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Interaction of Technology Choice and Financial Risk Management under Costly External Financing

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The Interaction of Technology Choice and Financial Risk Management: An Integrated Risk Management Perspective

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Abstract

This paper analyzes the integrated operational and financial risk management portfolio of a firm that determines whether to use flexible or dedicated technology and whether to undertake financial risk management or not. The risk management value of flexible technology is due to its risk pooling benefit under demand uncertainty. The financial risk management motivation comes from the existence of deadweight costs of external financing due to capital market imperfections. Financial risk management has a fixed cost, while technology investment incurs both fixed and variable costs. The firm's limited budget, which depends partly on a tradable asset, can be increased by borrowing from external markets, and its distribution can be altered with financial risk management. In a parsimonious model, we solve for the optimal risk management portfolio, and the related capacity, production, financial risk management and external borrowing levels, the majority of them in closed form. We characterize the optimal risk management portfolio as a function of firm size, technology and financial risk management costs, product market (demand variability and correlation) and capital market (external financing costs) characteristics. Our analysis contributes to the integrated risk management literature by characterizing the optimal risk management portfolio in terms of a more general set of operational and financial factors; providing the value and limitation of operational and financial risk management by explicitly modeling their costs and benefits; demonstrating the interactions between the two risk management strategies; and relating our theoretical results to empirical observations.

Key Words: Risk Management, Capacity Investment, Flexibility, Financing, Operational Hedging.

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1 Introduction

This paper is about integrating operational and financial risk management and characterizing the drivers of the optimal integrated risk management portfolio. The two means of risk management are motivated by the existence of different market imperfection costs and utilize different tools. On the operational side, firms are exposed to demand and supply uncertainties in product markets. These uncertainties, which we call forms of *product market imperfection*, impose supply-demand mismatch costs. To manage these costs, firms rely on different types of operational flexibility that provide a better response to product market imperfections and counterbalance the effect of supply-demand mismatch costs. On the financial side, firms do not always have sufficient internal cash flows to finance their operations and depend on external capital markets to raise funds. The transaction costs in capital markets (bankruptcy costs, taxes, underwriter fees, agency costs etc.), which are forms of *capital market imperfection*, impose deadweight costs of external financing on firms. To manage these costs, firms rely on different types of financial instruments written on tradable assets with which their cash flows are correlated. These financial instruments engineer the internal cash flows of firms to meet their optimal investment needs and counterbalance the effect of external financing costs.

Despite responding to two different types of market imperfection, operational and financial risk management interact with each other: The choice of operational risk management has implications for financial risk management and vice versa. Therefore, operational and financial risk management should be viewed as constituting an integrated risk management portfolio. In practice, most corporate-level risk management programs of non-financial firms focus only on financial risk management (Bodnar et al. 1998). At the same time, a number of large non-financial firms are becoming more interested in operational solutions to manage their risk exposures (Business Week 1998). Due to the existence of both product and capital market imperfections in practice, using both risk management tools – and doing so in an integrated fashion – is important.

The academic literature on risk management has largely documented the value and effectiveness of each risk management tool in isolation. Relatively little progress has been made in understanding their interactions and the main drivers of an optimal integrated risk management portfolio. The objective of this paper is to enhance our understanding of integrated risk management. Our main contributions are to model and analyze an integrated risk management problem that (i) yields structural results about the characteristics and drivers of an optimal risk management portfolio; (ii) provides managerial guidelines that can be used in designing risk management programs; and (iii) can be used to generate hypotheses that account for operational and product market characteristics to a greater extent than the existing empirical risk management literature.

To this end, we model a budget-constrained manufacturer who produces and sells two products. Product demands are random, which is the *product market imperfection*, and correlated. The firm chooses between flexible and dedicated technologies that incur fixed and variable costs, and determines the capacity level of the chosen technology. Because of its risk pooling benefit, the flexible technology is the firm's *operational risk management tool.* The firm's limited budget partially depends on a perfectly tradable asset. The firm can relax its budget constraint by borrowing from external markets, but borrowing incurs external financing costs that originate from *capital market imperfections.* Forwards written on the asset price can be used as the firm's *financial risk management tool* to alter the budget distribution and help counterbalance the effect of external financing costs. The fixed and variable investment costs of flexible technology are higher than those of dedicated technology, and financial risk management has a fixed cost. Therefore, it may be undesirable to use these tools despite their value. In this rich but parsimonious model, we answer the following research questions:

1. What is the optimal risk management portfolio of the firm (defined as choosing flexible versus dedicated technology, and engaging in financial risk management or not) as a function of firm size, technology and financial risk management costs, product market conditions (demand variability and correlation) and capital market conditions (external financing costs)?
2. What are the fundamental drivers of the optimal risk management portfolio?
3. Are financial and operational risk management complements or substitutes?
4. What are the consequences of the interaction between financial and operational risk management? What is the effect of financial risk management on operational decisions?
5. Can our results be used to support or refine existing empirical research?

We derive the optimal integrated risk management portfolio and the related capacity, production, financial risk management and external borrowing levels, the majority of them in closed form. Our analysis reveals that there are three fundamental drivers that explain the optimal portfolio choice: the robustness of the optimal capacity investment level to product market conditions, the level of reliance on external financing and the opportunity cost of financial risk management. These drivers work in opposite directions for large and small firms due to differences in their borrowing needs under financial risk management. As a result, the size of the firm is highly relevant – the same underlying conditions lead to different optimal portfolio choices as a function of firm size. Conversely, it may be optimal for small and large firms to choose the same optimal portfolio for entirely different reasons. These results generate managerial insights and guidelines for designing an integrated risk management program.

Our analysis clearly illustrates the intertwined nature of operational and financial risk management strategies. We show that firms can use financial risk management for speculative purposes with flexible technology, whereas they may prefer to hedge with dedicated technology. The reason is that firms with a limited internal budget can optimally increase their asset risk exposure to cover the higher fixed cost of flexible technology and invest in capacity to generate revenue. We demonstrate that engaging in financial risk management may induce the firm to change its technology decision; flexible technology and financial risk management can be complements or substitutes. This is a direct consequence of the difference between each technology regarding the counterbalancing value of financial risk management with respect to external financing costs.

