ALGORITHMS FOR SOLVING SPATIAL AND NON-STATIONARY HEAT CONDUCTION PROBLEMS IN MINING

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Considered an efficient algorithm for solving boundary problems of non-stationary heat conduction equations for complex areas of non-canonical form, which can be used for determining of temperature fields in the mountain ranges with the workings. Studied the three-dimensional problem of stationary heat conduction. Solutions of the respective boundary problems are obtained on the basis of the methods of the integral (potential) representation of the required quantities. Represented the results of the numerical implementation of the algorithm.

In mining, when driving various kinds of workings, there is a problem of assessment of the strength properties of mountain ranges. Stress concentration greatly depends on the shape, number and location of workings. Thermal stresses arising under the influence of heat flows in the mountain range have also great influence on the strength of the workings. The ability to find the temperature field and the stress state of the mountain range with the workings makes it possible to calculate the optimum shape and location of the workings in the mountain range to ensure its sufficient strength and efficiency.

The problems represented in this paper, in substance, are the problems of mathematical physics. Their solution can be obtained by different ways. One of these effective methods is a method of potentials. Let's study its application to some problems of three-dimensional and non-stationary heat conduction, which arise in mining.

Axisymmetric problems of mathematical physics for spatial bodies in fact are two-dimensional, and therefore they are not computationally more labor intensive than the planar problems. First, consider the algorithm for calculating the temperature distribution in the axisymmetric body of revolution, the axial cross-section of which is the stripe with the hole of a complex shape. Let be given an arbitrary temperature on the contour of the cavity.

Let us obtain Green's function for an axially symmetric heat conduction problem in the following formulation:

$$\frac{\partial^2 u(r,z)}{\partial z^2} + \frac{\partial^2 u(r,z)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,z)}{\partial r} = 0, \tag{1}$$

$$u(r,o) = \left.\frac{\partial u(r,z)}{\partial z}\right|_{z=a} = 0, \tag{2}$$

$$\left.\frac{\partial u(r,z)}{\partial r}\right|_{r=0} = 0, \; u(r,z)\big|_{r=\infty} = 0. \tag{3}$$

Assuming representation of the solution in the form of

$$u(r,z) = \sum_{m=1,3,5}^{\infty} u_m(r)\sin vz, \; v = \frac{m\pi}{2a},$$

for the function $u_m(r)$ we obtain the following boundary problem:

$$\frac{d^2 u_m(r)}{dr^2} + \frac{1}{r}\frac{du_m(r)}{dr} - v^2 u_m(r) = 0, \tag{4}$$

$$u_m^1(r)\big|_{r=0} = 0, \; u_m(r)\big|_{r=\infty} = 0. \tag{5}$$

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By making computation, similar to those in [1,6], we obtain the Green's function of the problem (1) - (3) in the form of:

$$G(r,z,\rho,\zeta) = \sum_{m=1,3,5}^{\infty} G_m(r,\rho)\sin\nu z\sin\nu\zeta,$$

where $G_m(r,\rho) = \begin{cases} \rho I_0(\nu r)K_0(\nu\rho) & 0\leq r\leq\rho<\infty, \\ \rho I_0(\nu\rho)K_0(\nu r) & 0\leq\rho\leq r<\infty, \end{cases}$ - Green's function of the problem (4) and (5), I_0, K_0 -

cylindrical functions of an imaginary argument.

