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Periodic Resource Reallocation in Two-Echelon Repairable Item Inventory Systems

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***Abstract*—Given an existing stock allocation in an inventory system, it is often necessary to perform reallocation over multiple time points to address inventory imbalance and maximize availability. In this paper, we focus on the situation where there are two opportunities to perform reallocation within a replenishment cycle. We derive a mathematical model to determine when and how to perform reallocation. Furthermore, we consider the extension of this model to the situation allowing an arbitrary number of reallocations. Experimental results show that the two-reallocation approach achieves better performance compared with the single-reallocation approach found in the literature. We also illustrate how to apply the proposed model to design cost-optimal periodic resupply policies.**

***Index Terms*—Two-Echelon Inventory, Periodic Resupply, Reallocation, Repairable Item, Military Logistics**

I. INTRODUCTION

We consider an arboreal inventory system in which a central depot serves n bases. Military systems such as aircrafts and tanks are deployed at the bases. These systems break down because the underlying components, which are called LRU (line replaceable units), are either worn out over time and/or damaged during usage. Stocks are allocated at the depot and bases to insure continuity of operations. When an LRU fails at a base, a spare replaces it if one is available; otherwise a backorder is incurred. In military practice, due to the limited space at the bases, the failed LRUs are usually sent back to the depot for repair. In the mean time, an order is placed by the base to the depot to send a spare. A spare will be sent to the base if one is available; otherwise there is a backorder at the depot. After the failure is repaired, it will be sent to the depot inventory to fulfill future demands.

As demands are stochastic, inventory imbalance will occur and tends to grow with time. This imbalance ultimately reaches a situation where some bases hold excess inventories, while others face critical shortage. To correct the imbalance, stocks need to be *reallocated*. To increase the efficiency and reduce unavailability, we also allow excess inventories at some bases to be reallocated laterally to others with shortage. In this paper, we are concerned with a *two-instant* reallocation scheme within a system replenishment cycle. This work is in response to the open challenge post by Cao and Silver [2] to consider two or more possible reallocations within a cycle.

In practice, this problem is also faced by planners who need determine the time between periodic reallocations. Although the (S-1,S) replenishment policy is generally assumed in the literature (i.e. depot will send a spare to the base once a failure occurs), this is a stylized situation since it is almost impossible to supply continuously in practice, especially in a naval environment. The depot has to send spares to offshore bases and bring the failures back for repair *periodically*. Hence, it is important to determine when and how to distribute stocks to the bases periodically in the cycle.

Given that we have two reallocation instants, it is important to determine when each reallocation should occur. If we perform the first reallocation too early, it may prevent early backorders but could lead to growth of backorders before the next reallocation or during the remaining time in the cycle. Conversely, performing reallocation later presents two problems: First, it may cause high levels of early backorders. Second, late reallocation may leave no time to perform the second reallocation. Furthermore, the time interval between the first and the second reallocations is also important: if it is too short, failures brought back to depot for repair may not have been completed and consequently the depot has too few spares to perform the second reallocation; if the time interval is too long, it causes higher levels of backorders at bases between reallocations. Therefore, the key issue is how best to synchronize the two reallocations.

Many papers have analyzed resource reallocation, periodic resupply and risk pooling effect. One example is the classic paper by Eppen and Schrage [4] which analyzed a multi-echelon inventory system considering external lead times and random demands, where the optimal allocation of stocks among multiple sites may not be feasible due to stock imbalance. In their model, the depot will order enough stocks from an outside supplier to ensure a certain level of system-wide inventory position, and then perform complete allocation of the received stock to the sites according to the demands during the external lead time. Jönsson and Silver [7] saw that

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this scheme has no reallocation possibility, and proposed a scheme that performs complete transshipment of all site inventories at a fixed instant, which is one period before the end of the order cycle – since their rationale is that stockouts primarily occur during the last periods of an order cycle. Jackson and Muckstadt [6] also considered a single, predetermined reallocation time and derive both exact and approximate optimality conditions that do not ignore the possibility of imbalance at the time of reallocation. One limitation of this work however is that, since they do not permit lateral resupply between sites, they encountered difficulties in trying to ascertain how much stock to allocate to each site. Another avenue of extension of the Eppen and Schrage [4] work, which also improves the situation given in Jackson and Muckstadt [6], is found in Jackson [5], which allows the central warehouse to hold stock and make allocations to the retailers in every period of the cycle. The proposed allocation policy is a "ship-up-to-S" policy: the warehouse makes shipments to restore the inventory position of each retailer to some predetermined value, S, in every period so long as the warehouse has sufficient stock. The concept of “pooled-risk period” was introduced, which refers to the latest period of allocation. Tsao and Enkawa [13] proposed a “two-phase push control policy” for considering the optimal reallocation instant in a two-echelon inventory system. Their method predetermines a fixed reallocation instant in all replenishment cycles, independent of the dynamic behavior of the inventories at the retailers. This situation was improved by Cao and Silver [2] recently, who proposed a heuristic method to dynamically determine the optimal reallocation instant in each replenishment cycle and perform reallocation at that instant.

The abovementioned papers deal mostly with consumable items. In a military context, it is important to perform reallocations of spare parts periodically because they are often very expensive and affect system availability greatly ([3], [9]). System availability is usually measured in terms of Expected Backorders (EBO) (e.g. [1], [8], [11]). To our knowledge, there are few works on redistribution of spare parts in multi-echelon systems, except a brief mention of the problem in [11] and a feature within a proprietary commercial tool OPUS [10].

This paper makes technical contributions in the following ways. First and foremost, we consider how to perform more than one reallocation within a replenishment cycle, and instead of fixing reallocation instants to certain time points, we propose how to determine the time point instants for reallocation. This is a response to the challenge post by Cao and Silver [2], which issued an open question for the problem of multiple reallocations within a cycle. Furthermore, we relax the classical assumptions in the following manner. The replenishment of stocks from the depot to bases is periodic, i.e. stocks and failures can be transported only at certain time points in a batch. The internal lead (or transport) time between the depot and bases is nonzero. Transshipments among bases are allowed.

The remainder of this paper is organized as follows. Section 2 gives the problem definition, assumptions and notations. Our mathematical model and approach are then presented in Section 3. Section 4 presents extensive experimental results. Section 5 shows how our model can be extended to design cost-optimal periodic resupply policy. Finally, conclusions and future work are provided in Section 6.

II. Problem Definition

Our resource reallocation problem is based on two important and simplifying assumptions. First we assume the internal lead (or transport) times for moving items from the central depot to each base is not negligible, but the lead time laterally between bases is negligible. In practice, the transport time between echelons is more important to military planners such that it usually cannot be ignored while the assumption of negligible lateral lead time is consistent with [2], [5], [13] which assume reallocation can be achieved instantly. Thus the spares should be transported from the depot ahead of internal lead time for the destination so that it can arrive on time for reallocation. Second, to simplify the problem, we assume all failures are only repairable at the depot which has infinite repair capacities. The repair time is exponentially distributed with mean T. Because of transport time, repair at the depot can only take place in lead time after reallocation.

The system replenishment cycle is H base periods, i.e. every H base periods, the central depot places orders to an outsider supplier. Therefore our reallocation decision time horizon is within this replenishment cycle. Demands over time at the bases are assumed to be independent, Poisson variables with mean λ_i at base i during each period.