We relate our theoretical findings to empirical observations concerning risk management practices of firms. Our results provide theoretical support for some observations and highlight additional trade-offs in others. For example, we establish that the value of financial risk management increases in external financing costs only for large firms and not for small firms. This is in contrast to existing understanding that this is true for any firm. We show that if firms use financial instruments only for hedging purposes, it is optimal for small firms to not undertake financial risk management; existing arguments attribute this observation only to the fixed cost of establishing a financial risk management program. The distinction we make between large and small firms, and our results related to the effect of technology and product market characteristics on the risk management portfolio provide new hypotheses that can be tested empirically.

We note that all of the results[1] obtained are analytical and are valid for any demand and asset price distribution with positive and bounded support. With these results, we contribute to the growing operations management literature that incorporates financial considerations in operational decision making. In the next section, we provide more detail about how our work contributes to the existing literature. In §3, we describe the model and discuss the basis for our assumptions. §4 analyzes the optimal strategy of the firm, culminating in a characterization of the optimal risk management portfolio. §5 and §6 flesh out the results of the previous section to describe the impact of various factors on the optimal portfolio choice. We analyze the value and effect of integrated decision making by comparing with the non-integrated benchmark in §7. In §8, we discuss the robustness of our results to our assumptions. §9 concludes.

2 Literature Review

In this section, we review the streams of literature related to our paper and delineate our contributions to each stream. The operations management literature has documented the risk management value of operational flexibility. Starting with the influential studies of Huchzermeier and Cohen (1996), Cohen and Huchzermeier (1999) and Kouvelis (1999), this stream delineates the value of various operational flexibilities (e.g. technology flexibility, geographical diversification, postponement) in the firm's network structure, referred as operational hedges, in managing demand-side product market imperfections (Van Mieghem 2003, 2006, Aytekin and Birge 2004, Kazaz et al. 2005). We refer the reader to Boyabatlı and Toktay (2004) for a recent review of papers in this stream. A number of papers take this analysis further and study the interaction between different operational flexibilities of firms (Bish and Wang 2004, Goyal and Netessine 2005, Chod et al. 2006a, Dong et al. 2006). This stream of papers (often implicitly) assumes perfect capital markets and hence there are neither deadweight costs of external financing nor any value for financial risk management. We demonstrate the effect of external financing costs and financial risk management on the value of operational risk management, and document several interactions between operational and financial risk management.

The finance literature on risk management, in turn, focuses on financial risk management (e.g. forwards, options, etc.) and typically does not consider product market imperfections and operational risk management. The majority of this literature i) provides different explanations for the existence of financial risk management that are based on different types of capital market imperfections; or ii) focuses on the optimal use of financial instruments in a variety of settings. Since the focus of these papers is financial risk management, the interactions between the two risk management strategies are not studied. We refer the reader to Fite and Pfleiderer (1995) for a review of the first stream and Brown and Toft (2001) for a review of the second.

There are a few theoretical papers that study the firm's integrated risk management portfolio choice. In operations, Chod et al. (2006b) and Ding et al. (2005) analyze the interaction between financial risk management and different types of operational flexibility, where financial risk management is motivated by the risk aversion of the decision maker. Chod et al. (2006b) analyze whether financial risk management complements or substitutes operational flexibility. They demonstrate that this depends on whether the optimal flexibility level increases or decreases with financial hedging. We show that financial and operational risk management can again be either complements or substitutes under external financing, but the driver is firm size. Ding et al. (2005) is closest to our paper in terms of its research objective. They study the integrated operational (postponement) and financial risk management (currency options) decisions of a multinational firm and delineate the value of each risk management strategy under demand and exchange rate uncertainty. In a numerical study, they show that engaging in financial risk management alters the robustness of operational decision variables (capacity) with respect to demand variability and changes the strategic decision variables (global supply chain structure). We demonstrate similar results analytically. In addition, we analyze the effect of external financing costs, demand correlation and firm size on the optimal risk management portfolio. Incorporating the costs of each risk management strategy enables us to also explore the limits of their use.

In finance, Mello et al. (1995) and Chowdry and Howe (1999) model a multinational firm that has sourcing flexibility (sourcing from both domestic and foreign production facilities is possible) and that uses financial instruments to manage the exchange rate risk. These papers demonstrate the value of sourcing flexibility in conjunction with financial risk management. The focus of these papers is mainly financial risk management, and they do not consider a detailed representation of the firm's operations. Our analysis generates a number of insights about integrated risk management in a more detailed model of firm operations.

All of these papers assume that financial risk management is costless, in which case financial risk management is trivially included in the optimal risk management portfolio since it has positive value. In contrast, the fixed cost of financial risk management (e.g. software and personnel costs) can be a deterrent in practice. Motivated by this observation, we incorporate a positive fixed cost for engaging in financial risk management. This makes whether to engage in financial risk management or not a nontrivial question. The answer to this question goes beyond a boundary invest/do not invest decision divorced of the other decision variables: Under a budget limit and external financing costs, the effective cost of financial risk management is larger than its fixed cost because the firm may need to borrow an additional amount as a result of incurring this fixed cost. Therefore, engaging in financial risk management has an impact on the level of other decisions variables. Similarly, the fixed cost of the technology investment has a subtle effect on the optimal portfolio. These interactions add interesting dimensions to the optimal risk management portfolio.

In contrast to the theoretical finance research, the empirical finance literature has paid more attention to operational risk management, as reviewed in Smithson and Simkins (2005). This literature either statistically or qualitatively attributes a number of empirical observations to the firm's operational risk management capabilities, which we discuss these observations in detail in §5 and §6. We contribute to this stream in a number of ways: We provide theoretical support for some empirical observations and delineate additional trade-offs in some others; we provide alternative explanations to some observations that are based on the interplay between the two risk management strategies; and we identify potential future empirical research avenues.

In summary, our major contribution is to the integrated risk management literature. We contribute to this literature by i) characterizing the optimal risk management portfolio in terms of a more general set of operational and financial factors; ii) providing the value and limitation of each risk management strategy by explicitly modelling the costs and benefits of each strategy; iii) demonstrating the interactions between the two risk management strategies; and iv) relating our theoretical predictions to empirical observations.

Note that we have made a distinction between papers that augment the financial risk management analysis with operational risk management versus operational decisions only. Up to this point, we focused on the former, which involves a type of flexibility that can be used for risk management (and subsumes a number of operational decisions). The latter focuses only on operational decisions in analyzing financial risk management.

