On Expressiveness of Halpern-Shoham Logic and its Horn Fragments*

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Abstract

Halpern and Shoham's modal logic of time intervals (HS in short) is an elegant and highly influential propositional interval-based logic. Its Horn fragments and their hybrid extensions have been recently intensively studied and successfully applied in real-world use cases. Detailed investigation of their decidability and computational complexity has been conducted, however, there has been significantly less research on their expressive power. In this paper we make a step towards filling this gap. We (1) show what time structures are definable in the language of HS, and (2) determine which HS fragments are capable of expressing: hybrid machinery, i.e., *nominals* and *satisfaction operators*, and *somewhere*, *difference*, and *everywhere* modal operators. These results enable us to classify HS Horn fragments according to their expressive power and to gain insight in the interplay between their decidability/computational complexity and expressiveness.

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1 Introduction

The aim of this paper is to investigate the expressive power of the temporal logic of Halpern and Shoham (HS in short) [13] and its Horn fragments [9, 8]. The latter are especially interesting due to their relatively low computational complexity [9, 17, 3] and the range of potential applications, e.g, real-world use cases in temporal ontology-based data access (OBDA) [14]. Although decidability and computational complexity of these fragments have been intensively studied [9, 17, 3], their expressive power is yet to be studied in any significant depth. Our research aims at filling this gap and enabling a better understanding of the interplay between decidability/computational complexity and expressive power of HS fragments.

Halpern-Shoham logic is a propositional multimodal logic which enables reasoning about relations between time-intervals in a one dimensional timeline. The HS language contains 12 modal operators, each corresponding to one of the Allen's binary relations between intervals [1], namely *adjacent to*, *begins*, *during*, *ends*, *later than*, *overlaps*, and their inverses (Allen's algebra contains also *identity* as the 13th relation). A model of HS is a linear ordering of time-points (a temporal frame), where propositional variables are interpreted by sets of intervals over this temporal frame. The HS language is very expressive and the satisfiability problem of its formulas is undecidable over a range of interesting linear orders including $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ [13]. As a result, restrictions on HS have been intensively investigated in order to establish fragments of relatively low computational complexity, whose expressive

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Table 1 Semantics definable in HS under irreflexive and reflexive semantics, respectively.

	Under irreflexive semantics:	**Under reflexive semantics:**
Definable semantics:	(Dis), (Den), (S), (Non-S)	(Non-S)

power is high enough for a variety of applications. A number of methods to specify HS fragments have been proposed, e.g., restricting the set of modal operators occurring in the language [11, 12, 6], softening semantics of modal operators [15], and restricting the nesting-depth of modal operators [7]. Recently, a twofold method has been proposed to obtain HS fragments [9]: firstly, by imposing restrictions on the use of classical propositional connectives in the language, giving rise to fragments called *Horn*, *Krom*, and *core*, and secondly, by additionally disallowing diamond (or box) modal operators. This new approach has led to the identification of tractable fragments (precisely P-complete) [3], which were already applied in real-world use cases within temporal OBDA [14]. The success of this approach motivated applying the same technique in other logics in order to establish their low complexity fragments. Namely, it was applied in modal logics K, T, K4, S4 [10], and in Metric Temporal Logic [5].

In this paper we investigate the expressiveness of HS and its Horn fragments, and show how it depends on the structure of a temporal frame. Namely, we will follow an idea from [8] and distinguish between HS-models (i) with irreflexive ($<$) (originally introduced by Halpern and Shoham in [13]) and reflexive ($\leq$) (obtained by softening semantics as described in [8, 7]) semantics of relations between intervals, (ii) over discrete (Dis) and dense (Den) frames, and (iii) under strict (S) (i.e., without punctual-intervals) and non-strict (Non-S) (i.e., with punctual-intervals) semantics. Combinations of the above 3 lines of distinction give us 8 distinct semantics. A precise description of HS, its Horn fragments, as well as all 8 semantics, is presented in Section 2. Our contributions are as follows.

First, in Section 3 we study which semantics are definable in the language of HS, where a semantics is definable by a formula, if the formula is true exactly in frames satisfying conditions imposed on this semantics. We show that if the irreflexive semantics of relations is assumed, then not only (Dis) and (Den) semantics are definable (as proved in [13]) but also (S), and (Non-S). Moreover, we show that under reflexive semantics (Non-S) is definable but it is an open question if (Dis), (Den), and (S) are definable or not – see Table 1.

Second, in Section 4 we study the expressive power of HS, Horn fragment HS_{horn}, and its further restrictions, namely $\mathsf{HS}^{\Diamond}_{horn}$ obtained by deleting box modal operators from the language and $\mathsf{HS}^{\Box}_{horn}$ in which diamond operators are deleted. We show which of the following expressions are expressible in these languages: *somewhere* p ($\mathsf{E}p$), *in a different interval* p ($\mathsf{D}p$), and *everywhere* p ($\mathsf{A}p$), where p is a propositional variable. Moreover, we show in which fragments *nominals* (i), and *satisfaction operators* ($@_i$) are expressible – see Table 2. Nominals and satisfaction operators constitute a standard hybrid machinery exploited in order to overcome the local nature of a modal language [2, 4].

Furthermore, we show that the expressive power of $\mathsf{HS}^{\Box,i}_{horn}$ (which stands for $\mathsf{HS}^{\Box}_{horn}$ whose language is extended with nominals) and $\mathsf{HS}^{\Box,i,@}_{horn}$ (i.e., $\mathsf{HS}^{\Box,i}_{horn}$ extended by satisfaction operators) is the same in any semantics. A Hasse diagram of HS fragments is depicted in Figure 1a where an arrow indicates a syntactical extension. Our research resulted in a classification of these fragments according to their expressive power. A map of expressiveness under all semantics except ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S) is presented in Figure 1b, where $\approx$ stands for the same expressive power and $\succcurlyeq$ for greater-or-equal expressive power.

Table 2 Summarized expressiveness results.

	Ep	Dp	Ap	i	i and $@_i$
HS	✓	✓*	✓	✓*	✓*
HS_{horn}	✓	?	✓	✓*	✓*
$\mathsf{HS}^{\Diamond}_{horn}$	✓	–	✓	✓*	✓*
$\mathsf{HS}^{\Box}_{horn}$	–*	–*	✓	–*	–*

✓ : definable in all semantics;
✓* : definable in all semantics except ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S);
– : undefinable in any semantics;
–* : undefinable in ($<$,Den,S), ($<$,Den,Non-S), ($\leq$,Dis,Non-S), ($\leq$,Den,S), ($\leq$,Den,Non-S);
? : unknown.

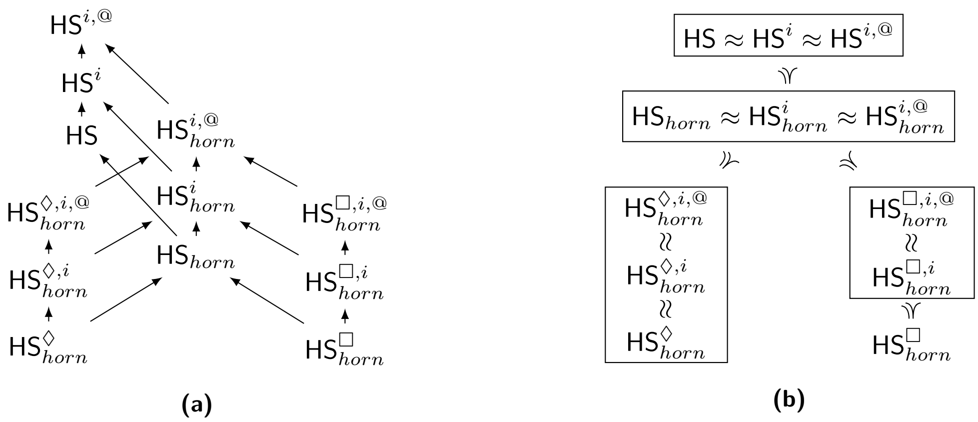


Figure 1 Syntactical dependencies of HS fragments (a) and expressive power dependencies in HS fragments under all semantics except ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S) (b).

2 Halpern-Shoham Logic and its Horn Fragments

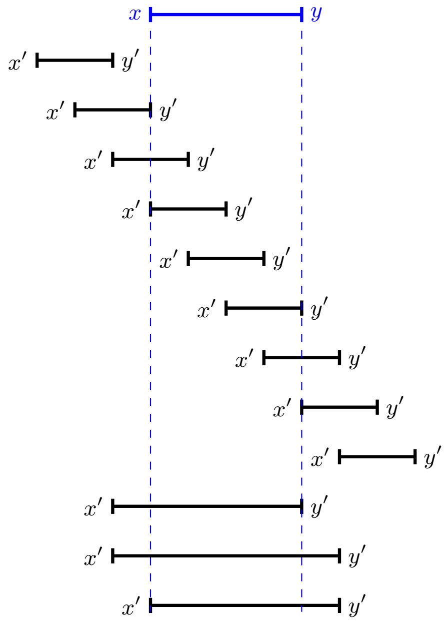
The language of Halpern-Shoham logic consists of a set of propositional variables PROP, propositional constants $\top$ (true) and $\bot$ (false), classical propositional connectives $\neg, \wedge, \vee, \rightarrow$, 12 modal operators of the form $\langle \mathsf{R} \rangle$, called *diamonds*, and their duals of the form $[\mathsf{R}]$, called *boxes*, where $\mathsf{R} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{D}, \overline{\mathsf{D}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{O}, \overline{\mathsf{O}}, \mathsf{A}, \overline{\mathsf{A}}, \mathsf{L}, \overline{\mathsf{L}}\}$ (in what follows, we denote this set by $\mathsf{HS}_{\mathsf{rel}}$). Well-formed HS-formulas are defined by the following abstract grammar:

$$\varphi := \top \mid \bot \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \langle \mathsf{R} \rangle \varphi \mid [\mathsf{R}]\varphi,$$