The main notations used in this paper are as follows.

i: index of site ($i = 0$ for the depot)
n: number of bases
S_i: initial stock level of LRU at site i
H: system replenishment cycle, in base periods
L: internal lead time, i.e. transport time between depot and base
T: mean repair time of LRU
$y_i(t)$: (Poisson random variable) demands in a single period t at base i
λ_i: mean value of $y_i(t)$
τ: index of the period at the end of which reallocation takes place
$I_i(\tau)$: stock level at site i instantly before reallocation at the end of period τ
$U_i(\tau)$: stock level at site i instantly after reallocation at the end of period τ
t_1: time point at which the first reallocation is performed
t_2: time point at which the second reallocation is performed
$EBO_i(t)$: expected backorder at the end of period t at base i (If reallocation takes place at t, it denotes the EBO instantly before reallocation)
$EBO(t)$: expected sum of backorders over all bases at the end of period t

$TE_i(t_1, t_2)$: total (i.e. aggregate) EBO at base i at time points t_1, t_2 and H

$TE(t_1, t_2)$: total EBO across all bases at time points t_1, t_2 and H.

Given the initial number of stocks at each site, we need to decide the variables t_1, t_2 at which reallocation takes place, as well as the number of stocks at each site after reallocation so as minimize the total EBO over all bases at three time points (at the end of period t_1 and t_2 just before the reallocations respectively and at the end of the cycle). We like to clarify at this stage that we use the term 'reallocation' to refer to three separate reallocation activities of different types of inventories at different time points within a cycle: a) Reallocation of spare items from the depot to bases at time points $t_1 - L$ and $t_2 - L$; b) Reallocation of spare items among bases at time points t_1 and t_2; and c) Sending failed items from bases to the depot at time point t t_1.

Note that the value of the objective function depends on the times at which the reallocations are carried out and how the reallocations are done. Notationally therefore, our aim is to find t_1, t_2, $U_i(t_1)$, $U_i(t_2)$ such that the function $TE(t_1,t_2) = \sum_{i=1}^{n} TE_i(t_1,t_2)$ is minimized, where

$$TE_i(t_1,t_2) = EBO_i(t_1) + EBO_i(t_2) + EBO_i(H) \qquad (1)$$

III. Mathematical Model

A. Before the First Reallocation

Given the initial stock allocations at all sites, the expected backorders over all bases at time t_1, $EBO(t_1)$, instantly before reallocation can be calculated according to the standard definition of EBO as follows:

$$EBO(t_1) = \sum_{i=1}^{n} EBO_i(t_1) = \sum_{i=1}^{n} \int_{S_i}^{\infty} (x_i - S_i) f(x_i) dx_i \qquad (2)$$

where x_i is a realization of random variable of $D_i(t_1)$. $D_i(t_1) = \sum_{t=1}^{t_1} y_i(t)$ denotes the demands of LRUs at base i during time interval $[0, t_1]$, which is a Poisson random variable with mean $t_1\lambda_i$ and $f(\cdot)$ is the probability density function of $D_i(t_1)$. In this paper, we approximate $D_i(t_1)$ by a normally distributed random variable with mean $E(D_i(t_1)) = t_1\lambda_i$, and variance $Var(D_i(t_1)) = t_1\lambda_i$ and hence the probability density function is $f(x_i) = \frac{1}{\sqrt{2\pi t_1 \lambda_i}} \exp\left\{-\frac{(x_i - t_1\lambda_i)^2}{2t_1\lambda_i}\right\}$. We justify this approximation as follows. Standard statistics have shown that this approximation is good when the mean value of the Poisson random variable is no smaller than 10. Furthermore, it turns out from our detailed numerical investigation that the approximation is still satisfactory for our purpose even with mean values no smaller than 4 (see Appendix A). In the military context, we witness a prolonged replenishment cycle, where the sum of demands arising in the interval between the start and the first allocation or between the two allocations exhibit relatively large mean values. This phenomenon is also seen in a variety of commercial settings reviewed in [14], such as copying machines and transportation equipment, which have relatively long product lifecycle, or electronics, which require a relatively large number of repairable items.

Hence, after standardization and computation, $EBO(t_1)$ can be expressed by:

$$EBO(t_1) = \sum_{i=1}^{n} \sqrt{t_1\lambda_i}\, G\left(\frac{S_i - t_1\lambda_i}{\sqrt{t_1\lambda_i}}\right) \qquad (3)$$

where $G(k) = \int_k^{\infty} (z-k)\phi(z)dz$ is the unit normal loss function and $\phi(z)$ is the probability density function of the standard normal distribution.

B. The First Reallocation

At time t_1, we will perform the first reallocation. The spares will be distributed from the depot to bases and among different bases while failures at all bases will be sent back to depot for repair. From our assumption, the spares at the depot will commence transportation for the bases at time $t_1 - L$ and arrive at time t_1 for reallocation, while transshipment among bases will occur instantly. On the other hand, repair for failed items will start at the depot at $t_1 + L$ because of the transport time. Given the inventory levels at all sites before reallocation $I_i(t_1)$ ($i = 0, 1, \ldots, n$), our goal is to find the inventory levels at all sites after reallocation $U_i(t_1)$ ($i = 0, 1, \ldots, n$) such that the EBO over all bases by t_2 just before the second reallocation will be minimized. That is, the reallocated spares will be used to last until the next reallocation. Mathematically, EBO over all bases by t_2 can be expressed as:

$$\begin{aligned} EBO(t_2) &= \min_{U_i(t_1)'s} \sum_{i=1}^{n} EBO_i(t_2) \\ &= \min_{U_i(t_1)'s} \sum_{i=1}^{n} \sqrt{(t_2 - t_1)\lambda_i}\, G\left(\frac{U_i(t_1) - (t_2 - t_1)\lambda_i}{\sqrt{(t_2 - t_1)\lambda_i}}\right) \end{aligned} \qquad (4)$$

And we have the constraints

$$\begin{aligned} &\sum_{i=1}^{n} U_i(t_1) + U_0(t_1) = \sum_{i=1}^{n} I_i(t_1) + I_0(t_1) \\ &U_i(t_1) \geq 0, (i = 0, \ldots, n) \end{aligned} \qquad (5)$$

Without transshipments, the inequality in constraints (5) should be changed into $U_i(t_1) \geq I_i(t_1)$. We know that those spares in the transport pipelines from the depot have no effect on the EBO at bases until they arrive at bases instantly before the reallocation. Therefore, we constrain $I_0(\tau) = I_0(\tau\text{-}L)$ and $U_0(\tau) = U_0(\tau\text{-}L)$ where reallocation takes place at the end of period τ. So we have $I_0(t_1) = I_0(t_1 - L) = S_0$ and we know $I_i(t_1) = S_i - D_i(t_1)$ for $i = 1, \ldots, n$ where $D_i(t_1) = \sum_{t=1}^{t_1} y_i(t)$ and $t_1 > L$. Hence, Equation (5) can be changed into:

$$\sum_{i=1}^{n} U_i(t_1) + U_0(t_1) = S_0 + \sum_{i=1}^{n} S_i - \sum_{i=1}^{n} D_i(t_1) \qquad (6)$$

Using a Lagrange multiplier λ, the optimization can be represented as

$$\min_{U_i(t_1)'s} \sum_{i=1}^{n} \sqrt{(t_2-t_1)\lambda_i}\, G\left(\frac{U_i(t_1)-(t_2-t_1)\lambda_i}{\sqrt{(t_2-t_1)\lambda_i}}\right) \qquad (7)$$

$$+\lambda(\sum_{i=1}^{n} U_i(t_1)+U_0(t_1)-\sum_{i=1}^{n} I_i(t_1)-I_0(t_1))$$

Differentiating with respect to $U_i(t_1)$ (i = 1, …, n) and setting the result to zero, we obtain

$$-\Psi\left(\frac{U_i(t_1)-(t_2-t_1)\lambda_i}{\sqrt{(t_2-t_1)\lambda_i}}\right)+\lambda=0 \quad (i = 1, \ldots, n) \qquad (8)$$

where $\Psi(k)=\int_k^{\infty}\phi(z)dz$ is the right-hand tail area of the standard normal distribution. So according to the property of standard normal distribution,

$$\frac{U_i(t_1)-(t_2-t_1)\lambda_i}{\sqrt{(t_2-t_1)\lambda_i}}=c \quad (i = 1, \ldots, n) \qquad (9)$$

where c is a constant, independent of i.

