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Inventory Model with Seasonal Demand: A Specific Application to Haute Couture

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Abstract

In the stochastic multiperiod inventory problem, a vast majority of the literature deals with demand volume uncertainty. Other dimensions of uncertainty have generally been overlooked. In this paper, we develop a newsboy formulation for the aggregate multiperiod inventory problem intended for products of short sales season and without replenishments. A distinguishing characteristic of our formulation is that it takes a time dimension of demand uncertainty into account. The proposed model is particularly suitable for applications in haute couture, i.e., high fashion industry. The model determines the time of switching primary sales effort from one season to the next as well as optimal order quantity for each season with the objective of maximizing expected profit over the planning horizon. We also derive the optimality conditions for the time of switching primary sales effort and order quantity. Furthermore, we show that if time uncertainty and volume uncertainty are independent, order quantity becomes the main decision over the interval of the primary selling season. Finally, we demonstrate that the results from the two-season case can be directly extended to the multi-season case and the limited resource multiple-item case.

Key Words: Inventory Model; Newsboy Formulation; Optimal Policies

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1. Motivation

A vast majority of the literature dealing with the stochastic multiperiod inventory problem focuses on demand volume uncertainty. Other dimensions of uncertainty are generally overlooked. In this paper, we incorporate a time dimension of demand uncertainty into the aggregate multiperiod inventory problem for products of short sales seasons that have no replenishment and long lead times. For products exhibiting a seasonal demand pattern, at least some portion of the demand in two consecutive seasons overlaps each other. Therefore, the decision on the time to switch the primary sales effort from one season to another is of critical interest to managers, along with the associated order quantity decision for each season. The "time" decision is particularly relevant to haute couture, i.e., high fashion industry.

While the functional aspects of haute couture products such as keeping warm are not entirely ignored; nevertheless, the key magnet of haute couture rests on the leading edge fashion. Hence, haute couture's sales effort is concentrated on how to stimulate their customers' fashion impulses. Most of the major markets of haute couture are located in regions with four distinctive seasons, including Europe, Japan, and North America. When a new season is approaching, haute couture often creates a fashion fad promoting the belief that "a new season is here." Based on our experience, a common practice in the haute couture industry is to create a fashion impulse for the new season products by making an *overnight* transition from the current season sale to the next season sale. For retailers of low-to-medium priced apparels—such as Target, JC Penny, and Macy's—this transition happens rather gradually. As a new season comes closer, more and more of the new season products are brought onto sales floors, and the new season products are generally on display along with the leftovers of the current season products. In the haute couture industry, on the other hand, this transition is very fast and occurs virtually overnight. This is because an effective means to stimulate an impulse is through a quick and complete transition.

It is frequently observed that sales of the new season product start much earlier than the actual season. For instance, a shop has the Spring Collections on display and for sale in the middle of late winter. The shop could have delayed the transition from the Winter Collections to the Spring Collections for a week or two and satisfied some late demands of winter clothing. On the other hand, if the shop had delayed the transition, the shop would have missed early demands of spring clothing. Therefore, it is important for haute couture managers to determine the optimal time to terminate the current season sale and to start a new season sale. This "switch timing decision"—when to switch from the current season sale to the next season sale—is more critical to haute couture managers, because they generally do not participate in the secondary market. Lee and Whang (2002) have investigated the impacts of a secondary market where resellers can buy and sell excess inventory. The secondary market is opened when the first period market ends. The equilibrium price of the secondary market is typically assumed to be lower than the price of the primary market. Haute couture companies, however, have no desire to create the secondary market. Some haute couture companies go even further, they actively engage in activities to prevent the secondary market from emerging. One of the key appeals to haute couture customers is exclusiveness, or so-called "being snob." Customers satisfy their personal esteems of being exclusive by knowing the fact that the prices of the products they just paid for are so high that not everyone can afford it. It is not desirable for brand loyalty if high fashion companies make products affordable to more people by cutting prices, even though these products are somewhat out-of-fashion or off-season.

When a firm is facing seasonal demands with non-negotiable secondary market price, the newsboy model has been extensively used as a tool to determine the order quantity for this stochastic inventory problem. Petruzzi and Dada (1999), Emmons and Gilbert (1998), Eppen and Iyer (1997), and Khouja (1996) are just a few examples. The existing literature of the newsboy problem unfortunately does not deal with the switching time decision. They only consider the order quantity decision when a specific switching time is given.