In the latter stream, we highlight Froot et al. (1993) from the finance literature since their modelling of the financial risk management motive is the same as in our paper. The authors use a concave increasing investment cost function to capture the operational dimension. They demonstrate that financial risk management adds value by generating sufficient internal funds to finance operational investments when there exist deadweight costs of external financing. We extend their framework by formalizing the operational investments (by incorporating product market characteristics, and technology and production decisions), and by imposing a cost for financial risk management. We illustrate that some of their predictions continue to hold, whereas some change due to the interplay between financial and operational decisions.

In the operations literature, Birge (2000), Chen et al. (2004), Gaur and Seshadri (2005), and Caldentey and Haugh (2005, 2006) document the value of financial risk management when the operating cash flows are correlated with a financial index. The financial risk management rationale is the risk-aversion of the decision maker in the latter three papers. Among these papers, we can link our paper to Caldentey and Haugh (2005) who motivate financial risk management by imposing a budget constraint on the firm, but without the possibility of external financing. This can be viewed as a special case of our model: When the external financing cost is sufficiently high, the firm never borrows. The external borrowing feature of our model is an important determinant of the risk management portfolio: the reliance on external borrowing determines the technology choice and the value of financial risk management with each technology.

Finally, our work is related to two other streams in operations management. The stochastic capacity investment literature analyzes the question of flexible versus dedicated technology choice with demand-side (uncertain demand) and supply-side (unreliable supply) product market imperfections. We refer readers to Van Mieghem (2003) for an excellent review and to Tomlin and Wang

(2005) for a specific focus on the supply-side imperfection. As highlighted in Van Mieghem (2003), stochastic capacity models (often implicitly) assume perfect capital markets. We demonstrate that under financing frictions, there exist additional trade-offs in technology choice: the level of reliance on external financing and the value of financial risk management with each technology.

A second stream relaxes the perfect capital market assumption and models the firm's joint financial and operational decisions (Lederer and Singhal 1994, Buzacott and Zhang 2004, Babich and Sobel 2004, Xu and Birge 2004 and Babich et al. 2006). The primary focus of these papers is to analyze the effect of external financing costs and the financing decision on operational decisions. They demonstrate the value of integrated financing and operational decision making. We extend the interaction argument in these papers by considering another facet of financial decisions, financial risk management. Our analysis reveals that the effect of external financing costs are largely dependent on the value of financial risk management and that technology choice is a key determinant of the firm's reliance on external markets: the higher investment cost of flexible technology requires higher external financing levels than dedicated technology.

3 Model Description and Assumptions

We consider a monopolist firm selling two products in a single selling season under demand uncertainty. The firm chooses the technology (dedicated versus flexible), the capacity investment level and the production level so as to maximize expected shareholder wealth. Differing from the majority of traditional stochastic technology and capacity investment problems, we model the firm as being budget constrained, where the budget partially depends on a hedgeable market risk. We allow the firm to undertake financial risk management to hedge this market risk, and to borrow from external markets to augment its budget. After operating profits are realized, the firm pays back its debt; default occurs if it is unable to do so.

We model the firm's decisions as a three-stage stochastic recourse problem under financial market and demand risk. In stage 0, the firm chooses its integrated risk management portfolio. The firm decides its technology choice (flexible or dedicated), whether to engage in financial risk management, and if so, its financial risk management level under demand and financial market risk. In stage 1, the financial market risk is resolved and the financial risk management contract (if any) is exercised; these two factors determine the internal cash level of the firm. The firm then determines the level of external borrowing and makes its capacity investment using its total budget (internal cash and borrowed funds). In stage 2, demand uncertainty is resolved and the firm chooses the production quantities for each product. Subsequently, the firm either pays back its debt or defaults. In the remainder of this section, we define the firm's objective and discuss the assumptions concerning each decision epoch in detail. We discuss the robustness of our results with respect to the majority of these assumptions in §8.

Assumption 1 *The firm maximizes the expected (stage 2) shareholder wealth by maximizing the expected value of equity. The shareholders are assumed to be risk-neutral and the risk-free rate* r_f *is normalized to 0. Shareholders have limited liability.*

The main goal of corporations is to maximize shareholder wealth. The expected shareholder wealth is a function of the expected cash flows to equity of the firm and the required rate of return of the shareholders. By assuming the risk neutrality of shareholders, we focus on maximizing the expected equity value of the firm. The required rate of return is the risk-free rate, which is normalized to 0 by assumption. Although the shareholders are risk-neutral, the existence of external financing costs creates an aversion to the downside volatility of the internal cash level in stage 1: The firm may be forced to underinvest in capacity at low internal cash level realizations because of external financing costs. This creates a motivation for undertaking firm-level financial risk management activities (Froot et al. 1993).

3.1 Stage 0

In this stage, the firm determines its technology choice $T \in \{D, F\}$, whether to use financial risk management, and if so, the financial risk management level H_T under financial market and demand uncertainty. The flexible technology (F) has a single resource that is capable of producing two products. The dedicated technology (D) consists of two resources that can each produce a single product.

Assumption 2 *Technology* T *has fixed* (F_T) *and variable* (c_T) *capacity investment costs. The fixed cost of the flexible technology is higher than that of the dedicated technology;* $F_F \geq F_D$*. The variable capacity investment cost of the two dedicated resources are identical. Both technologies are sold immediately at the end of the selling season at a reduced price of* $\gamma_T F_T$ *where* γ_T *is the salvage rate and* $0 \leq \gamma_T < 1$*. The firm commits to technology in this stage whose fixed cost is incurred in stage 1.*

Since flexible technology is generally more sophisticated than dedicated technology, the fixed cost of flexible technology is assumed to be higher. The stage 0 commitment of the firm to technology choice can be justified by the lead time of the acquisition (if outsourced) or the development time (if built in-house) of the technology. When the technology is resold, because of depreciation and liquidation costs, the fixed cost of the technology cannot be fully retrieved ($\gamma_T < 1$).

Assumption 3 *The firm uses a loan commitment contract to finance its capacity investment and to cover the fixed cost of the committed technology. The terms of the contract are known at stage 0, while borrowing takes place at stage 1.*

Loan commitment is a promise to lend up to a pre-specified amount at pre-specified terms. In practice, most short-term industrial and commercial loans in the US are made under loan commitment contracts (Melnik and Plaut 1986). At stage 0, the firm owns the right to a loan contract that can be exercised in stage 1. We discuss the characteristics of the loan commitment contract in Assumption 6 of stage 1.