Using (9) to sum over all bases and using (6) leads to

$$U_i(t_1)=(t_2-t_1)\lambda_i+\frac{\sqrt{\lambda_i}}{\sum_{j=1}^{n}\sqrt{\lambda_j}} \times[S_0+\sum_{i=1}^{n}S_i-\sum_{i=1}^{n}D_i(t_1)-U_0(t_1)-(t_2-t_1)\sum_{j=1}^{n}\lambda_j] \qquad (10)$$

C. The Second Reallocation

Using the same method as above, for the second reallocation, our purpose is to find $U_i(t_2)$ (i = 0, 1, …, n) such that the EBO over all bases at the end of the cycle will be minimized. However, due to the transport time L between the depot and bases, the repair cannot start until failures arrive at the depot at time $t_1 + L$ and similarly spares must be sent out to bases at $t_2 - L$. In order to have more spares for the second reallocation, we constrain $t_2 > t_1 + 2L$. Those failures coming out of the repair pipeline after $t_2 - L$ can only be used for reallocation next time if there are subsequent reallocation instants, but since there is no more opportunity to perform a third reallocation, we will completely redistribute all stocks on hand at the depot to bases by $t_2 - L$. Assuming complete redistribution at the second reallocation instant, we have $U_0(t_2) = U_0(t_2 - L) = 0$.

Our objective function is:

$$EBO(H)=\min_{U_i(t_2)'s}\sum_{i=1}^{n}EBO_i(H) =\min_{U_i(t_2)'s}\sum_{i=1}^{n}\sqrt{(H-t_2)\lambda_i}\,G\left(\frac{U_i(t_2)-(H-t_2)\lambda_i}{\sqrt{(H-t_2)\lambda_i}}\right) \qquad (11)$$

subject to the constraints

$$\sum_{i=1}^{n}U_i(t_2)=\sum_{i=1}^{n}I_i(t_2)+I_0(t_2) \qquad (12)$$

We know $I_i(t_2) = U_i(t_1) - D_i(t_2)$ for i = 1, …, n where $D_i(t_2)=\sum_{t=t_1+1}^{t_2}y_i(t)$. The number of available spares at the depot just before the second reallocation equals to the number of spares left at the depot after the first reallocation $U_0(t_1)$, plus those failed items (sent to depot during the first reallocation) which have finished repair and sent to depot inventory by time $t_2 - L$. Let R be the random variable that represents the number of such items. Hence, we have $I_0(t_2) = I_0(t_2-L) = U_0(t_1) + R$. Assuming the repair time at the depot follows an exponential distribution with mean T, the probability that a failed item has finished repair and sent to inventory by $t_2 - L$ is given by $1 - \exp[-(t_2 - t_1 - 2L)/T]$. Since we know $E(\sum_{i=1}^{n}D_i(t_1))=t_1\sum_{i=1}^{n}\lambda_i$ and $Var(\sum_{i=1}^{n}D_i(t_1))=t_1\sum_{i=1}^{n}\lambda_i$, this implies $E(R)=t_1\sum_{i=1}^{n}\lambda_i(1-e^{-(t_2-t_1-2L)/T})$ and $Var(R)=t_1\sum_{i=1}^{n}\lambda_i(1-e^{-(t_2-t_1-2L)/T})^2$. Therefore, Equation (12) can be rewritten as:

$$\sum_{i=1}^{n}U_i(t_2)=\sum_{i=1}^{n}U_i(t_1)-\sum_{i=1}^{n}D_i(t_2)+U_0(t_1)+R \qquad (13)$$

Using (6), Equation (13) can be further changed into:

$$\sum_{i=1}^{n}U_i(t_2)=S_0+\sum_{i=1}^{n}S_i-\sum_{i=1}^{n}D_i(t_1)-\sum_{i=1}^{n}D_i(t_2)+R \qquad (14)$$

As the above method, using a Lagrange multiplier and differentiating with respect to $U_i(t_2)$ (i = 1, …, n) and setting the result to zero, we obtain:

$$U_i(t_2)=(H-t_2)\lambda_i+\frac{\sqrt{\lambda_i}}{\sum_{j=1}^{n}\sqrt{\lambda_j}}\times [S_0+\sum_{i=1}^{n}S_i-Y-(H-t_2)\sum_{j=1}^{n}\lambda_j] \qquad (15)$$

where $Y = Y_1 + Y_2 - R$, $Y_1=\sum_{i=1}^{n}D_i(t_1)$, $Y_2=\sum_{i=1}^{n}D_i(t_2)$.

D. Compute Optimal EBO

Using (10) and (15), we can compute the optimal spare allocations of LRU after each reallocation. However, we have not specified how to compute $EBO(t_2)$ by (4) and $EBO(H)$ by (11). In addition, $U_0(t_1)$ is still included in (10), i.e. the inventory level at each base after the first reallocation depends on different inventory levels left at the depot.

From (15), we know that no matter how many spares are left at the depot after the first reallocation, under the complete redistribution assumption for the second reallocation, the term $U_0(t_1)$ will disappear, i.e. the inventory level is independent of $U_0(t_1)$. Therefore, in order to reduce EBO just before the second reallocation at t_2, $U_0(t_1)$ should be set to zero according to (10). That is, we also adopt complete redistribution for the first reallocation. Thus,

$$U_i(t_1)=(t_2-t_1)\lambda_i+\frac{\sqrt{\lambda_i}}{\sum_{j=1}^{n}\sqrt{\lambda_j}}\times [S_0+\sum_{i=1}^{n}S_i-Y_1-(t_2-t_1)\sum_{j=1}^{n}\lambda_j] \qquad (16)$$

where $Y_1=\sum_{i=1}^{n}D_i(t_1)=\sum_{t=1}^{t_1}\sum_{i=1}^{n}y_i(t)$.

