_K > 0$ and $\sigma > 1$, $\frac{f}{Lf_L}$ would rise indefinitely. But this cannot be. Once $g_K$ reaches $\overline{g}_K$, $g_K = \phi(\overline{g}_K) = 0$. So the only situation that can satisfy $g_K = \phi(g_K)\sigma \frac{f}{Lf_L}$ is $g_K = 0$ (the origin in Figure 1). This proves (i). $g_y^* = 0$ follows immediately from $g_K = \phi(\overline{g}_K) = 0$. Q.E.D.

Proposition 6 is a somewhat extreme result. It says that an economy accumulating capital "too fast" would miss learning opportunities entirely and degenerate to total stagnation. Although in the model this occurs when $\sigma > 1$, it is directly caused by assumption (12) and in particular $\phi(\overline{g}_K) = 0$. We will now alter this assumption in order to test the robustness of Propositions 5 and 6.

Assumption (12) contains two elements. We will retain the boundedness of learning, but relax $\phi(\overline{g}_K) = 0$. See also Figure 2.

ASSUMPTION.

$$g_A = \phi(g_K), \quad \phi(0) = 0, \quad \lim_{g_K \to \infty} \phi(g_K) = \Omega \quad 0 < \Omega < \infty; \\ \phi' > 0, \quad \phi'(0) = \infty, \quad \text{and} \quad \phi'' < 0. \tag{13}$$

The important change in our results concerns with the comparative statics of $\sigma$. Under assumption (12) the balanced growth rate rises and then falls with $\sigma$. Under (13), growth rate rises monotonically with $\sigma$.

PROPOSITION 5'. Assume $0<\sigma\le 1$, and the CES function $Y=A\left[K^{-\lambda}+L^{-\lambda}\right]^{-\frac{1}{\lambda}}$. Adopt assumption (13). Then

(i) a balanced growth path exists;

(ii) $\partial g_Y^*/\partial\sigma\ge 0$ ; $g_Y^*(1)=\max\left[g_Y^*(\sigma)\right]$.

PROOF. Using the same procedure as in Proposition 3 and 5 yields $g_y^*=g_A^*>0$. Using assumption (13), we have $0<g_y^*\le\Omega$. This proves (i) of the proposition. Part (ii) follows from inspection of Figure 2. Q.E.D.

Proposition 6 has to be revised also. Under assumption (12), the output growth rate falls to zero and the system is stagnant. Under (13), the system becomes highly productive. A constant-rate balanced growth path again fails to exist because the growth rate will become infinitely high.

PROPOSITION 6'. Assume (13), and $\sigma>1$. Then

(i) $\lim_{t\to\infty} g_K=\infty$;

(ii) $g_y^*=\infty$.

PROOF. From (9) we have $g_K=\phi(g_K)\sigma\frac{f}{Lf_L}$. Using (10), noting $g_K>0$ and $\sigma>1$, $\frac{f}{Lf_L}$ rises indefinitely. $\lim_{g_K\to\infty}\phi(g_K)=\Omega$, $\sigma>1$ so (i) follows immediately.

Again using the CES function $g_y \equiv \frac{\dot{y}}{y} = \frac{\dot{A}}{A} + \left[\frac{r^{-\lambda}}{r^{-\lambda}+1}\right] \cdot \frac{\dot{r}}{r}$. By L'Hôpital's rule, $\lim_{r\to\infty}\left[\frac{r^{-\lambda}}{r^{-\lambda}+1}\right] = 1$ for $-1 \leq \lambda < 0 \left(\text{i.e. } \sigma > 1\right)$. Hence $\left[\frac{r^{-\lambda}}{r^{-\lambda}+1}\right] \cdot \frac{\dot{r}}{r} = \infty$ using (i). This proves (ii) of the proposition.

Q.E.D.

We have to conclude that a balanced growth path, in general, may not exist in the case of $\sigma > 1$.

## III. Concluding Remarks

In the course of our investigation, two different effects of the elasticity of substitution are found. The first is a direct one. A higher elasticity slows down the fall of the marginal product of capital, which encourages capital accumulation and saving. Because the added incentive from elasticity is counteracted by the resulting larger stock of capital, the direct effect influences only income levels but not the balanced growth rate. The second effect is an indirect one. We assume capital accumulation to be related to learning by doing. More specifically, the speed of capital accumulation may raise or lower the speed of learning. The speed of learning, in turn, affects the rate of technological change. In this way, elasticity maintains a rate-of-change effect on growth.