where $p \in \mathsf{PROP}$ and $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$. We follow [8] and define an HS-model as a pair $(\mathbb{D}, V)$ such that $\mathbb{D} = (D, \leq)$, called a *temporal frame* (or simply a *frame*), is a non-strict linear order (reflexive, antisymmetric, transitive, and total relation) of time-points, $I(\mathbb{D})$ is a set of intervals over $\mathbb{D}$ (defined in what follows), and $V : \mathsf{PROP} \rightarrow \mathcal{P}(I(\mathbb{D}))$ assigns a set of intervals to each propositional variable (notice that the original definition from [13] of an HS-model is more general). In what follows, we also use "$x < y$" as an abbreviation for "$x \leq y$ and $x \neq y$". In the paper we restrict attention to orderings $\mathbb{D}$ which contain an infinitely ascending and descending chains. As proposed in [8], semantics of HS may be specified according to the following lines of division. The first distinction deals with a definition of binary relations between intervals, namely *begins* (rel_B), *during* (rel_D), *ends* (rel_E), *overlaps* (rel_O), *adjacent to* (rel_A), *later than* (rel_L), and opposite relations: $\mathsf{rel}_{\overline{\mathsf{B}}}$, $\mathsf{rel}_{\overline{\mathsf{D}}}$, $\mathsf{rel}_{\overline{\mathsf{E}}}$, $\mathsf{rel}_{\overline{\mathsf{O}}}$, $\mathsf{rel}_{\overline{\mathsf{A}}}$, $\mathsf{rel}_{\overline{\mathsf{L}}}$ ("dashed" relations are not always inverses of their "not dashed" counterparts, e.g., $\mathsf{rel}_{\overline{\mathsf{A}}}$ is not an inverse of rel_A in non-strict semantics):

(<) *Irreflexive* semantics: a relation between intervals $[x, y]$ and $[x', y']$ is defined as:

$[x, y]\mathsf{rel}_{\overline{\mathsf{L}}}[x', y']$	iff	$y' < x$
$[x, y]\mathsf{rel}_{\overline{\mathsf{A}}}[x', y']$	iff	$x' < y',\ y' = x$
$[x, y]\mathsf{rel}_{\overline{\mathsf{O}}}[x', y']$	iff	$x' < x < y' < y$
$[x, y]\mathsf{rel}_\mathsf{B}[x', y']$	iff	$x = x',\ y' < y$
$[x, y]\mathsf{rel}_\mathsf{D}[x', y']$	iff	$x < x',\ y' < y$
$[x, y]\mathsf{rel}_\mathsf{E}[x', y']$	iff	$x < x',\ y = y'$
$[x, y]\mathsf{rel}_\mathsf{O}[x', y']$	iff	$x < x' < y < y'$
$[x, y]\mathsf{rel}_\mathsf{A}[x', y']$	iff	$y = x',\ x' < y'$
$[x, y]\mathsf{rel}_\mathsf{L}[x', y']$	iff	$y < x'$
$[x, y]\mathsf{rel}_{\overline{\mathsf{E}}}[x', y']$	iff	$x' < x,\ y = y'$
$[x, y]\mathsf{rel}_{\overline{\mathsf{D}}}[x', y']$	iff	$x' < x,\ y < y'$
$[x, y]\mathsf{rel}_{\overline{\mathsf{B}}}[x', y']$	iff	$x = x',\ y < y'$



(≤) *Reflexive* semantics: each occurrence of "$<$" is replaced by "$\leq$" with respect to the definition of relations under irreflexive semantics.

The second distinction is between:

(Dis) *Discrete* frames: any $x \in D$ has an immediate $<$-successor, and an immediate $<$-predecessor;

(Den) *Dense* frames: for any $x, y \in D$ such that $x < y$ there is $z \in D$ such that $x < z < y$.

The third distinction differentiates between:

(S) *Strict* semantics: punctual intervals are disallowed, i.e., a set of all intervals over $\mathbb{D}$ is defined as $I(\mathbb{D}) = \{[x, y] \mid x, y \in D \text{ and } x < y\}$;

(Non-S) *Non-Strict* semantics: punctual intervals are allowed, i.e., a set of all intervals over $\mathbb{D}$ is defined as $I(\mathbb{D}) = \{[x, y] \mid x, y \in D \text{ and } x \leq y\}$;

where $[x, y] = \{z \mid z \in D \text{ and } x \leq z \leq y\}$. Independently of the semantics, the relations between non-identical intervals are exhaustive in the sense that between any two non-identical intervals necessarily holds some relation. In irreflexive semantics the relations are also disjoint, and consequently exactly one relation holds between any two non-identical intervals.

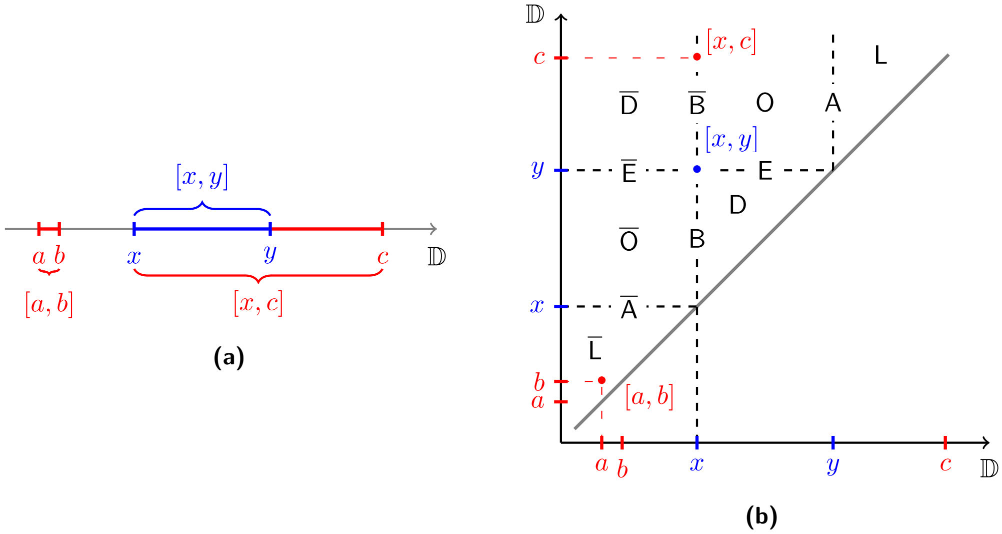


Figure 2 One-dimensional (a) and two-dimensional (b) representations of the same HS-model, in which $[x,y]\overline{\mathsf{L}}[a,b]$ ($[a,b]$ is earlier than $[x,y]$) and $[x,y]\overline{\mathsf{B}}[x,c]$ ($[x,c]$ is begun by $[x,y]$).

The satisfaction relation for an HS-model $\mathcal{M}$ and an interval $[x,y]$ is defined as follows:

$$
\begin{array}{lll}
\mathcal{M},[x,y] \models \top & & \text{for all } [x,y] \in I(\mathbb{D});\\
\mathcal{M},[x,y] \not\models \bot & & \text{for all } [x,y] \in I(\mathbb{D});\\
\mathcal{M},[x,y] \models p & \text{iff} & [x,y] \in V(p), \text{ for any } p \in \mathsf{PROP};\\
\mathcal{M},[x,y] \models \neg\varphi & \text{iff} & \mathcal{M},[x,y] \not\models \varphi;\\
\mathcal{M},[x,y] \models \varphi \wedge \psi & \text{iff} & \mathcal{M},[x,y] \models \varphi \text{ and } \mathcal{M},[x,y] \models \psi;\\
\mathcal{M},[x,y] \models \varphi \vee \psi & \text{iff} & \mathcal{M},[x,y] \models \varphi \text{ or } \mathcal{M},[x,y] \models \psi;\\
\mathcal{M},[x,y] \models \varphi \rightarrow \psi & \text{iff} & \text{if } \mathcal{M},[x,y] \models \varphi, \text{ then } \mathcal{M},[x,y] \models \psi;\\
\mathcal{M},[x,y] \models \langle \mathsf{R} \rangle\varphi & \text{iff} & \text{there exists } [x',y'] \in I(\mathbb{D}) \text{ such that } [x,y]\mathsf{rel}_{\mathsf{R}}[x',y']\\
& & \text{and } \mathcal{M},[x',y'] \models \varphi;\\
\mathcal{M},[x,y] \models [\mathsf{R}]\varphi & \text{iff} & \text{for every } [x',y'] \in I(\mathbb{D}) \text{ such that } [x,y]\mathsf{rel}_{\mathsf{R}}[x',y']\\
& & \text{it holds that } \mathcal{M},[x',y'] \models \varphi;
\end{array}
$$

for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$, and any HS-formulas φ, ψ. An HS-formula φ is true in an HS-model $\mathcal{M}$ (in symbols: $\mathcal{M} \models \varphi$) iff for all $[x,y] \in I(\mathbb{D})$ it holds that $\mathcal{M},[x,y] \models \varphi$. An HS-formula φ is true in a frame $\mathbb{D}$ (in symbols: $\mathbb{D} \models \varphi$) iff for any HS-model $\mathcal{M}$ based on $\mathbb{D}$ it holds that $\mathcal{M} \models \varphi$. An HS-formula φ is valid (in symbols: $\models \varphi$) iff for any HS-frame $\mathbb{D}$ we have $\mathbb{D} \models \varphi$.

A convenient representation of a temporal frame (e.g., for decidability and computational complexity proofs [8]) is obtained by treating an interval $[x,y]$ as a point in a two-dimensional Cartesian space $D \times D$, where the abscissa has a value x and the ordinate has a value y [16]. In the two-dimensional representation, intervals correspond to the points in the north-western half-plane of $D \times D$, and points on the diagonal correspond to punctual-intervals. In such a representation relations between intervals obtain a spatial interpretation. Let $[x,y]$ be a fixed interval, then a relation between $[x,y]$ and any other interval $[x',y']$ may be determined on the basis of a relative position of points (x,y) and (x',y') as presented in Figure 2.