Substituting $U_i(t_1)$ in (4) by (16), we can compute the

$EBO(t_2)$ just before the second reallocation for a given value of Y_1. However, Y_1 is a normally distributed random variable with mean $E(Y_1) = t_1 \sum_{i=1}^{n} \lambda_i$ and variance $Var(Y_1) = t_1 \sum_{i=1}^{n} \lambda_i$. Thus, weighting $EBO(t_2)$ for a given value of Y_1 by the density of Y_1 and integrating over Y_1, we have

$$EBO(t_2) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sqrt{(t_2 - t_1)\lambda_i}\, G\left(\frac{U_i(t_1) - (t_2 - t_1)\lambda_i}{\sqrt{(t_2 - t_1)\lambda_i}} \right) f(y_1) dy_1 \quad (17)$$

By substituting $\xi = \dfrac{Y_1 - t_1 \sum_{i=1}^{n} \lambda_i}{\sqrt{t_1 \sum_{i=1}^{n} \lambda_i}}$ and using (16), Equation (17) can be rewritten as

$$EBO(t_2) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sqrt{(t_2 - t_1)\lambda_i}\, G(a\xi + b)\varphi(\xi) d\xi \quad (18)$$

where $a = \dfrac{-\sqrt{t_1 \sum_{i=1}^{n} \lambda_i}}{\sqrt{t_2 - t_1} \sum_{i=1}^{n} \sqrt{\lambda_i}}$ and $b = \dfrac{S_0 + \sum_{i=1}^{n} S_i - t_2 \sum_{i=1}^{n} \lambda_i}{\sqrt{t_2 - t_1} \sum_{i=1}^{n} \sqrt{\lambda_i}}$

Using the result in [2] and [12] where $\int_{-\infty}^{\infty} G(ax + b)\phi(x) dx = \sqrt{1 + a^2}\, G\left(\dfrac{b}{\sqrt{1 + a^2}} \right)$, we obtain

$$EBO(t_2) = \sqrt{(t_2 - t_1)(\sum_{i=1}^{n} \sqrt{\lambda_i})^2 + t_1 \sum_{i=1}^{n} \lambda_i} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - t_2 \sum_{i=1}^{n} \lambda_i}{\sqrt{(t_2 - t_1)(\sum_{i=1}^{n} \sqrt{\lambda_i})^2 + t_1 \sum_{i=1}^{n} \lambda_i}} \right) \quad (19)$$

Respectively, using (11) and (15), we can also construct the formula of $EBO(H)$ for a given value of Y, in which Y is a normally distributed random variable. $Y = Y_1 + Y_2 - R$, $Y_1 = \sum_{i=1}^{n} D_i(t_1) = \sum_{t=1}^{t_1} \sum_{i=1}^{n} y_i(t)$, and $Y_2 = \sum_{i=1}^{n} D_i(t_2) = \sum_{t=t_1+1}^{t_2} \sum_{i=1}^{n} y_i(t)$. Thus, it has mean

$$E(Y) = E(Y_1) + E(Y_2) - E(R)$$
$$= t_1 \sum_{i=1}^{n} \lambda_i + (t_2 - t_1) \sum_{i=1}^{n} \lambda_i - t_1 \sum_{i=1}^{n} \lambda_i [1 - e^{-(t_2 - t_1 - 2L)/T}]$$
$$= t_2 \sum_{i=1}^{n} \lambda_i - t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})$$

and variance

$$Var(Y) = Var(Y_1) + Var(Y_2) + Var(R)$$
$$= t_1 \sum_{i=1}^{n} \lambda_i + (t_2 - t_1) \sum_{i=1}^{n} \lambda_i + t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})^2$$
$$= t_2 \sum_{i=1}^{n} \lambda_i + t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})^2$$

Weighting $EBO(H)$ for a given value of Y by the density of Y and integrating over Y, we have

$$EBO(H) = \int_{-\infty}^{\infty} \sum_{i=1}^{n} \sqrt{(H - t_2)\lambda_i}\, G\left(\frac{U_i(t_2) - (H - t_2)\lambda_i}{\sqrt{(H - t_2)\lambda_i}} \right) f(y) dy \quad (20)$$

and using the same method as $EBO(t_2)$, we obtain

$$EBO(H) = \sqrt{(H - t_2)(\sum_{i=1}^{n} \sqrt{\lambda_i})^2 + t_2 \sum_{i=1}^{n} \lambda_i + t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})^2} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - H \sum_{i=1}^{n} \lambda_i + t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})}{\sqrt{(H - t_2)(\sum_{i=1}^{n} \sqrt{\lambda_i})^2 + t_2 \sum_{i=1}^{n} \lambda_i + t_1 \sum_{i=1}^{n} \lambda_i (1 - e^{-(t_2 - t_1 - 2L)/T})^2}} \right) \quad (21)$$

E. Two-Allocation Approach

Using (3), (19) and (21), we can calculate our objective function TE, the total expected backorders over all bases at three time points for a given reallocation instant pair t_1 and t_2:

$TE(t_1, t_2) = EBO(t_1) + EBO(t_2) + EBO(H)$ (22)

Our purpose is to find such t_1 and t_2 ($0 < t_1 < t_2 < H$) that TE is minimized. Due to the transport time L from the depot to bases, the first reallocation (t_1) cannot take place earlier than the period L. In addition, the second reallocation (t_2) has to take place before the period H. The two reallocation times are constrained by $t_2 > t_1 + 2L$ as mentioned in Section 3.3. Therefore, the first reallocation (t_1) can take place at the interval $[L, H\text{-}2L\text{-}2]$ whereas the second reallocation (t_2) can take place at the interval $[t_1+2L+1, H\text{-}1]$. Hence, we can compute TE for possible pairs of (t_1, t_2) to determine the minimal TE and the corresponding (t_1, t_2).

However, if the second reallocation instant is the end of the cycle, we do not make good use of two reallocation opportunities. TE will be the same as that in [2] with only one reallocation because our objective function is the expected backorders just before reallocation. [2] proved that the TE value will decrease first and then increase as t_2 increases for a given t_1. Hence our heuristic strategy can be stated as a simple search procedure as follows:

```
for (t1 = L; t1 < H – 2L – 1; t1++)
    for (t2 = t1 + 2L + 1; t2 < H; t2++) {
        // perform 2nd reallocation later
        if (TE(t1, t2) > TE(t1, t2+1)) continue;
        // perform 2nd reallocation at this t2
        else store this value of TE(t1, t2) and break;
    }
Compare the stored TE(t1, t2) values for different values
of t1 and choose the minimum with corresponding t1 and
t2.
```

Computationally speaking, the worst case number of iterations is bounded by $(H\text{-}3L)*(H\text{-}3L\text{-}1)/2$, and hence the computational time complexity is $O(n*H^2)$, since each "if" statement requires $O(n)$ computation. We present the computation performance for various replenishment horizons H in Appendix B. These results show that the computation time for our approach is reasonably acceptable.

F. Extension to Multiple Reallocations

Based on the result of two-reallocation for the repairable item inventory system, the extension to M-reallocation is now presented. The EBO over all bases at ($M + 1$) time points for a given set of reallocation instants $\{t_1, t_2, \ldots, t_M\}$ are given as

$TE(t_1, t_2, \ldots, t_M) = EBO(t_1) + EBO(t_2) + \ldots$
$+ EBO(t_M) + EBO(H)$ (23)

