Towards a Quantum Theory of Geographic Fields

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Abstract

This paper proposes a framework that that allows for the possibility that multiple classically incompatible states are expressed simultaneously at a given point of a geographic field. The admission of such superposition states provides the basis for a new understanding of indeterminacy and ontological vagueness in the geographic world.

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1 Introduction

Classical geography (CG) presupposes that it is possible to identify and to analyze the distribution of geographic qualities on the surface of the Earth in a way that (a) the different kinds of geographic qualities can be analyzed and classified using the Aristotelian method of classification (see below) and (b) the ways in which geographic qualities are instantiated on the surface of the Earth allows for the delineation of regions 'on the ground' at which distinct geographic qualities are instantiated.

The Aristotelian method of classification [2] is based on the assumption that categories/kinds of geographic qualities are structured hierarchically in a tree-like manner. Such trees are called the taxonomic trees or taxonomic hierarchies. Categories farther from the root in the taxonomic hierarchy of geographic qualities are differentiated from categories closer to the root by additional additional more specific qualities. Those additional qualities determine what marks out instances of a more specific category/kind (or species) within the wider parent category (or genus) [2]. Ideally, the Aristotelian method of classification leads classification trees which leaf categories are jointly exhaustive and pairwise disjoint.

In the geographic context there are additional constraints that hold in at least some idealized sense: (i) regions of geographic space at which distinct categories of qualities that are at the same level of the hierarchy tree (e.g., the leafs of the taxonomic tree) are instantiated cannot overlap (ii) jointly the regions with geographic qualities of same level of the taxonomic hierarchy partition the underlying space. That is, the distribution/instantiation of geographic qualities on the surface of the Earth gives rise to geographic fields. At least at sufficiently coarse scales those fields will be smooth and relatively homogeneous.

In non-idealized situations geographic fields have granular features and, in addition, the distribution of geographic qualities displays inhomogeneities. The aspect of granularity is due to the fact that geographic qualities are instantiated at regions of certain scale, i.e., regions that are of some minimal size or larger. Geographic fields are subject to inhomogeneities in the sense that if a quality universal is instantiated at a region of geographic scale then this does not mean that every part of geographic scale of this region is an instance of that quality

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universal. There may be comparatively small (geographic-scale) regions in which different and possibly conflicting quality universals are instantiated.

Consider, for example, the geographic region called 'Central Great Plains' at which the land-surface form *Irregular plain* is instantiated. There are comparatively small (geographic-scale) regions in which different land-surface forms are instantiated. Analogously for climate types: There may, for example, exist comparatively small (geographic-scale) regions near larger water bodies where the average temperature (and thus the climate type) is different from the larger surroundings.

All of this seems to indicate that many geographic fields are subject to scale dependency, granularity, and inhomogeneities, all of which seem to be fundamentally vague. It is a fundamental assumption of this paper that an important aspect of understanding geographic fields is to understand the interrelations between scale dependency, granularity, inhomogeneity and the phenomenon of vagueness. This paper aims to contribute to the understanding of those aspects of geographic fields.

In conjunction scale dependency, granularity, inhomogeneities and vagueness of geographic fields seem to lead to a fundamental tradeoff between the classification and the delineation of geographic fields. Consider Fig. 1 in which two ways of delineating the contiguous USA into ecoregions of sub-regional scale are (partly) displayed. The bold black lines depict the delineation formed by the ecoregion sections identified by Bailey [1]. The non-bold black lines depict the delineation of formed by the collection level three regions identified by the EPA [3]. There is an obvious difference between the generalized and coarse character of Bailey's delineation (bold black boundaries) and the more fine-grainedness of the EPA delineation (non-bold black boundaries).

The smooth and highly generalized boundaries of Bailey's delineation seem to convey the intuition that there is a large degree of 'freedom' to place boundaries by fiat in the way that supports best the purpose of a map as a medium for conveying of information. As [11] puts it, to minimize information decoding error, map designers strive (i) for crisp (non-graduated) boundaries and (ii) for minimizing boundary complexity by drawing boundaries in highly generalized ways. Drawing boundaries in highly generalized ways avoids the misinterpretation of the delineation as realism of a map. By contrast, the non-bold black boundaries are the result of observations 'on the ground' that are aimed at identifying local variations of qualities and there by identifying boundaries that separate ecoregions of different kinds [7]. The fine-grainedness of the boundaries conveys the preciseness of the delineation.

Both approaches to identifying ecoregions on the surface of the Earth include classification and delineation operations. Surprisingly, the outcomes of both approaches are very different as can be seen in the maps displayed in Fig. 1.

► **Hypothesis 1.** *The reason for different outcomes of the operations of classification and delineation can be attributed to the fact that the sequence of the application of the operation of classification and delineation is significant. More precisely, Bailey applies the classification operation first and then the delineation on the ground second by contrast, the EPA applies the operation of delineation first and then the operation of classifying the delineated regions second. In technical terms this is to say that the different maps produced by Bailey and the EPA are an indication that the operations of classification and delineation do not commute. The non-commutativity of operations that act on field-like phenomena that are subject to granularity, inhomogeneity, and vagueness is a fundamental aspect of many types of geographic fields.*

If this hypothesis is true then it is a fundamental criterium for the adequacy of a theory of geographic fields that it is able to give a satisfying explanation for the non-commutativity and the tradeoff between classification and delineation.

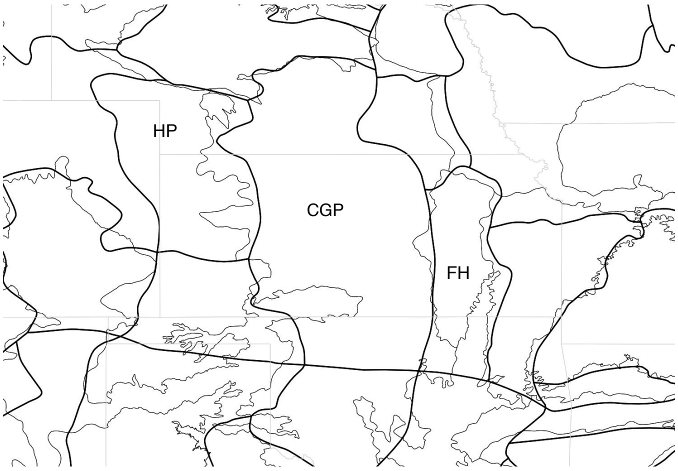


Figure 1 Classification and delineation of the central US into ecoregions [4] according to Bailey [1] (bold boundaries) and [3] (non-bold boundaries).

