On Corecursive Algebras for Functors Preserving Coproducts*

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Abstract

For an endofunctor H on a hyper-extensive category preserving countable coproducts we describe the free corecursive algebra on Y as the coproduct of the terminal coalgebra for H and the free H-algebra on Y. As a consequence, we derive that H is a *cia functor*, i.e., its corecursive algebras are precisely the cias (completely iterative algebras). Also all functors $H(-) + Y$ are then cia functors. For finitary set functors we prove that, conversely, if H is a cia functor, then it has the form $H = W \times (-) + Y$ for some sets W and Y.

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1 Introduction

Iteration and (co)recursion are of central importance in computer science. A formalism for iteration was proposed by Elgot [11] as iterative algebraic theories. Later Nelson [14] and Tiuryn [15] introduced iterative algebras for finitary signatures which yield an easier approach to iterative theories. For endofunctors H there are two related notions of algebras. *Corecursive algebras* introduced by Capretta et al. [9] are those algebras A such that every recursive equation expressed as a coalgebra for H has a unique solution (i.e., a coalgebra-to-algebra morphism into A). The other notion, *completely iterative algebras* (or *cia*, for short), introduced by the second author [13], are H-algebras A with the stronger property that every recursive equation with parameters in A has a unique solution (Definition 7). Corecursive algebras often fail to be cias. In the present paper we study endofunctors such that every corecursive algebra is a cia – we call them *cia functors*.

Our first result is that every endofunctor preserving countable coproducts and having a terminal coalgebra is a cia functor (Corollary 21). This is based on a description of the free cia on an object Y as a coproduct

$$\nu H + FY$$

of the terminal coalgebra and the free algebra on Y (Theorem 14). We deduce that, for H preserving countable coproducts and having a terminal coalgebra, we obtain cia functors

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$H(-) + Y$ for all objects Y (Corollary 24). All this holds in every *hyper-extensive* base category (Definition 1), e.g., in sets, posets, graphs and all presheaf categories.

In particular, if the base category is also cartesian closed, then $X \mapsto W \times X + Y$ is a cia functor for every pair of objects W and Y. For finitary set functors we prove a surprising converse: the only cia functors are those of the above form $X \mapsto W \times X + Y$.

2 Preliminaries

Throughout the paper H denotes an endofunctor on a hyper-extensive category (recalled below) having a terminal coalgebra

$$t : \nu H \to H(\nu H).$$

By the famous Lambek Lemma [12], the coalgebra structure t is invertible and its inverse makes νH an H-algebra.

We denote by $\mathsf{Alg}\, H$ the category of H-algebras and their morphisms.

▶ **Definition 1** ([2]). A category is called *hyper-extensive* if it has countable coproducts which are

(1) *universal*, i.e., preserved by pullbacks along any morphism,

(2) *disjoint*, i.e., coproduct injections are monomorphic and have pairwise intersection 0 (the initial object), and

(3) *coherent*, i.e., given pairwise disjoint morphisms $a_n : A_n \to A$, $n \in \mathbb{N}$, each of which is a coproduct injection, then their copairing $[a_n]_{n\in\mathbb{N}} : \coprod_{n\in\mathbb{N}} A_n \to A$ is also a coproduct injection.

▶ **Example 2.** The categories of sets, posets, graphs, and presheaf categories are hyper-extensive.

▶ Remark 3.

(1) We write $A+B$ for the coproduct of the objects A and B and denote coproduct injections by $\mathsf{inl} : A \to A + B$ and $\mathsf{inr} : B \to A + B$.

(2) Recall that a category with finite coproducts is *extensive* if it has pullbacks along coproduct injections and conditions (1) and (2) are satisfied [10]. Equivalently, in a diagram of the following form

$$\begin{tikzcd} X \arrow[r, "x"] \arrow[d, "f"'] & Z \arrow[d, "h"'] & Y \arrow[l, "y"'] \arrow[d, "g"] \\ A \arrow[r, "\mathsf{inl}"'] & A + B & B \arrow[l, "\mathsf{inr}"] \end{tikzcd}$$

the top row is a coproduct if and only if the squares are pullbacks. Another, more compact, equivalent characterization of extensivity states that the canonical functor $\mathcal{C}/A \times \mathcal{C}/B \to \mathcal{C}/(A+B)$ is an equivalence of categories for any pair of objects A and B.

(3) The somewhat technical condition (3) in Definition 1 is not a consequence of the other two. In fact, let $\mathcal{C}$ be the category of Jónsson-Tarski algebras, i.e., binary algebras A whose operation $A \times A \to A$ is a bijection. Then $\mathcal{C}$ has disjoint and universal countable (in fact, all) coproducts but is not hyperextensive [2].

▶ **Definition 4** ([9]). An algebra $a : HA \to A$ is called *corecursive* if for every coalgebra $e : X \to HX$ there exists a unique algebra-to-coalgebra morphism $e^\dagger : X \to A$:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX \arrow[r, "He^\dagger"'] & HA \arrow[u, "a"'] \end{tikzcd} \tag{1}$$

▶ **Examples 5.**

(1) The terminal coalgebra νH (considered as an algebra) is obviously corecursive. This is the initial corecursive algebra [9].
Furthermore, let Y be an object of $\mathcal{C}$ and assume that the functor $H(-) + Y$ has a terminal coalgebra TY. Then its structure

$$TY \xrightarrow{\alpha_Y} HTY + Y$$

has an inverse which is the copairing of two morphisms denoted by

$$HTY \xrightarrow{\tau_Y} TY \xleftarrow{\eta_Y} Y.$$

It follows that TY is a coproduct of HTY and Y with the above coproduct injections. It is easy to show that (TY, τ_Y) is a corecursive algebra.

(2) The trivial terminal algebra $H1 \to 1$ is corecursive, and if (A, a) is a corecursive algebra so is (HA, Ha) [9, Prop. 21]. Furthermore, if $\mathcal{C}$ has limits then corecursive algebras are closed under limits in the category of algebras for H [3, Prop. 2.4]. It follows that all members of the *terminal-coalgebra* chain

$$1 \longleftarrow H1 \longleftarrow HH1 \longleftarrow \cdots$$

are corecursive algebras.