Assumption 4 *At stage 0, the firm has rights to a known internal stage 1 endowment* (ω_0, ω_1)*. Here,* ω_0 *represents the cash holdings and* ω_1 *represents the asset holdings of the firm. The asset is a perfectly tradeable asset that has a known stage 0 price of* α_0 *and random stage 1 price of* α_1*. The random variable* α_1 *has a continuous distribution with positive support and bounded expectation* $\overline{\alpha}_1$*.*

With this assumption, in stage 0, the firm knows that the value of its endowment will be $\omega_0 + \alpha_1\omega_1$ in stage 1, where α_1 is random; this is the financial market risk in our model. This representation is consistent with practice: In general, firms hold both cash and tradable assets on their balance sheet, such as a multinational firm that has pre-determined contractual fixed payments denominated in both domestic and foreign currency, or a gold producer that produces a certain level of gold that is exposed to gold price risk. In these examples, the asset price α_1 represents the exchange rate and the gold price in stage 1, respectively. Although the cash and the asset holdings are certain, the price of the asset makes the stage 1 value of the internal endowment random. The firm can use financial risk management tools to alter the distribution of this quantity.

Assumption 5 *The firm uses forward contracts written on asset price* α_1 *to financially manage the market risk. There is a fixed cost* F_{FRM} *of engaging in financial risk management that is incurred in stage 0 by transferring the rights of the firm's claims* ω_0 *and* ω_1*, in proportions* β *and* $1 - \beta$*. Forward contracts are fairly priced. We restrict the number of forward contracts* H_T *such that the firm does not default on its financial transaction in stage 1.*

Forward contracts are the most prevalent type of financial derivatives used by non-financial firms (Bodnar et al. 1995). The fixed cost of financial risk management (F_{FRM}) includes the costs of hiring risk management professionals, and purchasing hardware and software for risk management; it is independent of the number of forward contracts used. In a recent survey, non-financial firms report this fixed cost as the second most important reason for not implementing a financial risk management program (Bodnar et al. 1998). Since we focus on loan commitment contracts and the firm can borrow from external markets only at stage 1, F_{FRM} is deducted in stage 0 from the firm's stage 1 endowment by transferring the rights of the claims ω_0 and ω_1 with β and $1-\beta$ proportions respectively. In other words, rights for βF_{FRM} of the cash holdings and $\frac{(1-\beta)F_{FRM}}{\alpha_0}$ of the asset holdings are transferred in stage 0. This leaves the firm with a stage 1 endowment of $(\omega_0^{FRM}, \omega_1^{FRM}) \doteq (\omega_0 - \beta F_{FRM}, \omega_1 - \frac{1-\beta}{\alpha_0} F_{FRM})$. The firm can only engage in financial risk management if these quantities are non-negative, or equivalently, if $F_{FRM} \leq \min\left(\frac{\omega_0}{\beta}, \frac{\alpha_0 \omega_1}{1-\beta}\right)$. Since the firm is exposed to external financing costs in stage 1, there is an opportunity cost associated with F_{FRM}: The firm has lower internal cash in stage 1 and may need to borrow more from external markets after paying for F_{FRM}. The fair-pricing assumption ensures that the firm can only affect the distribution of its budget in stage 1 – and not its expected value – by financial risk management. We restrict the feasible set of forwards to the range $\left[-\frac{\omega_0^{FRM}}{\overline{\alpha}_1}, \omega_1^{FRM}\right]$. Within this range of forwards the firm never defaults on its financial transaction in stage 1. This ensures that we can use default-free prices in forward transactions.

3.2 Stage 1

In stage 1, the market risk α_1 is resolved. The value of the firm's internal endowment and the exercise of the financial contract (if any) determine the firm's budget B. In this stage, the firm can raise external capital if the budget is not sufficient to finance the desired capacity investment. The firm determines the amount of external borrowing and the capacity investment level under demand uncertainty.

Assumption 6 *With the loan commitment contract, the firm can borrow up to credit limit E from a unit interest rate of $a > r_f = 0$. The face value of the debt $e_T(1+a)$ is repaid out of the firm's assets in stage 2. The firm has physical assets of value P (e.g. real estate) that are pledged to the creditor as collateral. The loan is secured (fully collateralized), i.e. $E(1+a) \leq P$. The physical assets are illiquid; they can only be liquidated with a lead time. The value of the physical assets P is sufficient to finance the budget-unconstrained optimal capacity investment level of the firm. The salvage value of technology ($\gamma_T F_T$) cannot be seized by the creditor among the firm's assets. Any possible costs that may be incurred in the borrowing process by the creditor (e.g. fixed bankruptcy costs) are charged ex-ante to the firm in a.*

We assume that the loan commitment is fully collateralized by the firm's physical assets P, i.e. $E(1+a) \leq P$, since most bank loans are secured by the company's assets (Weidner 1999) and modelled as such (Mello and Parsons 2000). Although the loan is fully collateralized, if the firm's final cash position is not sufficient to cover the face value of the debt, the firm cannot immediately liquidate the collateral assets to repay its debt since the physical assets are illiquid. Under limited shareholder liability, this leads to default, in which case the creditor can seize these physical assets, liquidate them and use their liquidation value to recover the loan. The salvage value of technology is assumed to be non-seizable; the creditor cannot use the salvage value to recover the face value of the loan. We also assume that the creditor's transaction costs associated with default (e.g. fixed bankruptcy costs) are charged to the firm ex-ante in the unit borrowing cost.

A positive unit financing cost ($a > 0$) and a credit limit less than the value of the collateralized asset ($E < P$) can be interpreted as the deadweight costs of external financing that arise from capital market imperfections: If the capital markets are perfect (i.e. there are no transaction costs, default related costs, information asymmetries), then the contract parameters are determined such that the loan is fairly valued in terms of its underlying default exposure. Since we focus on a collateralized loan, in the absence of default-related deadweight costs, there is no risk for the creditor associated with default. Consequently, in perfect capital markets, the fair unit financing cost of the loan commitment contract would be the risk-free rate ($a = 0$), and the credit limit would be the value of collateralized physical asset ($E = P$). If there are capital market imperfections, then $a > 0$ and $E < P$ would be obtained in a creditor-firm interaction. Therefore, although we assume that they are exogenous parameters in this paper, a positive unit financing cost ($a > 0$) and a credit limit less than the value of the collateralized asset ($E < P$) can be interpreted as capturing the deadweight costs of external financing that arise from capital market imperfections. This parallels the assumptions in Froot et al. (1993) who take the external financing costs as exogenous and state that they can be argued to arise from deadweight costs associated with capital market imperfections.

