Brief Announcement: A Note on Hardness of Diameter Approximation*

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Abstract

We revisit the hardness of approximating the diameter of a network. In the CONGEST model, $\tilde{\Omega}(n)$ rounds are necessary to compute the diameter [Frischknecht et al. SODA'12]. Abboud et al. [DISC 2016] extended this result to sparse graphs and, at a more fine-grained level, showed that, for any integer $1 \leq \ell \leq \text{polylog}(n)$, distinguishing between networks of diameter $4\ell+2$ and $6\ell+1$ requires $\tilde{\Omega}(n)$ rounds. We slightly tighten this result by showing that even distinguishing between diameter $2\ell+1$ and $3\ell+1$ requires $\tilde{\Omega}(n)$ rounds. The reduction of Abboud et al. is inspired by recent conditional lower bounds in the RAM model, where the orthogonal vectors problem plays a pivotal role. In our new lower bound, we make the connection to orthogonal vectors explicit, leading to a conceptually more streamlined exposition. This is suited for teaching both the lower bound in the CONGEST model and the conditional lower bound in the RAM model.

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1 Introduction

In distributed computing, the diameter of a network is arguably the single most important quantity one wishes to compute. In the CONGEST model, where in each round every vertex can send to each of its neighbors a message of size $O(\log n)$, it is known that $\tilde{\Omega}(n)$ rounds are necessary to compute the diameter [3] even in sparse graphs [1], where n is the number of vertices. With this negative result in mind, it is natural that the focus has shifted towards *approximating* the diameter. In this note, we revisit hardness of computing a diameter approximation in the CONGEST model from a *fine-grained* perspective.

The current fastest approximation algorithm [4], which is inspired by a corresponding RAM model algorithm [5], takes $O(\sqrt{n \log n} + D)$ rounds and computes a $\frac{3}{2}$-approximation of the diameter, i.e., an estimate $\hat{D}$ such that $\lfloor \frac{2}{3} D \rfloor \leq \hat{D} \leq D$, where D is the true diameter. In terms of lower bounds, Abboud, Censor-Hillel, and Khoury [1] showed that $\tilde{\Omega}(n)$ rounds are necessary to compute a $(\frac{3}{2} - \epsilon)$-approximation of the diameter for any constant $0 < \epsilon < \frac{1}{2}$. At a more fine-grained level, they show that, for any integer $1 \leq \ell \leq \text{polylog}(n)$, at least $\tilde{\Omega}(n)$ rounds are necessary to decide whether the network has diameter $4\ell + 2$ or $6\ell + 1$, thus ruling out any "relaxed" notions of $(\frac{3}{2} - \varepsilon)$-approximation that additionally allow small

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additive error. We tighten this result by showing that, for any integer $\ell \geq 1$, at least $\tilde{\Omega}(n)$ rounds are necessary to distinguish between diameter $2\ell+1$ and $3\ell+1$.

The reduction of Abboud et al. [1] is inspired by recent work on conditional lower bounds in the RAM model, where the *orthogonal vectors problem* plays a pivotal role. In particular, the Orthogonal Vectors Hypothesis (OVH) is a weaker "polynomial-time analogue" of the Strong Exponential Time Hypothesis (SETH); it is well-known that SETH implies OVH. In our new lower bound, we make the connection to orthogonal vectors explicit: we consider a communication complexity version of orthogonal vectors that we show to be hard *unconditionally* by a reduction from set disjointness and then devise a reduction from orthogonal vectors to diameter approximation. The latter reduction also has implications in the RAM model. We show that under OVH, for any integer $1 \leq \ell \leq n^{o(1)}$, there is no algorithm that distinguishes between graphs of diameter 2ℓ and 3ℓ with running time $O(m^{2-\delta})$ for some constant $\delta > 0$, where m is the number of edges of the graph. This tightens the result of Cairo, Grossi, and Rizzi [2], who provide the same lower bound under the stronger hardness assumption SETH. To summarize, our approach is more streamlined than in previous works [3, 2, 1], allowing for a more unified view of CONGEST model and RAM model lower bounds.

2 Reduction via Orthogonal Vectors

Set disjointness is a problem in communication complexity between two players, called Alice and Bob, in which Alice is given an n-dimensional bit vector x and Bob is given an n-dimensional bit vector y and the goal for Alice and Bob is to find out whether there is some index k at which both vectors contain a 1, i.e., such that $x[k] = y[k] = 1$. The relevant measure in communication complexity is the number of bits exchanged by Alice and Bob in any protocol that Alice and Bob follow to determine the solution. A classic result states that any such protocol requires Alice and Bob to exchange $\Omega(n)$ bits to solve set disjointness.

In the orthogonal vectors problem, Alice is given a set of bit vectors $L = \{l_1, \ldots, l_n\}$ and Bob is given a set of bit vectors $R = \{r_1, \ldots, r_n\}$, and the goal for them is to find out if there is a pair of orthogonal vectors $l_i \in L$ and $r_j \in R$ (i.e., such that, for every $1 \leq k \leq d$, $l_i[k] = 0$ or $r_j[k] = 0$). We give a reduction from set disjointness to orthogonal vectors.