(3) A particular instance of point (1) is given by a signature $\Sigma = (\Sigma_n)_{n<\omega}$ of operation symbols with prescribed arity and considering the corresponding polynomial endofunctor H_Σ on Set defined by

$$H_\Sigma X = \coprod_{n<\omega} \Sigma_n \times X^n.$$

For an operation symbol $\sigma \in \Sigma_n$ we write $\sigma(x_1, \ldots, x_n)$ in lieu of $(\sigma, (x_1, \ldots, x_n))$ for elements in the summand of $H_\Sigma X$ corresponding to $n < \omega$. The terminal coalgebra νH_Σ is carried by the set of all Σ-*trees*, i.e., rooted and ordered trees with nodes labeled in Σ such that every node with n children is labeled by an n-ary operation symbol. The algebraic operation of νH_Σ is *tree-tupling*: t^{-1} assigns to $\sigma(t_1, \ldots, t_n)$ with $\sigma \in \Sigma_n$ and $t_i \in \nu H_\Sigma$, $i = 1 \ldots, n$, the Σ-tree obtained by joining the Σ-trees $t_1, \ldots, t_n$ by a root node labeled by σ.

For every set Y we denote by

$$T_\Sigma Y$$

the algebra of all Σ-*trees over* Y, i.e., Σ-trees whose leaves are labeled by constant symbols in Σ_0 or elements of Y. This is the terminal coalgebra for $H_\Sigma(-) + Y$, and therefore it is a corecursive algebra.

▶ Remark 6. For a polynomial endofunctor H_Σ on Set we can view a coalgebra $e : X \to H_\Sigma X$ as a system of *recursive equations* over the set X of (recursion) variables: for every variable $x \in X$ we have a formal equation

$$x \approx \sigma(x_1, \dots, x_n) = e(x).$$

The map $e^\dagger$ in Definition 4 is then a *solution* of the system of equations in the Σ-algebra A: the commutative square (1) states that $e^\dagger$ turns the above formal equations into actual identities in A: $e^\dagger(x) = \sigma^A(e^\dagger(x_1), \dots, e^\dagger(x_n))$.

▶ **Definition 7 ([13]).** An algebra $a : HA \to A$ is called *completely iterative* (or *cia*, for short) if the algebra $[a, A] : HA + A \to A$ is corecursive for the endofunctor $H(-) + A$. That means that for every *(flat) equation morphism* $e : X \to HX + A$ there exists a unique *solution*, i.e., a unique morphism $e^\dagger$ such that square below commutes:

$$\begin{tikzcd} X \arrow[r, "e^\dagger"] \arrow[d, "e"'] & A \\ HX + A \arrow[r, "He^\dagger + A"'] & HA + A \arrow[u, "{[a,A]}"'] \end{tikzcd} \tag{2}$$

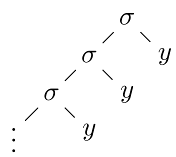
▶ **Examples 8.**

(1) If $H(-) + Y$ has a terminal coalgebra TY (cf. Example 5(1)), then (TY, τ_Y) is a cia. In fact, (TY, τ_Y) is a free cia on Y with the universal morphism η_Y [13].

(2) For a polynomial functor H_Σ on Set the above example states that the algebra $T_\Sigma Y$ of all Σ-trees over Y is the free cia on the set Y. Let us denote by

$$C_\Sigma Y$$

the subalgebra of $T_\Sigma Y$ given by all Σ-trees over Y which have only a finite number of leaves labeled in Y (and the remaining, possibly infinitely many, leaves are labeled in Σ_0). This algebra is corecursive but, whenever Σ contains an operation symbol of arity at least 2, not a cia. Moreover, $C_\Sigma Y$ is the free corecursive algebra on Y [3].
As a concrete example, consider the signature Σ consisting of a single binary operation σ. Then the equation morphism $e : \{x_1, x_2\} \to H_\Sigma\{x_1, x_2\} + \{y\}$ given by the recursive equations $x_1 \approx \sigma(x_1, x_2)$ and $x_2 \approx y$ has the unique solution $e^\dagger : \{x_1, x_2\} \to T_\Sigma\{y\}$ given as follows

$e^\dagger : x_1 \mapsto$  $x_2 \mapsto y$.

This demonstrates that $C_\Sigma\{y\}$ is not a cia because the above infinite Σ-tree is not contained in it.

▶ **Definition 9.** A *cia functor* is an endofunctor such that every corecursive algebra for it is a cia. (It the follows that cias and corecursive algebras coincide).

▶ **Notation 10.**

(1) If a free H-algebra on Y exists, we denote it by FY and its structure and universal morphism by

$$\varphi_Y : HFY \to FY \qquad \text{and} \qquad \eta_Y^F : Y \to FY,$$

respectively.

In the case of a polynomial set functor H_Σ, the free Σ-algebra $F_\Sigma Y$ is the subalgebra of $T_\Sigma Y$ on all finite Σ-trees over Y.

(2) If a free corecursive H-algebra on Y exists, we denote it by CY and its structure and universal morphism by

$$\psi_Y : HCY \to CY \qquad \text{and} \qquad \eta_Y^C : Y \to CY,$$

respectively.

3 Functors Preserving Countable Coproducts

▶ **Assumption 11.** In this and the subsequent section we assume that H is an endofunctor on a hyper-extensive category having a terminal coalgebra and preserving countable coproducts.

▶ **Fact 12** ([8]). *A free algebra on Y is*

$$FY = H^*Y = \coprod_{n<\omega} H^n Y \qquad \text{with coproduct injections } j_n : H^n Y \to H^* Y.$$

Its algebra structure and universal morphism are given by

$$\varphi_Y \cdot Hj_n = j_{n+1} \quad (n > 0) \qquad \text{and} \qquad \eta_Y^F = j_0 : Y \to H^*Y$$

using that $HFY = \coprod_{n<\omega} H^{n+1}Y$.

▶ **Notation 13.** We denote by

$$\sigma_Y : H^*Y = \coprod_{n<\omega} H^n Y \to Y + H\left(\coprod_{n<\omega} H^n Y\right) = Y + HH^*Y$$

the isomorphism inverse to $[\eta_Y^F, \varphi_Y] : Y + HH^*Y \to H^*Y$. It is defined by the following commutative diagrams:

$$\begin{tikzcd} Y \arrow[d, "j_0"'] \arrow[dr, "\mathsf{inl}"] & \\ H^*Y \arrow[r, "\sigma_Y"'] & Y + HH^*Y \end{tikzcd} \qquad \begin{tikzcd} H^n Y \arrow[r, "Hj_{n-1}"] \arrow[d, "j_n"'] & HH^*Y \arrow[d, "\mathsf{inr}"] \\ H^*Y \arrow[r, "\sigma_Y"'] & Y + HH^*Y \end{tikzcd} \quad \text{for } n > 0. \tag{3}$$

▶ **Theorem 14.** *The free cia on Y is*

$$CY = H^*Y + \nu H$$

with algebra structure $\varphi_Y + t^{-1} : H(H^*Y + \nu H) \cong HH^*Y + H(\nu H) \to H^*Y + \nu H$.