In this paper, we suggest a variation of the newsboy problem formulation that considers the order quantity and the switching time simultaneously. The rest of this paper is organized as follows. In Section 2, we review the relevant literature. In Section 3, we present the basic model of two-season inventory planning with a single product for each season. The optimal conditions for the order quantity and switching time have been derived. In Section 4, we discuss a special case and the general extensions of the model. The results and the future extension are concluded in Section 5.

2. Relevant Literature

Very few published papers have treated the interval of sales period as a decision variable. Instead, researchers have developed newsboy model applications or yield management techniques to study stochastic multiperiod inventory systems. In this section, we review inventory literature on the following two topics: newsboy applications related to the stochastic inventory research and yield management research involving a stopping time of a sales offer.

The newsboy problem has been a critical building block of the stochastic inventory theory (Petruzzi and Dada, 1999). As an application to retail inventory management, Khouja (1996) extends the newsboy problem to a case involving supplier's quantity discounts and retailer's progressive discounts. He demonstrates that the newsboy problem with multiple discounts gives a larger order quantity than the problem with only supplier discounts. He explains the result by arguing that multiple discounts lead to an increased demand at prices that are higher than the salvage value of the classical newsboy model.

Emmons and Gilbert (1998) examine the role of return policies for catalogue goods through a newsboy formulation. They refer "catalogue style goods" to a situation in which a retailer must commit to a fixed retail price for a substantial portion of the selling season for a particular item. They demonstrate that it is necessary to incorporate retailer's self-interest into manufacturer's pricing policy. Eppen and Iyer (1997) study backup agreements between a catalogue company and manufacturers. A backup agreement states that if the catalogue company commits to a number of units for the season, the manufacturer holds back a constant fraction of the commitment and delivers the remaining units before the start of the fashion season. Through Bayesian updating, their results indicate that the backup agreement, including penalty cost for each unit not taken from backup, may increase the committed quantity as well as the expected profit.

As an application to global supply chain management, Kouvelis and Gutierrez (1997) examine a transfer pricing policy for a producer of “style goods” who sells the goods to two different countries (known as the primary and secondary market countries) with non-overlapping selling season. They demonstrate a decentralized production control policy, with the production centres at each country are treated as independent profit centres and a constant transfer price is used to coordinate their production, may lead to sub-optimal solutions. They also show that much of the penalties from placing decentralized control policy can be eliminated by adopting their non-linear transfer pricing scheme.

Petruzzi and Dada (1999) examine an extension of the newsboy problem such that stocking quantity and selling price are determined simultaneously. Their work is remarkable in that it incorporates selling price, which have been typically taken as exogenous, into the newsboy model.

Facing the problem of selling a fixed stock of items over a finite horizon, maximizing revenue in an excess of salvage value has always been an important issue for industries such as airlines, hotels, and seasonal manufacturers (Feng and Gallego, 1995). Feng and Gallego (1995) address the problem of deciding the optimal timing of a single price change from a given initial price to a given second price. Their notion of “stopping time” denotes the moment of stopping selling products at the initial price, that is, the time of changing price. Their “stopping time” is determined by the number of unsold units and the time-left to the end of the season. More recently, Feng and Xiao (1999) present a risk sensitive pricing model to maximize sales revenue for perishable commodities with fixed capacity. They add a variance term to the objective function in the form of penalty (or premium). As another stream of application, Bitran, Caldentey, and Mondschein (1998) and Smith and Achabal (1998) study the clearance pricing policy for retail chains. In summary, the core issue in the yield management literature is about adjusting selling price according to the level of unsold inventory and time left to the end of the season.

As discussed above, the traditional newsboy models mainly deal with the quantity decision to minimize the inventory cost while the yield management researches tend to focus on the pricing policy for maximizing the revenue. This paper is the first to combine the decision of quantity and timing together in a newsboy setup. The model has potential applications in seasonal product industries, particularly in high fashion haute couture environment.

