Separators in Region Intersection Graphs

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Abstract

For undirected graphs $G = (V, E)$ and $G_0 = (V_0, E_0)$, say that G is a *region intersection graph over* G_0 if there is a family of connected subsets $\{R_u \subseteq V_0 : u \in V\}$ of G_0 such that $\{u, v\} \in E \iff R_u \cap R_v \neq \emptyset$.

We show if G_0 excludes the complete graph K_h as a minor for some $h \geq 1$, then every region intersection graph G over G_0 with m edges has a balanced separator with at most $c_h\sqrt{m}$ nodes, where c_h is a constant depending only on h. If G additionally has uniformly bounded vertex degrees, then such a separator is found by spectral partitioning.

A string graph is the intersection graph of continuous arcs in the plane. String graphs are precisely region intersection graphs over planar graphs. Thus the preceding result implies that every string graph with m edges has a balanced separator of size $O(\sqrt{m})$. This bound is optimal, as it generalizes the planar separator theorem. It confirms a conjecture of Fox and Pach (2010), and improves over the $O(\sqrt{m}\log m)$ bound of Matoušek (2013).

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1 Introduction

Consider an undirected graph $G_0 = (V_0, E_0)$. A graph $G = (V, E)$ is said to be a *region intersection graph (rig) over* G_0 if the vertices of G correspond to connected subsets of G_0 and there is an edge between two vertices of G precisely when those subsets intersect. Concretely, there is a family of connected subsets $\{R_u \subseteq V_0 : u \in V\}$ such that $\{u, v\} \in E \iff R_u \cap R_v \neq \emptyset$. For succinctness, we will often refer to G as a *rig over* G_0.

Let $\mathsf{rig}(G_0)$ denote the family of all finite rigs over G_0. Prominent examples of such graphs include the intersection graphs of pathwise-connected regions on a surface (which are intersection graphs over graphs that can be drawn on that surface).

For instance, *string graphs* are the intersection graphs of continuous arcs in the plane. It is easy to see that every finite string graph G is a rig over some planar graph: By a simple compactness argument, we may assume that every two strings intersect a finite number of times. Now consider the planar graph G_0 whose vertices lie at the intersection points of strings and with edges between two vertices that are adjacent on a string (see Figure 1). Then $G \in \mathsf{rig}(G_0)$. It is not too difficult to see that the converse is also true; see Section 4.

To illustrate the non-trivial nature of such objects, we recall that there are string graphs on n strings that require $2^{\Omega(n)}$ intersections in any such representation [14]. The recognition problem for string graphs is NP-hard [13]. Decidability of the recognition problem was established in [25], and membership in NP was proved in [24]. We refer to the recent survey [22] for more of the background and history behind string graphs.

Even when G_0 is planar, the rigs over G_0 can be dense: Every complete graph is a rig over some planar graph (in particular, every complete graph is a string graph). It has been

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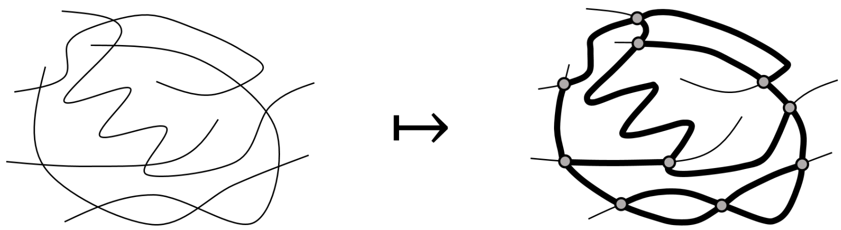

Figure 1 A string graph as a rig over a planar graph.

conjectured by Fox and Pach [5] that every m-edge string graph has a balanced separator with $O(\sqrt{m})$ nodes. Fox and Pach proved that such graphs have separators of size $O(m^{3/4}\sqrt{\log m})$ and presented a number of applications of their separator theorem. Matoušek [21] obtained a near-optimal bound of $O(\sqrt{m}\log m)$. In the present work, we confirm the conjecture of Fox and Pach, and generalize the result to include all rigs over graphs that exclude a fixed minor. This extended abstract contains mostly theorem statements; for detailed proofs and further arguments, we refer to the full paper [15].

▶ **Theorem 1.** *If $G \in \mathsf{rig}(G_0)$ and G_0 excludes K_h as a minor, then G has a $\frac{2}{3}$-balanced separator of size at most $c_h\sqrt{m}$ where m is the number of edges in G. Moreover, one has the estimate $c_h \leq O(h^3\sqrt{\log h})$.*

In the preceding statement, an ϵ-balanced separator of $G = (V, E)$ is a subset $S \subseteq V$ such that in the induced graph $G[V \setminus S]$, every connected component contains at most $\epsilon|V|$ vertices.