The non-commutative nature of certain operations is the hallmark of Quantum Mechanics [6]. For example, Heisenberg's uncertainty principle is a consequence of the non-commutativity of operators that determine the position and the momentum of a particle [9]. It is the aim of this paper to present an adequate theory of geographic fields by applying ideas and techniques from quantum mechanics to geographic fields. This set of ideas and techniques applied to geographic phenomena will be called *Quantum Geography* (QG).

The idea of exploring the quantum nature of the geographic world has been discussed previously [13, 5]. This paper goes beyond those discussions by showing how a quantum theory of geographic fields actually could look like in the specific context of Ecoregion classification and delineation. The author feels that in this more specific context it will be easier to go beyond mostly philosophical discussions towards developing scientific theories which predictions can be tested empirically.

2 (Geographic) fields as fiber bundles

In what follows the language of fiber bundles (e.g., [10, 15]) is used to express a theory of geographic fields. The reasons for this choice are as follows:

Firstly, geographic fields can have the form of scalar fields (temperature, elevation, etc.) and vector fields (air flow, hydrological flow, etc.), co-vector (gradient) fields (rates of changes of scalar as well as vector fields) such as slope fields (direction of largest rate of changes of elevation at every point), temperature gradient fields (direction of largest change of temperature at every point), etc. Fiber bundles are general enough to include scalar as well as vector and co-vector fields. Secondly, formalizations of geographic fields need to be able to integrate multitudes of local descriptions of field phenomena into a global framework. For example, fiber bundles provide powerful means to deal with globally curved spaces using locally flat reference systems. Thirdly, a scientific theory of geographic fields must be able to talk about the class of all geographically possible fields in an efficient manner. Fiber bundles provide means to formally characterize what it means for a field to be geographically possible. Finally, 'Classical' descriptions of geographic fields based on fiber bundles can be naturally generalized to the descriptions of geographic fields within the framework of quantum geography. In this respect the paper will mostly follow [10].

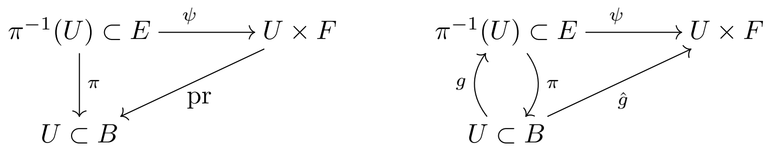


Figure 2 Fibers of a fiber bundle (left); (local) section of a fiber bundle (right).

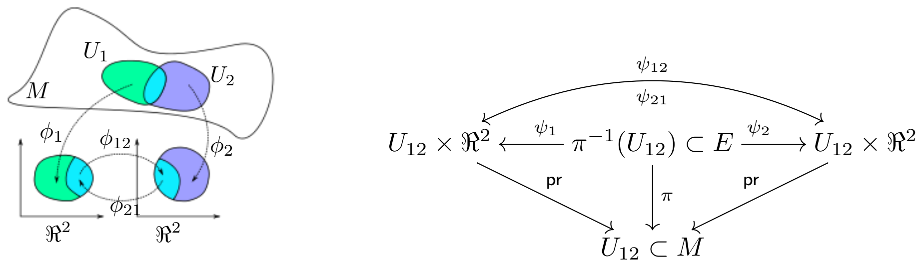


Figure 3 Bundle atlas (left: Image licensed under Creative Commons Attribution-Share Alike 3.0 via Wikimedia Commons).

2.1 Fiber bundles

The literature on fiber bundles is vast. In this subsection some basic definitions have been collected from [10, 15]. A fiber bundle is a structure (E, B, π, F), where E, B, and F are topological spaces and $\pi : E \to B$ is a continuous surjection satisfying a local triviality condition: for every $e \in E$, there is an open (trivializing) neighborhood $U \subset B$ of $\pi(e)$ such that there is a homeomorphism $\psi : \pi^{-1}(U) \to (U \times F)$ such that the diagram in the left of Figure 2 commutes.

The space B is the base space of the bundle, E is the total space, and F is the abstract fiber. The map π is the bundle projection; B is assumed to be topologically connected; $(U \times F)$ is a product space; $\mathrm{pr} : U \times F \to U$ is the natural projection; and the trivialization map is $\psi : \pi^{-1}(U) \to U \times F$. In a fiber bundle (E, B, π, F) every fiber $\pi^{-1}(x) \in E$ over $x \in B$ is homomorphic to some abstract fiber F. In particular ψ is defined as:

$$\psi : e \in E \mapsto (\pi(e), f(e)) \in U \times F \quad \text{with } f : \pi^{-1}(x) \to F \tag{1}$$

such that f is a homomorphism.

An *open covering* of a fiber bundle (E, B, π, F) is a system $\{U_\alpha\}$ of open subsets of B together with a trivialization maps $\psi_\alpha : \pi^{-1}(U_\alpha) \to U_\alpha \times F$. The system $\psi = \{(U_\alpha, \psi_\alpha)\}$ is a *bundle atlas*. If $\{(U_1, \psi_1), (U_2, \psi_2), \ldots\}$ is an atlas then the trivializations for overlapping members $U_{12} = U_1 \cap U_2 \neq \emptyset$ of the covering are compatible such that for all $e \in \pi^{-1}(U_{12})$: $\psi_{12} \circ \psi_1 = \psi_2^{-1}$ as illustrated in Fig. 3.

A *local section* of (E, B, π, F) is a continuous map $g : U \to E$ where U is an open set in B and $\pi(g(x)) = x$ for all $x \in U$. For a local section g there is a map $\hat{g} : U \to (U \times F)$ such that $\hat{g} = \psi \circ g$. This is displayed in the right of Fig. 2. If (U, ψ) is a local trivialization chart then local sections always exist over U. Given an atlas, local sections can be combined to cover the base space of the fiber bundle as a whole – *covering sections*. The set of all covering sections over the fiber bundle (E, B, π, F) is denoted by $\mathsf{Sect}(E)$.

2.2 Fiber bundles, determinable and determinate qualities

Fiber bundles provide formal means to represent smoothly distributed qualities (quality fields). They also very naturally incorporate the ontological distinctions between quality determinables, quality determinates [14]. Intuitively, a quality field as a whole corresponds to a quality determinable such as energy, temperature, ecoregion domain, etc. At given points of the base space then quality determinates such as 10 Joule, 72 degree Fahrenheit, Dry domain, etc. are instantiated.