Proofsketch. In view of Example 8 it suffices to prove that the terminal coalgebra for $Y + H(-)$ is $H^*Y + \nu H$ with the following coalgebra structure

$$H^*Y + \nu H \xrightarrow{\sigma_Y + t} Y + HH^*Y + H(\nu H) \cong Y + H(H^*Y + \nu H).$$

That means that for a given coalgebra $e : X \to Y + HX$ there exists precisely one coalgebra morphism $h : X \to H^*Y + \nu H$. This morphism h is defined by an iterative construction using pullbacks and (hyper-)extensivity that we now explain.

Let $X_0 = X$ and $e_0 = e$ and denote the coproduct injections of $Y + HX$ by $i_0 : HX \to Y + HX$ and $\bar{i}_0 : Y \to Y + HX$. Next form the pullbacks of e along these injections:

$$\begin{tikzcd} X_1 \arrow[r, "i_1"] \arrow[d, "e_1"'] & X_0 \arrow[d, "e_0"] & \overline{X}_1 \arrow[l, "\bar{i}_1"'] \arrow[d, "\bar{e}_1"] \\ HX \arrow[r, "i_0"'] & Y + HX & Y \arrow[l, "\bar{i}_0"] \end{tikzcd} \tag{4}$$

By extensivity, $X = X_1 + \overline{X}_1$ with injections i_1 and $\bar{i}_1$. The component $\overline{h}_1 := h \cdot \bar{i}_1$ of h at $\overline{X}_1$ is defined by

$$h \cdot \bar{i}_1 = \left(\overline{X}_1 \xrightarrow{\bar{e}_1} Y \xrightarrow{j_0} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H\right).$$

In order to analyze the complementary coproduct component $h \cdot i_1$, we form the pullbacks of e_1 along the coproduct injections of $HX_0 = HX_1 + H\overline{X}_1$:

$$\begin{tikzcd} X_2 \arrow[r, "i_2"] \arrow[d, "e_2"'] & X_1 \arrow[d, "e_1"] & \overline{X}_2 \arrow[l, "\bar{i}_2"'] \arrow[d, "\bar{e}_2"] \\ HX_1 \arrow[r, "Hi_1"'] & HX_0 & H\overline{X}_1 \arrow[l, "H\bar{i}_1"] \end{tikzcd}$$

Then $X_1 = X_2 + \overline{X}_2$ and the component $\overline{h}_2 = h \cdot i_1 \cdot \bar{i}_2$ of h at $\overline{X}_2$ is defined by

$$h \cdot i_1 \cdot \bar{i}_2 = \left(\overline{X}_2 \xrightarrow{\bar{e}_2} H\overline{X}_1 \xrightarrow{H\bar{e}_1} HY \xrightarrow{j_1} H^*Y \xrightarrow{\mathsf{inl}} H^*Y + \nu H\right).$$

We continue this process recursively: given a coproduct $X_n \xrightarrow{i_n} X_{n-1} \xleftarrow{\bar{i}_n} \overline{X}_n$ and a morphism $e_n : X_n \to HX_{n-1}$ we form its pullbacks along the coproduct injection of $HX_{n-1} = HX_n + H\overline{X}_n$:

$$\begin{tikzcd} X_{n+1} \arrow[r, "i_{n+1}"] \arrow[d, "e_{n+1}"'] & X_n \arrow[d, "e_n"'] & \overline{X}_{n+1} \arrow[l, "\bar{i}_{n+1}"'] \arrow[d, "\bar{e}_{n+1}"] \\ HX_n \arrow[r, "Hi_n"'] & HX_{n-1} & H\overline{X}_n \arrow[l, "H\bar{i}_n"] \end{tikzcd} \tag{5}$$

Since compositions of coproduct injections are always coproduct injections, we obtain coproduct injections

$$\bar{i}^*_{n+1} = \left(\overline{X}_{n+1} \xrightarrow{\bar{i}_{n+1}} X_n \xrightarrow{i_n} X_{n+1} \xrightarrow{i_{n-1}} \cdots \xrightarrow{i_1} X\right) \qquad (n < \omega) \tag{6}$$

and morphisms

$$\widehat{e}_{n+1} = \left(\overline{X}_{n+1} \xrightarrow{\bar{e}_{n+1}} H\overline{X}_n \xrightarrow{H\bar{e}_n} H^2\overline{X}_{n-1} \xrightarrow{H^2\bar{e}_{n-1}} \cdots \xrightarrow{H^n\bar{e}_1} H^nY\right) \qquad (n < \omega). \tag{7}$$

The component $\overline{h}_{n+1} := (\overline{X}_{n+1} \xrightarrow{\bar{i}^*_{n+1}} X \xrightarrow{h} H^*Y + \nu H)$ of h at $\overline{X}_{n+1}$ is defined by the commutativity of the following square

$$\begin{tikzcd} \overline{X}_{n+1} \arrow[rr, "\bar{i}^*_{n+1}"] \arrow[d, "\widehat{e}_{n+1}"'] & & X \arrow[d, "h"] \\ H^nY \arrow[r, "j_n"'] & H^*Y \arrow[r, "\mathsf{inl}"'] & H^*Y + \nu H \end{tikzcd} \tag{8}$$