The proof of Theorem 1 is constructive, as it is based on solving and rounding a linear program; it yields a polynomial-time algorithm for constructing the claimed separator. In the case when there is a bound on the maximum degree of G, one can use the well-known spectral bisection algorithm (see Section 1.5).

Since planar graphs exclude K_5 as a minor, Theorem 1 implies that m-edge string graphs have $O(\sqrt{m})$-node balanced separators. Since the graphs that can be drawn on any compact surface of genus g exclude a K_h minor for $h \leq O(\sqrt{g+1})$, Theorem 1 also applies to string graphs over any fixed compact surface.

In addition, it implies the Alon-Seymour-Thomas [1] separator theorem[1] for graphs excluding a fixed minor, for the following reason. Let us define the *subdivision of a graph G* to be the graph $\dot{G}$ obtained by subdividing every edge of G into a path of length two. Then every graph G is a rig over $\dot{G}$, and it is not hard to see that for $h \geq 1$, G has a K_h minor if and only if $\dot{G}$ has a K_h minor.

1.1 Applications in topological graph theory

We mention two applications of Theorem 1 in graph theory. In [6], the authors present some applications of separator theorems for string graphs. In two cases, the tight bound for separators leads to tight bounds for other problems. The next two theorems confirm conjectures of Fox and Pach; as proved in [6], they follow from Theorem 1. Both results are tight up to a constant factor.

[1] Note that Theorem 1 is quantitatively weaker in the sense that [1] shows the existence of separators with $O(h^{3/2}\sqrt{n})$ vertices. Since every K_h-minor-free graph has at most $O(nh\sqrt{\log h})$ edges [12, 27], our bound is $O(h^{7/2}(\log h)^{3/4}\sqrt{n})$.

▶ **Theorem 2.** *There is a constant $c > 0$ such that for every $t \geq 1$, it holds that every $K_{t,t}$-free string graph on n vertices has at most $cnt(\log t)$ edges.*

A *topological graph* is a graph drawn in the plane so that its vertices are represented by points and its edges by curves connecting the corresponding pairs of points.

▶ **Theorem 3.** *In every topological graph with n vertices and $m \geq 4n$ edges, there are two disjoint sets, each of cardinality*

$$\Omega\left(\frac{m^2}{n^2 \log \frac{n}{m}}\right) \tag{1}$$

so that every edge in one set crosses all edges in the other.

This improves over the bound of $\Omega\left(\frac{m^2}{n^2(\log \frac{n}{m})^c}\right)$ for some $c > 0$ proved in [7], where the authors also show that the bound (1) is tight. Before we conclude this section, let us justify the observation made earlier.

▶ **Lemma 4.** *Finite string graphs are precisely finite region intersection graphs over planar graphs.*

Proof. We have already argued that string graphs are planar rigs. Consider now a planar graph $G_0 = (V_0, E_0)$ and a finite graph $G = (V, E)$ such that $G \in \mathsf{rig}(G_0)$. Let $\{R_u \subseteq V_0 : u \in V\}$ be a representation of G as a rig over G_0.

Since G is finite, we may assume that each region R_u is finite. To see this, for $v \in V_0$, let its *type* be the set $T(v) = \{u \in V : v \in R_u\}$. Then since G is finite, there are only finitely many types. For any region $R_u \subseteq V_0$, let $\tilde{R}_u$ be a finite set of vertices that exhausts every type in R_u, and let $\hat{R}_u$ be a finite spanning tree of $\tilde{R}_u$ in the induced graph $G_0[R_u]$. Then the regions $\{\hat{R}_u : u \in V\}$ are finite and connected, and also form a representation of G as a rig over G_0.

When each region R_u is finite, we may assume also that G_0 is finite. Now take a planar drawing of G_0 in $\mathbb{R}^2$ where the edges of G_0 are drawn as continuous arcs, and for every $u \in V$, let $T_u \subseteq \mathbb{R}^2$ be the drawing of the spanning tree of R_u. Each T_u can be represented by a string (simply trace the tree using an in-order traversal that begins and ends at some fixed node), and thus G is a string graph. ◀

1.2 Balanced separators and extremal spread

Since complete graphs are string graphs, we do not have access to topological methods based on the exclusion of minors. Instead, we highlight a more delicate structural theory. The following fact is an exercise.

Fact: If $\dot{G}$ is a string graph, then G is planar.