Let (E, B, π, F) be a fiber bundle over the manifold B. The set of all possible instances of the universal `field of type` $\mathcal{E}$ `over` B is the set of all covering sections $\mathsf{Sect}(E)$ of (E, B, π, F) such that

- The fiber $\pi^{-1}(x) \subset E$ is the class of quality determinates that fall under the quality determinable $\mathcal{E}$ and that can possibly be instantiated in the neighborhood of $x \in B$.
- If g is a covering section of the base space B then there is a field that is geographically possible and on this possibility it holds that for all $x \in B : \; g(x) = e$ iff the quality determinate $e \in \pi^{-1}(x)$ is instantiated in the neighborhood of $x \in B$. Neighborhood in this context does not mean infinitesimal neighborhood but neighborhood in the geographic sense. Such neighborhoods are specified in the context of the local trivializations.1
- The local representation of the possible $\mathcal{E}$-field g is the function $\hat{g} : U \to (U \times F)$ as depicted in Fig. 2 (right). Here $F = F' \times \Gamma \times \Delta$ where F' is a representation of the quality determinate instantiated in the neighborhood of a given location $x \in U \subseteq B$. Γ is the minimal diameter of a region at which an instance of this quality determinate can be instantiated. Δ is the distance from x at which F' is actually instantiated. (This will be important to capture the possibility of inhomogeneities.)

That is, every $\mathcal{E}$-field-universal in conjunction with its possible instantiating fields (possible $\mathcal{E}$-field-particulars) can be thought of as having the structure of a fiber bundle (E, B, π, F). Possible $\mathcal{E}$-field-particulars correspond to (covering) sections of the underlying fiber bundle. A $\mathcal{E}$-field-particular is such that in the neighborhood of every location $x \in U$ exactly one of the quality determinates of $\pi^{-1}(x)$ is instantiated in a way that is consistent with the constraints that reflect the granular and locally inhomogeneous nature of geographic fields. This will be discussed in the next section.

3 Geographic fields in Quantum geography

It is fundamental to classical geography that geographically possible fields are sections in fiber bundles that take every point in the base space to a member of a set of disjoint possibilities that constitutes the fiber over that base point. By contrast, in *Quantum geography* (QG) the state of a geographic field at a given point of the base space of is not just a point in the space of disjoint possible states collected in the fiber over that point.

3.1 Superposition states

In QG the state of a geographic field at a given point of the base space is represented by a vector which is the sum of the base vectors that span the vector space that forms the fiber over that base point.

▶ **Example 1.** Consider Fig. 4. The quality determinable *Ecoregion domain* has four subclasses as determinates. In classical geography exactly one of those of those four possibilities can be instated at (in the neighborhood of) a given point. In the language of fiber bundles the class of all possible fields of type *Ecoregion domains* are sections in a fiber bundle which

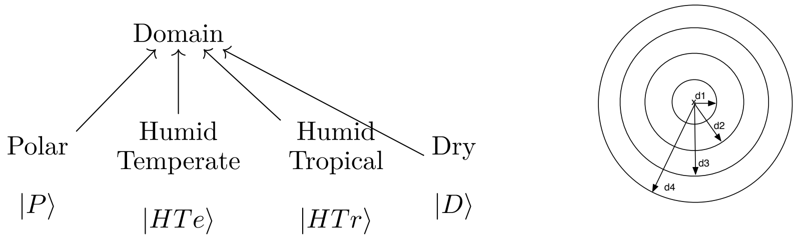


Figure 4 Quality base $\mathcal{Q} = \{|P\rangle, |HTe\rangle, |HTr\rangle, |D\rangle\}$ (left); Localization base $\mathcal{X} = \{|\Delta_1\rangle, |\Delta_2\rangle, |\Delta_3\rangle, |\Delta_4\rangle\}$ (right).

fibers over every point are constituted by the set { Polar, Humid Temperate, Humid Tropical, Dry }.

What are points in the space of possibilities (the fibers) in classical geography are dimensions (base vectors) in a *vector space* of possible states in QG. The fibers of fields of type *Ecoregion domains* in QG form a four-dimensional (complex) vector space with base vectors labeled Polar, Humid Temperate, Humid Tropical, and Dry. In QG a geographic field at a given point of the base space can be in a *superposition* of multiple classically incompatible states. A field of type *Ecoregion domains* at a given location x can be in the state $\sqrt{0.1}$ Polar + $\sqrt{0.4}$ Humid Temperate + $\sqrt{0.3}$ Humid Tropical + $\sqrt{0.2}$ Dry.

▶ **Postulate 1.** *Geographic fields in QG are sections of* vector bundles*. Every fiber has the structure of a vector space. The bases (dimensions) of the vector space of each fiber correspond to what in CG are the points in the space of possible states. In QG states of geographic fields at given points in the base space include superpositions (vector sums) of what in CG are distinct states.*

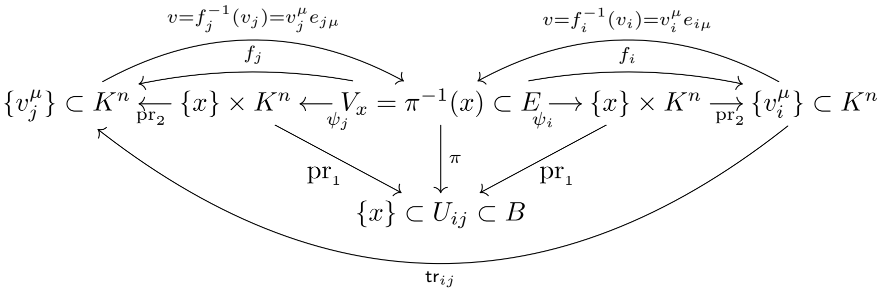
The admission of superposition states constitutes a major departure from CG because it allows for the possibility that multiple classically incompatible states are expressed simultaneously at a given point of a geographic field. In what follows the existence of superposition states will provide the basis for a new understanding indeterminacy and ontological vagueness in the geographic world. To develop the links between ontological vagueness and superposition states some more technical apparatus about vector bundles is needed.