Fig. 1a shows the level lines of the dimensionless stationary temperature field in the axisymmetric layer with a spherical cavity $(a=1, \quad Q=1)$; Fig. 1b - the stationary temperature field in the axisymmetric layer with a cut, on the boundary of which a dimensionless temperature
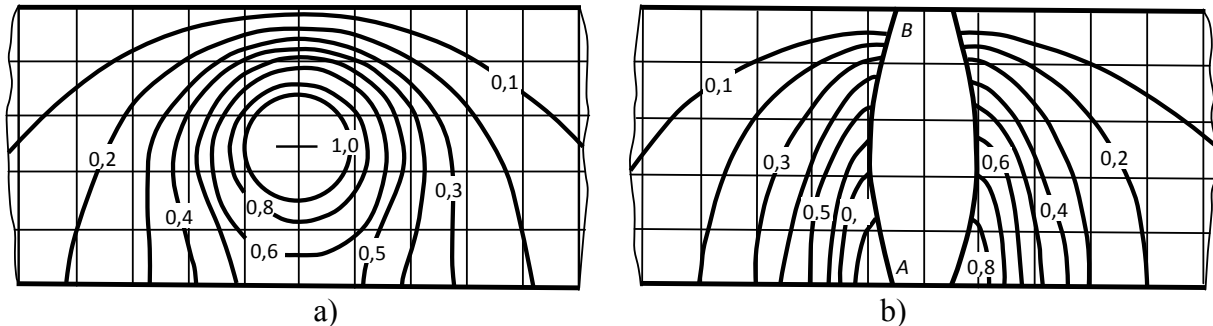

Fig. 1. Level lines of the dimensionless temperature field in the stationary axisymmetric layer

from the point A to the point B varies from 1 to 0 as $\cos(\pi z/ra)$. The error of satisfaction of the boundary condition on the contour of the cavity, ranges from 0.5% to 1.5% depending on the shape of the latter.

Note that there are no fundamental difficulties in considering axially symmetric radial heterogeneity of the studied layer. This raises an ordinary differential equation, different from (4), and to obtain its fundamental system it is necessary, in general, the use of a numerical procedure.

Consider three-dimensional stationary heat conduction problem.

Development of effective algorithms for solving three-dimensional mathematical physics problems meets with considerable computational difficulties. Some methods (e.g., methods of the theory of functions of complex variable, which represent powerful apparatus for solving two-dimensional boundary problems) do not allow the extension to the case of three independent variables, and the methods, such as of finite elements, variation approaches lead to cumbersome large-scale numerical arrays.

Consider the application of the potentials method to solving spatial boundary problems of the stationary heat conduction for the layer with a finite number of cylindrical cavities S_j. Formulate for such a multiply connected region D (Fig.2.), the following boundary problem:
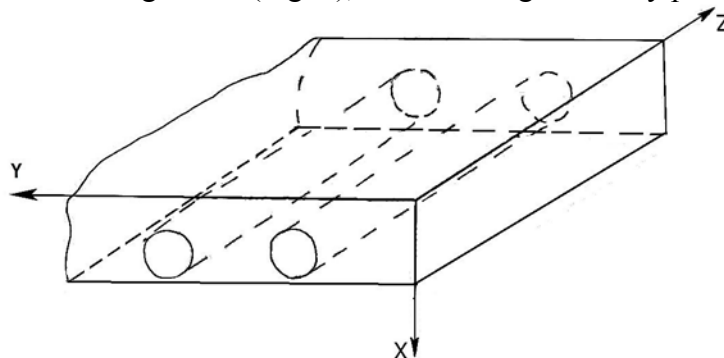

Fig. 2. Layer with a finite number cylindrical cavities

$$\Delta u(x, y, z) = F(x, y, z), \tag{6}$$

$$u\big|_{x=0} = \frac{\partial u}{dx}\bigg|_{x=0} = 0, \tag{7}$$

$$\frac{\partial u}{dy}\bigg|_{y=0;\infty} = 0, \tag{8}$$

$$B_j u\big|_{S_j} = Q_j(x, y, z). \tag{9}$$

Suppose, in addition, the following conditions of symmetry with respect to the variable z to be satisfied:

$$\frac{\partial u}{dz}\bigg|_{z=0;\acute{n}} = 0. \tag{10}$$

Under this assumption we can write the following natural expansions:

$$u(x, y, z) = \sum_{n=0}^{\infty} u_n(x, y)\cos\frac{n\pi z}{C}, \qquad F(x, y, z) = \sum_{n=0}^{\infty} F_n(x, y)\cos\frac{n\pi z}{C},$$