In a creditor-lender equilibrium, the (endogenous) contract parameters need not be identical for each technology. In §8, we discuss conditions under which our results with identical contract parameters are valid in a general equilibrium setting, and refer the reader to Boyabatlı and Toktay (2006) for an analysis of equilibrium contract (a_T^*, E_T^*) for each technology in a creditor-firm Stackelberg game.

To conclude, we note that our external financing cost structure provides a parsimonious model that is consistent with real-life practices; allows us to implicitly capture capital market imperfections and enables us to preserve tractability.

3.3 Stage 2

In this stage, demand uncertainty is resolved. The firm then chooses the production quantities (equivalently, prices) to satisfy demand optimally. If the firm is able to repay its debt from its final cash position, it does so and terminates by liquidating its physical assets. Otherwise, default occurs.

In this case, because of the limited liability of the shareholders, the firm goes to bankruptcy. The cash on hand and the ownership of the collateralized physical assets are transferred to the creditor. The firm receives the remaining cash after the creditor covers the face value of the debt from the seized assets of the firm.

Assumption 7 *Price-dependent demand for each product is represented by the iso-elastic inverse-demand function* $p(q_i;\xi_1) = \xi_i q_i^{1/b}$ *for* $i = 1, 2$. *Here,* $b \in (-\infty, -1)$ *is the constant elasticity of demand, and* p *and* q *denote price and quantity, respectively.* ξ_i *represents the idiosyncratic risk component.* (ξ_1, ξ_2) *are correlated random variables with continuous distributions that have positive support and bounded expectation* $(\overline{\xi}_1, \overline{\xi}_2)$ *with covariance matrix* $\mathbf{\Sigma}$, *where* $\mathbf{\Sigma}_{ii} = \sigma_i^2$ *and* $\mathbf{\Sigma}_{ij} = \rho\sigma_1\sigma_2$ *for* $i \neq j$ *and* ρ *denotes the correlation coefficient.* (ξ_1, ξ_2) *and* α_1 *have independent distributions. The marginal production costs of each product at stage 2 are 0.*

4 Analysis of the Firm's Optimal Risk Management Portfolio

In this section, we describe the optimal solution for the firm's technology choice, and the levels of financial risk management, external borrowing, capacity investment and production. A realization of the random variable s is denoted by $\tilde{s}$ and its expectation is denoted by $\overline{s}$. Bold face letters represent vectors of the required size. Vectors are column vectors and $'$ denotes the transpose operator. Vector exponents are taken componentwise. $\mathbf{xy}$ denotes the componentwise product of vectors $\mathbf{x}$ and $\mathbf{y}$ with identical dimensions. We use the following vectors throughout the text: $\boldsymbol{\xi}' = (\xi_1, \xi_2)$ (product market demand), $\mathbf{K_F} = K_F$ (flexible capacity investment) and $\mathbf{K_D'} = (K_D^1, K_D^2)$ (dedicated capacity investment). Pr denotes probability, $\mathbb{E}$ denotes the expectation operator, $\chi(.)$ denotes the indicator function with $\chi(\varpi) = 1$ if ϖ is true, $(x)^+ \doteq \max(x, 0)$ and $\Omega^{01} \doteq \Omega^0 \bigcup \Omega^1$. Monotonic relations (increasing, decreasing) are used in the weak sense otherwise stated. Table 1 summarizes the decision variables. Table 6 that summarizes other notation and all proofs are provided in Appendix A. We solve the problem by using backward induction starting from stage 2.

Stage	Name	Meaning
Stage 0	$T \in \{D, F\}$	Technology choice, dedicated or flexible
	H_T	Number of forwards with technology T
Stage 1	e_T	Borrowing level with technology T
	$\mathbf{K_T}$	Capacity investment level with technology T
Stage 2	$\mathbf{Q_T}$	Production quantity with technology T

Table 1: Decision variables by stage

4.1 Stage 2: Production Decision

In this stage, the firm observes the demand realization $\tilde{\boldsymbol{\xi}}$ and determines the production quantities $\mathbf{Q_T}' = (q_T^1, q_T^2)$ within the existing capacity limits to maximize the stage 2 equity value.

Proposition 1 *The optimal production quantity vector in stage 2 with technology $T \in \{D, F\}$ for given $\mathbf{K_T}$ and $\tilde{\boldsymbol{\xi}}$ is given by*

$$\mathbf{Q_D^*} = \mathbf{K_D}, \quad \mathbf{Q_F^*} = \frac{K_F}{\tilde{\xi}_1^{-b} + \tilde{\xi}_2^{-b}} \; \tilde{\boldsymbol{\xi}}^{-b}.$$

Since the unit production cost is zero, the firm optimally utilizes the entire available capacity. With dedicated technology, the optimal individual production quantities are equal to the available capacity levels for each product. With flexible technology, the firm allocates the available capacity K_F between each product in such a way that the marginal profits for each product are equal.

4.2 Stage 1: Capacity Choice and External Financing

In this stage, the firm exercises the forward contract H_T (if the firm has already decided to engage in financial risk management at stage 0) and observes the asset price $\tilde{\alpha}_1$. With fair pricing, the strike price of the forward is equal to $\overline{\alpha}_1$. The stage 1 budgets with and without financial risk management are therefore $B_{FRM}(\tilde{\alpha}_1, H_T) \doteq \omega_0^{FRM} + \tilde{\alpha}_1(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T$ and $B_{-FRM}(\tilde{\alpha}_1) \doteq \omega_0 + \tilde{\alpha}_1\omega_1$, respectively. We henceforth suppress $\tilde{\alpha}_1$ and H_T and denote the available budget realization by $\tilde{B} \in [0, \infty)$. For given $\tilde{B}$ and T, the firm determines the optimal capacity investment level $\mathbf{K_T^*}(\tilde{B})$ and the optimal external borrowing level $e_T^*(\tilde{B})$.