Observe also that by composing pullback squares we obtain the following pullback:

$$
\begin{tikzcd}
\overline{X}_n \arrow[r, "\bar{i}_n"] \arrow[d, "\bar{e}_n"'] \arrow[rrrrrrr, bend left, "\bar{i}_n^*"] & X_{n-1} \arrow[r, "i_{n-1}"] \arrow[d, "e_{n-1}"'] & X_{n-2} \arrow[r, "i_{n-2}"] \arrow[d, "e_{n-2}"'] & \cdots \arrow[r, "i_3"] & X_2 \arrow[r, "i_2"] \arrow[d, "e_2"'] & X_1 \arrow[r, "i_1"] \arrow[d, "e_1"'] & & X_0 = X \arrow[d, "e"] \\
H\overline{X}_{n-1} \arrow[r, "H\bar{i}_{n-1}"'] \arrow[rrrrr, bend right, "H\bar{i}^*_{n-1}"'] & HX_{n-2} \arrow[r, "Hi_{n-2}"'] & HX_{n-3} \arrow[r, "Hi_{n-3}"'] & \cdots \arrow[r, "Hi_2"'] & HX_1 \arrow[r, "Hi_1"'] & HX_0 \arrow[rr, "i_0"'] & & Y + HX
\end{tikzcd}
\tag{9}
$$

Now the coproduct injections in (6) are clearly pairwise disjoint. Therefore, by hyper-extensivity, we have a coproduct injection $[\bar{i}^*_{n+1}]_{n<\omega}$ which we denote by

$$\overline{X}_\infty \xrightarrow{\bar{i}_\infty} X \qquad \text{for} \qquad \overline{X}_\infty := \coprod_{n<\omega} \overline{X}_{n+1},$$

and $h \cdot \bar{i}_\infty$ is defined coponentwise by (8). By hyper-extensivity we can consider the complementary coproduct component $i_\infty : X_\infty \to X$, i.e., we have the coproduct

$$\overline{X}_\infty \xrightarrow{\bar{i}_\infty} X \xleftarrow{i_\infty} X_\infty.$$

Since the pullbacks (9) have pairwise disjoint coproduct injections as their upper arrows, they form together the pullback on the left below:

$$
\begin{tikzcd}
\overline{X}_\infty = \overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \cdots \arrow[r, "\bar{i}_\infty"] \arrow[d, "\coprod \bar{e}_n"'] & X \arrow[d, "e"] & X_\infty \arrow[l, "i_\infty"'] \arrow[dd, "e_\infty"] \\
Y + H\overline{X}_1 + H\overline{X}_2 + \cdots \arrow[r, "Y + H\bar{i}_\infty"] \arrow[d, equal] & Y + HX \arrow[d, equal] & \\
Y + H\overline{X}_\infty \arrow[r, "\mathsf{inl}"'] & Y + H\overline{X}_\infty + HX_\infty & HX_\infty \arrow[l, "\mathsf{inr}"] \arrow[ul, "\mathsf{inr} \cdot Hi_\infty"']
\end{tikzcd}
\tag{10}
$$

By extensivity, we obtain a morphism $e_\infty : X_\infty \to HX_\infty$ complementary to $\coprod \bar{e}_n$. This morphism is the structure of an H-coalgebra on X_∞. Thus, we define $h \cdot i_\infty$ to be the unique coalgebra morphism from X_∞ to νH.

One now verifies that the morphism $h : X \to H^*Y + \nu H$ so defined is a unique coalgebra morphism for $Y + H(-)$ as desired (see the full version [5] of our paper for details). ◀

▶ **Example 15.**

(a) It is well-known that the identity functor on Set has the free cias (equivalently, final coalgebras for $(-) + Y$) $TY = \mathbb{N} \times Y + 1$ where $\mathbb{N}$ is the set of natural numbers. It follows from Theorem 14 that the same formula holds in every hyper-extensive category with a terminal object 1. To see this, one first shows that

$$N := \coprod_{n<\omega} 1 \quad \text{with} \quad 1 \xrightarrow{\mathsf{in}_0} N \xleftarrow{[\mathsf{in}_{n+1}]_{n<\omega}} N$$

forms a natural number object, i.e., an initial algebra for $1 + (-)$. Using distributivity we see that for any object Y the free algebra Id^*Y is

$$\mathsf{Id}^*Y = \coprod_{n<\omega} Y \cong \left(\coprod_{n<\omega} 1 \right) \times Y = N \times Y. \tag{11}$$

Finally, we clearly have $\nu\mathsf{Id} = 1$. By Theorem 14, we thus obtain

$$TY \cong N \times Y + 1.$$

(b) For the above formula giving the free cia for Id on every Y it is *not* sufficient that $\mathcal{C}$ be an extensive category. As a counterexample consider the category $\mathcal{C} = \mathsf{CHaus}$ of compact Hausdorff spaces. Its limits and finite coproducts are created by the forgetful functor into Set, thus CHaus is extensive. However, it is not hyper-extensive since countable coproducts are not universal. For $Y = 1$ (the one point space) the formula (11) gives an uncountable space since $\coprod_{n<\omega} 1$ is the Stone-Čech compactification of an infinite discrete space. However, in the notation of Example 5, $T1$ is a countable space; for the terminal ω^{op}-chain

$$1 \leftarrow 1+1 \leftarrow 1+1+1 \leftarrow \cdots$$

of the functor $\mathsf{Id} + 1$ on CHaus has the corresponding underlying chain in Set. The limit in Set is countable, giving the set $N + 1$. The limit in CHaus is then a compact space on this set, in fact, it is the one-point compactification of the discrete space on N. Since the functor $X \mapsto X + 1$ preserves this limit, it is its terminal coalgebra. That means that $T1$ is countable.

▶ **Example 16.** Extending Example 15(a), we know that the functor $HX = \Sigma \times X$ on Set has the free cias $TY = \Sigma^* \times Y + \Sigma^\omega$, where Σ^* and Σ^∞ are the usual sets of strings (words) and sequences (streams) on Σ.

It follows from Theorem 14 that the same formula holds in every hyper-extensive category $\mathcal{C}$ with finite products commuting with countable coproducts. Examples of such categories are presheaf categories, posets, graphs and unary algebras.

Given an object Σ of $\mathcal{C}$, the functor $HX = \Sigma \times X$ has the terminal coalgebra

$$\Sigma^\omega = \lim_{n<\omega} \Sigma^n$$

which is the limit of the ω^{op}-chain of projections as follows:

$$1 \xleftarrow{!} \Sigma \xleftarrow{\Sigma\times !} \Sigma \times \Sigma \xleftarrow{\Sigma\times\Sigma\times !} \Sigma \times \Sigma \times \Sigma \leftarrow \cdots$$

The free algebras H^*Y are obtained as follows: define

$$\Sigma^* = \coprod_{n<\omega} \Sigma^n.$$

Then $H^*Y = \Sigma^* \times Y$. Thus, according to Theorem 14, the free cia for H on Y is given by

$$TY = \Sigma^* \times Y + \Sigma^\omega.$$

Similarly, given another object W of $\mathcal{C}$, the functor $H'X = W + \Sigma \times X$ has the free cias $T'Y = \Sigma^* \times (W + Y) + \Sigma^\omega$.