3.2 Vector bundles

A vector bundle (E, M, π, K^n) has fibers with the structure of vector spaces. Local trivializations are of the form $U_{ij} \times K^n$ where K^n is assumed to be R^n or C^n. Consider Diag. 2. If $V_x =_{df} \pi^{-1}(x)$ is a concrete fiber (i.e., an internal space) over M with vector elements v, the linear map $f_i : \pi^{-1}(x) \to K^n$ is equivalent to choosing a base $e_{i\mu}$ and to express the vectors $v \in V_x$ as components with respect to the base $e_{i\mu}$. That is, $f_i(v) = v_i^\mu$, such that

$$v_i^0 e_{i0} + \ldots + v_i^{n-1} e_{in-1} \;\equiv\; \sum_{\mu=0}^{n-1} v_i^\mu e_{i\mu} \;\equiv\; v_i^\mu e_{i\mu} = v \in V_x.$$

Usually, there are a multitude of possible bases for a given vector space. Thus it makes sense to transform the representation of a vector space in one base to a representation of the same space in another base. A change of basis can be defined as $v_j e_j = (v_i e_i)\mathsf{tr}_{ij}$ where tr_{ij} is a linear map that transforms the coordinate v_i in the base e_i to the coordinate v_j in the base e_j.



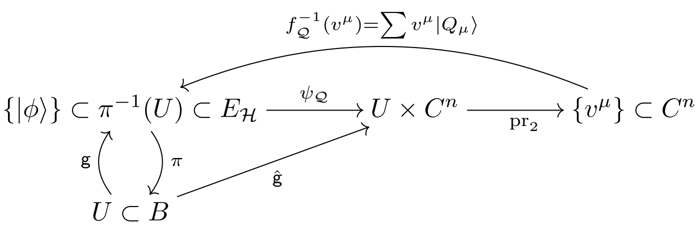
(2)

3.3 Hilbert bundles

Like quantum mechanics, quantum geography requires that the vector spaces that constitute the fibers of the vector bundles are Hilbert spaces [10]. A Hilbert space $\mathcal{H}$ is a *complex* vector space with an *inner product*. In what follows Dirac's notation for vectors in Hilbert spaces [6] is used. The members of a Hilbert space $\mathcal{H}$ are written as ket vectors of the form $|\phi\rangle$ where ϕ is a name/label. As vector spaces Hilbert spaces are closed under vector addition and scalar multiplication. That is if $|\phi\rangle, |\psi\rangle \in \mathcal{H}$ then $\alpha|\phi\rangle + \beta|\psi\rangle \in \mathcal{H}$, where α and β are complex numbers that modify the length of a vector via scalar multiplication and $+$ is the vector addition. The inner product $\langle\psi|\phi\rangle$ of the vectors $|\psi\rangle, |\phi\rangle \in \mathcal{H}$ (defined below) is a complex number.

A base $\mathcal{Q} = |Q_1\rangle, \ldots, |Q_n\rangle$ of a n-dimensional Hilbert space $\mathcal{H}$ is a system of vectors such that every member of $\mathcal{H}$ can be expressed as a vector sum of the base vectors. A base is orthonormal if the inner product of distinct base vectors is zero and all base vectors are of unit length, i.e., $\langle Q_i|Q_j\rangle = 1$ if $i = j$ and $\langle Q_i|Q_j\rangle = 0$ otherwise. If the vector $|\phi\rangle = \alpha_1|Q_1\rangle + \ldots + \alpha_n|Q_n\rangle$ then there exists a dual vector $\langle\phi| = \overline{\alpha}_1|Q_1\rangle + \ldots + \overline{\alpha}_n|Q_n\rangle$ where $\overline{\alpha}_i$ is the complex conjugate of α_i. Thus if $|\phi\rangle = \alpha_1|Q_1\rangle + \ldots + \alpha_n|Q_n\rangle$ and $|\psi\rangle = \beta_1|Q_1\rangle + \ldots + \beta_n|Q_n\rangle$ then the inner product of $|\phi\rangle$ and $|\psi\rangle$ designated by $\langle\psi|\phi\rangle$ is the sum of the products of the components of $\langle\psi|$ and $|\phi\rangle$ computed as $\sum_i \overline{\beta_i}\alpha_i$. In what follows $|\overline{q}_i q_i|$ is an abbreviation for the squared modulus of the scalar product $|\langle Q_i|Q_i\rangle|^2$.[1]

A *Hilbert bundle* is a vector bundle $(E_{\mathcal{H}}, B, \pi, C^n)$ with base space B, bundle space $E_{\mathcal{H}}$ and abstract fiber $\mathcal{H}_{\mathcal{Q}}$ – the Hilbert space $\mathcal{H}$ in the $\mathcal{Q}$-base. A section of a Hilbert bundle $(E_{\mathcal{H}}, B, \pi, C^n)$ is a mapping of signature $U \subset B \to U \times C^n$.



(3)

▶ **Postulate 2.** *The representation of a geographic field* g *in the $\mathcal{Q}$-base of a Hilbert space $\mathcal{H}$ is a smooth section* $\mathsf{g} \in \mathsf{Sect}(E_{\mathcal{H}}^{\mathcal{Q}})$ *of a Hilbert bundle $(E_{\mathcal{H}}, B, \pi, C^n)$. The section $\hat{\mathsf{g}}$ is a vector field such that for every $x \in B$, $\hat{\mathsf{g}}(x) \in \mathcal{H}_x$ is a vector of unit length of expressed in the base $\mathcal{Q}$, i.e., $\hat{\mathsf{g}} = \{x \in U \mapsto \sum \alpha_i |Q_i\rangle \in \mathcal{H}_x \mid \sum_i |\overline{\alpha}_i\alpha_i| = 1\}$.*

[1] Details can be found in any text book on quantum mechanics. The classic reference is [6].

The constraint $\sum_i |\overline{\alpha_i}\alpha_i| = 1$ ensures that possible states of geographic fields are such that the contributions of all the orthogonal possibilities captured in the system of base vectors $\mathcal{Q}$ jointly add up to 1.

▶ Remark. There are more restrictions needed to ensure that the Hilbert bundles that are intended to represent geographic fields are 'well behaved' in the sense that the Hilbert spaces at neighboring points are compatible, that the section representing the geographic fields are smooth, and others more. In particular the notion of 'connection' [10] is needed to compare vectors in different fibers of the fiber bundle. This goes beyond the scope of this paper. A very good discussion of many relevant aspects can be found in [10]. Whether or not those requirements are necessary/sufficient in the context of geographic fields is still an open question.