$$Q_j(x, y, z) = \sum_{n=0}^{\infty} Q_{jn}(x, y)\cos\frac{n\pi z}{C}.$$

Therefore, we obtain for the coefficients $u_n(x, y)$ the following boundary problem:

$$\left(\Delta - v^2\right)u_n(x, y) = F_n(x, y), \qquad u_n\big|_{x=0} = \frac{\partial u_n}{\partial x}\bigg|_{x=a} = 0, \quad \frac{\partial u_n}{\partial y}\bigg|_{y=0;\infty} = 0,$$

$$B_j u_n\big|_{L_j} = Q_{jn}(x, y), \quad v = \frac{n\pi}{C}, \quad (L_j \text{ is the directrix of the cavity } S_j).$$

Thus, the matter is reduced to a sequence of independent problems, for which the algorithm of solution has been described above.

Fig. 3 shows the dimensionless stationary temperature field in the layer with an elliptic cavity, obtained at the following initial data:

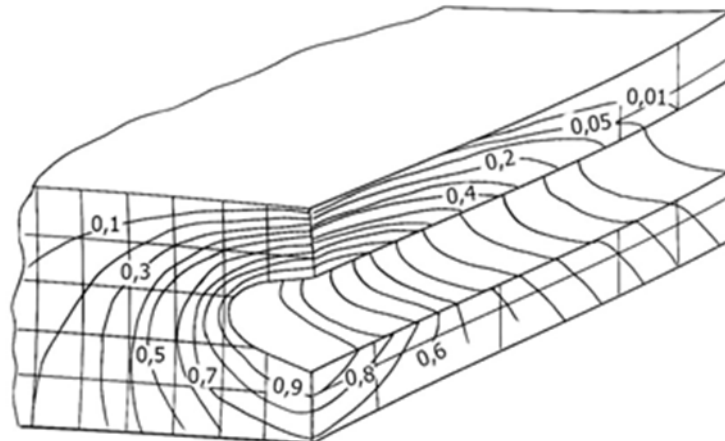$$F = 0, \ Q_j = \cos^3\frac{\pi z}{C}, \ c = 3, \ B_j \equiv I.$$

Fig. 3. Stationary temperature field in the layer with an elliptic cavity

The accuracy of satisfaction of the boundary condition (9) is in the range 97-99% (depending on the number and shape of the cavities).

A lot of engineering, including mining problems, are reduced to the non-stationary heat conduction problems for the regions of the non-canonical form.

Two-dimensional non-stationary heat conduction boundary problems for the bodies of complex geometry represent very complicated three-dimensional problems of mathematical physics (the third independent variable here is the time).

The advantages of potential methods in solving boundary problems of the shell theory [2 - 4], theory of elasticity [5] and stationary heat conduction [6, 7] are well-known. First, the problem reduces to the integral equations, and this affects the accuracy of the approximate solutions. Secondly, the dimension of the problem is reduced.

In this connection, it is expedient to use the heat potentials to obtain a similar effect in the non-stationary boundary problems of heat conduction. Moreover, in this paper, such attempt is carried out for the regions which boundaries do not fully coincide with the coordinate lines of the original system (non-classical regions) that allows showing more saliently the advantage of the method.

First, consider the simplest problem of finding the temperature field $u = u(x, y, F_0)$ in a rectangular plate $\Omega(0 \leq x \leq 0; 0 \leq y \leq b)$ with a cut or hole of a complex contours L

$$\frac{\partial u}{\partial F_0} = \Delta u - \sigma^2 u , \tag{11}$$

$$u\big|_{B=0} = \psi(x, y), \tag{12}$$

$$\frac{\partial u}{\partial x}\bigg|_{x=0} = u\big|_{x=\pm a} = 0, \tag{13}$$

$$\frac{\partial u}{\partial y}\bigg|_{y=0} = u\big|_{y=\pm b} = 0, \tag{14}$$

$$Bu\big|_L = Q. \tag{15}$$

Here, F_0 is the Fourier criterion (dimensionless time); B - operator of the boundary conditions of the first, second or third kind; Q -continuous and smooth function on the domain L.