Decidability of the HS-formulas satisfiability problem (HS-satisfiability) depends on the semantics. However, for most interesting frames it is undecidable, e.g., it was already shown in [13] that the problem is undecidable under irreflexive and non-strict semantics for any class of temporal frames that contains an infinite ascending chain (e.g., $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$). These negative results motivated searching for decidable and tractable fragments of HS which would be still interesting from the expressiveness point of view.

As observed in [8], any HS-formula can be transformed into an equisatisfiable formula defined by the following grammar, and vice versa:

$$\varphi := \lambda \mid \neg\lambda \mid [\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \to \lambda \vee \ldots \vee \lambda) \mid \varphi \wedge \varphi, \tag{1}$$

where $[\mathsf{U}]$ is the universal modality, i.e., $[\mathsf{U}]\psi$ is satisfied iff ψ is satisfied in every $[x, y] \in I(\mathbb{D})$ whereas λ, the so-called *positive temporal literal*, is a formula defined by the grammar:

$$\lambda := \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda \mid [\mathsf{R}]\lambda, \tag{2}$$

where $p \in \mathsf{PROP}$ and $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$. Horn fragments of HS (proposed in [9, 8]) are obtained by imposing limitations on the grammars (1) and (2) as follows:

- First, the limitation of the grammar (1) gives rise to an HS fragment denoted by HS_{horn}, in which (1) is restricted to the grammar:

  $$\varphi := \lambda \mid [\mathsf{U}](\lambda \wedge \ldots \wedge \lambda \to \lambda) \mid \varphi \wedge \varphi.$$

- Second, additional limitations on the grammar of positive temporal literals (2) in HS_{horn} give rise to fragments denoted by $\mathsf{HS}^{\Diamond}_{horn}$ and $\mathsf{HS}^{\Box}_{horn}$. In the case of $\mathsf{HS}^{\Diamond}_{horn}$ the grammar of positive temporal literals is restricted to:

  $$\lambda := \top \mid \bot \mid p \mid \langle \mathsf{R} \rangle \lambda,$$

  whereas in the case $\mathsf{HS}^{\Box}_{horn}$ grammar (2) is restricted to:

  $$\lambda := \top \mid \bot \mid p \mid [\mathsf{R}]\lambda.$$

$\mathsf{HS}^{\Box}_{horn}$ is particularly interesting, as it is both decidable and tractable (P-complete) under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S) semantics [9, 8].

3 Defining Semantics

A formula φ defines a semantics *Sem* if the following are equivalent:

- φ is true in a frame $\mathbb{D}$;
- $\mathbb{D}$ satisfies the conditions imposed by *Sem*.

We will show that under the standard, i.e., irreflexive semantics (<), the semantics (Dis), (Den), (S), and (Non-S) are definable but under the softened semantics, i.e., reflexive (≤) we only know how to define (Non-S). We treat existence and non-existence of punctual-intervals as a property of a frame. At first assume that the semantics is irreflexive. As showed in [13], (Dis) is definable in the language of HS by the formula:

$$[\mathsf{B}]\bot \vee (\langle \mathsf{B} \rangle \top \wedge [\mathsf{B}][\mathsf{B}]\bot) \vee \Big(\langle \mathsf{B} \rangle (\langle \mathsf{B} \rangle \top \wedge [\mathsf{B}][\mathsf{B}]\bot) \wedge \langle \mathsf{E} \rangle (\langle \mathsf{B} \rangle \top \wedge [\mathsf{B}][\mathsf{B}]\bot)\Big);$$

and (Den) by the formula:

$$\neg(\langle \mathsf{B} \rangle \top \wedge [\mathsf{B}][\mathsf{B}]\bot).$$

We show that (S) and (Non-S) are also definable in the language of HS.

▶ **Theorem 1.** (S) *is definable in the language of* HS *under irreflexive semantics by:*

$$\varphi_{<,S} := \langle \mathsf{A} \rangle [\mathsf{B}]\bot \vee [\mathsf{A}]\neg \langle \mathsf{B} \rangle [\mathsf{B}]\bot.$$

Proof. Fix any HS-frame $\mathbb{D}$. First, assume that $\mathbb{D} \models \varphi_{<,\mathsf{S}}$. We will show that the semantics is strict. Fix any time-point y. To show that $[y, y]$ is not an interval. Fix any $x < y$ and any HS-model $\mathcal{M}$ based on $\mathbb{D}$. It follows that $\mathcal{M}, [x, y] \models \varphi_{<,\mathsf{S}}$.

(Case 1) $\mathcal{M}, [x, y] \models \langle \mathsf{A} \rangle [\mathsf{B}] \bot$. Hence, there is $z > y$ such that $\mathcal{M}, [y, z] \models [\mathsf{B}] \bot$. If $[y, y]$ was an interval, we would have $\mathcal{M}, [y, z] \not\models [\mathsf{B}] \bot$. Hence, $[y, y]$ cannot be an interval.

(Case 2) $\mathcal{M}, [x, y] \models [\mathsf{A}] \neg \langle \mathsf{B} \rangle [\mathsf{B}] \bot$. Suppose that $[y, y]$ is an interval, so $\mathcal{M}, [y, y] \models [\mathsf{B}] \bot$. Hence, for any $z > y$ we have $[y, z] \mathsf{rel}_{\mathsf{B}} [y, y]$. Then $\mathcal{M}, [x, y] \models [\mathsf{A}] \langle \mathsf{B} \rangle [\mathsf{B}] \bot$. Since $\mathbb{D}$ contains an infinite ascending chain, we have reached a contradiction, so $[y, y]$ is not an interval.

As a result, for any time-point y, $[y, y]$ is not an interval, hence the semantics is strict.

Second, assume that the semantics is non-strict. We will show that $\mathbb{D} \not\models \varphi_{<,\mathsf{S}}$. Fix any punctual-interval $[y, y]$ and any HS-model $\mathcal{M}$ based on $\mathbb{D}$. To show that $\mathcal{M}, [y, y] \not\models \varphi_{<,\mathsf{S}}$, i.e., $\mathcal{M}, [y, y] \not\models \langle \mathsf{A} \rangle [\mathsf{B}] \bot \vee [\mathsf{A}] \neg \langle \mathsf{B} \rangle [\mathsf{B}] \bot$.

$\mathcal{M}, [y, y] \not\models \langle \mathsf{A} \rangle [\mathsf{B}] \bot$, because an interval $[u, w]$ is in relation $\mathsf{rel}_{\mathsf{A}}$ with $[y, y]$ whenever $u = y$ and $w > y$. It follows that $w > u$, so $\mathcal{M}, [u, w] \not\models [\mathsf{B}] \bot$. $\mathcal{M}, [y, y] \not\models [\mathsf{A}] \neg \langle \mathsf{B} \rangle [\mathsf{B}] \bot$, because if $z > y$ then $[y, y] \mathsf{rel}_{\mathsf{A}} [y, z]$, $[y, z] \mathsf{rel}_{\mathsf{B}} [y, y]$, and $\mathcal{M}, [y, y] \models [\mathsf{B}] \bot$. ◀

▶ **Theorem 2.** (Non-S) *is definable in the language of* HS *under irreflexive semantics by:*

$$\varphi_{<,\mathit{Non\text{-}S}} := [\mathsf{A}] \neg [\mathsf{B}] \bot \wedge [\mathsf{A}] \langle \mathsf{B} \rangle [\mathsf{B}] \bot.$$

Proof. Fix any HS-frame $\mathbb{D}$. First, assume that $\mathbb{D} \models \varphi_{<,\mathsf{Non\text{-}S}}$. To show that the semantics is non-strict. Fix any time-point y. We will show that $[y, y]$ is an interval. Fix any HS-model $\mathcal{M}$ based on $\mathbb{D}$ and $x < y$. It follows that $\mathcal{M}, [x, y] \models \varphi_{<,\mathsf{Non\text{-}S}}$.

From the one hand, $\mathcal{M}, [x, y] \models [\mathsf{A}] \neg [\mathsf{B}] \bot$, hence for any $z > y$ we have $\mathcal{M}, [y, z] \not\models [\mathsf{B}] \bot$. On the other hand, $\mathcal{M}, [x, y] \models [\mathsf{A}] \langle \mathsf{B} \rangle [\mathsf{B}] \bot$. Fix any $w > y$. Then $\mathcal{M}, [y, w] \models \langle \mathsf{B} \rangle [\mathsf{B}] \bot$. Hence, for all u such that $y < u < w$ we have $\mathcal{M}, [y, u] \models \langle \mathsf{B} \rangle [\mathsf{B}] \bot$, which leads to a contradiction

Second, assume that the semantics is strict. To show that $\mathbb{D} \not\models \varphi_{<,\mathsf{Non\text{-}S}}$. Fix any interval $[x, y]$ and any HS-model $\mathcal{M}$ based on $\mathbb{D}$. To show that $\mathcal{M}, [x, y] \not\models \varphi_{<,\mathsf{Non\text{-}S}}$.

(Case 1) y has an immediate $>$-successor. Let z be the immediate $>$-successor of y. Then $\mathcal{M}, [y, z] \models [\mathsf{B}] \bot$, so $\mathcal{M}, [x, y] \not\models [\mathsf{A}] \neg [\mathsf{B}] \bot$. Hence $\mathcal{M}, [x, y] \not\models \varphi_{<,\mathsf{Non\text{-}S}}$.

(Case 2) y does not have an immediate $>$-successor. Hence, there is no $z > y$ such that $\mathcal{M}, [y, z] \models [\mathsf{B}] \bot$. As a result $\mathcal{M}, [x, y] \not\models [\mathsf{A}] \langle \mathsf{B} \rangle [\mathsf{B}] \bot$, so $\mathcal{M}, [x, y] \not\models \varphi_{<,\mathsf{Non\text{-}S}}$. ◀

Finally, we show that under reflexive semantics HS is expressive enough to define non-strict semantics.