▶ **Definition 2.** Consider a geographic field $\hat{\mathsf{g}}$ in the $\mathcal{Q}$-base: $\hat{\mathsf{g}}$ is maximally determinate with respect to the $\mathcal{Q}$-base at x iff $\hat{\mathsf{g}}(x) = 1\,|Q_i\rangle$ for the i-th base vector and $0\,|Q_j\rangle$ for $i \neq j$. The field $\hat{\mathsf{g}}$ is minimally determinate with respect to the $\mathcal{Q}$-base at x iff $\hat{\mathsf{g}}(x) = \frac{1}{\sqrt{n}}\,|Q_1\rangle + \ldots + \frac{1}{\sqrt{n}}\,|Q_n\rangle$.

4 Quality base vs. localization base

Given the structure of the Hilbert spaces that form the fibers of a Hilbert bundle, every geographic field $\mathsf{g} \in \mathsf{Sect}(E_{\mathcal{H}})$ can be expressed in (at least) two *complimentary* systems of base vectors: the quality base and the localization base. Roughly, when g is expressed in the quality base, then the field is a map of signature $\hat{\mathsf{g}} : U \to C^n_{\mathcal{Q}}$ taking locations of the base space to superpositions of qualities that are represented by the n-tuples $C^n_{\mathcal{Q}}$. By contrast, if g is expressed in the localization base, then the field is a map of signature $\hat{\mathsf{g}} : U \to C^n_{\mathcal{X}}$ taking locations of the base space to superpositions of possible deviations from x that are represented by the n-tuples $C^n_{\mathcal{X}}$. That is, in the quality base $\hat{\mathsf{g}}$ maps locations of the base space to information about quality pattern while in the localization base $\hat{\mathsf{g}}$ maps locations of the base space to information about the (metric) closeness to which the information about the quality pattern contained in $\mathsf{g}(x)$ is linked to the location x of the base space (details in Sec. 4.2). In what follows the term 'deviation' is used to describe the distance between the point x to which the quality q is attributed by the field to the closest point at which q is actually expressed. To say that the quality and localization bases are complimentary is to say that if the information expressed in the quality base is maximally determinate then the information expressed in the localization base is minimally determinate and vice versa.

▶ Remark. Examples 3 and 4 below will illustrate that, when compared with the quantum mechanics of a free particle, the quality of a geographic field at a given position in the base space of the underlying fiber bundle is like the position of a free particle. The (degree of) localization a geographic field at that position in the base space is like the momentum of a free particle. Usually the quantum mechanics of a free particle allows for a continuum of possible positions and momenta. For the purpose of this paper it will be sufficient to consider discrete quality spaces and finitely many possible distinct qualities. Roughly, to go from the discrete to the continuous case is to replace sums by integrals.

QG allows for geographic fields that are in indeterminate states and thereby provides means for the expression of ontological vagueness. According to QG ontological vagueness has two interrelated aspects: quality indeterminacy and localization indeterminacy.

4.1 Geographic fields in the quality base

Let $Q_1, \ldots, Q_n$ be quality determinates that are pairwise disjoint and jointly exhaust some quality determinable ξ. A geographic ξ-field is a section in a Hilbert bundle (E_H, B, π, C^n) such that fibers $\pi^{-1}(x) = \mathcal{H}_x$ over each point $x \in B$ have the structure of a Hilbert space. The vectors in this space are expressed in the quality base $\mathcal{Q} = |Q_1\rangle, \ldots, |Q_n\rangle$ of the abstract fiber of the associated local trivialization. The base vectors $|Q_1\rangle, \ldots, |Q_n\rangle$ could, for example, be qualities at the same level of a universal hierarchy identified by the Aristotelian method of classification. Consider the left of Fig. 4. The geographic field associated with the quality determinable *Ecoregion Domain* has at every point of the base space a Hilbert space $\mathcal{H}_x^D$ which vectors can be expressed in the quality base $\mathcal{Q} = \{|P\rangle, |HTe\rangle, |HTr\rangle, |D\rangle\}$.

▶ **Postulate 3.** *A ξ-field of the form $x \in B \mapsto q_1 |Q_1\rangle + \ldots + q_n |Q_n\rangle \in \mathcal{H}_x$ is interpreted as: The quality Q_i is expressed in the neighborhood of x to the degree $|\overline{q}_i q_i|$ in a way such that $\sum_j |\overline{q}_j q_j| = 1$.*

As pointed out above, in QG it is possible that in the neighborhood of a given point x a combination of incompatible qualities are expressed at any given time. This captures the aspect of quality indeterminacy of the underlying ontological vagueness. For example, for the geographic field associated with the quality determinable *Ecoregion Domain* the maximally indeterminate state in the quality base at a given point of the base space is $\frac{1}{\sqrt{4}}(|P\rangle + |HTe\rangle + |HTr\rangle + |D\rangle)$. By contrast, maximally determinate states in the quality base include the state $1|P\rangle + 0|HTe\rangle + 0|HTr\rangle + 0|D\rangle$. A state of intermediate indeterminacy is $\sqrt{0.1}\,|P\rangle + \sqrt{0.4}\,|HTe\rangle + \sqrt{0.3}\,|HTR\rangle + \sqrt{0.2}\,|D\rangle$.

Classification

Classification in QG, is the assignment of a determinate classification value to all the locations of the base space of a geographic field. That is, classification is an operation that takes as input a geographic field $\hat{\mathfrak{g}}$ that, when represented in the quality base, is in a state in which superpositions of classically contradicting qualities are expressed in the neighborhoods of the points in the base space. The classification operation then maps $\hat{\mathfrak{g}}$ to a field $\hat{\mathfrak{g}}'$ in the same fiber bundle in which exactly one of the possible qualities is exclusively expressed in the respective neighborhoods.