Proposition 2 *The optimal capacity investment vector $\mathbf{K_T^*}(\tilde{B})$ and the optimal external borrowing level $e_T^*(\tilde{B})$ for technology $T \in \{D, F\}$ with a given budget level $\tilde{B}$ are*

$$\mathbf{K_T^*}(\tilde{B}) = \begin{cases} \mathbf{K_T^0} & \text{if } \tilde{B} \in \Omega_T^0 \doteq \{\tilde{B} : \tilde{B} \geq c_T \mathbf{1}'\mathbf{K_T^0} + F_T\} \\ \overline{\mathbf{K}}_\mathbf{T} & \text{if } \tilde{B} \in \Omega_T^1 \doteq \{\tilde{B} : c_T \mathbf{1}'\mathbf{K_T^1} + F_T \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^0} + F_T\} \\ \mathbf{K_T^1} & \text{if } \tilde{B} \in \Omega_T^2 \doteq \{\tilde{B} : \tilde{B} \geq \widehat{B_T},\ c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E_T \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^1} + F_T\} \\ \overline{\overline{\mathbf{K}}}_\mathbf{T} & \text{if } \tilde{B} \in \Omega_T^3 \doteq \{\tilde{B} : \widehat{B_T} \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E_T\} \\ \mathbf{0} & \text{if } \tilde{B} \in \Omega_T^4 \doteq \{\tilde{B} : 0 \leq \tilde{B} < \widehat{B_T}\} \end{cases} \tag{1}$$

$$e_T^*(\tilde{B}) = \left(c_T \mathbf{1}'\mathbf{K_T^*}(\tilde{B}) + F_T - \tilde{B}\right)^+ \chi\left(\tilde{B} > \widehat{B}_T\right). \tag{2}$$

Here, $\chi(.)$ is the indicator function and $\widehat{B_T}$ is the unique budget threshold for technology $T \in \{F, D\}$ such that the firm optimally does not borrow ($e_T^*(\tilde{B}) = 0$) and does not invest in capacity ($\mathbf{K_T^*}(\tilde{B}) = \mathbf{0}$) for $\tilde{B} \leq \widehat{B_T}$.

The explicit expressions for the capacity vectors in the proposition are given in (28) in the proof. $\mathbf{K_T^0}$ is the optimal capacity investment in the absence of a budget constraint (the "budget-unconstrained optimal capacity"). If the budget realization is high enough to cover the corresponding cost $F_T + c_T \mathbf{1}'\mathbf{K_T^0}$ ($\tilde{B} \in \Omega_T^0$), then $\mathbf{K_T^*}(\tilde{B}) = \mathbf{K_T^0}$ with no borrowing. Otherwise, for each budget level $\tilde{B} \in \Omega_T^{1234}$, the firm determines to borrow or not by comparing the marginal revenue from investing in an additional unit of capacity over its available budget with the marginal cost of that investment including the external financing cost, $(1+a)c_T$. For $\tilde{B} \in \Omega_T^1$, the budget is insufficient to cover $\mathbf{K_T^0}$, and the marginal revenue of capacity is lower than its marginal cost. Therefore, the firm optimally does not borrow, and only purchases the capacity level $\overline{\mathbf{K}}_\mathbf{T}$ that fully utilizes its budget $\tilde{B}$. For $\tilde{B} \in \Omega_T^{23}$, the marginal revenue of capacity is higher than its marginal cost $(1 + a)c_T$. Therefore, the firm optimally borrows from external markets to invest in capacity. $\mathbf{K_T^1}$ is the optimal capacity investment with borrowing, in the absence of a credit limit (the "credit-unconstrained optimal capacity"). If the budget realization and the credit limit can jointly cover its cost, $\mathbf{K_T^1}$ is the optimal capacity investment; otherwise, the firm purchases the capacity level $\overline{\overline{\mathbf{K}}}_\mathbf{T}$ that fully utilizes its budget and its credit limit. For $\tilde{B} \in \Omega_T^4$, the firm must borrow to be able to invest in technology, but the total cost of the capacity that can be purchased with the remaining $\tilde{B} + e_T - F_T$ cannot be covered by the expected revenue it generates for any e_T. Therefore, the firm optimally does not borrow and does not invest in capacity. Appendix B characterizes $\widehat{B}_T$ and provides a closed-form expression for a subset of parameter values.

The optimal external borrowing level $e_T^*(\tilde{B})$ is such that the firm borrows exactly what it needs to cover its capacity investment. Since production is costless, the firm does not incur any further costs beyond this stage. Moreover, since the face value of the debt is always deducted from the firm's assets, the firm cannot transfer wealth from the creditor to shareholders by borrowing more money than what is needed for its capacity investment. Therefore, the firm only borrows for funding the capacity investment, which yields (2).

The optimal expected (stage 1) equity value of the firm with a given budget level $\tilde{B}$, $\pi_T(\tilde{B})$, is obtained in closed form (Equation 34 in Appendix A).

Corollary 1 $\pi_T(\tilde{B})$ *strictly increases in* $\tilde{B}$ *for* $\tilde{B} \geq 0$, *and is concave in* $\tilde{B}$ *on* $[\widehat{B_T}, \infty)$. *It is not concave in* $\tilde{B}$ *on* $[0, \infty)$.

As we will see in 4.3.1, this structure has implications for the optimal financial risk management level.

4.3 Stage 0: Financial Risk Management Level and Technology Choice

In this stage, the firm decides on the technology choice $T \in \{D, F\}$, whether to engage in financial risk management (FRM) and if so, the financial risk management level H_T, the number of forward contracts written on the stage 1 asset price α_1. The optimal expected (stage 0) equity value $\Pi^*(\mathbf{W})$ as a function of the internal (stage 1) endowment $\mathbf{W}' = (\omega_0, \omega_1)$ is

$$\Pi^*(\mathbf{W}) = \max\left\{\Lambda^{-FRM}, \Lambda^{FRM}, \omega_0 + \overline{\alpha}_1\omega_1 + P\right\}. \tag{3}$$

Here, Λ_{FRM} and Λ_{-FRM} denote the expected (stage 0) equity value of the better technology with and without financial risk management (FRM), respectively, where Λ_{FRM} is calculated at the optimal risk management level H_T^*. In (3), the firm compares these equity values with $\omega_0 + \overline{\alpha}_1\omega_1 + P$, the expected (stage 0) equity value of not investing in any technology. §4.3.1 derives H_T^*, §4.3.2 characterizes the optimal technology choice with and without FRM, and §4.3.3 characterizes the solution to (3). This characterization is valid for any continuous α_1 and $\boldsymbol{\xi}$ distribution with positive support and bounded expectation.