Expand functions u and ψ in series

$$u(x, y, F_0) = \sum_{n=1,3,5}^{\infty} u_n(x, F_0) \cos \nu y,$$

$$\psi(x, y) = \sum_{n=1,3,5}^{\infty} \psi_n(x) \cos \nu y, \quad \nu = \frac{n\pi}{2b}. \tag{16}$$

Then for the functions $u_n(x, F_0)$ we obtain the following boundary problem:

$$\frac{\partial u_n}{\partial F_0} = \frac{\partial^2 u_n}{\partial x^2} - \left(\nu^2 + \sigma^2\right) u_n, \tag{17}$$

$$\left.\frac{\partial u_n}{\partial x}\right|_{x=0} = \left.u_n\right|_{x=a} = 0, \tag{18}$$

$$\left.u_n\right|_{F_0=0} = \psi_n. \tag{19}$$

The solution of the last we find, using the method of separation of variables

$$u_n(x,F_0) = \sum_{k=1,3,5}^{\infty} A_k e^{-\lambda_{nk}F_0} \cos\frac{k\pi x}{2a},$$

where $\lambda_{nk} = \dfrac{k^2\pi^2}{4a^2} + \dfrac{n^2\pi^2}{4b^2} + \sigma^2.$

Having satisfied the initial conditions (19), we obtain

$$\psi_n(x) = \sum_{k=1,3,5}^{\infty} A_k \cos\frac{k\pi x}{2a}.$$

It follows from the last representation and formulae Fourier that

$$A_k = \frac{2}{a}\int_0^a \psi_n(\xi)\cos\frac{k\pi\xi}{2a}d\xi.$$

Then $u_n(x,F_0) = \displaystyle\int_o^a \frac{2}{a}\sum_{k=1,3,5}^{\infty} e^{-\lambda_{nk}F_0}\cos\frac{k\pi\xi}{2a}\cos\frac{k\pi x}{2a}\psi_n(\xi)d\xi.$

Considering the second of the expansions (16), we have

$$\psi_n(\xi) = \frac{2}{b}\int_0^b \psi(\xi,\eta)\cos\frac{n\pi\eta}{2b}d\eta.$$

Substituting $\psi_n(\xi)$ in the expression for $u_n(x,F_0)$, and $u_n(x,F_0)$ in the first of the expansions (16), we get

$$u_n(x,y,F) = \int_o^a\int_o^b \frac{4}{ab}\sum_{k=1,3}^{\infty}\sum_{n=1,3,}^{\infty} e^{-\lambda_{nk}F_0}\psi(\xi,\eta)\cos\frac{k\pi x}{2a}\cos\frac{k\pi\xi}{2a}\cos\frac{n\pi y}{2b}\cos\frac{n\pi\eta}{2b}d\eta d\xi.$$

Therefore, when $F_0 > 0$ the Green's function can be represented as follows:

$$G(x,y,\xi,\eta,F_0) = \frac{4}{ab}\sum_{k=1,3}^{\infty}\sum_{n=1,3,}^{\infty} e^{-\lambda_{nk}F_0}\cos\frac{k\pi x}{2a}\cos\frac{k\pi\xi}{2a}\cos\frac{n\pi y}{2b}\cos\frac{n\pi\eta}{2b}.$$

The solution of the initial problem when $B = I$, $\psi(x,y) = 0$ we get as follows:

$$u(x,y,F_0) = \int_L\int_0^{F_0} G(x,y,\xi,\eta;F_0-\tau)\mu(\xi,\eta,\tau)d\tau d_{\xi,\eta}L,$$

where $\mu(\xi,\eta,\tau)$ is determined from the equation of the form

$$Q(x,y,F_0) = \int_L\int_0^{F_0} G(x,y,\xi,\eta;F_0-\tau)\mu(\xi,\eta,\tau)d\tau d_{\xi,\eta}L.$$