▶ **Example 17.** In Theorem 14 it is not sufficient that H preserves finite coproducts. In fact, consider the ultrafilter functor $U : \mathsf{Set} \to \mathsf{Set}$ which assigns to every set X the set of all ultrafilters on X and to a map $f : X \to Y$ the map Uf sending an ultrafilter $\mathcal{A}$ on X to $\{B \subseteq Y \mid f^{-1}(B) \in \mathcal{A}\}$. It preserves finite coproducts and $\nu U = 1$. But for Y infinite, $Y + U(-)$ has no fixed points; for suppose that $TY \cong Y + UTY$, then TY must be infinite since Y is so and therefore $|TY| < |UTY|$ contradicting the isomorphism.

4 Corecursiveness vs. Complete Iterativity

Under Assumption 11 we prove in this section that H is a cia functor, i.e., every corecursive algebra is a cia. Let $a : HA \to A$ be a fixed algebra.

▶ **Notation 18.**

(1) Define morphisms

$$a^n : H^n A \to A$$

by the following induction:

$$a^0 = \mathsf{id}_A \qquad \text{and} \qquad a^{n+1} = (H^{n+1}A = HH^nA \xrightarrow{Ha^n} HA \xrightarrow{a} A).$$

(2) For every equation morphism $e : X \to HX + A$ we use the notation of the proof of Theorem 14, except that Y is replaced by A everywhere (and the order of summands is swapped). Thus we use the morphisms

$$i_n, \bar{i}_n, e_n, \bar{e}_n, e_\infty, i_\infty, \bar{i}_\infty, \widehat{e}_n, \text{ and}, \bar{i}^*_n$$

as in that proof.

▶ **Construction 19.** Let $a : HA \to A$ be an algebra. Given an equation morphism $e : X \to HX + A$ and a coalgebra-to-algebra morphism $s : X_\infty \to A$:

$$\begin{tikzcd} X_\infty \arrow[r, "s"] \arrow[d, "e_\infty"'] & A \\ HX_\infty \arrow[r, "Hs"'] & HA \arrow[u, "a"'] \end{tikzcd} \tag{12}$$

we define a morphism $e^\dagger_s : X \to A$ on the components of the coproduct $X = \left(\coprod_{n\geq 1} \overline{X}_n\right) + X_\infty$ (with injections $\bar{i}^*_n$, for every $n \geq 1$, and i_∞) separately as follows:

$$\begin{tikzcd} \overline{X}_n \arrow[r, "\widehat{e}_n"] \arrow[d, "\bar{i}^*_n"'] & H^{n-1}A \arrow[d, "a^{n-1}"] \\ X \arrow[r, "e^\dagger_s"'] & A \end{tikzcd} \quad \text{for } n \geq 1, \text{ and} \quad \begin{tikzcd} X_\infty \arrow[d, "i_\infty"'] \arrow[dr, "s"] & \\ X \arrow[r, "e^\dagger_s"'] & A \end{tikzcd} \tag{13}$$

▶ **Proposition 20.** *The morphism $e^\dagger_s$ is a solution of e. Moreover, every solution of e is of the form $e^\dagger_s$ for some coalgebra-to-algebra morphism s.*

▶ **Corollary 21.** *The functor H is a cia functor.*

Indeed, if (A, a) is a corecursive H-algebra and $e : X \to HX + A$ is a given equation morphism, we have a unique s as in (12). Now note that Proposition 20 establishes a bijective correspondence between solutions of e and coalgebra-to-algebra morphisms from e_∞ to a, and therefore there exists a unique solution of e.

▶ **Example 22.** For the ultrafilter functor U of Example 17 consider the subfunctor U_0 of all ω-complete ultrafilters, i.e., those closed under countable intersections. This functor preserves countable coproducts and $\nu U_0 = 1$. Assume that a proper class of measurable cardinals n

exists (i.e., for each n we have an ω-complete ultrafilter P on a set X not containing any subset of X of less than n elements). This is quite a strong assumption in set theory, but we make it here to derive a strong property of U_0: it is a non-accessible cia functor! Indeed, the latter follows from Corollary 21, and U_0 is not accessible: for every measurable cardinal n it does not preserve the n-filtered colimit of all subsets Y of X of cardinality less than n, since P lies in U_0X but not in U_0Y if $|Y| < n$. This is a surprising example in view of Theorem 37 which shows that such a complex example does not exist among finitary set functors.

Finally, note that both cias and corecursive algebras form full subcategories of the category of all algebras for H. Thus Corollary 21 establishes an isomorphism of categories between the categories of cias and corecursive algebras for H.

The following proposition needs no assumptions on H or the base category except that binary coproducts exist.

▶ **Proposition 23.** *If H is a cia functor, then so is $H(-) + Y$ for every object Y.*

▶ **Corollary 24.** *Let H be a functor having a terminal coalgebra and preserving countable coproducts. Then $H(-) + Y$ is a cia functor for every object Y.*

5 Finitary Set Functors

We have seen above that for every functor H on a hyper-extensive category preserving countable coproducts, the functors $H(-) + Y$ are cia functors (i.e., every corecursive algebra is a cia). In particular, if $\mathcal{C}$ is cartesian closed, then the functor $X \mapsto W \times X + Y$ is a cia functor. For $\mathcal{C} = \mathsf{Set}$ and H finitary we now prove the converse: if H is a cia functor then it has the form $X \mapsto W \times X + Y$ for some sets W and Y.

▶ **Assumption 25.** Throughout this section H denotes a standard, finitary set functor.

Recall from [6] that H is *finitary* iff for every set X we have $HX = \bigcup HY$ where the union ranges over finite subsets $Y \subseteq X$. An example of a finitary functor on Set is the polynomial functor H_Σ, see Example 5(3).

Standard means that H preserves

(1) inclusions, i.e., $X \subseteq Y$ implies $HX \subseteq HY$ and the H-image of the inclusion map $X \hookrightarrow Y$ is the inclusion map $HX \hookrightarrow HY$, and

(2) finite intersections.