With the assumption about complete distribution of items at central depot at each reallocation instant, the order-up-to level at each reallocation instant and the corresponding expected backorder from this reallocaton to the next reallocation can be similarly calculated as Two-reallocaton problem stated before. Here, we only present the formula on the order-up-level at final reallocation instant t_M and the corresponding expected backorder from t_M till the end of replenishment cycle as follows:

$$U_i(t_M) = (H - t_M)\lambda_i + \frac{\sqrt{\lambda_i}}{\sum_{j=1}^{n}\sqrt{\lambda_j}}[S_0 + \sum_{i=1}^{n} S_i - Y - (H - t_M)\sum_{j=1}^{n}\lambda_j] \quad (24)$$

where $Y = \sum_{t=1}^{t_M}\sum_{i=1}^{n} y_i(t) - \sum_{i=2}^{M} R_i$ and R_i (i = 2, 3, …, M) denotes the arriving repaired items at reallocation instant t_i from the depot to bases. We use X_k to denote the number of failures generated during period [t_{k-1}, t_k] and we know $E(X_k) = (t_k - t_{k-1})\sum_{i=1}^{n}\lambda_i$ and $Var(X_k) = (t_k - t_{k-1})\sum_{i=1}^{n}\lambda_i$ for all k (k=1,…, M) if we assume $t_0 = 0$. Thus, we can calculate R_i as follows:

$$R_i = \sum_{k=1}^{i-2} X_k e^{-[(t_{i-1}-t_k)-2L]/T}(1-e^{-(t_i-t_{i-1})/T}) + X_{i-1}[1-e^{-(t_i-t_{i-1}-2L)/T}]$$

$$E(R_i) = (1-e^{-(t_i-t_{i-1})/T})\sum_{k=1}^{i-2}\sum_{j=1}^{n}(t_k - t_{k-1})\lambda_j e^{-[(t_{i-1}-t_k)-2L]/T} + (t_{i-1}-t_{i-2})\sum_{j=1}^{n}\lambda_j[1-e^{-(t_i-t_{i-1}-2L)/T}] \quad (25)$$

$$Var(R_i) = \sum_{k=1}^{i-2}\sum_{j=1}^{n}(t_k - t_{k-1})\lambda_j e^{-2[(t_{i-1}-t_k)-2L]/T}(1-e^{-(t_i-t_{i-1})/T})^2 + (t_{i-1}-t_{i-2})\sum_{j=1}^{n}\lambda_j[1-e^{-(t_i-t_{i-1}-2L)/T}]^2$$

Furthermore, the mean and variance of Y can be calculated as follows:

$$E(Y) = \sum_{t=1}^{t_M}\sum_{i=1}^{n} E[y_i(t)] - \sum_{i=2}^{M} E(R_i) = t_M\sum_{i=1}^{n}\lambda_i - \sum_{i=2}^{M} E(R_i)$$

and

$$Var(Y) = \sum_{t=1}^{t_M}\sum_{i=1}^{n} Var(y_i(t)) + \sum_{i=2}^{M} Var(R_i) = t_M\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{M} Var(R_i).$$

Weighting $EBO(H)$ for a given value of Y by the density of Y and integrating over Y, we have

$$EBO(H) = \int_{-\infty}^{\infty}\sum_{i=1}^{n}\sqrt{(H-t_M)\lambda_i}\, G\left(\frac{U_i(t_M)-(H-t_M)\lambda_i}{\sqrt{(H-t_M)\lambda_i}}\right) f(y)dy \quad (26)$$

and using the same method as the two-reallocation problem, we obtain

$$EBO(H) = \sqrt{(H-t_M)(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + t_M\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{M} Var(R_i)} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - H\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{M} E(R_i)}{\sqrt{(H-t_M)(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + t_M\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{M} Var(R_i)}}\right) \quad (27)$$

Again, our purpose is to find such set of reallocation instants $\{t_1, t_2, …, t_M\}$ ($0 < t_1 < … < t_M < H$) that TE is minimized. Due to the transport time L from the depot to bases, the first reallocation (t_1) cannot take place earlier than the period L. In addition, final reallocation (t_M) has to take place before the period H. Two consecutive reallocation instants are constrained by $t_i > t_{i-1} + 2L$ ($2 \le i \le M$) for the same reason as the two-reallocation problem (stated in Section 3.3). Due to this constraint, it is also easy to see that Normal distribution approximates Poisson distribution well. Therefore, the first reallocation t_1 can take place at the interval [L, H-(M-1)*(2L+1)-1] whereas the reallocation instant t_i ($2 \le i \le M$) can take place at the interval [t_{i-1}+2L+1, H-(M-i)*(2L+1)-1]. The worst-case total iterations is thus $\binom{H-2LM+L}{M}$, and hence the computational time complexity can be measured as $O(\frac{n*(H-M)^M}{M!})$.

IV. Experimental Result and Sensitivity Analysis

Our experimental results are presented in this section. In Section 4.1, we use test cases to compare the results under two reallocation instants with those having single instant in [2]. In Section 4.2, we perform extensive sensitivity analysis to show the effects of each independent parameter on the values of total EBO and reallocation instants.

A. Comparison

First, we use test case to determine when and how to reallocate for two reallocation instants, and compare the results with those allowing single reallocation, as seen in [2]. In our experiment, we have one depot supports 5 bases (n=5). The length of the replenishment cycle is 30 periods (H=30). In order to make meaningful comparison between the two-reallocation and the single-reallocation scheme proposed in [2], we set both the internal lead time and the mean repair time to be zero (L=0, T=0). First we focus on the identical independent demand distributions at all bases. We set λ_i=4 for all i (i = 1, …, n). The stock level at each base i is set to S_i=H*λ_i =120. We determine the stock level at the depot to be $S_0 = 2.33\sqrt{H\sum_{i=1}^{n}\lambda_i} = 57.07$, where 2.33 represents the probability of 1% probability that the total system demands in a cycle H exceeds the total stocks at the depot and all bases.

We implement both our method and that of [2] so that we can compare them on the effect of two reallocations during a cycle. The results are shown in Fig. 1, together with Fig. 2 (which provides an enlarged view on the comparison among two-reallocations over different first-reallocation time instants). From Fig. 2, we can see firstly that given the first reallocation instant, the total EBO TE decreases and then increases as the second reallocation instant t_2 increases. This is consistent with what we mentioned in the above section. Therefore, the time interval between reallocations can be neither too short because of fewer repaired failures at the depot nor too long because of more failures at bases.