4.3.1 Financial Risk Management

The expected direct gain from the financial contract is 0 due to the fair pricing assumption. At the same time, financial risk management affects the distribution of the stage 1 budget $B_{FRM}(\alpha_1, H_T)$, which is used to finance the firm's capacity investment after paying for the fixed cost commitment. In choosing H_T, the goal of the firm is to engineer its budget to maximize the expected gain from the technology commitment made in stage 0. When $H_T > 0$ ($H_T < 0$), the firm decreases (increases) its exposure to the asset price risk α_1. Following Hull (2000, p.12), we refer to the first case as *financial hedging*, and to the second as *financial speculation*. We call $H_T = \omega_1^{FRM}$ *full hedging* because it isolates the budget from the underlying risk exposure. We call $H_T = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$ *full speculation* because it maximizes the firm's asset risk exposure within the feasible range of forward contracts. Proposition 3 characterizes H_T^*.

Proposition 3 *There exists a unique technology fixed cost threshold $\overline{F}_T$ such that*

(i) If $F_T \leq \overline{F}_T$, then the firm fully hedges ($H_T^* = \omega_1^{FRM}$).

(ii) If $F_T > \overline{F}_T$ then

1. *if $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \leq \widehat{B}_T$, then full speculation is optimal ($H_T^* = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$);*
2. *if $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} > \widehat{B}_T$, $H_T^* \in \left\{\left\{H_T < \frac{\widehat{B}_T - \omega_0^{FRM}}{\overline{\alpha}_1}\right\} \cup \left\{\omega_1^{FRM}\right\}\right\}$ and is distribution dependent.*

The structure of π_T is key to these results. If π_T is a concave function of the available budget $\tilde{B}$ on $[0, \infty)$, then full hedging is optimal. This follows by Jensen's inequality: For concave π_T, $\mathbb{E}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right] \leq \pi_T(E[B_{FRM}(\alpha_1, H_T]) = \pi_T(\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM})$, the equity value under full hedging. However, π_T is not concave if $\Omega_T^4 \neq \emptyset$, i.e. if there is a budget range in which the firm would not invest in capacity in stage 1 despite having made the technology investment in stage 0. This happens when the fixed cost of the technology investment is too high to leave sufficient funds for a profitable capacity investment.

Below the fixed cost threshold $\overline{F}_T$, $\Omega_T^4 = \emptyset$, π_T is concave, and full hedging is optimal. Above this threshold, H_T^* depends on the expected value of the internal (stage 1) endowment $\omega_0^{FRM} + \overline{\alpha}\omega_1^{FRM}$, which is also the budget available to the firm under full hedging. When this value is lower than $\widehat{B_T}$, the firm would optimally not invest in capacity if it were to fully hedge. Instead, the firm optimally chooses to increase its exposure as much as possible so as to maximize the probability of realizing high-budget states in which it is able to invest in capacity and generate revenue from its technology investment. (This also increases the probability of realizing low-budget states, but the outcome in those states does not change - no capacity investment is optimal.) For $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} > \widehat{B}_T$, the optimal risk management level is distribution dependent and a full characterization is not possible without making further assumptions.

4.3.2 Technology Choice

We now turn to the technology selection problem with and without financial risk management. The choice T^* between flexible versus dedicated technology is determined by a unit cost threshold that makes the firms indifferent between the two technologies.

Proposition 4 *For given technology cost parameters (F_T, γ_T) and financing cost scheme (a, E), and under the financial risk management level H_T^* for each technology, there exists a unique variable cost threshold $\overline{c}_F(c_D, \mathbf{H}^*)$ such that when $c_F < \overline{c}_F(c_D, \mathbf{H}^*)$ it is more profitable to invest in flexible technology ($T^* = F$). Without financial risk management, there is a parallel threshold $\overline{c}_F(c_D, \mathbf{0})$. These thresholds increase in c_D, F_D, γ_F and demand variability (σ), and they decrease in F_F, γ_D and the demand correlation $(\rho)^2$. With symmetric fixed costs and salvage rates,*

$$\overline{c}_F(c_D, \mathbf{H}^*) = \overline{c}_F(c_D, \mathbf{0}) = \overline{c}_F^S(c_D) = c_D \left(\frac{\mathbb{E}^{-b}\left[\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}\right]}{\mathbb{E}^{-b}[\xi_1] + \mathbb{E}^{-b}[\xi_2]} \right)^{-\frac{1}{b+1}} \geq c_D, \tag{4}$$

where the equality only holds if the product markets are deterministic ($\sigma = 0$), or the product markets are perfectly positively correlated ($\rho = 1$) and $\boldsymbol{\xi}$ has a proportional bivariate distribution.

The comparative statics results developed here are used in §6 to analyze the drivers of the firm's optimal risk management portfolio. The threshold $\overline{c}_F^S(c_D)$ is independent of unit financing cost a, credit limit E, and engaging in financial risk management. Although these factors do have an effect on the equity value of each technology, the differential value of this effect is never sufficient to induce the firm to alter its technology decision. This threshold is independent of α_1 and valid for any distribution of $\boldsymbol{\xi}$. The threshold $\overline{c}_F^S(c_D)$ is a variant of the mix flexibility threshold in Chod et al. (2006a), and has the same structure. It is interesting to note that the same threshold structure is valid despite the existence of external financing costs and financial risk management policy in the symmetric cost case.

Due to the risk pooling benefit of flexible technology, we have $\overline{c}_F^S(c_D) \geq c_D$. Proposition 4 shows that there is no risk pooling benefit ($\overline{c}_F^S(c_D) = c_D$) only if the product market demand is deterministic, or the multiplicative demand uncertainty is perfectly positively correlated and it has a proportional bivariate distribution ($\rho = 1$, $\sigma_1 = k\sigma_2$ and $\overline{\xi}_1 = k\overline{\xi}_2$ for $k > 0$). Flexible technology can have risk pooling value even if the product markets are perfectly positively correlated. This observation is in the spirit of Proposition 6 in Van Mieghem (1998), which is based on the price-differential of two products in a price-taking newsvendor setting. In our case, the value comes from the fact that for non-proportional bivariate distributions, the optimal production quantities with the flexible technology in stage 2 are state dependent such that there is still value from production switching at different $\boldsymbol{\xi}$ realizations.