In our inquiry, we have paid particular attention to the elasticity of substitution being less than 1. The case of elasticity greater than 1 generally led to the non-existence of a balanced path. There is some indication in the literature that the elasticity is empirically more likely to be less than 1 (see footnote 1 above). It is hard to believe, however, that a country with substitution elasticity greater than one will necessarily be heading for an explosive growth or stagnation. More work is perhaps needed in this direction.

Again, for future research directions, it would be rewarding to investigate empirically the relations between elasticity, income levels, and income growth rates. Our results suggest such relations may not be monotonic. Furthermore, only learning by doing is investigated here. The relation between elasticity and human capital growth, for instance, seems an interesting topic for future research.

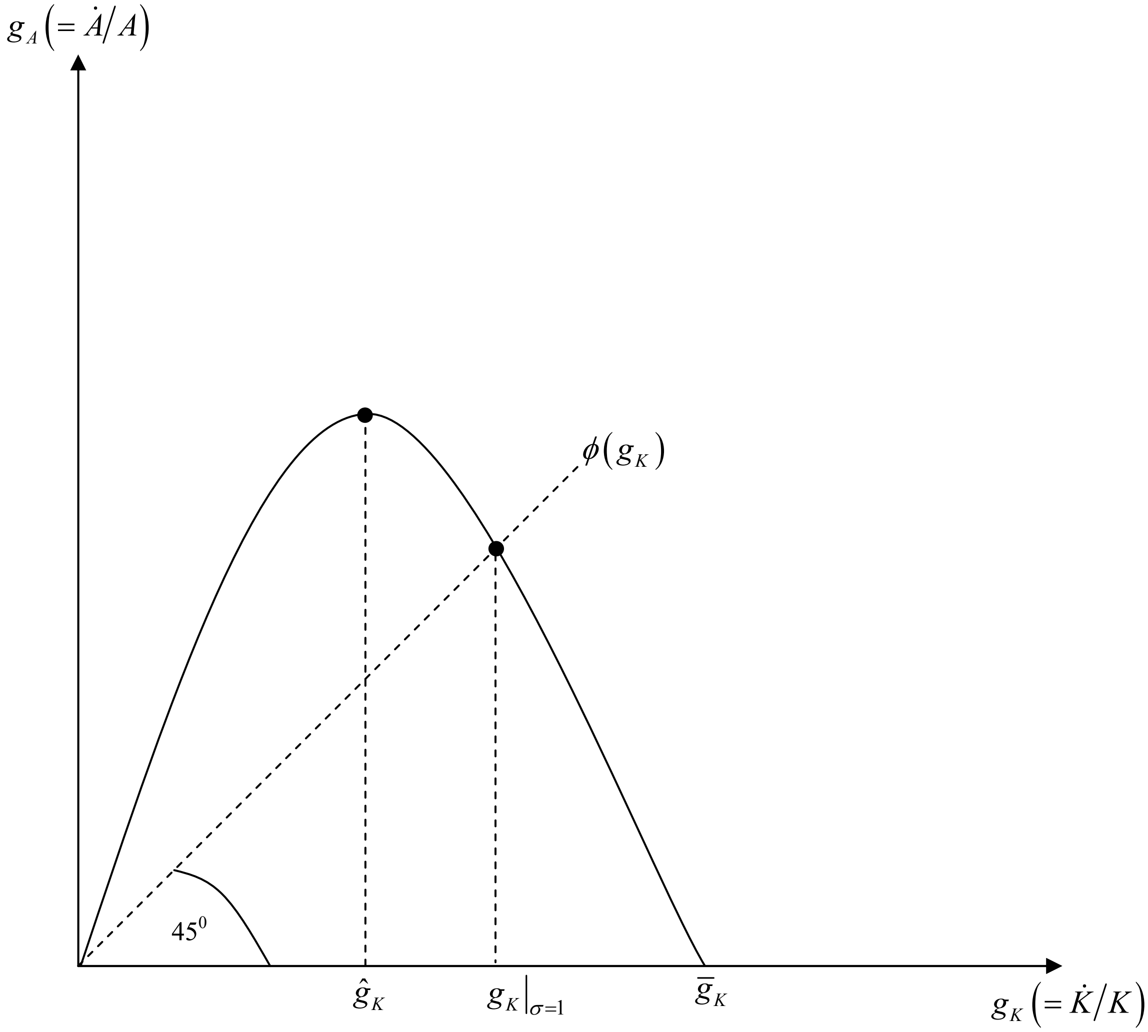


Figure 1. The learning-investment curve and growth according to assumption (12).