3. Model

To illustrate the problem more clearly, we first examine the two-season inventory planning with a single product for each season. Let a random variable X_i^t be the demand volume of season *i* product at time *t*. X_i^t follows a joint density function $f_i(x_i, t_i)$, where a random variable x_i represents the demand volume and random variable t_i represents timing of demand occurrence ($x_i \in [0,\infty)$, $t_i \in [0,\infty)$). The marginal density function of x_i, $f_{x_i}(x_i) = \int_0^\infty f_i(x_i, t_i) dt_i$ is equivalent to a typical demand volume distribution. On the other hand, the marginal density function of t_i, $f_{t_i}(t_i) = \int_0^\infty f_i(x_i, t_i) dx_i$, represents the demand behaviour over time. At least in theory, it is a straightforward statistical task to find an empirical distribution of $f_{t_i}(t_i)$ from the historical data. Suppose 10% of the demand takes place until time unit 10 and 50% of demand occurs until time unit 30. Then, $F_{t_i}(t)$, c.d.f. of $f_{t_i}(t_i)$, is defined as $F_{t_i}(10) = 0.1$ and $F_{t_i}(30) = 0.5$, respectively. Once the demand is estimated as a function of t_i, the density function $f_{t_i}(t_i)$ is derived by taking the first order derivative of $F_{t_i}(t)$ with respect to t_i.

Let Q_i be the order quantity for season *i* product and $P_i(x_i, Q_i)$ be the profit function of season *i*. The profit function $P_i(x_i, Q_i)$ is defined as a typical newsboy formulation.

$$P_i(x_i, Q_i) = \begin{cases} p_i x_i - (Q_i - x_i)Co_i, & \text{if} \quad x_i \le Q_i \\ p_i Q_i - (x_i - Q_i)Cu_i, & \text{if} \quad x_i > Q_i \end{cases} \tag{1}$$

where p_i, Co_i, and Cu_i are unit profit, overstock cost, and under stock cost of season *i* product, respectively.

With (1), the objective function is stated as:

$$\begin{aligned} \text{Max} \quad TP &= \int_0^\infty \int_0^d P_1(x_1, Q_1) f_1(x_1, t_1) dt_1 dx_1 + \int_0^\infty \int_d^\infty P_2(x_2, Q_2) f_2(x_2, t_2) dt_2 dx_2 \\ &= \int_0^{Q_1} \int_0^d [p_1 x_1 - (Q_1 - x_1)Co_1] f_1(x_1, t_1) dt_1 dx_1 \\ &+ \int_{Q_1}^\infty \int_0^d [p_1 Q_1 - (x_1 - Q_1)Cu_1] f_1(x_1, t_1) dt_1 dx_1 \\ &+ \int_0^{Q_2} \int_d^\infty [p_2 x_2 - (Q_2 - x_2)Co_2] f_2(x_2, t_2) dt_2 dx_2 \end{aligned}$$

$$+\int_{Q_2}^{\infty}\int_{d}^{\infty}[p_2Q_2-(x_2-Q_2)Cu_2]f_2(x_2,t_2)dt_2dx_2 \tag{2}$$

where *TP* is the expected total profit, and *d* is the time that the firm switches its sales effort from season 1 to season 2. In other words, season 1 lasts during the time period of $[0,d]$, and season 2 covers the time period of $[d, \infty]$.

Optimality Conditions for *d**

The first order optimality condition of *d** is derived by taking the first order derivative of (2) with respect to *d*.

$$\int_0^{\infty} P_1(x_1,Q_1)f_1(x_1,d^*)dx_1 = \int_0^{\infty} P_2(x_2,Q_2)f_2(x_2,d^*)dx_2 \tag{3}$$

The interpretation of Equation (3) is quite intuitive. It implies *d** is the time that equates the expected profit of season 1 to the expected profit of season 2.

The marginal expected profit of season 1 at *d** and marginal expected loss of season 2 at *d** are defined as $P_1(x_1,Q_1)\int_0^{\infty} x_1 \frac{\partial f_1(x_1,d^*)}{\partial d}dx_1$ and $P_2(x_2,Q_2)\int_0^{\infty} x_2 \frac{\partial f_2(x_2,d^*)}{\partial d}dx_2$, respectively. It can be easily seen that the following condition should hold for *d** to be an optimal solution.

$$P_1(x_1,Q_1)\int_0^{\infty} x_1 \frac{\partial f_1(x_1,d^*)}{\partial d}dx_1 < P_2(x_2,Q_2)\int_0^{\infty} x_2 \frac{\partial f_2(x_2,d^*)}{\partial d}dx_2 \tag{4}$$

Inequality (4) leads the following theorem.