▶ **Theorem 3.** (Non-S) *is definable in the language of* HS *under reflexive semantics by:*

$$\varphi_{\leq,\mathit{Non\text{-}S}} := [\mathsf{E}] p \rightarrow \langle \mathsf{A} \rangle p.$$

Proof. Fix any HS-frame $\mathbb{D}$. First, assume that the semantics is non-strict. To show that $\mathbb{D} \models \varphi_{\leq,\mathsf{Non\text{-}S}}$. Fix any interval $[x, y]$ and any HS-model $\mathcal{M}$ based on $\mathbb{D}$. Assume that $\mathcal{M}, [x, y] \models [\mathsf{E}] p$. It follows that $\mathcal{M}, [y, y] \models p$. Since $[x, y] \mathsf{rel}_{\mathsf{A}} [y, y]$, we have $\mathcal{M}, [x, y] \models \langle \mathsf{A} \rangle p$. It follows that $\mathbb{D} \models \varphi_{\leq,\mathsf{Non\text{-}S}}$.

Second, assume that the semantics is strict. We will show that $\mathbb{D} \not\models \varphi_{\mathsf{S}}$. Fix any interval $[x, y]$ and any HS-model $\mathcal{M} = (\mathbb{D}, V)$ such that $V(p) = \{[z, y] \mid z \geq x\}$. To show that $\mathcal{M}, [x, y] \not\models \varphi_{\geq,\mathsf{Non\text{-}S}}$. By the definition of V it follows that $\mathcal{M}, [x, y] \models [\mathsf{E}] p$. However, $\mathcal{M}, [x, y] \not\models \langle \mathsf{A} \rangle p$, so $\mathcal{M}, [x, y] \not\models \varphi_{\geq,\mathsf{Non\text{-}S}}$. ◀

4 Hybrid Machinery and Additional Operators

In this section we will study whether HS and its Horn fragments are expressive enough to define hybrid machinery (*nominals* and *satisfaction operators*) and the *somewhere* (E), *difference* (D), and *universal* (A) operators (notice that, following the standard notation, we use symbols E, D, and A in two meanings, namely as elements of $\mathsf{HS}_{\mathsf{rel}}$ and as above mentioned modal operators). The satisfaction relation in an HS-model $\mathcal{M}$ and an interval $[x, y]$ is defined for E, D, and A as follows:

$$\begin{array}{lll} \mathcal{M}, [x,y] \models \mathsf{E}\varphi & \text{iff} & \text{there is } [x',y'] \in I(\mathbb{D}) \text{ such that } \mathcal{M}, [x',y'] \models \varphi; \\ \mathcal{M}, [x,y] \models \mathsf{D}\varphi & \text{iff} & \text{there is } [x',y'] \in I(\mathbb{D}) \text{ such that} \\ & & [x',y'] \neq [x,y] \text{ and } \mathcal{M}, [x',y'] \models \varphi; \\ \mathcal{M}, [x,y] \models \mathsf{A}\varphi & \text{iff} & \text{for all } [x',y'] \in I(\mathbb{D}) \text{ it holds that } \mathcal{M}, [x',y'] \models \varphi; \end{array}$$

where φ is any HS-formula. Notice that A has the same meaning as [U] occurring in (1).

Hybrid machinery is obtained by enriching the language with the second sort of atoms, called *nominals* (we denote the set of all nominals by NOM) and *satisfaction operators* $@_i$ indexed by nominals. Intuitively, a nominal is a special kind of atom which is satisfied in exactly one interval, whereas an expression of the form $@_i\varphi$ is satisfied if φ is satisfied in the interval in which i is satisfied.

Formally, a hybrid HS-model $\mathcal{M}$ is a pair $(\mathbb{D}, V)$, such that $V : \mathsf{ATOM} \to \mathcal{P}(I(\mathbb{D}))$ assigns a set of intervals to each atom ($\mathsf{ATOM} = \mathsf{PROP} \cup \mathsf{NOM}$) with an additional restriction that $V(i)$ is a singleton for any $i \in \mathsf{NOM}$. The satisfaction relation conditions for nominals and @ operators are as follows:

$$\begin{array}{lll} \mathcal{M}, [x,y] \models i & \text{iff} & V(i) = \{[x,y]\}, \text{ for any } i \in \mathsf{NOM}; \\ \mathcal{M}, [x,y] \models @_i\varphi & \text{iff} & \mathcal{M}, [x',y'] \models \varphi, \text{ where } V(i) = \{[x',y']\} \text{ and } i \in \mathsf{NOM}. \end{array}$$

Hybrid extensions of HS fragments were introduced in [17] and their Hasse diagram is depicted in Figure 1a, where "i" in the superscript of a fragment's symbol means that an expression of the form i (for $i \in \mathsf{NOM}$) is added to the grammar of positive temporal literals, whereas "$i, @$" in the superscript denotes a further extension obtained by adding an expression of the form $@_i\lambda$ (for $i \in \mathsf{NOM}$) to the grammar of positive temporal literals.

Interestingly, it has been shown in [17] that the satisfiability problems in $\mathsf{HS}^{\Box,i,@}_{horn}$ and in $\mathsf{HS}^{\Box,i}_{horn}$ are NP-complete under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S) semantics, whereas the satisfiability problem in $\mathsf{HS}^{\Box}_{horn}$ is known to be P-complete under these semantics [9, 8]. However, differences in expressive power of $\mathsf{HS}^{\Box}_{horn}$, $\mathsf{HS}^{\Box,i}_{horn}$, and $\mathsf{HS}^{\Box,i,@}_{horn}$ have, thus far, not been investigated.

4.1 The Case of Full Halpern-Shoham Logic

Our aim is to determine if the hybrid machinery is definable in the full HS language, and whether the choice of semantics affects the answer to this problem. We start by recalling a general result established in [2], stating that we can eliminate all occurrences of nominals and @ operators in a hybrid modal logic formula by simulating them using the D operator .

▶ **Theorem 4** (Areces, Blackburn, Marx [2]). *There is a polynomial reduction which preserves satisfaction from any hybrid language enabling unrestricted use of classical propositional connectives, containing nominals and @ operators, to the fragment without nominals and @ operators but enabling to polynomially define* D.

Theorem 4 holds if there are no restrictions on the use of classical connectives. As noticed in [2] in the case of HS under irreflexive semantics, i.e., (<,Dis,S), (<,Den,S), (<,Dis,Non-S), and (<,Den,Non-S), operator the D is definable as follows:

$$\mathsf{D}\varphi := \bigvee_{\mathsf{R}\in\mathsf{HS}_{\mathsf{rel}}} \langle\mathsf{R}\rangle\varphi. \tag{3}$$

Hence, by Theorem 4 nominals and @ operators are definable in the language of HS under any irreflexive semantics. However, in the reflexive semantics the definition (3) would not be correct because it is not the case that $\mathsf{rel}_{\mathsf{R}}$ is irreflexive for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$. Nevertheless, under (≤,Dis,S) and (≤,Den,S) we can define D as follows:

$$\mathsf{D}\varphi := \langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\langle\mathsf{A}\rangle\varphi \vee \langle\overline{\mathsf{A}}\rangle\langle\overline{\mathsf{B}}\rangle\langle\mathsf{B}\rangle\varphi \vee \langle\mathsf{A}\rangle\langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\varphi \vee \langle\overline{\mathsf{E}}\rangle\langle\mathsf{E}\rangle\langle\overline{\mathsf{A}}\rangle\varphi. \tag{4}$$

The key is to observe that under (≤,Dis,S) and (≤,Den,S) if $[x,y]\mathsf{rel}_{\mathsf{A}}[x',y']$ then $y' > y$. Moreover, if $[x,y]\mathsf{rel}_{\overline{\mathsf{A}}}[x',y']$ then $x' < x$. To describe what formula (4) means let the current interval be $[x,y]$. Then (4) states that φ holds in some $[x',y']$ such that (i) $x' > x$, or (ii) $x' < x$, or (iii) $y' > y$, or (iv) $y' < y$. As a result, (4) states that φ holds in some $[x',y']$ such that $[x',y'] \neq [x,y]$.

Next, we will show that in the remaining semantics, i.e., (≤,Dis,Non-S) and (≤,Den,Non-S) neither D nor nominals are definable. For this purpose, we will construct a *bisimulation* (see, e.g., [4, Chapter 2]), which is a relation between models, which makes them indistinguishable by any modal formula. We say that HS-models $\mathcal{M} = (\mathbb{D}, V)$ and $\mathcal{M}' = (\mathbb{D}', V')$ are bisimilar (in symbols $\mathcal{M} \leftrightarrows \mathcal{M}'$) if there is a relation (a bisimulation) $Z \subseteq I(\mathbb{D}) \times I(\mathbb{D}')$, which satisfies the following conditions:

(atom) For any intervals $[x,y]$, $[x',y']$ such that $[x,y]Z[x',y']$ it holds that $\mathcal{M},[x,y] \models p$ iff $\mathcal{M}',[x',y'] \models p$, for any $p \in \mathsf{PROP}$;

(zig) If $[x,y]Z[x',y']$ and $[x,y]\mathsf{rel}_{\mathsf{R}}[u,w]$, then there exists $[u',w']$ (in $\mathcal{M}'$) such that $[x',y']\mathsf{rel}_{\mathsf{R}}{}'[u',w']$ and $[u,w]Z[u',w']$, for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$;

(zag) If $[x,y]Z[x',y']$ and $[x',y']\mathsf{rel}_{\mathsf{R}}{}'[u',w']$, then there exists $[u,w]$ (in $\mathcal{M}$) such that $[x,y]\mathsf{rel}_{\mathsf{R}}[u,w]$ and $[u,w]Z[u',w']$, for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$.

It is easy to show by induction on an HS-formula construction that HS is *invariant for bisimulation* in the following sense.

▶ **Lemma 5.** *Let $\mathcal{M}$, $\mathcal{M}'$ be any* HS*-models, and Z a bisimulation between them. For any intervals $[x,y]$, $[x',y']$ if $[x,y]Z[x',y']$, then for any* HS*-formula φ the following are equivalent:*

1. *$\mathcal{M},[x,y] \models \varphi$;*
2. *$\mathcal{M}',[x',y'] \models \varphi$.*

▶ **Lemma 6.** *Nominals are not definable in* HS *under* (≤,Dis,Non-S) *and* (≤,Den,Non-S).