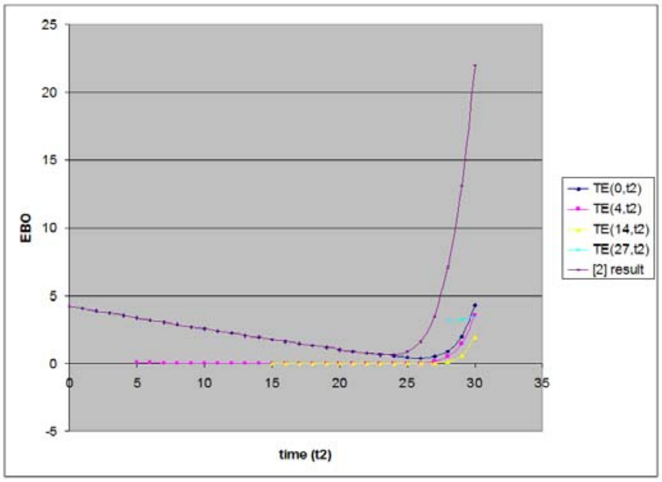

Fig. 1. Comparison of EBO vs. time between single- and two-reallocation

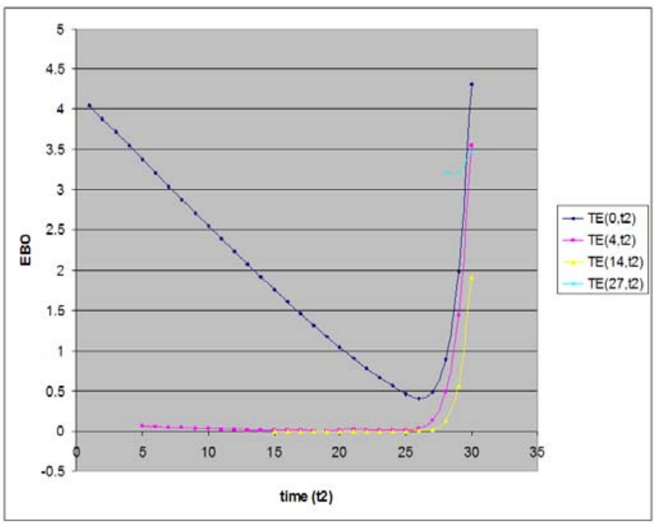

Fig. 2. Comparison of EBO vs. time among two-reallocations

Secondly, we can see from Fig. 1 that *TE* decreases and then increases as the first reallocation instant t_1 increases. This is indicated by comparing $TE(0, t_2)$, $TE(4, t_2)$ and $TE(27, t_2)$. The curve of $TE(4, t_2)$ is below that of $TE(0, t_2)$, indicating it better to perform the first reallocation at t_1=4 than t_1=0 while the curve of $TE(27, t_2)$ is above that of $TE(4, t_2)$, indicating it worse to perform the first reallocation at t_1=27 than t_1=4. Thirdly, Fig. 2 shows that the curve of $TE(27, t_2)$ intersects with that of $TE(0, t_2)$. This indicates that it should not always delay the first reallocation and perform the second one in a hurry. In fact, there is also an intersection between the curve of $TE(4, t_2)$ and the curve of $TE(27, t_2)$ although not obvious. Comparing all combinations of (t_1, t_2), we find the optimal reallocation instants pair is (14, 20) with *TE* = 3.7e-14. Fourthly, we can see from Fig. 1 that multiple reallocations can reduce the total EBO compared with single reallocation. From Fig. 1, the optimal reallocation instant in [2] is at *t*=24 with *TE* = 0.6670. We can also see that our first reallocation instant should be before *t*=24. This is because if we reallocate at later than *t*=24, there will be a large number of backorders at bases. More interestingly, we compare [2]'s result with those whose first reallocation takes place at t_1=24. The results are shown in Table I. From Table I, we can see that when the first reallocation instant is *t*=24, if we perform the second reallocation immediately after the first one, *TE* can also be improved because more failures can be repaired as more failures are brought back the depot under the assumption of infinite repair capacities. However, if we perform the second reallocation at the end of the cycle, it is equivalent to reallocate only once (recall our objective function is that just before reallocation). Hence, *TE* should be the same as that of that presented in [2], which is proved to be 0.6670 in Table I.

TABLE I
TOTAL EBO WHEN THE FIRST REALLOCATION INSTANT IS 24 (*H*=30)

(t_1, t_2)	TE
(24, 25)	0.1147
(24, 26)	0.1147
(24, 27)	0.1164
(24, 28)	0.1190
(24, 29)	0.1857
(24, 30)	0.6670

Next, we set lower stock levels at bases to investigate what will happen under a higher level of EBO. The results are shown in Fig. 3, where we multiply the stock level at each base in the previous case by 0.8. Fig. 4 is to highlight the comparison within two-reallocations when the first reallocation takes place at different time instant. Cao and Silver [2] claim that because generally it is more costly to delay allocation beyond the best time than to perform it somewhat early, it tends to hedge against the higher penalties by committing to an earlier allocation time. Fig. 3 shows firstly that the optimal reallocation instant for [2] is *t*=18 with *TE*=63.9580. From Fig. 4, our optimal reallocation instant pair is (14, 19) with *TE*=3.68e-5 correspondingly, the second one being brought forward. This is consistent with the claim in [2]. Furthermore, Fig. 4 shows secondly that under the same reallocation instant for the first time, the second reallocation instant is also brought forward. In the previous case, when t_1=0, the optimal t_2 is t_2=26 while in the current case the optimal t_2 is t_2=21. Similarly when t_1=4, t_2=19 in the previous case while t_2=22 in the current one.

Next, we run a test case by relaxing the assumption of identical demand distributions at bases while demand is still assumed to be independent. For non-identical demand distributions at bases, we use the same method as in [2], setting the mean demand of each base *i* by $\lambda_i = 2i*\lambda/(n+1)$ ($i = 1, \ldots, n$, λ=4). The results are shown in Table II. From Table II, we observe that this change does not bring about consistent effect on the results at high stock level and low stock level with the method in [2]. However, using our method, either at high stock level or at low stock level, the non-identical demand will incurs higher expected backorder than identical demand. It is probably due to the higher *CV* at some bases with non-identical demand.

TABLE II
TE FOR IDENTICAL DEMAND DISTRIBUTIONS VS. NON-IDENTICAL DEMAND DISTRIBUTIONS

Case	[2]-1	Ours-1	[2]-0.8	Ours-0.8
Identical	0.6670	3.7e-14	63.9580	3.68e-5
Non-identical	0.7624	2.02e-8	55.3050	1.63e-3

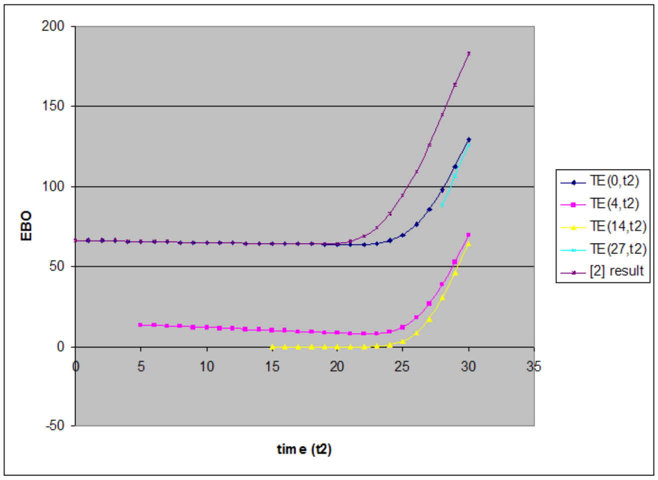

Fig. 3. Comparison of EBO vs. time between single- and two-reallocation with fewer stocks

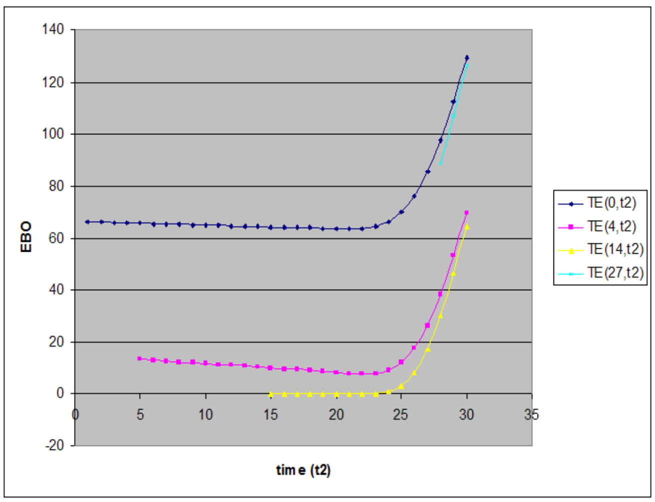

Fig. 4. Comparison of EBO vs. time among two-reallocations with fewer stocks

B. Sensitivity Analysis

As presented in [2], in this subsection we will show the effects of each independent parameter on the values of total EBO (i.e. *TE*) and reallocation instants for two reallocation instants cases. We use the parameters in Section 4.1, plus repair time and internal lead time set by us (see Table III) and focus on the situation of low stock level as stated in Section 4.1. We run the test cases for both identical and non-identical demand distributions while demand is still assumed to be independent. For non-identical demand distributions at bases, we use the same method as in [2], setting the mean demand of each base i as $\lambda_i = 2i*\lambda / (n+1)$ $(i = 1, \ldots, n)$.