Assuming that H is standard is without loss of generality because for every set functor H there exist a standard set functor H' naturally isomorphic to H on the full subcategory of all nonempty sets [7, Theorem 3.4.5]. (And the change of value at $\emptyset$ is irrelevant for us since corecursive algebras and cias, respectively, for H are in bijective correspondence with those for H').

▶ **Definition 26.**

(1) By a *presentation* of H is meant a finitary signature Σ and natural epitransformation $\varepsilon : H_\Sigma \to H$, i.e., every component ε_X is a surjective map.

(2) An *ε-equation* is an expression $\sigma(x_1, \dots x_n) = \tau(z_1, \dots, z_m)$ where σ is an n-ary operation symbol and τ an m-ary one such that ε_X merges the two elements of $H_\Sigma X$ where $X = \{x_1, \dots, x_n, z_1, \dots, z_n\}$.

▶ Remark 27. All ε-equations form an equivalence relation. More precisely, for any set X all ε-equations with variables replaced by elements of X form precisely the kernel equivalence of ε_X. Moreover, the elements of HX may be regarded as equivalence classes of the elements $\sigma(x_1, \dots, x_n)$ of $H_\Sigma X$ modulo this equivalence.

▶ **Example 28.** The finite power-set functor $\mathcal{P}_\mathsf{f}$ has a presentation with Σ having a single n-ary operation for every n, and ε sending $\sigma(x_1, \dots, x_n)$ to $\{x_1, \dots, x_n\}$.

The following lemma was proved in [7]. We present a (short) proof since we refer to it later.

▶ **Lemma 29.** *Every finitary set functor has a presentation $\varepsilon : H_\Sigma \to H$, and the category* $\mathsf{Alg}\, H$ *is isomorphic to the variety of all Σ-algebras satisfying all ε-equations.*

Proof. Define a signature $\Sigma = (\Sigma_n)_{n<\omega}$ by $\Sigma_n = Hn$ where we regard n as the finite ordinal $\{0, \dots, n-1\}$ for all n. By the Yoneda lemma we have a natural transformation $\varepsilon_X : H_\Sigma X \to HX$ assigning to every $\sigma(x_1, \dots, x_n)$ represented as a function $x : n \to X$ the element $Hx(\sigma)$. Since H is finitary, ε_X is surjective.

Every H-algebra $a : HA \to A$ defines the corresponding Σ-algebra $a \cdot \varepsilon_A : H_\Sigma A \to A$ which clearly satisfies all ε-equations. This defines a full embedding of $\mathsf{Alg}\, H$ into $\mathsf{Alg}\, H_\Sigma$ (which is identity on morphisms). We now easily prove that every Σ-algebra satisfying all ε-equations has the above form $(A, a \cdot \varepsilon_A)$. Indeed, given $a^\Sigma : H_\Sigma A \to A$ satisfying all ε-equations, define $a : HA \to A$ by $a([\sigma(a_1, \dots, a_n]) = a^\Sigma(\sigma(a_1, \dots, a_n))$. Since we know from Remark 27 that a^Σ merges all pairs in the kernel of ε_A, this is well-defined and we clearly have $a^\Sigma = a \cdot \varepsilon_A$. Thus, our full embedding defines the desired isomorphism between H-algebras and Σ-algebras satisfing all ε-equations. ◀

▶ Remark 30.

(1) Denote by C_1 the constant functor with value $1 = \{c\}$, and by $C_{0.1}$ its subfunctor with $C_{0,1}\emptyset = \emptyset$ and $C_{0,1}X = 1$ else. For every natural transformation $\alpha : C_{0,1} \to H$ there exists a unique extension to $\alpha' : C_1 \to H$.
Indeed, since H is standard, it preserves the (empty) intersection of the coproduct injections $\mathsf{inl}, \mathsf{inr} : 1 \to 1+1$. Since $H\mathsf{inl}(\alpha_1(c)) = \alpha_{1+1}(c) = H\mathsf{inr}(\alpha_1(c))$, there exists a unique element t of $H\emptyset$ such that the inclusion map $v : \emptyset \to 1$ fulfils $\alpha_1(c) = Hv(t)$. We put $\alpha'_\emptyset(c) = t$.

(2) All constants in our presentation of H are *explicit*. That means that whenever some n-ary symbol σ has the property that some ε-equation has the form $\sigma(x_1, \dots, x_n) = \sigma(z_1, \dots, z_n)$, where the variables x_i are pairwise distinct and none of them equals some z_j, then there exists a constant symbol τ in Σ for which we have the following ε-equation: $\sigma(x_1, \dots, x_n) = \tau$. Indeed, for every set $X \neq \emptyset$ we have an element

$$\alpha_X = \varepsilon_X(\sigma(a_1, \dots, a_n)) \in HX$$

independent of the choice of $a_1, \dots, a_n$ in X. This defines a natural transformation $\alpha : C_{0,1} \to H$. Let $\alpha' : C_1 \to H$ be its extension according to item (1). The element $\alpha'_\emptyset(c)$ of $H\emptyset$ has, since ε is an epitransformation, the form $\varepsilon_\emptyset(\tau)$ for some nullary symbol τ. Then the desired ε-equation holds because for $X = \{x_1, \dots, x_n\}$ and the unique empty map $u : \emptyset \to X$ we have

$$\varepsilon_X(\sigma(x_1, \dots, x_n)) = \alpha_X(c) = \alpha'_X(c) = Hu \cdot \alpha'_\emptyset(c) = Hu \cdot \varepsilon_\emptyset(\tau) = \varepsilon_X \cdot Hu(\tau) = \varepsilon_X(\tau).$$

▶ **Definition 31.** A presentation $\varepsilon : H_\Sigma \to H$ is *reduced* provided that for every ε-equation

$$\sigma(x_1, \dots, x_n) = \tau(z_1, \dots, z_m)$$

the following hold:

(1) if $x_1, \dots, x_n$ are pairwise distinct, then they all lie in $\{z_1, \dots, z_n\}$, and
(2) if, moreover, $z_1, \dots, z_n$ are also pairwise distinct, then $\sigma = \tau$.

▶ **Proposition 32.** *Every finitary set functor has a reduced presentation.*

▶ **Notation 33.** From now on we assume that a reduced presentation of H is given.