► **Theorem 1.** *Any b-bit protocol for the orthogonal vectors problem in which Alice and Bob each hold n vectors of dimension $d = 2\lceil \log n \rceil + 3$, gives a b-bit protocol for the set disjointness problem where Alice and Bob each hold an n-dimensional bit vector.*

► **Corollary 2.** *Any protocol solving the orthogonal vectors problem with n vectors of dimension $d = 2\lceil \log n \rceil + 3$, requires Alice and Bob to exchange $\Omega(n)$ bits.*

We now establish hardness of distinguishing between networks of diameter $2\ell + q$ and $3\ell + q$, where $\ell \geq 1$ and in the CONGEST model $q \geq 1$, whereas in the RAM model $q \geq 0$. To unify the cases of odd and even ℓ, we introduce an additional parameter $p \in \{0, 1\}$ and change the task to distinguishing between networks of diameter $4\ell' - 2p + q$ and $6\ell' - 3p + q$ for integers $\ell' \geq 1$, $q \geq 0$, and $p \in \{0, 1\}$. This covers the original question: if ℓ is even, then set $\ell' := \ell/2$ and $p := 0$ and if ℓ is odd, then set $\ell' := \lceil \ell/2 \rceil$ and $p := 1$.

Given an orthogonal vectors instance $\langle L := \{l_1, \ldots, l_n\}, R := \{r_1, \ldots, r_n\}\rangle$ of d-dimensional vectors and parameters $\ell \geq 1$, $q \geq 0$, and $p \in \{0, 1\}$, we define an unweighted undirected graph $G := G_{L,R,\ell,p,q}$ as follows. The graph G contains the following *exterior* vertices: $u_1^L, \ldots, u_n^L$, $u_1^R, \ldots, u_n^R$, $v_1^L, \ldots, v_n^L$, $v_1^R, \ldots, v_n^R$, $w_1^L, \ldots, w_d^L$, $w_1^R, \ldots, w_d^R$, x^L, x^R, y^L, and y^R. These exterior vertices are connected by paths as depicted in Figure 1, where

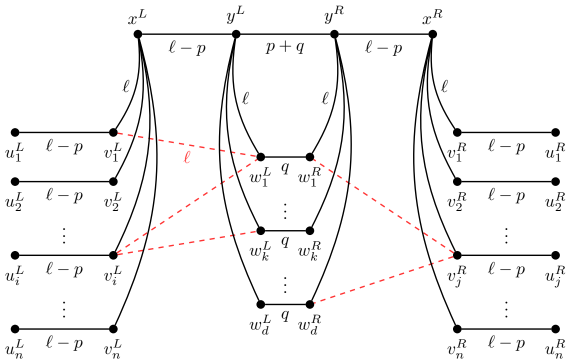

Figure 1 Visualization of the graph $G := G_{L,R,\ell,p,q}$ used in our reduction from orthogonal vectors to diameter distinction. The red, dashed edges encode the orthogonal vectors instance.

each path introduces a separate set of interior vertices. In particular, the instance $\langle L, R\rangle$ is encoded as follows: for every $1 \leq i \leq n$ and every $1 \leq k \leq d$, if $l_i[k] = 1$, then add a path from v_i^L to w_k^L of length ℓ, and if $r_i[k] = 1$, then add a path from v_i^R to w_k^R of length ℓ.

▶ **Theorem 3.** *Let $\langle L, R\rangle$ be an orthogonal vectors instance of two sets of d-dimensional vectors of size n each and let $\ell \geq 1$, $p \in \{0, 1\}$, and $q \geq 0$ be integer parameters. Then the unweighted, undirected graph $G := G_{L,R,\ell,p,q}$ has $O(nd\ell + dq)$ vertices and edges and its diameter D has the following property: if $\langle L, R\rangle$ contains an orthogonal pair, then $D = 6\ell - 3p + q$, and if $\langle L, R\rangle$ contains no orthogonal pair, then $D = 4\ell - 2p + q$.*

For the CONGEST model, observe that G has a small cut of size $d + 1$ between its left hand side and its right hand side. A standard simulation argument, where communication between Alice and Bob is limited to messages sent along the small cut, yields our main result.

▶ **Corollary 4.** *In the CONGEST model, any algorithm distinguishing between graphs of diameter $2\ell + q$ and $3\ell + q$ when $\ell \geq 1$ and $q \geq 1$ requires $\Omega(n/((\ell + q)\log^3 n))$ rounds.*

In the RAM model, the Orthogonal Vectors Hypothesis (OVH) states that there is no algorithm that decides whether a given orthogonal vectors instance contains an orthogonal pair in time $O(n^{2-\delta}\,\mathrm{poly}(d))$ for some constant $\delta > 0$. Under this hardness assumption, our reduction has the following straightforward implication.

▶ **Corollary 5.** *In the RAM model, under OVH, there is no algorithm distinguishing between graphs of diameter $2\ell + q$ and graphs of diameter $3\ell + q$ when $\ell \geq 1$ and $q \geq 0$ in time $O(m^{2-\delta}/(\ell + q)^{2-\delta})$ for any constant $\delta > 0$.*

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