TABLE III: PARAMETERS FOR SENSITIVITY ANALYSIS

Parameter	Values
k	1.645, 2.33
λ	4, 6, 10
H	30, 50, 100
T	0, 10, 20, 30, 100
L	0, 2, 5, 8

Instead of using graphics as in [2], which seems to give a visual illusion that the relationship is linear, we use tables to show the changes of *TE* with different independent parameters and the changes of reallocation instants with repair time and internal lead time. Tables IV to VI illustrate that *TE* decreases as k (the safety factor) increases, that *TE* decrease as λ (the average demand level) increases and that *TE* decreases with H (the system cycle length), all of which are consistent with corresponding ones in [2] for single reallocation cases.

TABLE IV: EFFECTS OF k ON *TE*

k	1.645	2.33
Identical	2.4e-4	3.68e-5
Non-identical	3.3e-3	1.63e-3

TABLE V: EFFECTS OF λ ON *TE*

λ	4	6	10
Identical	3.68e-5	3.13e-7	3.98e-11
Non-identical	1.63e-3	8.47e-5	4.53e-7

TABLE VI: EFFECTS OF H ON *TE*

H	30	50	100
Identical	3.68e-5	3.11e-8	5.2e-16
Non-identical	1.63e-3	1.37e-4	1.43e-8

More interestingly, we show the effects of repair time and internal lead time on the total EBO *TE* and reallocation instants (t_1, t_2), which is not in the model of [2]. It is not surprising that *TE* increases as T (repair time) increases (Table VII) since it takes more time to repair a failure so that fewer items come out given a certain time interval. From Table VII, it is interesting for us to find that as T increases, the time interval between two reallocations increases (8 when T=10, 9 when T=20, 10 when T=30 for identical case and 8 when T=10, 10 when T=20, 11 when T=30 for non-identical case). However, when T=100, it recommends that the first reallocation should be shifted earlier (t_1=13) and then increases the time interval between two reallocations (11 for identical case). In Table IX, it is also not surprising that *TE* increases as L (internal lead time) increases since it takes more time to transport items back and forth except that the effect is not obvious when L is small (*TE* remains same for L = 0 and L = 2.). In Table X, we show the interesting effect of L on reallocation instants especially t_1. From Table X, we know that the first optimal reallocation instant is 14 if transport time (L) is zero and repair will take place at the depot instantly after the first instant. But if L=2, repair can only take place 2 time periods after the first instant. When mean repair time is assumed to be zero, this change on transport time will not affect reallocation instants. However, when transport time becomes larger like L=5, the first instant has to be put earlier so as to let the second reallocation not so hurry. In addition, transport must take place from the depot to the bases at the beginning of the cycle (t=0) for reallocation in such a way that the first reallocation will not be done earlier than t_1=8 as shown in Table X.

TABLE VII: EFFECTS OF T ON TE

T	0	10	20	30	100
Identical	3.68e-5	0.0866	2.0439	7.4840	39.1864
Non-identical	1.63e-3	0.0828	1.7130	6.1837	33.0417

TABLE VIII: EFFECTS OF T ON REALLOCATION INSTANTS (t_1, t_2)

T	0	10	20	30	100
Identical	(14,19)	(14,22)	(15,24)	(14,24)	(13,24)
Non-identical	(13,19)	(14,22)	(14,24)	(13,24)	(14,25)

TABLE IX: EFFECTS OF L ON TE

L	0	2	5	8
Identical	3.68e-5	3.68e-5	1.05e-2	4.6977
Non-identical	1.63e-3	1.63e-3	8.99e-3	3.8779

TABLE X: EFFECT OF L ON REALLOCATION INSTANTS (t_1, t_2)

L	0	2	5	8
Identical	(14, 19)	(14, 19)	(10,21)	(8, 25)
Non-identical	(13, 19)	(13, 19)	(10, 21)	(8, 25)

V. APPLICATION: PERIODIC RESUPPLY POLICY

In most studies on periodic resupply policies, one usually determines the optimal allocation of inventories under a given *fixed* time interval between periodic allocations, for example, every h time units. The question we need to ask is whether this h time unit is cost-optimal. Note that without considering costs, it would be optimal to reallocate as frequently as possible, i.e. let the time interval between reallocations tend to be infinitely small so that it approximates continuous resupply as in [1], [8]-[11]). Given that reallocation is costly, frequent reallocation will incur high operating cost, while shortage of stocks will also incur penalty costs. It is therefore important to find the right value for h so as to balance between the costs of reallocations and shortages.

In this section, we show how our reallocation model can be extended and applied to derive a period resupply policy. To achieve this purpose, we first extend our proposed model to multiple (more than 2) reallocations, where the challenge is to determine the time interval between reallocations assuming the intervals are the same for a given time horizon. This would pave the way for the design of cost-optimal periodic resupply policies by studying the balance between the costs of reallocations and shortages. We introduce more notations based on those in Section 2. Instead of random reallocation instants, here reallocations are assumed to take place every h period so that the total number of reallocations is $m = [(H - 1)/h]$ (it is unnecessary to reallocate at the end of the cycle), where $[x]$ is the maximal integer number not greater than x. We also assume h is greater than $2L$, the lead time to transport back and forth between the depot and bases. C_r is the unit cost of a reallocation and C_s is the unit cost of a shortage. Thus, our objective is to minimize the total cost;

$$\min \text{TC} = \text{Cr} \times \text{m} + \text{Cs} \times \text{TE} \tag{28}$$

Extended from (27), TE here is the expected backorders over all bases at a series of reallocation instants and at the end of the cycle, i.e.

$$TE = \sum_{p=1}^{m} \sum_{i=1}^{n} EBO_i(ph) + \sum_{i=1}^{n} EBO_i(H) = \sum_{p=1}^{m+1} EBO(t_p) \tag{29}$$

where $EBO(t_p) = \sum_{i=1}^{n} EBO_i(ph)$ (p=1,…,m) and $EBO(t_{m+1}) = \sum_{i=1}^{n} EBO_i(H)$

According to (3), the expected backorders over all bases at time h, just before the first reallocation is

$$EBO(t_1) = \sum_{i=1}^{n} \sqrt{h\lambda_i}\, G\left(\frac{S_i - h\lambda_i}{\sqrt{h\lambda_i}}\right) \tag{30}$$

Respectively according to (19), the expected backorders over all bases at time $2h$, just before the second reallocation is

$$EBO(t_2) = \sqrt{h(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + h\sum_{i=1}^{n}\lambda_i} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - 2h\sum_{i=1}^{n}\lambda_i}{\sqrt{h(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + h\sum_{i=1}^{n}\lambda_i}}\right) \tag{31}$$