Fig. 4a and 4b shows the level lines of non-dimensional temperature field in a rectangular plate with a circular hole at $F_0 = 0.02$ and $F_0 = 0.1$, respectively.

a) $F_0 = 0.02$ b) $F_0 = 0.1$
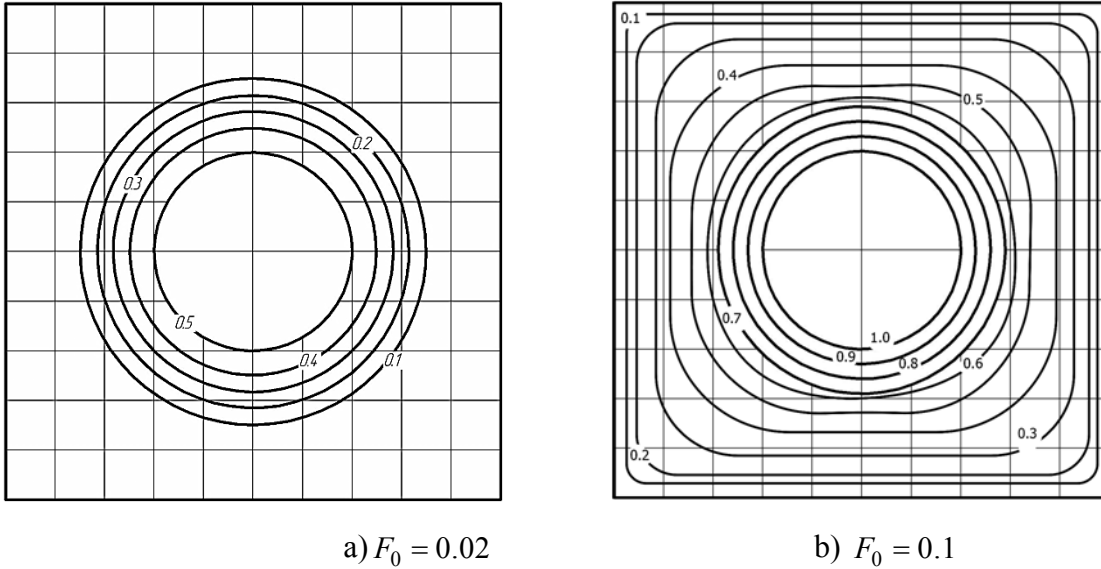
Fig. 4. Level lines of non-dimensional temperature field in a rectangular plate with a circular hole.

Here $\psi \equiv 0$, $Q(F_0) = 1 - \exp\left(-1000 F_0^2\right)$; $a = b = 1/2$; $\sigma = 0$.

It is apparent, for example, that at the value of the Fourier criterion $F_0 = 0.02$ a field as if «does not have time to feel» the impact of the external border of the plate, i.e., for small values of the Fourier criterion in the problem (11) - (15) we can neglect the influence of the outer contour, and treat it as to external with respect to the hole L. Whereas if $F_0 = 0.1$, isotherms, starting with 0.5, reveal this effect (they are gradually straightened on approaching the outer contour).

Let us now consider non-stationary heat conductivity problem for a strip with the holes of a complex shape, periodically located along its length:

$$\frac{\partial u}{\partial F_0} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}, \quad u\big|_{x=0} = 0, \quad \frac{\partial u}{\partial y}\bigg|_{y=0;\pm b} = 0,$$

$$\left(\frac{\partial u}{\partial x} + \alpha u\right)\bigg|_{x=a} = 0, \quad u\big|_{F_0=0} = \psi(x, y), \quad u\big|_L = \Phi(x, y, F_0).$$