Proof. Fix an HS-model $\mathcal{M} = (\mathbb{D}, V)$ under (≤,Dis,Non-S) or (≤,Den,Non-S) such that in the case of the former semantics $\mathbb{D}$ is $\mathbb{Z}$ (i.e., the standard ordering of integers), whereas in the latter case $\mathbb{D}$ is $\mathbb{Q}$ (i.e., the standard ordering of rational numbers). Fix a nominal and towards a contradiction suppose that there is an HS-formula φ which under (≤,Dis,Non-S) or (≤,Den,Non-S) enables us to simulate this nominal. It follows that φ is satisfied in exactly one interval in $\mathcal{M}$, say $[a,b]$.

In what follows, we will construct an HS-model $\mathcal{M}' = (\mathbb{D}', V')$ and a bisimulation Z between $\mathcal{M}$ and $\mathcal{M}'$ such that $[a,b]$ is bisimilar with more than one interval. Then, by Lemma 5, φ is satisfied in more than one interval in $\mathcal{M}'$ hence we will obtain a contradiction with the statement that φ simulates a nominal.

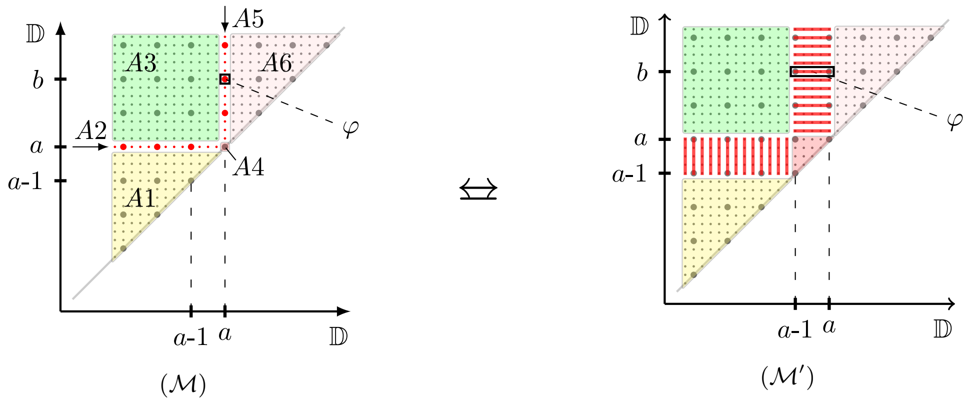


Figure 3 Bisimulation Z between models $\mathcal{M}$ and $\mathcal{M}'$.

First, we divide $\mathbb{D}$ into areas $A1$–$A6$ as follows:

$[x, y] \in A1$ iff $(x < a$ and $y < a)$;
$[x, y] \in A2$ iff $(x < a$ and $y = a)$;
$[x, y] \in A3$ iff $(x < a$ and $y > a)$;
$[x, y] \in A4$ iff $(x = a$ and $y = a)$;
$[x, y] \in A5$ iff $(x = a$ and $y > a)$;
$[x, y] \in A6$ iff $(x > a$ and $y > a)$.

Let $\mathcal{M}' = (\mathbb{D}', V')$ be such that $\mathbb{D}' = \mathbb{D}$. Then, we exploit areas $A1$–$A6$ to define the intended bisimulation $Z \subseteq I(\mathbb{D}) \times I(\mathbb{D}')$ between intervals in $\mathcal{M}$ and $\mathcal{M}'$ (see Figure 3) as follows (we use a standard functional notation below, i.e., $Z([x, y]) = \{[x', y'] \mid [x, y]Z[x', y']\}$):

$$\begin{aligned}
&\forall [x, y] \in A1 \quad Z([x, y]) = \{[x-1, y-1]\};\\
&\forall [x, y] \in A2 \quad Z([x, y]) = \{[x, y'] \mid a-1 \leq y' \leq a\};\\
&\forall [x, y] \in A3 \quad Z([x, y]) = \{[x-1, y]\};\\
&\forall [x, y] \in A4 \quad Z([x, y]) = \{[x', y'] \mid a-1 \leq x' \leq a, \text{ and } a-1 \leq y' \leq a\};\\
&\forall [x, y] \in A5 \quad Z([x, y]) = \{[x', y] \mid a-1 \leq x' \leq a\};\\
&\forall [x, y] \in A6 \quad Z([x, y]) = \{[x, y]\}.
\end{aligned}$$

To finish defining $\mathcal{M}'$ let V' be such that for any $[x, y] \in I(\mathbb{D})$ and any $p \in \mathsf{PROP}$:

$$[x, y] \in V'(p) \quad \text{iff} \quad Z^{-1}[x, y] \in V(p).$$

Let's check if Z is a bisimulation between $\mathcal{M}$ and $\mathcal{M}'$. Condition (atom) follows directly from the definition of V'. To show that (zig) and (zag) hold for Z, observe that we may restrict the set of modal operators in the language of HS to $\langle \mathsf{R} \rangle$ and $[\mathsf{R}]$ such that $\mathsf{R} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$. Other operators are definable as follows:

$\langle \mathsf{D} \rangle \varphi := \langle \mathsf{E} \rangle \langle \mathsf{B} \rangle \varphi$; $\langle \mathsf{O} \rangle \varphi := \langle \mathsf{E} \rangle \langle \overline{\mathsf{B}} \rangle \varphi$; $\langle \mathsf{L} \rangle \varphi := \langle \mathsf{A} \rangle \langle \mathsf{E} \rangle \varphi$;
$\langle \overline{\mathsf{D}} \rangle \varphi := \langle \overline{\mathsf{E}} \rangle \langle \overline{\mathsf{B}} \rangle \varphi$; $\langle \overline{\mathsf{O}} \rangle \varphi := \langle \mathsf{B} \rangle \langle \overline{\mathsf{E}} \rangle \varphi$; $\langle \overline{\mathsf{L}} \rangle \varphi := \langle \overline{\mathsf{A}} \rangle \langle \mathsf{B} \rangle \varphi$;

where φ is any HS-formula and the translation for box modalities is obtained by replacing $\langle \mathsf{R} \rangle$ with $[\mathsf{R}]$ for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$ in the above definitions. Hence it remains to perform a routine inspection of all $\mathsf{rel}_{\mathsf{R}}$ such that $\mathsf{R} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$ against (zig) and (zag) conditions. We leave this inspection to the reader.

Hence, φ does not simulate a nominal. It follows that a nominal cannot be defined in HS under ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S). ◀

Let us summarize the results obtained so far in this subsection.

▶ **Theorem 7.** *Nominals and @ operators are definable in* HS *under all semantics except* (≤,Dis,Non-S) *and* (≤,Den,Non-S) *in which even nominals are not definable.*

Notice that discreteness/density of a time frame has no influence on definability of the hybrid machinery in HS. Next, we examine whether expressions of the form Ep, Dp, and Ap, where p is a propositional variable are definable in HS and if the choice of semantics affects the answer to this problem.

▶ **Theorem 8.** *For any $p \in$ PROP expressions of the form* Ep *and* Ap *are definable in* HS *under all semantics.* Dp *is definable in* HS *under all semantics except* (≤,Dis,Non-S) *and* (≤,Den,Non-S).

Proof. Expressions of the form Ep and Ap are definable in HS as follows:

$$\mathsf{E}p := \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p; \tag{5}$$

$$\mathsf{A}p := [\mathsf{L}][\overline{\mathsf{L}}]p. \tag{6}$$

An expression of the form Dp is definable in HS under (<,Dis,S), (<,Den,S), (<,Dis,Non-S), and (<,Den,Non-S) by (3), whereas under (≤,Dis,S) and (≤,Den,S) by (4).

To show that Dp is not definable in HS under (≤,Dis,Non-S) and (≤,Den,Non-S) suppose that Dp is definable in these semantics. Then by Theorem 4 we obtain that nominals are definable in HS under (≤,Dis,Non-S) and (≤,Den,Non-S). However, by Lemma 6 we know that it is not the case, so we obtain a contradiction. ◀

4.2 The Case of Horn Fragments of Halpern-Shoham Logic

In what follows, we show that hybrid machinery is definable in $\mathsf{HS}^{\Diamond}_{horn}$ under all semantics except (≤,Dis,Non-S) and (≤,Den,Non-S). However, we can no longer use Theorem 4, which holds if there are no restrictions on the use of classical propositional connectives (¬, ∧, ∨, →). As a result, we need to conduct the proof exploiting other techniques.

▶ **Lemma 9.** *Nominals and satisfaction operators are definable in the language of* $\mathsf{HS}^{\Diamond}_{horn}$ *under* (<,Dis,S), (<,Den,S), (<,Dis,Non-S), (<,Den,Non-S), (≤,Dis,S), *and* (≤,Den,S).

Proof. Let φ be a formula in the language of $\mathsf{HS}^{\Diamond,i,@}_{horn}$. We show how to construct under (<,Dis,S), (<,Den,S), (<,Dis,Non-S), (<,Den,Non-S), (≤,Dis,S), and (≤,Den,S) an equisatisfiable formula φ' in the language of $\mathsf{HS}^{\Diamond}_{horn}$ of size linear with respect to $|\varphi|$ (where $|\varphi|$ is the length of φ).