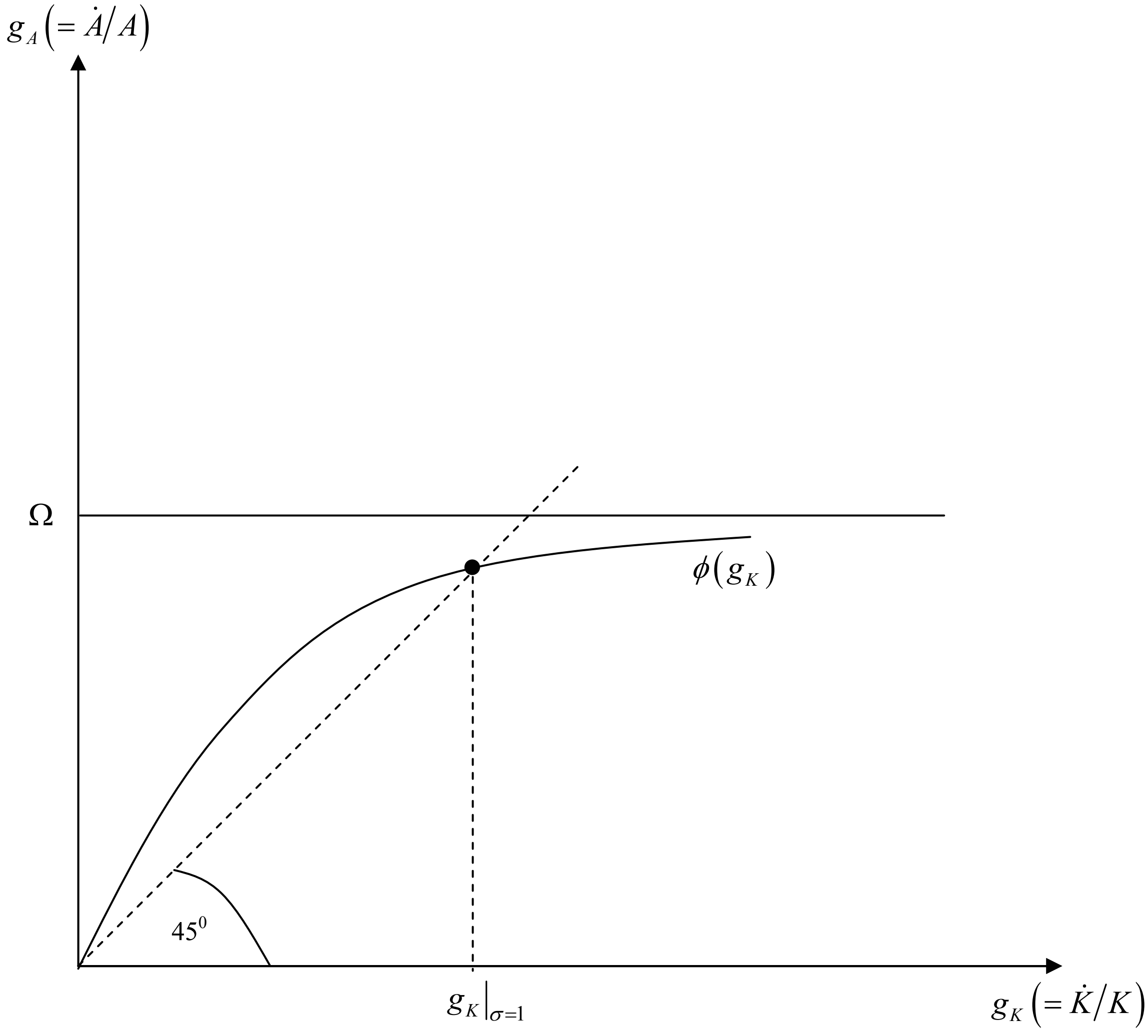


Figure 2. The learning-investment curve and growth according to assumption (13).

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