▶ **Postulate 4.** *A classification operator $\hat{\mathcal{C}}_x$ is an operator on the Hilbert space $\mathcal{H}_x^{\mathcal{Q}}$ with the following properties: (a) $\hat{\mathcal{C}}_x$ is a self-adjoint[2] operator on $\mathcal{H}_x^{\mathcal{Q}}$; (b) the set of base vectors of the quality base $\mathcal{Q} = |Q_1\rangle, \ldots, |Q_n\rangle \subset \mathcal{H}_x^{\mathcal{Q}}$ are eigenvectors of $\hat{\mathcal{C}}_x$ such that $\hat{\mathcal{C}} |Q_i\rangle = Q_i |Q_i\rangle$ where Q_i is a non-complex number; and (c) $\hat{\mathcal{C}}_x = |\phi\rangle \in \mathcal{H}_x^{\mathcal{Q}} \mapsto |Q_i\rangle \in \mathcal{Q}$ for some i.*

Here Q_i is a classification value and $|Q_i\rangle$ is the state in which the quality determinate that is represented by the number Q_i is exclusively expressed in the neighborhood of x associated with the underlying Hilbert space $\mathcal{H}_x^{\mathcal{Q}}$. That is, the quality values that a field in a determinate state can possibly have at a given location of the base space are given by the eigenvalues of the classification operator. The base vectors of the Hilbert spaces in the quality base are the eigenvectors of the classification operator. If a geographic field is in an eigenstate at

[2] A self-adjoint operator on a complex vector space $\mathcal{H}$ with inner product $\langle\cdot|\cdot\rangle$ is a linear map $\hat{A}$ from $\mathcal{H}$ to itself with a unique corresponding operator $\hat{A}^\dagger$ such that: $(\langle\phi| \hat{A}^\dagger) |\psi\rangle = \langle\phi| (\hat{A} |\psi\rangle)$ for all $|\phi\rangle, |\psi\rangle \in \mathcal{H}$. If $\hat{A}$ is represented by a square matrix with complex values, then $\hat{A}^\dagger$ is the matrix obtained from $\hat{A}$ by complex conjugation and transposition.

a given location x of the base space then the field value at this point is the corresponding eigenvalue. According to the classification the quality corresponding to this eigenvalue is exclusively expressed in the neighborhood assigned to x.

▶ **Example 3.** Consider[3] the Hilbert space $\mathcal{H}_y^{\mathcal{X}}$ with the base vectors $|P\rangle, |HTe\rangle, |HTr\rangle, |D\rangle$. In analogy to the position base in QM let $X = \{0, 1, 2, 3\}$ be the set of possible qualities (possible positions in QM) such that 0 stands for Polar Domain, 1 stands for Humid Temperate Domain, 2 stands for Humid Tropical Domain, and 3 stands for Dry Domain. Let the base vectors be functions of the form $|P\rangle : X \to \{0, 1\}$ where $|P\rangle \equiv \lambda x.\ x = 0$, $|HTe\rangle \equiv \lambda x.\ x = 1$, $|HTr\rangle \equiv \lambda x.\ x = 2$, and $|D\rangle \equiv \lambda x.\ x = 3$. For example, $|D\rangle : X \to \{0, 1\}$ is a function that yields 1 if the value of its argument is 3 ($3 = 3$ is true) and 0 otherwise (e.g., $3 = 2$ is false). Functions of this kind form a Hilbert space the members of which are all the functions that can be formed by adding the complex multiples of the functions that save as base vectors. In analogy to the position operator in QM, the classification operator $\mathcal{C}$ is defined as $(\mathcal{C}\ \ |\phi\rangle)$ the multiplication of the state vector $|\phi\rangle$ with the location x at which the operator $\mathcal{C}$ is evaluated. If the operator is evaluated at position x of the underlying quality space then one has $(\mathcal{C}\ \ |\phi\rangle)x \equiv (x\,|\phi\rangle)x$. If x takes its values from the set $X = \{0, 1, 2, 3\}$ then the eigenvalues are $0, 1, 2$ and 3 such that 0 stands for Polar Domain, 1 stands for Humid Temperate Domain, 2 stands for Humid Tropical Domain, and 3 stands for Dry Domain.

On this view, classification is an operation that takes as input a geographic field that is subject to indeterminacy and maps it to a different field that is not subject to indeterminacy. That is, the classification operator $\hat{\mathcal{C}}$ collapses a superposition state $|\phi\rangle$ into one of its eigenstates $|Q_i\rangle$. According to Quantum Mechanics the collapse of a superposition state into an eigenstate as the result of a measurement (in the widest sense) is, on the standard (the 'Kopenhagen') interpretation, inherently indeterministic in nature (e.g., [12]). In QM this indeterminacy is expressed in a probabilistic way. If one were to follow this view in QG one would have:

▶ **Hypothesis 2.** *If $|\phi\rangle = q_1 |Q_1\rangle + \ldots + q_n |Q_n\rangle \in \mathcal{H}_x^{\mathcal{Q}}$ is the state of a geographic field in the neighborhood of x then the probability that the neighborhood of x is classified as Q_i is $|\overline{q}_i q_i|$.*

That is, the more dominant the expression of Q_i in the neighborhood of x the more likely it is that this neighborhood is classified as having the quality Q_i.

Unlike measurement in Physics, classification in geography seems to be more the result of the deliberative actions of cognitive agents that include a certain degree of fiat. To understand the deliberative action of a cognitive agent as a random process may not be appropriate. An alternative way of interpreting the collapse a superposition state $|\phi\rangle$ into one of its eigenstates $|Q_i\rangle$ could be:

▶ **Hypothesis 3.** *If $|\phi\rangle = q_1 |Q_1\rangle + \ldots + q_n |Q_n\rangle \in \mathcal{H}_x^{\mathcal{Q}}$ is the state of a geographic field in the neighborhood of x then this neighborhood is classified as Q_i if a maximum $|\overline{q}_i q_i| = \max\{|\overline{q}_j q_j| \mid 1 \leq j \leq n\}$ exists.*

On this interpretation one assumes that a cognitive agent is able to perceive the degrees to which certain qualities are expressed at a certain location of a geographic field. One also

[3] This example in conjunction with Example 4 is intended to illustrate the ways in which the math of QM/QG with their operators, eigenvectors and eigenvalues is designed to achieve a formalism with properties that (at least in the case of QM) yields surprisingly accurate predictions. Details can be found in any introductory textbook on QM (e.g., [8]).

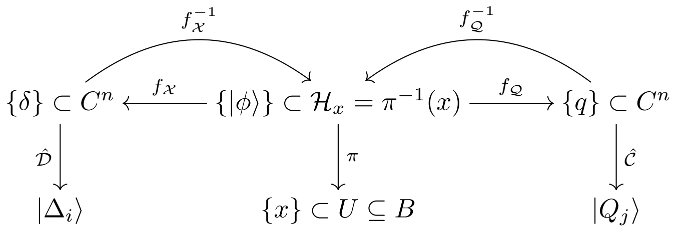
allows for the possibility that the cognitive agent is unable or unwilling to make a judgement when there is no unique maximum.

Whether or not the first or the second hypothesis or neither of them is actually true of the geographic world is in the opinion of the author an empirical question and can (at least in principle) be determined by experiments.