More generally, we recall that H *is a minor of* G if H can be obtained from G by a sequence of edge contractions, edge deletions, and vertex deletions. If H can be obtained using only edge contractions and vertex deletions, we say that H *is a strict minor of* G.

▶ **Lemma 5.** *If $G \in \mathsf{rig}(G_0)$ and $\dot{H}$ is a strict minor of G, then H is a minor of G_0.*

This topological structure of (forbidden) strict minors in G interacts nicely with "conformal geometry" on G. Consider the family of all pseudo-metric spaces that arise from a finite graph G by assigning non-negative lengths to its edges and taking the induced shortest path distance. Certainly if we add an edge to G, the family of such spaces can only grow (since by giving the edge length equal to the diameter of the space, we effectively remove it from consideration). In particular, if $G = K_n$ is the complete graph on n vertices, then every n-point metric space is a path metric on G.

A significant tool will be the study of extremal conformal metrics on a graph G. Unlike in the edge-weighted case, the family of path distances coming from conformal metrics can be well-behaved even if G contains arbitrarily large complete graph minors. As a simple example, let $K_{\mathbb{N}}$ denote the complete graph on countably many vertices. Then every distance arising from a conformal metric on $K_{\mathbb{N}}$ is bi-Lipschitz to an ultrametric.

1.3 Vertex expansion and observable spread

Fix a graph $G = (V_G, E_G) \in \mathsf{rig}(G_0)$ with $n = |V_G|$ and $m = |E_G|$. Since the family $\mathsf{rig}(G_0)$ is closed under taking induced subgraphs, a standard reduction allows us to focus on finding a subset $U \subseteq V_G$ with small isoperimetric ratio: $\frac{|\partial U|}{|U|} \lesssim \frac{\sqrt{m}}{n}$, where

$$\partial U = \{v \in U : E_G(v, V_G \setminus U) \neq \emptyset\},$$

and $E_G(v, V_G \setminus U)$ is the set of edges between v and vertices outside U. Also define the interior $U^\circ = U \setminus \partial U$.

Let us define the *vertex expansion constant* of G as

$$\phi_G = \min\left\{\frac{|\partial U|}{|U|} : \emptyset \neq U \subseteq V_G, |U^\circ| \leq \frac{|V_G|}{2}\right\}. \tag{2}$$

In [4], it is shown that this quantity is related to the concentration function (in the sense of Lévy and Milman; see also Gromov's observable diameter [8]) of extremal conformal metrics on G .

For a finite metric space (X, dist), we define the *spread of* X as the quantity

$$\mathfrak{s}(X, \mathrm{dist}) = \frac{1}{|X|^2} \sum_{x,y \in X} \mathrm{dist}(x, y).$$

Define the *observable spread of* X by

$$\mathfrak{s}_{\mathrm{obs}}(X, \mathrm{dist}) = \sup_{f: X \to \mathbb{R}} \left\{\frac{1}{|X|^2} \sum_{x,y \in X} |f(x) - f(y)| : f \text{ is 1-Lipschitz}\right\}. \tag{3}$$

▶ Remark. We remark on the terminology: In general, it is difficult to "view" a large metric space all at once; this holds both conceptually and from an algorithmic standpoint. If one thinks of Lipschitz maps $f : X \to \mathbb{R}$ as "observations" then the observable spread captures how much of the spread can be "seen."

We then define the L^1*-extremal observable spread of* G as

$$\bar{\mathfrak{s}}_{\mathrm{obs}}(G) = \sup_{\omega: V_G \to \mathbb{R}_+} \left\{\mathfrak{s}_{\mathrm{obs}}(V_G, \mathrm{dist}_\omega) : \|\omega\|_{L^1(V_G)} \leq 1\right\}, \tag{4}$$

where $\|\omega\|_{L^1(V_G)} := \frac{1}{|V_G|} \sum_{v \in V_G} \omega(v)$. We recall the following theorem from [4] that relates expansion to the observable spread.

▶ **Theorem 6** ([4]). *For every finite graph* G*,*

$$\tfrac{1}{2}\bar{\mathfrak{s}}_{\mathrm{obs}}(G) \leq \frac{1}{\phi_G} \leq 3\bar{\mathfrak{s}}_{\mathrm{obs}}(G).$$

▶ **Example 7.** If G is the subgraph of the lattice $\mathbb{Z}^d$ on the vertex set $\{0, 1, \ldots, L\}^d$, then $\phi_G \asymp 1/L$ and $\bar{\mathfrak{s}}(G) \asymp L$. This can be achieved by taking $\omega \equiv 1$ and defining $f : V_G \to \mathbb{R}$ by $f(x) = x_1$.