Recall the notation TY, FY and CY from Examples 8 and Notation 10. All these objects exist since H is finitary (and therefore so are all $H(-)+Y$). The corresponding notation for H_Σ is $T_\Sigma Y$, $F_\Sigma Y$ and $C_\Sigma Y$. The monad units of T and C are denoted by η and η^C, respectively.

As mentioned above, $T_\Sigma Y$ can be described as the algebra of all Σ-trees over Y. And $C_\Sigma Y$ and $F_\Sigma Y$ are its subalgebras on all trees with finitely many leaves labeled in Y, or all finite trees, respectively.

Since TY is a corecursive algebra, there exists a unique homomorphism of H-algebras

$$m_Y : CY \to TY$$

with $m_Y \cdot \eta^C_Y = \eta_Y$. The corresponding H_Σ-algebra morphism is denoted by

$$m^\Sigma_Y : C_\Sigma Y \to T_\Sigma Y.$$

▶ Remark 34. In [4] we described FY and TY as the following quotient of the Σ-algebras $F_\Sigma Y$ and $T_\Sigma Y$, respectively. Recall from Lemma 29 that every H-algebra $a : HA \to A$ may be regarded as the H_Σ-algebra with structure $a \cdot \varepsilon_A : H_\Sigma A \to A$.

(1) $FY = F_\Sigma Y/{\sim_Y}$, where $\sim_Y$ is the congruence of finite application of ε-equations. That is, the smallest congruence with $\sigma(x_1,\dots,x_n) \sim_Y \tau(z_1,\dots,z_m)$ for every ε-equation

$$\sigma(x_1,\dots,x_n) = \tau(z_1,\dots,z_m)$$

over Y. The universal map $\eta^F_Y : Y \to FY$ is the composition of the one of $F_\Sigma Y$ with the canonical quotient map $F_\Sigma Y \twoheadrightarrow F_\Sigma Y/{\sim_Y}$.

(2) $TY = T_\Sigma Y/{\sim^*_Y}$, where $\sim^*_Y$ is the congruence of (possibly infinitely many) applications of ε-equations. The universal map is $\widehat{\eta}_Y = \widehat{\varepsilon}_Y \cdot \eta^\Sigma_Y$, where $\eta^\Sigma_Y : Y \to T_\Sigma Y$ is the universal map of the free cia for H_Σ on Y and $\widehat{\varepsilon}_Y : T_\Sigma Y \twoheadrightarrow T_\Sigma Y/{\sim^*_Y}$ is the canonical quotient map.

The definition of a possibly *infinite application of ε-equations* is based on the concept of *cutting* a Σ-tree at level k: the resulting finite Σ-tree $\partial_k t$ is obtained from t by deleting all nodes of depth larger than k and relabeling all nodes at level k by a symbol $\perp \notin Y$. Then we define, for Σ-trees t and s in $T_\Sigma Y$,

$$t \sim^*_Y s \quad \text{iff} \quad \partial_k t \sim_{Y\cup\{\perp\}} \partial_k s \quad \text{for every } k < \omega.$$

Not surprisingly, CY can be described analogously:

▶ **Proposition 35.** *The free corecursive H-algebra CY is the quotient of the Σ-algebra $C_\Sigma Y$ modulo the application of ε-equations: $CY = C_\Sigma Y/{\sim^*_Y}$.*

Proof. This is based on the following description of CY presented in [3]: denote by $\oplus$ the binary coproduct of H-algebras in $\mathsf{Alg}\, H$. By Lemma 29, this is, equivalently, the coproduct in the variety of all Σ-algebras satisfying all ε-equations. Then we have

$$CY = \nu H \oplus FY.$$

Analogously, if $\boxplus$ denotes the binary coproduct of Σ-algebras, we of course have

$$C_\Sigma Y = \nu H_\Sigma \boxplus F_\Sigma Y.$$

For arbitrary H-algebras A and B we know that $A \oplus B$ is the quotient of $A \boxplus B$ modulo the application of ε-equations. Moreover, we have $T = T_\Sigma/{\sim^*}$ and $FY = F_\Sigma Y/{\sim}$. It follows immediately that $T \oplus FY = (T_\Sigma \boxplus F_\Sigma Y)/{\sim^*}$, as claimed. ◀

▶ **Lemma 36.** *Suppose that CY is a cia for H. For every equation morphism $e : X \to H_\Sigma X + Y$ with the unique solution $e^\ddagger : X \to T_\Sigma Y$ we can form an equation morphism*

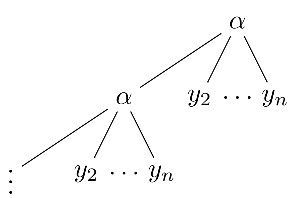
$$\overline{e} = (X \xrightarrow{e} H_\Sigma X + Y \xrightarrow{\varepsilon_X + \eta_Y^C} HX + CY).$$

Then we have $(X \xrightarrow{\overline{e}^\dagger} CY \xrightarrow{m_Y} TY) = (X \xrightarrow{e^\ddagger} T_\Sigma Y \xrightarrow{\hat{\varepsilon}_Y} TY)$.

▶ **Theorem 37.** *For a finitary set functor H the following conditions are equivalent:*
(1) *H is a cia functor,*
(2) *$H = H_0(-) + Y$ where H_0 preserves countable coproducts and Y is a set, and*
(3) *$H = W \times (-) + Y$ for some sets W and Y.*