4.2 Geographic fields in the localization base

In QG, a state vector $|\phi\rangle \in \mathcal{H}_x$, which in the quality base takes the form $q_1 |Q_1\rangle + \ldots + q_n |Q_n\rangle$, not only encodes information about the expression of geographic qualities $Q_1, \ldots, Q_n$ in the neighborhood of the position x in the base space. The state vector $|\phi\rangle$ also encodes information about possible inhomogeneities that may affect the expression of these qualities in the neighborhood of x. The information about possible inhomogeneities of the field g at x that is encoded in the state vector $|\phi\rangle \in \mathcal{H}_x$ is accessible when $|\phi\rangle$ is expressed in the localization base. The localization base $\mathcal{X}$ of $\mathcal{H}_x^{\mathcal{X}}$ is a set $\mathcal{X} = |\Delta_1\rangle, \ldots, |\Delta_n\rangle$ such that every field state $|\phi\rangle \in \mathcal{H}_x$ can be expressed in this base as $|\phi\rangle = \delta_1 |\Delta_1\rangle + \ldots + \delta_n |\Delta_n\rangle \in \mathcal{H}_x^{\mathcal{X}}$ with the additional normalization constraint $\sum_i |\overline{\delta}_i \delta_i| = 1$. The idea that the state $|\phi\rangle$ of a geographic field g at a given location $x \in B$ can be described in the quality base $\mathcal{Q}$ as well as the localization base $\mathcal{X}$ is visualized in Diag 4.

state vector in the localization base $\hat{\mathsf{g}}(x) = \sum_i \delta_i |\Delta_i\rangle$	state vector $\mathsf{g}(x) = |\phi\rangle$	state vector in the classification base $\hat{\mathsf{g}}(x) = \sum_i q_i |Q_i\rangle$	(4)



Intuitively, the base vectors $|\Delta_i\rangle$ can be thought of as the various degrees of inhomogeneity that are possible for the field at a given location of the base space. That is, the localization base vectors $|\Delta_i\rangle$ represent states of the field g at $x \in U \subseteq B$ where possible ranges of deviation from x due to inhomogeneities of the field g at that location corresponds to a collection of nested rings that are centered at x (Fig. 4 right). In general the state $|\phi\rangle = \sum_i \delta_i |\Delta_i\rangle$ will be a superposition state that is subject to the constraint $\sum_i |\overline{\delta}_i \delta_i| = 1$. That is, to the degree quantified by the value of the expression $|\overline{\delta}_i \delta_i|$ all of the possible ranges of deviation Δ_i are realized. These superpositions are expressions of the indeterminacy associated with the underlying ontological vagueness of the inhomogeneities of geographic fields. In the localization base a section of the fiber bundle is a mapping of the form

$$\hat{\mathsf{g}} : x \in U \subset B \mapsto \delta_1 |\Delta_1\rangle + \ldots + \delta_n |\Delta_n\rangle \in \mathcal{H}_x^{\mathcal{X}}.$$

Delineation

Delineation in Quantum Geography, is the assignment of determinate localization information to all the points of the base space of a geographic field. That is, delineation is an operation that takes a geographic field $\hat{\mathsf{g}}$ that is represented in the localization base and which is in superposition states at all (or many) points in the base space. The delineation operator $\hat{\mathcal{D}}$ maps the field $\hat{\mathfrak{g}}$ to a field $\hat{\mathfrak{g}}''$ in the same fiber bundle. At all positions of the base space the field $\hat{\mathfrak{g}}$ is in in a state that corresponds to one of the members of the localization base.

► **Postulate 5.** *If $\hat{\mathcal{D}}_x$ is a delineation operator on the Hilbert space $\mathcal{H}_x^{\mathcal{X}}$ then: (a) $\hat{\mathcal{D}}$ is a self-adjoint operator on $\mathcal{H}_x^{\mathcal{X}}$; (b) the vectors of the localization base $\mathcal{X} = |\Delta_1\rangle, \ldots, |\Delta_n\rangle$ are eigenvectors of the operator $\hat{\mathcal{D}}$ such that $\hat{\mathcal{D}}\,|\Delta\rangle_i = d_i\,|\Delta\rangle_i$ for the eigenvalues d_i; and (c) $\hat{\mathcal{D}} = |\phi\rangle \in \mathcal{H}_x^{\mathcal{X}} \mapsto |\Delta_i\rangle \in \mathcal{X}$.*

As depicted in Fig. 4 (right), d_i is a distance range from $x \in U$ and $|\Delta_i\rangle$ is the state in which the qualities associated with $|\Delta_i\rangle$ when expressed in the quality base is definitively expressed within the range of distances from $x \in U$ that are associated with d_i. The possible deviation from the base point is due to inhomogeneities of the underlying geographic field. Numerically possible deviation correspond to the eigenvalues d_i of the delineation operator $\hat{\mathcal{D}}$. The base vectors $|\Delta_i\rangle$ of the Hilbert spaces in the localization base are the eigenvectors of $\hat{\mathcal{D}}$. If a geographic field is in an localization eigenstate at a given location x in the base space then the field value expressed in the localization base at this point will be the eigenvalue corresponding to this state.

► **Example 4.** Consider the Hilbert space $\mathcal{H}_y^{\mathcal{X}}$ at the point $y \in B$ of the field's base space. Assume that the $\mathcal{H}_y^{\mathcal{X}}$ is expressed in the localization base formed by the vectors $|\Delta_1\rangle, |\Delta_2\rangle, |\Delta_3\rangle, |\Delta_4\rangle$. In analogy to the definition of the momentum eigenstates in QM these vectors have the form $|\Delta_k\rangle \equiv \lambda x.\frac{1}{\sqrt{2\pi\Omega}} e^{\frac{i}{\Omega} d_k x}$ for $1 \leq k \leq 4$ where $i = \sqrt{-1}$ the imaginary unit and x ranges over the possible locations in quality space, i.e., the eigenvalues of the classification operator of Example 3. The constant Ω is a scale factor and plays the role of Plank's constant in QM (more on Ω in Sec. 5). Functions of this kind form a Hilbert space analogous to Example 3. The delineation operator is defined in analogy to the momentum operator in QM as: $(\mathcal{D}\ |\phi\rangle) \equiv (-i\frac{\partial}{\partial x}\,|\phi\rangle)$. The eigenvalues are given by: $\mathcal{D}\,|\Delta_k\rangle = -i\frac{\partial}{\partial x}\frac{1}{\sqrt{2\pi\Omega}} e^{\frac{i}{\Omega} d_k x} = \frac{d_k}{\Omega\sqrt{2\pi\Omega}} e^{\frac{i}{\Omega} d_k x} = \frac{d_k}{\Omega}\,|\Delta_k\rangle$. On the intended interpretation the eigenvalues d_k label the concentric circles in Fig. 4 (right) that indicate possible degrees of deviation due to the inhomogeneities in the underlying field.