In light of Theorem 6, to prove Theorem 1, it suffices to give a lower bound on $\bar{\mathfrak{s}}_{\mathrm{obs}}(G)$. It is natural to compare this quantity to the L^1-*extremal spread of* G:

$$\bar{\mathfrak{s}}(G) := \max \left\{ \frac{1}{|V_G|^2} \sum_{u,v \in V_G} \mathrm{dist}_\omega(u, v) : \|\omega\|_{L^1(V_G)} \le 1 \right\} . \tag{5}$$

Let us examine these two notions for planar graphs using the theory of circle packings.

▶ **Example 8** (Circle packings)**.** Suppose that G is a finite planar graph. The Koebe-Andreev-Thurston circle packing theorem asserts that G is the tangency graph of a family $\{D_v : v \in V_G\}$ of circles on the unit sphere $\mathbb{S}^2 \subseteq \mathbb{R}^3$. Let $\{c_v : v \in V_G\} \subseteq \mathbb{S}^2$ and $\{r_v > 0 : v \in V_G\}$ be the centers and radii of the circles, respectively. An argument of Spielman and Teng [26] (see also Hersch [9] for the analogous result for conformal mappings) shows that one can take $\sum_{v \in V_G} c_v = \mathbf{0}$.

If we define $\omega(v) = r_v$ for $v \in V_G$, then $\mathrm{dist}_\omega \ge \mathrm{dist}_{\mathbb{S}^2} \ge \mathrm{dist}_{\mathbb{R}^3}$ on the centers $\{c_v : v \in V_G\}$. (The latter two distances are the geodesic distance on $\mathbb{S}^2$ and the Euclidean distance on $\mathbb{R}^3$, respectively).

Using the fact that $\sum_{v \in V_G} c_v = \mathbf{0}$, we have

$$\sum_{u,v \in V_G} \|c_u - c_v\|_2^2 = 2n \sum_{u \in V_G} \|c_v\|^2 = 2n^2 . \tag{6}$$

This yields

$$\sum_{u,v \in V_G} \mathrm{dist}_\omega(u, v) \ge \sum_{u,v \in V_G} \|c_u - c_v\| \ge \frac{n^2}{2} .$$

Moreover,

$$\|\omega\|_{L^1(V_G)} \le \|\omega\|_{L^2(V_G)} = \sqrt{\frac{1}{n} \sum_{v \in V_G} r_v^2} \le \sqrt{\frac{\mathrm{vol}(\mathbb{S}^2)}{\pi n}} = \sqrt{\frac{4}{n}} .$$

It follows that $\bar{\mathfrak{s}}(G) \ge \frac{\sqrt{n}}{4}$.

Observe that the three coordinate projections $\mathbb{R}^3 \to \mathbb{R}$ are all Lipschitz with respect to dist_ω, and one of them contributes at least a $1/3$ fraction to the sum (6). We conclude that $\bar{\mathfrak{s}}_{\mathrm{obs}}(G) \ge \frac{\sqrt{n}}{12}$. Combined with Theorem 6, this yields a proof of the Lipton-Tarjan separator theorem [18]. Similar proofs of the separator theorem based on circle packings are known (see [23]), and this one is not new (certainly it was known to the authors of [26]).

We will prove Theorem 1 in two steps: By first giving a lower bound $\bar{\mathfrak{s}}(G) \gtrsim n/\sqrt{m}$ and then establishing $\bar{\mathfrak{s}}_{\mathrm{obs}}(G) \gtrsim \bar{\mathfrak{s}}(G)$.

For the first step, we follow [21, 4, 2]. The optimization (5) is a linear program, and the dual optimization is a maximum multi-flow problem in G. Matoušek shows that a low-congestion multi-flow can be used to draw the complete graph in the plane with few edge crossings. Since this is impossible by a simple double-counting argument, one concludes that there is no low-congestion flow, providing a lower bound on $\bar{\mathfrak{s}}(G)$ via LP duality. We extend this argument to rigs over K_h-minor-free graphs using the flow crossing framework of [2].

1.4 Spread vs. observable spread

Our major departure from [21] comes in the second step: Rounding a fractional separator to an integral separator by establishing that $\bar{\mathfrak{s}}_{\mathrm{obs}}(G) \geq C_h \cdot \bar{\mathfrak{s}}(G)$ when G is a rig over a K_h-minor-free graph. Matoušek used the following result that holds for any metric space. It follows easily from the methods of [3] or [17] (see also [20, Ch. 15]).