<u>Theorem 1</u>

Let d^* be the solution of (3). If the marginal expected loss of season 2 at d^* is greater than marginal expected profit of season 1 at d^*, then d^* maximizes (2).

Proof of ***Theorem 1*** is omitted since it is a straight-forward interpretation of (4). Infeasibility in (3) may arise,

$$\text{if } P_1(x_1,Q_2)\int_0^\infty x_1 f_1(x_1,d)dx_1 > P_2(x_2,Q_2)\int_0^\infty x_2 f_2(x_2,d)dx_2\,, \quad \text{for } d \in [0,\infty)$$

$$\text{or } P_1(x_1,Q_2)\int_0^\infty x_1 f_1(x_1,d)dx_1 < P_2(x_2,Q_2)\int_0^\infty x_2 f_2(x_2,d)dx_2\,, \quad \text{for } d \in [0,\infty)$$

The former case implies that the sales of season 1 product is always more profitable than the sales of season 2 product. The latter case suggests the opposite. In other terms, $d^* = \infty$ for the former case and $d^* = 0$ for the latter case.

Optimality Conditions for Order Quantity

At any given d, the first order optimality condition for Q_1 is derived from (2).

$$\frac{\partial}{\partial Q_1}\{\int_0^{Q_1}\int_0^{d}[p_1x_1-(Q_1-x_1)Co_1]f_1(x_1,t_1)dt_1dx_1+\int_{Q_1}^{\infty}\int_0^{d}[p_1Q_1-(x_1-Q_1)Cu_1]f_1(x_1,t_1)dt_1dx_1\}=0$$

After algebraic manipulation, the above simplifies to:

$$(p_1+Cu_1)F_{t_1}(d)-(Co_1+Cu_1+p_1)\int_0^{d}\int_0^{Q_1}f_1(x_1,t_1)dx_1dt_1=0$$

By substituting $f_1(x_1,t_1)$ for $f_{x_1}(x_1|t_1)f_{t_1}(t_1)$, we obtain

$$F_{x_1|d}(Q_1^*)=\frac{p_1+Cu_1}{p_1+Cu_1+Co_1} \tag{5a}$$

$$Q_1{*} = F_{x_1|d}^{-1}\left(\frac{p_1 + Cu_1}{p_1 + Cu_1 + Co_1}\right) \tag{5b}$$

It is not surprising to see that (5b) possesses a typical form of the newsboy solution, except it is defined on a conditional distribution of x_i at given *d*. The next theorem shows that Q_1 * is indeed optimal.

Theorem 2

Q_1 * in (5b) maximizes (2).

Proof

Q_1 * satisfies the first order optimality condition, since it is a solution of (5b). From the second order optimality condition, we have

$$\frac{d^2 TP}{dQ_1{*}^2} \;=\; -(p_1 + Cu_1 + Co_1)\int_0^d f_1(Q_1{*}, t_1)dt_1 \le 0$$

The inequality direction follows from the observations below.

- $f_1(Q_1{*}, t) \ge 0$ since it is a probability density function,
- an integration of a non-negative term yields a non-negative value, and
- it is implicitly assumed that $p_1, Co_1, Cu_1 > 0$ by common practice.

We now need to show $\int_0^d f_1(Q_1{*}, t_1)dt_1 > 0$. Suppose $\int_0^d f_1(Q_1{*}, t_1)dt_1 = 0$. This equality holds if and only if $Q_1{*} = \infty$ and $d < \infty$. However, the assumptions of $p_1, Co_1, Cu_1 > 0$ imply $\frac{p_1 + Cu_1}{p_1 + Cu_1 + Co_1} < 1$. Then, $Q_1{*} < \infty$ by (5b). ■

Similarly,

$$Q_2{}^* = F_{x_2|d}{}^{-1}\left(\frac{p_2 + Cu_2}{p_2 + Cu_2 + Co_2}\right) \tag{5c}$$

Since Ignall and Veinott (1969), it has been well known that a myopic policy leads to an optimal order quantity for a multiple time period problem to minimize the inventory relevant costs. The first order derivative of (2) with respect to Q_1 is only defined in terms of Q_1. It implies that $Q_1{}^*$ is independent of $Q_2{}^*$ at any given *d*.