We will simulate any nominal i occurring in φ with a propositional variable p_i. We introduce a formula ψ_i expressing that p_i is satisfied in exactly one interval. In the case of any irreflexive semantics define:

$$\psi_i := \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_i \wedge \tag{7}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{B}} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i \to \bot) \wedge \tag{8}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{E}} \rangle \langle \overline{\mathsf{B}} \rangle \langle \mathsf{B} \rangle p_i \to \bot). \tag{9}$$

In the case of (Dis,≤,S) and (Den,≤,S) define ψ_i as:

$$\psi_i := \langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_i \wedge \tag{10}$$

$$[\mathsf{U}](p_i \wedge \langle \mathsf{A} \rangle \langle \overline{\mathsf{E}} \rangle \langle \mathsf{E} \rangle p_i \to \bot) \wedge \tag{11}$$

$$[\mathsf{U}](p_i \wedge \langle \overline{\mathsf{A}} \rangle \langle \overline{\mathsf{B}} \rangle \langle \mathsf{B} \rangle p_i \to \bot). \tag{12}$$

Under any irreflexive semantics (7) states that p_i is satisfied somewhere, (8) expresses that p_i cannot be satisfied in any two intervals $[x, y]$, $[x', y']$ such that $y' > y$, whereas (9) disallows p_i being satisfied in any two intervals $[x, y]$, $[x', y']$ such that $x' < x$. Formulas (10)–(12) have the analogous meaning under ($\leq$,Dis,S) and ($\leq$,Den,S). As a result, (7)–(9) as well as (10)–(12) enable us to simulate a nominal i with a propositional variable p_i.

Importantly, in ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S) none of the above encodings enable us to state that p_i holds in exactly one interval. (7)–(9) would not work because in reflexive semantics $\mathsf{rel}_{\mathsf{E}}$, $\mathsf{rel}_{\overline{\mathsf{E}}}$, $\mathsf{rel}_{\mathsf{B}}$, and $\mathsf{rel}_{\overline{\mathsf{B}}}$ are reflexive. On the other hand, (10)–(12) would also fail because under ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S) relations $\mathsf{rel}_{\mathsf{A}}$ and $\mathsf{rel}_{\overline{\mathsf{A}}}$ are reflexive in punctual-intervals, i.e., for any interval of the form $[x, x]$ we have $[x, x]\mathsf{rel}_{\mathsf{A}}[x, x]$, and $[x, x]\mathsf{rel}_{\overline{\mathsf{A}}}[x, x]$.

Let φ_1 be obtained from φ by replacing each occurrence of a nominal i (except symbols of nominals occurring in the index of a satisfaction operator) by a propositional variable p_i and by adding to the obtained formula a conjunction of the form $\bigwedge_{i \in \mathsf{NOM}(\varphi)} \psi_i$, where $\mathsf{NOM}(\varphi)$ is a set of nominals occurring in φ.

Next, we show how to replace occurrences of satisfaction operators $@_i$ in φ_1 in order to obtain an equisatisfiable formula in the language of $\mathsf{HS}^{\Diamond}_{horn}$. The construction is by induction on the number of $@_i$ operators occurring in φ_1. In each step of the construction choose any positive temporal literal λ in the so far constructed formula, such that some satisfaction operator occurs in λ. Find the left-most satisfaction operator occurring in λ, i.e., an operator $@_i$ such that $\lambda = \langle \mathsf{R}_1 \rangle \dots \langle \mathsf{R}_n \rangle @_i \eta$ with $\langle \mathsf{R}_1 \rangle \dots \langle \mathsf{R}_n \rangle$ being a (possibly empty) sequence of diamond modal operators and η a subformula of λ. Replace this occurrence of $@_i \eta$ by $\langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_{@_i \eta}$, where $p_{@_i \eta}$ is a fresh propositional variable which did not occur in the so far constructed formula. Moreover, add to the constructed formula the following conjunction:

$$[\mathsf{U}](p_i \wedge \eta \rightarrow p_{@_i \eta}) \wedge [\mathsf{U}](p_{@_i \eta} \rightarrow p_i) \wedge [\mathsf{U}](p_{@_i \eta} \rightarrow \eta). \tag{13}$$

Formula (13) states that $p_{@_i \eta}$ is satisfied exactly in an interval in which $p_i \wedge \eta$ is satisfied. Hence, $\langle \mathsf{L} \rangle \langle \overline{\mathsf{L}} \rangle p_{@_i \eta}$ – stating that $p_{@_i \eta}$ is satisfied somewhere – is equisatisfiable with $@_i \eta$.

The construction terminates when all occurrences of satisfaction operators are eliminated and we denote the finally obtained formula by φ'. It is easy to see that φ' is equisatisfiable with φ and, since we have eliminated all occurrences of nominals and satisfaction operators, φ' is in the language of $\mathsf{HS}^{\Diamond}_{horn}$. Moreover, the size of φ' is linear in the size of φ because within the construction of φ' for each occurrence of a nominal and a satisfaction operator we have added only a constant number of symbols to the formula. ◀

As a corollary to Lemma 6 we obtain that nominals are not definable in $\mathsf{HS}^{\Diamond}_{horn}$ under ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S). Hence, we get the following result.

▶ **Theorem 10.** *Nominals and satisfaction operators are definable in* $\mathsf{HS}^{\Diamond}_{horn}$ *under all semantics except* ($\leq$,Dis,Non-S) *and* ($\leq$,Den,Non-S) *in which nominals are not definable.*

Interestingly, although nominals and satisfaction operators are definable in $\mathsf{HS}^{\Diamond}_{horn}$ (except ($\leq$,Dis,Non-S) and ($\leq$,Den,Non-S)), we will show now that D is not.

▶ **Theorem 11.** *For any* $p \in \mathsf{PROP}$ *expressions of the form* $\mathsf{E}p$ *and* $\mathsf{A}p$ *are definable in* $\mathsf{HS}^{\Diamond}_{horn}$ *but* $\mathsf{D}p$ *is not. This result holds for all semantics.*

Proof Sketch. An expression of the form $\mathsf{E}p$ is definable by (5), whereas $\mathsf{A}p$ is definable as follows: $\mathsf{A}p := [\mathsf{U}](\top \rightarrow p)$. To show that an expression of the form $\mathsf{D}p$ is not definable

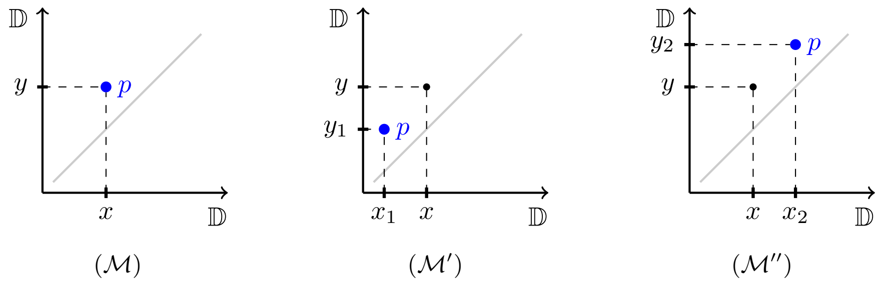


Figure 4 Isomorphic HS-models $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$.

in $\mathsf{HS}^{\Diamond}_{horn}$ consider HS-models $\mathcal{M} = (\mathbb{D}, V)$, $\mathcal{M}' = (\mathbb{D}', V')$, and $\mathcal{M}'' = (\mathbb{D}'', V'')$ (in any semantics), where $\mathbb{D} = (D, \leq)$, $\mathbb{D}' = (D', \leq)$, $\mathbb{D}'' = (D'', \leq)$, and:

$$V(p) = \{[x, y]\}; \qquad V'(p) = \{[x_1, y_1]\}; \qquad V''(p) = \{[x_2, y_2]\};$$

where $V(q) = V'(q) = V''(q) = \emptyset$ for any propositional variable $q \neq p$. D' is an isomorphic translation by an integer $c < 0$ (to the left) of D ($[x_1, y_1]$ is an image of $[x, y]$ with respect to this translation). Analogously, D'' is an isomorphic translation by an integer $c > 0$ (to the right) of D ($[x_2, y_2]$ is an image of $[x, y]$ with respect to this translation) – see Figure 4. Towards a contradiction suppose that there is an $\mathsf{HS}^{\Diamond}_{horn}$-formula $\varphi_{\mathsf{D}p}$ expressing $\mathsf{D}p$. Hence, $\mathcal{M}, [x, y] \not\models \varphi_{\mathsf{D}p}$, $\mathcal{M}', [x, y] \models \varphi_{\mathsf{D}p}$, and $\mathcal{M}'', [x, y] \models \varphi_{\mathsf{D}p}$. We will show that $(\star)$ for any $\mathsf{HS}^{\Diamond}_{horn}$-formula φ if $\mathcal{M}', [x, y] \models \varphi$ and $\mathcal{M}'', [x, y] \models \varphi$, then $\mathcal{M}, [x, y] \models \varphi$ which will give rise to a contradiction and finish the proof.

Let ψ be any conjunct of φ. Then ψ is of a form $[\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$ or λ, where λ_i and λ are generated by the grammar $\lambda := \top \mid \bot \mid r \mid \langle \mathsf{R} \rangle \lambda$. (Case 1): ψ is of a form $[\mathsf{U}](\lambda_1 \wedge \ldots \wedge \lambda_n \rightarrow \lambda_{n+1})$. Since the formula is preceded by $[\mathsf{U}]$ and models $\mathcal{M}$, $\mathcal{M}'$, and $\mathcal{M}''$ are isomorphic, then ψ is true in all three models or false in all of them, hence $(\star)$ holds. (Case 2): ψ is of the form $\top$, $\bot$, $\langle \mathsf{R}_1 \rangle \ldots \langle \mathsf{R}_n \rangle \top$, $\langle \mathsf{R}_1 \rangle \ldots \langle \mathsf{R}_n \rangle \bot$, p, q, or $\langle \mathsf{R}_1 \rangle \ldots \langle \mathsf{R}_n \rangle q$, where $q \neq p$ and $\mathsf{R}_i \in \mathsf{HS}_{\mathsf{rel}}$. Then it is easy to see that $(\star)$ holds. (Case 3) $\psi = \langle \mathsf{R}_1 \rangle \ldots \langle \mathsf{R}_n \rangle p$ for any $\mathsf{R}_1, \ldots, \mathsf{R}_n \in \mathsf{HS}_{\mathsf{rel}}$. As showed previously we may consider only $\mathsf{R}_i \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$. Assume that $\mathcal{M}', [x, y] \models \psi$ and $\mathcal{M}'', [x, y] \models \psi$. To show that $\mathcal{M}, [x, y] \models \psi$ it suffices to prove the following statement:

$(\star\star)$ For any sequence $\mathsf{R}_1, \ldots, \mathsf{R}_n$ of relations from $\{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$ the following holds:
if for some intervals $[x, y]$, $[x', y']$, $[x'', y'']$ we have $[x, y] \mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n} [x', y']$ and $[x, y] \mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n} [x'', y'']$, then for any interval $[s, t]$ such that $\min(x', x'') \leq s \leq \max(x', x'')$, and $\min(y', y'') \leq t \leq \max(y', y'')$, and $t - s \geq \min(y' - x', y'' - x'')$ it holds that $[x, y] \mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n} [s, t]$,

where $\circ$ is the composition operator and for any interval $[x, y]$ we use "$x - y$" to denote the number of time-points between x and y (notice that in the case of a dense frame $x - y$ equals 0 or infinity). Because of space limits we leave the proof of $(\star\star)$ to the reader. The proof may be conducted by an induction on the number n. Showing that $(\star\star)$ finishes the proof. ◀

In the remaining part of this subsection we consider expressiveness of $\mathsf{HS}^{\Box}_{horn}$. We will show that under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S), $\mathsf{HS}^{\Box}_{horn}$ is not expressive enough to define the *somewhere* and *difference* modalities, as well as nominals. However, the *universal* modality is still definable in $\mathsf{HS}^{\Box}_{horn}$.