Proof.
(2) ⇒ (3). Since H is finitary, so is H_0, by the description of finitarity following Assumptions 25. Therefore, H_0 preserves all coproducts. Trnková proved [16, Theorem IX.8], that every coproduct-preserving set functor preserves colimits, thus it is a left adjoint. It is well known that the only right adjoint set functors R are the representable ones: for given $L \dashv R$, put $W = L1$, then the elements $1 \to RY$ bijectively correspond to the maps $W \to Y$, thus, R is naturally isomorphic to $\mathsf{Set}(W, -)$. Consequently, H_0 is left adjoint to $\mathsf{Set}(W, -)$, hence it is naturaly isomorphic to $W \times (-)$.
(3) ⇒ (1). This follows from Corollary 24.
(1) ⇒ (2). Let $\varepsilon : H_\Sigma \to H$ be a reduced presentation.
(a) We prove below that all arities in Σ are 1 or 0. Let W be the set of all unary symbols and Y that of all constants. Then $H_\Sigma X = W \times X + Y$. Furthermore, we show that ε is a natural isomorphism. Indeed, each ε_X is, besides being surjective, also injective: it cannot merge distinct elements (w, x) and (w', x') of $W \times X$ because this would yield an ε-equation $w(x) = w'(x')$. Since the presentation is reduced, this implies $w = w'$ and $x = x'$. Analogously for all other pairs of elements of $H_\Sigma X$.
(b) Assume that some symbol α of Σ has arity at least 2. Then we derive a contradiction to H being a cia functor. Given a Σ-tree t we call a node r *pure* if the trees $t_1, \dots, t_n$ rooted at the children of r are pairwise distinct. Observe that an ε-equation applicable to a pure node r must have the form $\sigma(x_1, \dots, x_n) = \tau(y_1, \dots, y_m)$ for some $\tau \in \Sigma_m$, where $x_1, \dots, x_n$ are pairwise distinct.
Consider the following equation morphism $e : X \to H_\Sigma X + Y$ with $X = \{x_1, \dots, x_n\}$ and $Y = \{y_2, \dots, y_n\}$: $e(x_1) = \alpha(x_1, y_2, y_n)$ and $e(x_i) = y_i$, for $i = 2, \dots, n$. Then the unique solution $e^\ddagger : X \to T_\Sigma Y$ assigns to x_1 the Σ-tree below:

$e^\ddagger(x_1) =$



Next consider the equation morphism $\overline{e} = (X \xrightarrow{e} H_\Sigma X + Y \xrightarrow{\varepsilon_X + \eta_Y^C} HX + CY)$. Since CY is a cia, this has a unique solution $e^\dagger : X \to CY$. It assigns to x_1 an element of CY which by Proposition 35 has the form $\overline{e}^\dagger(x_1) = \overline{\varepsilon}_Y(s)$ for some $s \in C_\Sigma Y$, where $\overline{\varepsilon}_Y : C_\Sigma Y \twoheadrightarrow C_\Sigma Y/{\sim^*} \cong CY$ denotes the canonical quotient map. From Lemma 36 we know that

$$\widehat{\varepsilon}_Y(t) = \widehat{\varepsilon}_Y \cdot e^\ddagger(x_1) = m_Y \cdot \overline{e}^\dagger(x_1) = m_Y \cdot \overline{\varepsilon}_Y(s).$$

Therefore, we obtain $t \sim_Y^* s$.

We derive the desired contradiction by proving that every tree obtained from t by a finite application of ε-equations has a leaf labeled by y_2 at every positive level. From this we conclude immediately that the same holds for all trees obtained by an infinite application of ε-equations from t. However, $t \sim_Y^* s$ where s has only finitely many leaves labeled by y_2.

(b1) Assume that a single ε-equation is applied to t and let t' be the resulting tree. Let r be the node of t at which the application takes place. Then r is not a leaf labeled in Y; for recall that all ε-equations have operation symbols on both sides, thus, they are not applicable to leaves labeled in Y. Therefore, r is a pure node labeled by α. The ε-equation in question thus has the form $\alpha(u_1, \dots, u_n) = \tau(z_1, \dots, z_m)$ for some $\tau \in \Sigma_m$ and with the u_i pairwise distinct.

If r has depth k, then the tree t' has label y_2 at all levels $1, \dots, k$, since those leaves of t are unchanged. Furthermore, we have $u_2 = z_p$ for some $p = 1, \dots, m$ since ε is a reduced presentation. Therefore, y_2 occurs at level $k+1$ since the p-th child of r in t' is a leaf labeled by y_2. For the levels greater than $k+1$ we use that $u_1 = z_q$ holds for some $q = 1, \dots, m$, again because ε is a reduced presentation. Since the first subtree of r in t is t itself, it follows that the q-th child of r in t' is t itself. Thus, a label y_2 of depth n in t yields a label y_2 of depth $k+1+n$ of t'.

(b2) Assume that two ε-equations are applied to t. The resulting tree t'' can be obtained from t' in (b1) by a single application of an ε-equation. Let r' be the node of t' at which the application takes place. We can assume $r \neq r'$ (for if $r = r'$ we can obtain t'' from t by a single application on an ε-equation; this follows from Remark 27). If r' does not lie in the subtree of t' with root r, then r' is a pure node labeled by α and we argue as in (b1).

Suppose therefore that r' lies in the subtree rooted at r. If this is the q-th subtree from (b1) above (the one with $u_1 = z_q$), then we also argue as in (b1) using that the q-th subtree is t itself. Otherwise, if r' lies in any other subtree of r, then the labels y_2 of the q-th subtree are unchanged.

The remaining cases of three and more applications of ε-equations are completely analogous. This yields the desired contradiction: if $t \sim^* \bar{t}$, then $\bar{t}$ has label y_2 at every level $1, 2, 3, \dots$, thus $t \sim_Y^* s$ cannot be true. ◀

6 Conclusions and Open Problems

For endofunctors H preserving countable coproducts and having a terminal coalgebra we have described the free corecursive algebra on an object Y as $\nu H + \coprod_{n<\omega} H^n Y$. In addition, we have shown that H is a cia functor, i.e., every corecursive algebra for H is a cia. For this we assumed that the base category has well-behaved countable coproducts, i.e., the category is hyper-extensive. It is an open problem whether our results hold in more general categories, e.g., in all extensive locally presentable ones.

For accessible functors H on locally presentable categories, the free corecursive algebra on Y was described in previous work [3] as the coproduct of FY (the free algebra on Y) and νH (considered as an algebra) in the category $\mathsf{Alg}\, H$. If H preserves countable coproducts, this is quite similar to the above desciption of the free cia, since coproducts of algebras are then formed on the level of the underlying category and therefore $FY = \coprod_{n<\omega} H^n Y$. But the proof techniques are completely different, and a common generalization of the two results is open.

We have also characterized all cia functors among finitary set functors: they are precisely the functors $X \mapsto W \times X + Y$ for some sets W and Y. In Example 22 we have seen that the same result does not hold for all, not necessarily finitary, set functors. But that example required an assumption about set theory. It is an open problem whether that assumption was really necessary.

Our results can be stated in terms of corecursive monads [3] and completely iterative ones [1] as follows: a functor H having a terminal coalgebra νH and preserving countable coproducts has a free corecursive monad of the form $\coprod_{n<\omega} H^n(-) + \nu H$, and this is also the free completely iterative monad on H.

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