4. A Special Case and Extensions of the Model

4.1 A Special Case: x_i and t_i are Independent

To obtain the optimal solutions of d^*, $Q_1{}^*$, and $Q_2{}^*$, Equations (3), (5a), and (5b) are to be solved simultaneously. If x_i and t_i are independently distributed, further analytic results could be obtained from (3), (5a), and (5b). One interpretation of independence between x_i and t_i is the behaviour of demand spread overtime does not give any information on demand volume, and vice versa. Let's take college textbooks as an example. Publishers know most of the demand occurs in the first few days of a semester, say $F_{t_i}(first\_few\_days) = 0.9$. To estimate the demand volume, however, the publishers have to look at completely different sources of information, such as past class enrollments, ratio of students who actually purchase text book, etc. More specifically, from $F_{t_i}(first\_few\_days) = 0.9$, the publishers expect 90% of demand volume would occur in the first few days of a semester. The publishers have to look at different sources of information, however, in order to estimate how many textbooks would be ordered by students for a semester. Therefore, in some cases, it might be reasonable to assume that x_i and t_i are independently distributed.

The following two theorems show that order quantity is the primary decision to make over the switching time *d*, if x_i and t_i are independently distributed.

Theorem 3

If x_i and t_i are independent, Q_i^* $(i = 1,\ 2)$ are independent of *d**.

Proof

Since x_i and t_i are independent, (5b) and (5c) are reduced to

$$Q_i^* = F_{x_i}^{-1}\left(\frac{p_i + Cu_i}{p_i + Cu_i + Co_i}\right)$$

Clearly, Q_i^* is not defined as a function of *d**. ■

Theorem 4

If x_i and t_i are independent, *d** is determined as a function of Q_i^*.

Proof

Let $\delta_i(Q_i^*) = \int_0^\infty P_i(x_i, Q_i^*) f_{x_i}(x_i) dx_i$. From (2), *d** is the value which satisfies the following equality.

$$\delta_1(Q_1^*) f_{t_1}(d^*) = \delta_2(Q_2^*) f_{t_2}(d^*) .$$

Equivalently,

$$\frac{f_{t_1}(d^*)}{f_{t_2}(d^*)} = \frac{\delta_2(Q_2^*)}{\delta_1(Q_1^*)} \tag{6}$$

As it is shown, the left hand side of (6) is a function of Q_i *. ■

Theorem 3 is quite intuitive. If x_i and t_i are independent, order quantity decision is reduced to a typical newsboy problem. The switching time is then obtained by solving (6) numerically.

4.2. Model For Multiple Season Planning with a Single Product for Each Season

Let n denote the number of seasons in a planning horizon. The objective function is then formulated as the following.

$$\text{Max} \quad TP \;= \sum_{i=1}^{n} \int_{0}^{\infty} \int_{d_{i-1}}^{d_i} P_i(x_i, Q_i) f_i(x_i, t) dt dx_i \tag{7}$$

The season i is defined as the period during $[d_{i-1}, d_i]$. It should be noted that $d_0 = 0$ and $d_n = \Omega$, where Ω is the planning horizon.

As in two-season case, a myopic policy leads to the optimal solutions of Q_i *. The first order optimality condition for Q_i * is defined with respect to single i:

$$\frac{\partial}{\partial Q_i} \int_{d_{i-1}^*}^{d_i^*} \int_{0}^{\infty} P_i(x_i, Q_i) f_i(x_i, t_i) dx_i dt_i = 0 \tag{8}$$

where d_{i-1} * and d_i * are the solutions of (7) for season i and season $(i+1)$, respectively. The formula for Q_1 * is the same as (5b). The first order optimality conditions for Q_i * when $i \geq 2$ is stated as:

$$F_{x_i|d_i*}(Q_i^*|d_i^*)F_{t_i}(d_i^*) - F_{x_i|d_{i-1}*}(Q_i^*|d_{i-1}^*)F_{t_i}(d_{i-1}^*)$$

$$= \frac{(p_i + Cu_i)(F_{t_i}(d_i^*) - F_{t_i}(d_{i-1}^*))}{(p_i + Cu_i + Co_i)} \tag{9}$$

Further simplification of (9) is not possible unless more specific assumptions are made on demand distribution. If x_i and t_i are independently distributed, however, ***Theorems 3*** and ***4*** hold for the multiple season planning model. The proof is omitted since it is essentially identical the proofs of ***Theorem 3*** and ***Theorem 4***.

The next theorem extends the applicability of a myopic policy into the optimal solutions of d_i^*.