▶ **Theorem 12.** *Nominals and expressions of the form* $\mathsf{E}p$ *and* $\mathsf{D}p$ *for* $p \in \mathsf{PROP}$ *are not definable in* $\mathsf{HS}^{\Box}_{horn}$ *under* (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), *and* (≤,Den,Non-S). *Whereas,* $\mathsf{A}p$ *is definable in* $\mathsf{HS}^{\Box}_{horn}$ *under all semantics.*

Proof. An expression of the form $\mathsf{A}p$ is definable as $\mathsf{A}p := [\mathsf{U}](\top \to p)$. To show that nominals, $\mathsf{E}p$, and $\mathsf{D}p$ are not definable under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S) we will use a result from [8], where under these semantics a construction of a *canonical* $\mathsf{HS}^{\Box}_{horn}$-model $\mathcal{K}^{[a,b]}_{\varphi}$ of an HS-formula φ in an interval $[a,b]$ is presented. The model is canonical in the following sense [8, Theorem 3.2]:

(a) If in some HS-model $\mathcal{M}$ it holds that $\mathcal{M},[a,b] \models \varphi$, then $\mathcal{K}^{[a,b]}_{\varphi},[a,b] \models \varphi$, and

(b) For any interval $[x,y]$ and any $p \in \mathsf{PROP}$ if $\mathcal{K}^{[a,b]}_{\varphi},[x,y] \models p$, then in any HS-model $\mathcal{M}$ such that $\mathcal{M},[a,b] \models \varphi$ we have $\mathcal{M},[x,y] \models p$.

Towards a contradiction let us suppose that nominals, $\mathsf{E}p$, and $\mathsf{D}p$ are definable in $\mathsf{HS}^{\Box}_{horn}$ under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S). Let φ be an $\mathsf{HS}^{\Box}_{horn}$-formula expressing that (i) $p \in \mathsf{PROP}$ simulates a nominal i, or (ii) $\mathsf{E}p$, or (iii) $\mathsf{D}p$. We will reach a contradiction no matter in which of these forms φ is.

Let $\mathcal{M} = (\mathbb{D}, V)$ and $\mathcal{M}' = (\mathbb{D}, V')$ be HS-models such that $V(p) = \{[x,y]\}$, $V'(p) = \{[x',y']\}$, and $[x,y] \neq [x',y']$. Let $[a,b]$ be an interval distinct from $[x,y]$ and $[x',y']$. Hence, $\mathcal{M},[a,b] \models \varphi$ and $\mathcal{M}',[a,b] \models \varphi$. By (a) $\mathcal{K}^{[a,b]}_{\varphi},[a,b] \models \varphi$. From the definition of φ – see (i), (ii), and (iii) – it follows that there is $[u,w]$ such that $\mathcal{K}^{[a,b]}_{\varphi},[u,w] \models p$. Then by (b) we obtain that $\mathcal{M},[u,w] \models p$ and $\mathcal{M}',[u,w] \models p$. (Case 1): $[u,w] \neq [x,y]$. Then $\mathcal{M},[u,w] \not\models p$. (Case 2): $[u,w] \neq [x',y']$. Then $\mathcal{M}',[u,w] \not\models p$. We have obtained a contradiction in both cases, hence nominals, $\mathsf{E}p$, and $\mathsf{D}p$ are not definable in $\mathsf{HS}^{\Box}_{horn}$ under (<,Den,S), (<,Den,Non-S), (≤,Dis,Non-S), (≤,Den,S), and (≤,Den,Non-S). ◀

As a corollary to the above theorem we obtain that adding nominals to the language of $\mathsf{HS}^{\Box}_{horn}$ (which results in obtaining $\mathsf{HS}^{\Box,i}_{horn}$) strictly increases expressive power of the language (in the listed semantics). In what follows, we will show that the further extension of $\mathsf{HS}^{\Box,i}_{horn}$ obtained by adding satisfaction operators (i.e., reaching $\mathsf{HS}^{\Box,i,@}_{horn}$) does not increase its expressiveness in any semantics.

▶ **Theorem 13.** *@ operators are definable in the language of* $\mathsf{HS}^{\Box,i}_{horn}$ *under all semantics.*

Proof. Let φ be an $\mathsf{HS}^{\Box,i,@}_{horn}$-formula. Construction of an equisatisfiable $\mathsf{HS}^{\Box,i}_{horn}$-formula φ' is by induction on the number of $@_i$ operators occurring in φ. In each step of the construction choose any positive temporal literal λ in which occurs a satisfaction operator and find the left-most satisfaction operator occurring in λ, i.e., an operator $@_i$ such that $\lambda = [\mathsf{R}_1]\dots[\mathsf{R}_n]@_i\eta$ with $[\mathsf{R}_1]\dots[\mathsf{R}_n]$ being a (possibly empty) sequence of box modal operators and η a subformula of λ. We replace this occurrence of $@_i\eta$ by $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$, where $p_{@i\eta}$ is a fresh propositional variable that did not occur in the so far constructed formula. Moreover, add to the constructed formula the following conjunction:

$$[\mathsf{U}](i \land \eta \to [\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}) \land [\mathsf{U}]([\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta} \land i \to \eta), \tag{14}$$

where (14) allows us to simulate $@_i\eta$ with $[\mathsf{L}][\overline{\mathsf{L}}]p_{@i\eta}$. The construction terminates when all occurrences of satisfaction operators are eliminated. The resulting formula is in the language of $\mathsf{HS}^{\Box,i}_{horn}$ and is equisatisfiable with φ. ◀

5 Conclusions

Recently, decidability and computational complexity of Horn fragments of HS have been intensively investigated. However, significantly less research has focused on their expressive power. In this paper we address this gap by showing which semantics are definable in full HS and which operators (nominals, satisfaction operators, *somewhere*, *difference*, and *everywhere* operators) are definable in various Horn fragments of HS. Our results on the relative expressive power of HS Horn fragments are summarised diagrammatically in Figure 1b.

There are still numerous open problems concerning the expressive power of HS fragments. For instance, not much is known about the expressive power of *core* fragments of HS [8, 9], which are obtained by imposing further restrictions on Horn fragments. In particular, an interesting open question is whether the expressive power diagram for such fragments is analogous to the one for Horn fragments presented in Figure 1b.

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A Appendix (proofs)

Proof of Lemma 5. Fix HS-models $\mathcal{M}$, $\mathcal{M}'$, a bisimulation Z between them and bisimilar intervals $[x, y]$, $[x', y']$, i.e., intervals such that $[x, y]Z[x', y']$. We show by induction on a construction of an HS-formula that for any HS-formula φ the following conditions are equivalent:

1. $\mathcal{M}, [x, y] \models \varphi$;
2. $\mathcal{M}', [x', y'] \models \varphi$.

(Case 1) $\varphi \in \mathsf{PROP}$. By (atom) it follows that $\mathcal{M}, [x, y] \models \varphi$ iff $\mathcal{M}', [x', y'] \models \varphi$.

(Case 2) $\varphi = \neg\psi$ for any HS-formula ψ. By the inductive assumption $\mathcal{M}, [x, y] \models \psi$ iff $\mathcal{M}', [x', y'] \models \psi$. As a result, $\mathcal{M}, [x, y] \models \varphi$ iff $\mathcal{M}', [x', y'] \models \varphi$.

(Case 3) $\varphi = \psi \wedge \xi$ for any HS-formulas ψ, ξ. By the inductive assumption $\mathcal{M}, [x, y] \models \psi$ iff $\mathcal{M}', [x', y'] \models \psi$, and $\mathcal{M}, [x, y] \models \xi$ iff $\mathcal{M}', [x', y'] \models \xi$. Hence, $\mathcal{M}, [x, y] \models \varphi$ iff $\mathcal{M}', [x', y'] \models \varphi$.

(Case 4) $\varphi = \langle \mathsf{R} \rangle \psi$ for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$ and any HS-formula ψ.

(1 $\Rightarrow$ 2) Assume $\mathcal{M}, [x, y] \models \varphi$. Then, there is $[u, w]$ such that $[x, y]\mathsf{rel}_{\mathsf{R}}[u, w]$ and $\mathcal{M}, [u, w] \models \psi$. $[x, y]Z[x', y']$ so by (zig) there is $[u', w']$ such that $[x', y']\mathsf{rel}_{\mathsf{R}}'[u', w']$ and $[u, w]Z[u', w']$. Therefore, by the inductive assumption $\mathcal{M}', [u', w'] \models \psi$, so $\mathcal{M}', [x', y'] \models \varphi$.

(1 $\Leftarrow$ 2) Is proved analogously as (1 $\Rightarrow$ 2) but using (zag) instead of (zig).

(Case 5) $\varphi = [\mathsf{R}]\psi$ for any $\mathsf{R} \in \mathsf{HS}_{\mathsf{rel}}$ and any HS-formula ψ.