From the second reallocation onward, we must consider those failures that have finished repair and are sent to depot inventory in time to be transported to bases for the k^{th} reallocation, R_k. We use X_t to denote the number of failures generated during period $[(t - 1)h, th]$ and we know $E(X_t) = h\sum_{i=1}^{n}\lambda_i$ and $Var(X_t) = h\sum_{i=1}^{n}\lambda_i$ for all t (t=1, …, m) Hence, for the second reallocation, we have $R_2 = X_1[1 - e^{-(h-2L)/T}]$, $E(R_2) = h\sum_{i=1}^{n}\lambda_i\,[1 - e^{-(h-2L)/T}]$ and $Var(R_2) = h\sum_{i=1}^{n}\lambda_i\,[1 - e^{-(h-2L)/T}]^2$. For the third reallocation, we have $R_3 = X_1 e^{-(h-2L)/T}(1 - e^{-h/T}) + X_2[1 - e^{-(h-2L)/T}]$, $E(R_3) = h\sum_{i=1}^{n}\lambda_i\,[1 - e^{-(2h-2L)/T}]$ and $Var(R_3) = h\sum_{i=1}^{n}\lambda_i\,\{e^{-(2h-4L)/T}(1 - e^{-h/T})^2 + [1 - e^{-(h-2L)/T}]^2\}$. And in general, for the k^{th} (k = 2, …, m) reallocation, we have

$$\begin{aligned} R_k &= \sum_{t=1}^{k-2} X_t e^{-[(k-1-t)h-2L]/T}(1-e^{-h/T}) + X_{k-1}[1-e^{-(h-2L)/T}] \\ E(R_k) &= h\sum_{i=1}^{n}\lambda_i[1-e^{-[(k-1)h-2L]/T}] \\ Var(R_k) &= h\sum_{i=1}^{n}\lambda_i \times \{\sum_{t=1}^{k-2} e^{-2[(k-1-t)h-2L]/T}(1-e^{-h/T})^2 + [1-e^{-(h-2L)/T}]^2\} \end{aligned} \tag{32}$$

Hence, according to (26),

$$EBO(t_p) = \sqrt{h(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + (p-1)h\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{p-1}Var(R_i)} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - ph\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{p-1}E(R_i)}{\sqrt{h(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + (p-1)h\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{p-1}Var(R_i)}}\right) \tag{33}$$

(p = 3, …, m) and

$$EBO(t_{m+1}) = \sqrt{(H-mh)(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + mh\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{m}Var(R_i)} \times G\left(\frac{S_0 + \sum_{i=1}^{n}S_i - H\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{m}E(R_i)}{\sqrt{(H-mh)(\sum_{i=1}^{n}\sqrt{\lambda_i})^2 + mh\sum_{i=1}^{n}\lambda_i + \sum_{i=2}^{m}Var(R_i)}}\right) \quad (34)$$

Using (28)–(34), we can compute the total cost consisting of reallocation cost and shortage penalty cost for a given time interval between periodic reallocations. Mathematically, this total cost is a function of *h*. Thus intuitively, we can compute the first derivative and set the result to be zero (i.e. d*TC*/d*h* = 0) to obtain the optimal periodic policy in terms of the right *h* value. However, due to the complexity in the equation for *EBO* as defined in (33) and (34) (especially the unit normal loss function *G*), it is computationally intensive to compute the optimal *h* by using the first derivative. To overcome computational inefficiency, the following heuristic approach may be applied to compute EBO:

```
for (h = 2L+1; h < H ; h++) {
   Compute EBO(t1),…,EBO(tm+1) by (30)–(34).
   Compute total EBO TE by (29).
   Compute TC(h+1) by (28).
   if (TC(h) ≤ TC(h+1))
      output the optimal periodic policy h and corresponding
      stock reallocations
}
```

VI. Conclusion and Further Research

In this paper, we were interested in analyzing the performance of reallocation within a multi-echelon inventory system. During the replenishment cycle, we have *two* opportunities to reallocate the spares by redistributing the depot stocks to the bases and by lateral transshipment. We developed a mathematical model and use a Lagrange multiplier to determine how to reallocate the spares to achieve a minimized total expected backorders under a given reallocation instant pair. Then we derive a dynamic reallocation method to determine when to perform the first and second reallocation respectively. Experimental results show that two-reallocation is better than single-reallocation. The logic of our approach is easy to implement and efficiently computed.

Several possible avenues of extension of our work are worth considering:

1) Echelon Structure. We have considered the two-echelon tree structure. A natural extension is to handle more than two echelons where reallocation instants at different echelons can be different. Moreover, one can extend the supply chain structure from a tree structure to a network (graph).
2) Demand distribution. It is also interesting to consider nonstationary demand distributions, i.e. demands at each period are not identical with time-varying mean and standard deviation.
3) Cost consideration. The objective of this paper is to consider the timing of reallocation that seeks to improve efficiency and reduce unavailability. It is obvious that more frequent reallocation yields better performance compare to single reallocation if the model does not incorporate the cost of reallocation.
4) Periodic Resupply. Finally, from the practice standpoint, it is interesting to experiment on the idea of computing optimal periodic resupply proposed in Section 5.

Appendix

A. Experiments on Approximation of Poisson distribution by Normal distribution

In the following, we investigate the effect of approximation of Poisson distribution by a Normal distribution when the mean of Poisson random variable (in our case, demand) is smaller than 10. More precisely for the purpose of our paper, we are concerned with the approximation error on the value of the expected backorder. It is clear that the sum of Poisson demands with mean λ during the interval [0, t] still follows a Poisson distribution with mean $\lambda*t$. Given initial inventory *S*, the expected backorder based on Poisson demand distribution (denoted *EBO_Poisson*) is calculated as $e^{-\lambda t}\sum_{k=S}^{\infty}\frac{(\lambda t)^k}{k!}(k-S)$. If we approximate the Poisson distribution with a Normal distribution, then the expected backorder based on Normal demand distribution (denoted *EBO_Normal*) is calculated as $\sqrt{\lambda t}G(\frac{S-\lambda t}{\sqrt{\lambda t}})$. Thus, the relative approximation error can be calculated as (*EBO_Normal* - *EBO_Poisson*)/*EBO_Poisson.*

In Table A.1, we compare the expected backorder values as well as the relative approximation error computed under the two demand distributions using different values of $\lambda*t$ and assuming *S* to be equal to $\lambda*t$. We observe that the approximation error is very small (2% or less) when the mean of Poisson distribution is no smaller than 4. We conclude that the Normal distribution can effectively approximate Poisson distribution, for the purpose of this work.

TABLE A1: Relative approximation error on expected backorder

λ*t	***EBO_Normal***	***EBO_Poisson***	***Relative approximation error***
1	0.3989	0.3679	0.084
2	0.5642	0.5413	0.042
3	0.6910	0.6721	0.028
4	0.7979	0.7815	0.021
5	0.8921	0.8773	0.017
6	0.9772	0.9637	0.014
7	1.0555	1.0430	0.012
8	1.1284	1.1167	0.010
9	1.1968	1.1858	0.009
10	1.2616	1.2511	0.008

B. *Experiments on Computational performance of our two-reallocation algorithm*

Using the same experimental setup as Section 4.1, we measure the computational time required to execute the two-reallocation algorithm for different replenishment horizons *H* on a machine with CPU 1.66GHz and RAM 1GB. The result is shown as follows:

TABLE A2
COMPUTATION PERFORMANCE FOR DIFFERENT REPLENISHMENT HORIZONS

Replenishment horizon *H* (period)	Computation time (millisecond)
30	47
50	94
80	156
100	219
200	641

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