Theorem 5

Let d_i^* be an optimal solution of (7). d_i^* is independent of d_j^* for $i \neq j$.

Proof

For *i* = 1, it follows ***Theorem 1***. For $i \geq 2$, d_i^* satisfies the following equation.

$$\frac{\partial TP}{\partial d_i} = \int_0^\infty P_i(x_i, Q_i) \frac{\partial}{\partial T_i} \int_{d_{i-1}}^{d_i} f_i(x_i, t_i) dt_i dx_i + \int_0^\infty P_{i+1}(x_{i+1}, Q_{i+1}) \frac{\partial}{\partial T_i} \int_{d_i}^{d_{i+1}} f_{i+1}(x_{i+1}, t_{i+1}) dt_{i+1} dx_{i+1}$$

$$= \int_0^\infty P_i(x_i, Q_i) f_i(x_i, d_i^*) dx_i - \int_0^\infty P_{i+1}(x_{i+1}, Q_{i+1}) f_{i+1}(x_{i+1}, d_i^*) dx_{i+1} = 0 \tag{9}$$

Since the first order optimality condition on d_i^* is only defined on itself, d_i^* is independent of d_j^* for $i \neq j$. ■

(5b) and (5c) have shown that a myopic policy leads to an optimal order quantity. ***Theorem 5*** shows that a myopic policy also leads to the optimal solutions of $d_i *$.

4.3. Model for Multiple Items with Limited Resource

It is not just in the high fashion industry that all items of a season share the common sales interval. College textbooks of a term, for example, also share a common sales interval. Here, a model for multiple items with a single resource constraint is presented. The model is an extension of Silver and Peterson (1985) that present the budget limitation problem.

Let i denote season index $(i = 1,...,n)$ and j denote item index ($j = 1,...,m_i$). The objective function is stated as:

$$\text{Max} \quad TP \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{m_i} \int_{0}^{\infty}\int_{d_{i-1}}^{d_i} P_{ij}(x_{ij}, Q_{ij}) f_{ij}(x_{ij}, t_{ij}) dt_{ij} dx_{ij}$$

$$\text{s.t.} \qquad \sum_{j=1}^{m_i} s_{ij} Q_{ij} \le S_i, \quad \text{for } \forall i, \tag{10}$$

where $d_0 = 0, d_n = \Omega$, S_i is the amount of resource available in season *i*, and s_{ij} is the units of resource consumed by item *j* of season *i*. Restating equation (10) by applying a Lagrange multiplier, we have:

$$\text{Max} \quad TP \quad = \quad \sum_{i=1}^{n}\sum_{j=1}^{m_i} \int_{0}^{\infty}\int_{d_{i-1}}^{d_i} P_{ij}(x_{ij}, Q_{ij}) f_{ij}(x_{ij}, t_{ij}) dt_{ij} dx_{ij} - \lambda_i (\sum_{j=1}^{m_i} s_{ij} Q_{ij} - S_i) \tag{11}$$

The first order derivative of (11) with respect to λ_i yields the following, which imposes the resource constraint.

$$\frac{\partial(11)}{\partial\lambda_i} = \sum_{j=1}^{m_i} s_{ij} Q_{ij} - S_i = 0, \quad \text{for } \forall i$$

From the first order optimality condition of d_i in (11), we obtain:

$$\sum_{j=1}^{m_i} \int_0^{\infty} P_{ij}(x_{ij}, Q_{ij}) f_{ij}(x_{ij}, d_i^*) dx_{ij} = \sum_{j=1}^{m_{i+1}} \int_0^{\infty} P_{i+1,j}(x_{i+1,j}, Q_{i+1,j}) f_{i+1,j}(x_{i+1,j}, d_i^*) dx_{i+1,j} \tag{12}$$