Using the expansion (16), for $u_n(x, F_0)$ we obtain the boundary problem

$$\frac{\partial u_n}{\partial F_0} = \frac{\partial^2 u_n}{\partial x^2} - \nu^2 u_n,$$

$$u_n\big|_{x=0} = 0, \quad \left(\frac{\partial u_n}{\partial x} + \alpha u_n\right)\bigg|_{x=a} = 0, \quad u_n\big|_{F_0=0} = \psi_n(x), \quad \nu = \frac{n\pi}{b}.$$

To solve this problem apply the method of separation of variables

$$u_n(x, F_0) = X_n(x) T_n(F_0).$$

After substituting into the equation, it turns out

$$\frac{T_n'(F_0)}{T_n(F_0)} = \frac{X_n''(x)}{X_n(x)} - \nu^2 = -\lambda_n.$$

Hence, $X_n(x)$ is the solution of the following boundary problem:

$$X_n^{''}(x) + \omega_n^2 X_n(x) = 0, \; X_n(0) = 0, \; X_n^{'}(a) + \alpha X_n(a) = 0,$$

where $\omega_n^2 = \lambda_n - \nu^2$.

Having resolved the latter, we get

$$X_{nk}(x) = \sin \frac{\mu_{nk}}{a} x,$$

where μ_{nk} is the solution of the transcendental equation

$$tg\,\mu_{nk} = -\frac{\mu_{nk}}{\alpha a}, \; \mu_{nk} = \omega_{nk} a.$$

Therefore

$$\lambda_{nk} = \frac{\mu_{nk}^2}{a^2} + \nu^2 = \frac{\mu_{nk}^2}{a^2} + \frac{n^2\pi^2}{b^2}.$$

The solution $T_n(F_0)$ of the respective problem will be

$$T_{nk}(F_0) = A_k e^{-\lambda_{nk} F_0}.$$

It gives

$$u_n(x, F_0) = A_k e^{-\lambda_{nk} F_0} \sin \omega_{nk} x.$$

Through further calculations, similar to the previous ones, for the required Green's function we obtain the following expression:

$$G(x, y, \xi, \eta, F_0) = \frac{4}{ab} \sum_{k=1,3}^{\infty} \sum_{n=0}^{\infty} e^{-\lambda_{nk} F_0} \sin \omega_{nk} x \sin \omega_{nk} \xi \cos \nu y \cos \nu \eta.$$

Using the calculation scheme similar to the above one can obtain the non-stationary temperature distribution for an infinite strip with a periodic series of holes of the rather complex shape. The computational procedure in this case is completely identical to that in the calculation of the fields shown in Fig. 4.

Thus, the use of two-dimensional heat potentials, according to the described scheme enables to solve effectively the boundary problems of non-stationary heat conduction for the areas of the complex geometry.

Bibliography

1. Азанов Б.Г., Долгова И.М., Мельников Ю.А. Разработка эффективных алгоритмов решения граничных задач термоупругости для областей сложной формы. - Изв. АН СССР. Мех.тверд.тела, 1975, № 5, с. 205.

2. Гавеля С.П. О сохранении разрешимости граничных задач теории пологих оболочек при приведении их к интегральным уравнениям. - Изв. высш. учебн. заведений. Математика, 1969, № 5, с.14-19.

3. Гавеля С.П. Періодичні задачі для пологих оболонок довільної кривизни з отворами. - ДАН УРСР, 1969, А, № 8, с.703-708.

4. Гавеля С.П. Про один спосіб побудови матриць Гріна для зчленованих оболонок. - ДАН УРСР, 1969, А, № 12, с.1107-1111.

5. Долгова И.М., Мельников Ю.А. Построение функций и матриц Грина для уравнений и систем эллиптического типа. – ПММ, 1978, т.42, № 4, с.695-700.

6. Мельников Ю.А., Долгова И.М. Об одном эффективном подходе к решению двумерных задач теплопроводности для многосвязных составных тел сложной формы. - Инженерно-физический журнал, 1976, т.30, № 1, с.152-159.

7. Мельников Ю.А., Долгова И.М. Интегральное представление решений неклассических задач теплопроводности. - Инженерно-физический журнал, 1976, т.31, № 2, с.376.