▶ **Theorem 9.** *For any finite metric space (X, d) with $|X| \geq 2$, it holds that*

$$\mathfrak{s}_{\mathrm{obs}}(X, d) \geq \frac{\mathfrak{s}(X, d)}{O(\log |X|)} .$$

In particular, for any graph G on $n \geq 2$ vertices,

$$\bar{\mathfrak{s}}_{\mathrm{obs}}(G) \geq \frac{\bar{\mathfrak{s}}(G)}{O(\log n)} .$$

Instead of using the preceding result, we employ the graph partitioning method of Klein, Plotkin, and Rao [11]. Those authors present an iterative process for repeatedly partitioning a metric graph G until the diameter of the remaining components is bounded. If the partitioning process fails, they construct a K_h minor in G.

Since rigs over K_h-minor-free graphs do not necessarily exclude any minors, we need to construct a different sort of forbidden structure. This is the role that Lemma 5 plays in [15]. In order for the argument to work, it is essential that we construct induced partitions: We remove a subset of the vertices which induces a partitioning of the remainder into connected components. After constructing a suitable random partition of G, standard methods from metric embedding theory allow us to conclude

1.5 Eigenvalues and L^2-extremal spread

The methods presented here can be used to control eigenvalues of the discrete Laplacian on rigs. Consider the linear space $\mathbb{R}^{V_G} = \{f : V_G \to \mathbb{R}\}$. Let $\mathcal{L}_G : \mathbb{R}^{V_G} \to \mathbb{R}^{V_G}$ be the symmetric, positive semi-definite linear operator given by

$$\mathcal{L}_G f(v) = \sum_{u:\{u,v\}\in E_G} (f(v) - f(u)) .$$

Let $0 = \lambda_0(G) \leq \lambda_1(G) \leq \cdots \leq \lambda_{|V_G|-1}(G)$ denote the spectrum of $\mathcal{L}_G$.

Define the *L^p-extremal spread* of G as

$$\bar{\mathfrak{s}}_p(G) = \max_{\omega:V_G\to\mathbb{R}_+} \left\{ \frac{1}{|V_G|^2} \sum_{u,v\in V_G} \mathrm{dist}_\omega(u, v) : \|\omega\|_{L^p(V_G)} \leq 1 \right\} . \tag{7}$$

In [2], the L^2-extremal spread is used to give upper bounds on the first non-trivial eigenvalue of graphs that exclude a fixed minor. In [10], a stronger property of conformal metrics is used to bound the higher eigenvalues as well. Roughly speaking, to control the kth eigenvalue, one requires a conformal metric $\omega : V_G \to \mathbb{R}_+$ such that the spread on every subset of size $\geq |V_G|/k$ remains large. Combining their main theorems with our methods yields the following.

▶ **Theorem 10.** *Suppose that $G \in \mathsf{rig}(G_0)$ and G_0 excludes K_h as a minor for some $h \geq 3$. If $d_{\max}$ is the maximum degree of G, then for any $k = 1, 2, \ldots, |V_G| - 1$, it holds that*

$$\lambda_k(G) \leq O(d_{\max}^2 h^6 \log h) \frac{k}{|V_G|} .$$

In particular, the bound on $\lambda_1(G)$ shows that if $d_{\max}(G) \leq O(1)$, then recursive spectral partitioning (see [26]) finds an $O(\sqrt{n})$-vertex balanced separator in G.

1.6 Additional applications

Treewidth approximations. Bounding $\bar{\mathfrak{s}}_{\mathrm{obs}}(G)$ for rigs over K_h-minor-free graphs leads to some additional applications. Combined with the rounding algorithm implicit in Theorem 6 (and explicit in [4]), this yields an $O(h^2)$-approximation algorithms for the vertex uniform Sparsest Cut problem. In particular, it follows that if $G \in \mathsf{rig}(G_0)$ and G_0 excludes K_h as a minor, then there is a polynomial-time algorithm that constructs a tree decomposition of G with treewidth $O(h^2 \mathrm{tw}(G))$, where $\mathrm{tw}(G)$ is the treewidth of G. This result appears new even for string graphs. We refer to [4].

Lipschitz extension. Our results on padded decomposability of conformal metrics on string graphs combine with the Lipschitz extension theory of [16] to show the following. Suppose that (G, ω) is a conformal graph, where G is a rig over some K_h-minor free graph. Then for every Banach space Z, subset $S \subseteq V_G$, and L-Lipschitz mapping $f : S \to Z$, there is an $O(h^2 L)$-Lipschitz extension $\tilde{f} : V_G \to Z$ with $\tilde{f}|_S = f$. See [19] for applications to flow and cut sparsifiers in such graphs.

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