(1 $\Rightarrow$ 2) Assume $\mathcal{M}, [x, y] \models \varphi$. To show $\mathcal{M}', [x', y'] \models \varphi$. Fix any $[u', w']$ such that $[x', y']\mathsf{rel}_{\mathsf{R}}'[u', w']$. To show that $\mathcal{M}', [u', w'] \models \psi$. $[x, y]Z[x', y']$ so by (zag) there is $[u, w]$ such that $[x, y]\mathsf{rel}_{\mathsf{R}}[u, w]$ and $[u, w]Z[u', w']$. $\mathcal{M}, [x, y] \models [\mathsf{R}]\psi$, so $\mathcal{M}, [u, w] \models \psi$. By the inductive assumption $\mathcal{M}', [u', w'] \models \psi$, hence we obtain $\mathcal{M}', [x', y'] \models \varphi$.

(1 $\Leftarrow$ 2) Is proved analogously as (1 $\Rightarrow$ 2) but using (zig) instead of (zag). ◀

Theorem 4 was stated in [2] without a proof. In what follows, we present a proof of this theorem.

Proof of Theorem 4. Let $\mathcal{L}$ be a modal language in which the difference operator D is polynomially definable. Then, let $\mathcal{L}^{ext}$ be an extension of $\mathcal{L}$ with nominals and @ operators. Fix any $\varphi^{ext} \in \mathcal{L}^{ext}$. To prove the theorem we show a polynomial (with respect to the size of φ^{ext}) translation of φ^{ext} into $\varphi \in \mathcal{L}$ such that φ^{ext} is satisfiable iff φ is.

First, we show that E and A are polynomially definable in $\mathcal{L}$. Indeed, they are defined as:

$$\mathsf{E}\psi := \psi \vee \mathsf{D}\psi;$$

$$\mathsf{A}\psi := \neg\mathsf{E}\neg\psi.$$

Now, we show a step by step construction of φ. For any i occurring in φ^{ext} as an nominal or an index of some @ operator occurring in φ^{ext}, we introduce a fresh propositional variable p_i (i.e., not occurring in φ^{ext}) such that all p_i's are pairwise different. Then we proceed as follows.

First, inductively apply the following procedure to φ^{ext}. Let $@_i\psi$ be any subformula of φ^{ext} such that ψ does not contain any occurrence of @ operators. Construct φ' by replacing $@_i\psi$ with a new variable q_k and let:

$$\begin{aligned}\theta_k &= (\mathsf{A}q_k \lor \mathsf{A}\neg q_k) \land \\ &\quad (\mathsf{A}q_k \to \mathsf{E}(p_i \land \psi)) \land \\ &\quad (\mathsf{A}\neg q_k \to \mathsf{A}\neg\psi)\,; \\ \chi_i &= \mathsf{E}p_i \land \mathsf{A}(p_i \to \neg\mathsf{D}p_i).\end{aligned}$$

It is easy to see that φ^{ext} is satisfiable iff $\varphi' \land \theta_k \land \chi_i$ is. The procedure finishes when no more @ operators are in the constructed formula. Let φ'' be a conjunction of the finally constructed formula with all θ_k's and all χ_i's constructed so far. Obviously, φ'' does not contain @ operators and φ^{ext} is satisfiable iff φ'' is.

Second, for any nominal j occurring in φ'' replace all its occurrences in φ'' by p_j and let

$$\gamma_j = \mathsf{E}p_j \land \mathsf{A}(p_j \to \neg\mathsf{D}p_j).$$

A conjunction of the obtained formula with all γ_j's is the final formula φ.

Formula φ does not contain @ operators nor nominals, and φ^{ext} is satisfiable iff φ is. Furthermore, the translation of φ^{ext} to φ is polynomial. More precisely, it is at most quadratic in the size of φ^{ext}. ◀

Completion of the proof of Theorem 11. To finish the proof it remains to prove the following statement:

(⋆⋆) For any sequence $\mathsf{R}_1, \ldots, \mathsf{R}_n$ of relations from $\{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$ the following holds: if for some intervals $[x,y]$, $[x',y']$, $[x'',y'']$ we have $[x,y]\mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n}[x',y']$ and $[x,y]\mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n}[x'',y'']$, then for any interval $[s,t]$ such that $\min(x',x'') \leq s \leq \max(x',x'')$, and $\min(y',y'') \leq t \leq \max(y',y'')$, and $t-s \geq \min(y'-x', y''-x'')$ it holds that $[x,y]\mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_n}[s,t]$,

where $\circ$ is the composition operator and for any interval $[x,y]$ we use "$x-y$" to denote a number of time-points between x and y (in the case of dense time frame $x-y$ equals 0 or infinity).

We will prove (⋆⋆) by induction on the number n of relations. Let us fix an interval $[x,y]$ and assume that the statement holds for k relations, i.e., for any $\mathsf{R}_1, \ldots, \mathsf{R}_k \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$. Let

$$X = \{[x',y'] \mid [x,y]\mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_k}[x',y']\}.$$

We will shows that the statement holds for any $k+1$ relations, i.e., for any $\mathsf{R}_1, \ldots, \mathsf{R}_{k+1} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$. Let us fix any $\mathsf{R}_{k+1} \in \{\mathsf{B}, \overline{\mathsf{B}}, \mathsf{E}, \overline{\mathsf{E}}, \mathsf{A}, \overline{\mathsf{A}}\}$ and define

$$X' = \{[x',y'] \mid [x,y]\mathsf{rel}_{\mathsf{R}_1} \circ \ldots \circ \mathsf{rel}_{\mathsf{R}_{k+1}}[x',y']\}.$$

Fix any intervals $[x',y'], [x'',y''] \in X'$ and any interval $[s,t]$ such that $\min(x',x'') \leq s \leq \max(x',x'')$ and $\min(y',y'') \leq t \leq \max(y',y'')$. We need to show that $[s,t] \in X'$. In what follows we will assume that the semantics is irreflexive but for any reflexive semantics the proof is analogous (namely by replacing "$<$" with "$\leq$" in the remaining part of the proof).

(Case 1) $\mathsf{R}_{k+1} = \mathsf{B}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\mathsf{B}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\mathsf{B}}[x'', y'']$. By the definition of $\mathsf{rel}_{\mathsf{B}}$ it follows that $u' = x'$, $w' > y'$, $u'' = x''$, and $w'' > y''$. Let $z = \max(w', w'')$, then by the inductive assumption we obtain $[s, z] \in X$. It follows that $z > y'$ and $z > y''$, so $z > t$. Then $[s, z]\mathsf{rel}_{\mathsf{B}}[s, t]$, so $[s, t] \in X'$.

(Case 2) $\mathsf{R}_{k+1} = \overline{\mathsf{B}}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\overline{\mathsf{B}}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\overline{\mathsf{B}}}[x'', y'']$. By the definition of $\mathsf{rel}_{\overline{\mathsf{B}}}$ it follows that $u' = x'$, $w' < y'$, $u'' = x''$, and $w'' < y''$. Let $[s, z]$ be such an interval that $z - s = \min(w' - u', w'' - u'')$, then by the inductive assumption $[s, z] \in X$. Since $w' < y'$ and $w'' < y''$, $z - s < \min(y' - x', y'' - x'')$. Hence $z - s < t - s$, so $z < t$. It follows that $[s, z]\mathsf{rel}_{\overline{\mathsf{B}}}[s, t]$ and so $[s, t] \in X'$.

(Case 3) $\mathsf{R}_{k+1} = \mathsf{E}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\mathsf{E}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\mathsf{E}}[x'', y'']$. By the definition of $\mathsf{rel}_{\mathsf{E}}$ it follows that $u' < x'$, $w' = y'$, $u'' < x''$, and $w'' = y''$. Let $z = \min(u', u'')$, then by the inductive assumption we obtain $[z, t] \in X$. It follows that $z < x'$ and $z < x''$, so $z < s$. Then $[z, t]\mathsf{rel}_{\mathsf{E}}[s, t]$, so $[s, t] \in X'$.

(Case 4) $\mathsf{R}_{k+1} = \overline{\mathsf{E}}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\overline{\mathsf{E}}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\overline{\mathsf{E}}}[x'', y'']$. By the definition of $\mathsf{rel}_{\overline{\mathsf{E}}}$ it follows that $u' > x'$, $w' = y'$, $u'' > x''$, and $w'' = y''$. Let $[z, t]$ be such an interval that $t - z = \min(w' - u', w'' - u'')$, then by the inductive assumption $[z, t] \in X$. Since $u' > x'$ and $u'' > x''$, $t - z < \min(y' - x', y'' - x'')$. Hence $t - z < t - s$, so $s < z$. It follows that $[z, t]\mathsf{rel}_{\overline{\mathsf{E}}}[s, t]$ and so $[s, t] \in X'$.

(Case 5) $\mathsf{R}_{k+1} = \mathsf{A}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\mathsf{A}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\mathsf{A}}[x'', y'']$. By the definition of $\mathsf{rel}_{\mathsf{A}}$ it follows that $w' = x'$ and $w'' = x''$. Let $z = \min(u', u'')$, then by the inductive assumption we obtain $[z, s] \in X$. Then $[z, s]\mathsf{rel}_{\mathsf{A}}[s, t]$, so $[s, t] \in X'$.

(Case 6) $\mathsf{R}_{k+1} = \overline{\mathsf{A}}$. Then there are intervals $[u', w'], [u'', w''] \in X$ such that $[u', w']\mathsf{rel}_{\overline{\mathsf{A}}}[x', y']$ and $[u'', w'']\mathsf{rel}_{\overline{\mathsf{A}}}[x'', y'']$. By the definition of $\mathsf{rel}_{\overline{\mathsf{A}}}$ it follows that $u' = y'$ and $u'' = y''$. Let $z = \max(w', w'')$, then by the inductive assumption we obtain $[t, z] \in X$. Then $[t, z]\mathsf{rel}_{\overline{\mathsf{A}}}[s, t]$, so $[s, t] \in X'$. ◀