Similarly to the collapse of a superposition state in the quality base, the collapse of a superposition state $|\phi\rangle$ into an eigenstate $|\Delta_i\rangle \in \mathcal{X}$ can be understood in a probabilistic way as follows:

► **Hypothesis 4.** *If $|\phi\rangle = \Sigma_i\ \delta_i\,|\Delta_i\rangle$ is the state of a geographic field in the neighborhood of x then the probability that the localization deviation associated with $|\phi\rangle$ has the value d_i is $|\overline{\delta}_i\delta_i|$.*

On this hypothesis the value $|\overline{\delta}_i\delta_i|$ specifies the probability that the quality pattern associated with the state $|\phi\rangle$ that is attributed to the point $x \in B$ can actually deviate from x by the distance range corresponding to d_i.

Similar to classification, delineation in geography seems to be more the result of the deliberative actions of cognitive agents that include a certain degree of fiat. The degree of fiat was particularly persuasive in the way in which the boundaries in Fig. 1 were drawn as the result of delineation operations. But, again, despite the indeterminacy there does not seem to be a random process at the heart of the collapse the superposition states. An alternative way of interpreting the collapse a superposition state $|\phi\rangle$ into one of its eigenstates $|\Delta_i\rangle$ could be the following:

▶ **Hypothesis 5.** *If* $|\phi\rangle = \Sigma_i\ \delta_i\ |\Delta_i\rangle$ *is the state of a geographic field in the neighborhood of* x *then the localization deviation associated with* $|\phi\rangle$ *has the degree* d_i *if there is a maximum of the form* $|\bar{\delta}_i \delta_i| = \max\{|\bar{\delta}_j \delta_j| \mid 1 \leq j \leq n\}$.

On this interpretation one assumes that a cognitive agent is able to perceive the localization deviation when drawing boundaries that delineate inhomogeneous regions with distinct qualities. This interpretation also allows for the possibility that the cognitive agent is unable or unwilling to make a judgement.

As in the case of classification, whether or not the first or the second hypothesis or neither is actually true of the geographic world is an empirical question and can (at least in principle) be determined by experiments.

4.3 Classification and delineation do not commute

After performing a classification or delineation operation on a geographic field, intermediate fields are generated (to a certain degree by fiat) that are such that all locations of the base space of the underlying fiber bundle are mapped to eigenstates of the classification or delineation operators that correspond to the classification of delineation values assigned to the point of the base space.

$$
\begin{array}{ccccccc}
 & & & \text{Fourier transform} & & & \\
\{\delta\} \subset C^n & \xrightarrow{f_X^{-1}} & \{|\phi\rangle\} \subset \mathcal{H}_x = \pi^{-1}(x) & \xleftarrow{f_Q^{-1}} & \{q\} \subset C^n & \xrightarrow{\hat{\mathcal{C}}} & |Q_i\rangle \\
\downarrow \hat{\mathcal{D}} & & & & & & \downarrow \hat{\mathcal{D}} \\
|\Delta_i\rangle & \xrightarrow{\hat{\mathcal{C}}} & \sum_k q_k\, |Q_k\rangle & \overset{\hat{\mathcal{C}}\hat{\mathcal{D}} \neq \hat{\mathcal{D}}\hat{\mathcal{C}}}{\Longleftrightarrow} & & & \sum_k \delta_k\, |\Delta_k\rangle
\end{array} \tag{5}
$$

Consider Diag. 5. If classification and delineation operations are performed in sequence (in either order) then the input of the second operation is a field that at all locations of the base space is in eigenstates of the preceding operator. States represented in the localization base are related to equivalent descriptions in the quality base via (discrete) Fourier transforms and vice versa [8]. States related by Fourier transforms are *complementary* in the sense that if one state is maximally determinate then its Fourier transform will be minimally determinate [8]. Consequently, the order of the sequence of the application of classification and delineation operations ($\hat{\mathcal{C}}\hat{\mathcal{D}}$ vs. $\hat{\mathcal{D}}\hat{\mathcal{C}}$) to geographic fields is significant. Performing the classification operation first will result in precise classification and an imprecise delineation. By contrast, performing the delineation operation first will result in a precise delineation and an imprecise classification. That is, the operators representing classification and delineation operations do not commute. That the operators representing classification and delineation operations do not commute is captured in the formalism of QG as follows:

$$\forall\, |\phi\rangle \in \mathcal{H}_x :\ (\hat{\mathcal{C}}\hat{\mathcal{D}} - \hat{\mathcal{D}}\hat{\mathcal{C}})\, |\phi\rangle =_{df} [\hat{\mathcal{C}}, \hat{\mathcal{D}}]\, |\phi\rangle \geq i\Omega \tag{6}$$

As above, $i = \sqrt{-1}$ is the imaginary unit and Ω is a scale factor. Equation (6) a restatement of Heisenberg's uncertainty principle [8] for classification and delineation operations.

5 More open questions . . .

An important open question is how to understand the role of the scale factor Ω in QG. In concluding the following preliminary ideas are offered in contexts where geographic fields are considered at time scales at which they can assumed to be in *stationary states*. In such a stationary state a geographic field does not change in ways that affect its classification and delineation. That is, structurally complex (but real *valued*) terms of the form $|\overline{q}_i q_i|$ and $|\overline{\delta}_i \delta_i|$ remain unchanged in stationary states. By contrast, their constituting terms – the complex *valued* q_i and δ_i – are not constant and change in ways that is governed by the time-independent Schrodinger Equation [8]. According to this equation the phase factors of the complex values q_i and δ_i change and form a standing wave which frequency depends on the scale factor Ω in ways that mirrors Plank's constant in QM. This suggests that the complex values of a geographic field in a stationary state corresponds to the frequency of the processes that give rise to the geographic quality determinables associated with the underlying geographic field. This would fundamentally link the aspects of granularity and inhomogeneity of geographic fields with the frequency of the underlying processes that give rise to those fields.

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