To derive the optimality conditions of $Q_{ij}*$, the profit function is expanded as below:

$$P_{ij}(x_{ij}, Q_{ij}) = \begin{cases} p_{ij} x_{ij} - (Q_{ij} - x_{ij}) Co_{ij}, & if \quad x_{ij} \le Q_{ij} \\ p_{ij} Q_{ij} - (x_{ij} - Q_{ij}) Cu_{ij}, & if \quad x_{ij} > Q_{ij} \end{cases}$$

where $P_{ij}(x_{ij}, Q_{ij})$, p_{ij}, Co_{ij}, Cu_{ij} are the profit function, unit profit, overstock cost, and under stock cost of item *j* of season *i*, respectively. Substituting the above profit function into (11), the first order optimality condition of Q_{ij} is derived as:

$$F_{x_{ij}|T_i*}(Q_{ij}^* \mid d_i^*) F_{t_{ij}}(d_i^*) - F_{x_{ij}|T_{i-1}*}(Q_{ij}^* \mid d_{i-1}^*) F_{t_{ij}}(d_{i-1}^*)$$

$$= \frac{(p_{ij} + Cu_{ij})[F_{t_{ij}}(d_i^*) - F_{t_{ij}}(d_{i-1}^*)]}{Co_{ij} + Cu_{ij} + p_{ij}} - \frac{\lambda_i s_{ij}}{Co_{ij} + Cu_{ij} + p_{ij}} \tag{13}$$

Further simplification of (13) for $Q_{ij}*$ is not possible unless more specific assumptions are made on the demand distribution.

It is straightforward to show that ***Theorem 3*** and ***4*** still hold for (11), if x_{ij} and t_{ij} are independently distributed. Further, it is easy to show that $Q_{ij}*$ decreases, as λ_i, s_{ij} increase. If x_{ij} and t_{ij} are independent, Equation (13) is simplified to:

$$F_{x_{ij}}(Q_{ij}^*) = \frac{p_{ij} + Cu_{ij} - \alpha_{ij}^{-1}\lambda_i s_{ij}}{Co_{ij} + Cu_{ij} + p_{ij}}$$

where $\alpha_{ij} = F_{t_{ij}}(d^*_i) - F_{t_{ij}}(d_{i-1}^*)$ and $0 \le \alpha_{ij} \le 1$. Then,

$$Q_{ij}^* = F_{x_{ij}}^{-1}\left(\frac{p_{ij} + Cu_{ij} - \alpha_{ij}^{-1}\lambda_i s_{ij}}{Co_{ij} + Cu_{ij} + p_{ij}}\right)$$

Clearly, as λ_i, s_{ij} increase, Q_{ij}^* decreases. λ_i has the similar interpretation of shadow price of additional unit of resource in season *i*. Also, λ_i is non-increasing in S_i, and Q_{ij}^* is non-increasing in S_i. Similar to (6), if x_{ij} and t_{ij} are independently distributed, (12) can be simplified into the following equation to obtain d_i^* numerically.

$$\sum_{j=1}^{m_i}[\int_o^\infty P_{ij}(x_{ij}, Q_{ij}^*) f_{x_{ij}}(x_{ij}) dx_{ij} f_{t_{ij}}(d_i^*)] = \sum_{j=1}^{m_{i+1}}[\int_o^\infty P_{i+1j}(x_{+1ij}, Q_{i+1j}^*) f_{x_{i+1j}}(x_{i+1j}) dx_{i+1j} f_{t_{i+1j}}(d_i^*)] \quad (14)$$

The Interpretation of (14) is similar to ***Theorems 3*** and ***4***.

5. Conclusions

The switching time decision has attracted very little attentions from the stochastic inventory literature. In this paper, we develop a newsboy formulation for the aggregate multiperiod inventory problem intended for products of short sales season and without replenishments. A distinguishing feature of our formulation is that it takes a time dimension of demand uncertainty into consideration. The proposed model is important because it represents one of the first to investigate the switching time and quantity decisions simultaneously and is easy to understand and thus to apply. It is especially suitable for applications in haute couture, i.e., high fashion industry. The model determines the time of switching primary sales effort from one season to the next as well as optimal order quantity for each season with the aim of maximizing expected profit over the planning horizon. We also derive the optimality conditions for the time of switching primary sales effort and order quantity. Furthermore, we show that if time uncertainty and volume uncertainty are independent, order quantity becomes the major decision over the interval of the primary selling season. Finally, we demonstrate that the results from the two-season case can be directly extended to the multi-season case and the limited resource multiple-item case.

Future extensions include incorporating the pricing and progressive switching decisions into the proposed model. Contrary to haute couture, many firms have opportunities to maximize revenue by changing prices over the selling period. Hence, another future research avenue is to incorporate the optimal timing of price change and optimal price level into the switching time decision. Without doubt, these additions will on one hand make the model more complicated. On the other hand, the applicability of the proposed model will be substantially broadened.

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