4.3.3 Optimal Portfolio Choice

The cost thresholds developed in Proposition 4 reveal which technology is more profitable with and without financial risk management, but we need several more elements to fully characterize the solution to (3). Four more cost thresholds achieve this purpose. These thresholds are summarized in Table 2 and derived in the Appendix A.

The "algorithm" to solve (3) is as follows: We use the variable cost thresholds derived in Proposition 4 to determine the optimal technologies yielding Λ_{FRM} and Λ_{-FRM}. Using the fixed technology cost thresholds $\underline{F}_T^{-FRM}$ and $\underline{F}_T^{FRM}$, if we determine that not investing in any technology dominates either exactly one or both of Λ_{FRM} and Λ_{-FRM}, (3) is solved. Otherwise, we need to compare Λ_{FRM} and Λ_{-FRM}. If the same technology is optimal in both cases, then the fixed financial risk management cost threshold $\underline{F}_{FRM}^T$ is used to determine whether FRM or no FRM is optimal with that technology and (3) is solved. If different technologies are optimal with and without FRM, then $\overline{c}_T(c_{-T}, H_T^*, 0)$ is used to determine the optimal solution. This completes the characterization of the optimal portfolio. The next three sections highlight and discuss a series of

Threshold	Usage
$\overline{c}_F(c_D, \mathbf{0})$	Comparison between technologies without engaging in FRM
$\overline{c}_F(c_D, \mathbf{H}^*)$	Comparison between technologies with optimal FRM
$\overline{c}_F^S(c_D)$	Comparison between technologies with symmetric F_T and γ_T
$\underline{F}_T^{-FRM}$	Comparison between investing in T without FRM and not investing in any technology
$\underline{F}_T^{FRM}$	Comparison between investing in T with FRM and not investing in any technology
$\underline{F}_{FRM}^T$	Comparison between FRM and no FRM with technology T
$\overline{c}_T(c_{-T}, H_T^*, 0)$	Comparison between technology T with FRM and the other technology $(-T)$ without FRM

Table 2: Thresholds used in solving for the firm's optimal strategy. The first three were derived in Proposition 4 and the last four are derived in Propositions 11, 12 and 13 in the Appendix.

insights that can be obtained from this analysis.

5 Observations Concerning the Optimal Risk Management Portfolio

In this section, we make several observations about the structure of the optimal risk management portfolio and its managerial implications. We start with an observation that illustrates the limits of the value of each risk management strategy.

Corollary 2 *If capital markets are perfect, $\overline{F}_{FRM}^F = \overline{F}_{FRM}^D = 0$: financial risk management has no value. If product markets are perfect, and absent a fixed cost or salvage value advantage, $\overline{c}_F(c_D, \mathbf{H}^*) = \overline{c}_F(c_D, \mathbf{0}) = c_D$: flexible technology has no value.*

Without capital market imperfections, the firm is not exposed to deadweight costs of external financing, as discussed in Assumption 6. In this case, financial risk management does not have any value. This is consistent with the decoupling of operational and financial decisions in perfect capital markets (Modigliani and Miller 1958). If there is no demand uncertainty ($\mathbf{\Sigma} = \mathbf{0}$), the product markets are perfect, and the firm is not exposed to supply-demand mismatch costs. Absent a fixed cost or salvage value advantage, flexible technology does not have any value. Observation 2 confirms our intuition about the risk management role of each strategy in counterbalancing the effects of costs that originate from product and capital market imperfections.

Corollary 3 *The firm can optimally speculate with forward contracts. Flexible technology can trigger speculative behavior.*

While firms frequently use financial derivatives for hedging purposes, Bodnar et al. (1998) document that some firms take speculative positions with financial derivatives. Froot et al. (1993)

show that speculation may indeed be optimal when there is an external financing cost and the return on the operational investments and the risk variable are statistically correlated. They also conclude that in the absence of such correlation, the firm optimally fully hedges. In Proposition 3, we prove that the full-hedging conclusion need not hold if there are fixed costs of technology investment: Firms with limited expected internal endowment may optimally speculate to be able to invest in capacity. The majority of empirical papers assume that firms use financial derivatives for hedging purposes (Geczy et al. 1997). Observation 3 illustrates that such an assumption can be problematic in industries with fixed cost requirements.

It is interesting to note that speculation can be triggered by investment in flexible technology. The higher investment cost of flexible technology induces the firm to speculate while it uses forward contracts for hedging purposes with dedicated technology. This illustrates the intertwined nature of the integrated risk management portfolio. Engaging in operational risk management (flexible technology) may have a structural effect (going from hedging to speculation) on financial risk management.

Firms may limit their usage of financial risk management to hedging only, since speculation is typically not viewed as a desired strategy. Non-speculative use of financial risk management imposes a hedging constraint on the feasible set of forwards by imposing $H_T \geq 0$, which yields the following outcome:

Proposition 5 *If the firm uses forward contracts for hedging purposes only, then the firm optimally may not engage in financial risk management even if it is costless* ($F_{FRM} = 0$).

The intuition of this result is similar to the full speculation case above, obtained in the case of low expected internal endowment value. The firm is better off by leaving the exposure to asset price as high as possible (this corresponds to $H_T^* = 0$) to be able to invest in capacity. Empirical studies unanimously demonstrate more widespread usage of financial risk management among large firms, and this observation is attributed to the fixed costs of establishing a financial risk management program (Allayannis and Weston 1999). Proposition 5 proposes another possible explanation: the no-speculation constraint on financial derivative usage. With this constraint, small firms (that have low internal endowments) do not engage in financial risk management.

In a recent empirical study, Guay and Kothari (2003) find no significant usage of financial risk management among non-financial firms, and suggest that these firms may be using operational hedges instead to manage their risks. We observe that indeed, firms can rely only on operational hedges in an integrated risk management framework.

Corollary 4 *Any risk management portfolio can be optimal. Financial risk management is not a panacea. Firms can rely only on flexible technology for risk management purposes.*

If financial risk management was costless, it would always be in the optimal risk management portfolio. Our analysis finds two reasons why firms may not use financial risk management: i) Its fixed cost is high. Since non-financial firms do not have as much expertise as financial firms in financial risk management, its effective fixed cost could be higher for them, which provides support for the observed difference in usage. ii) The firm limits itself to only hedging even if it is costless. Thus, not only the investment cost of financial risk management, but also the interplay between financial and operational decisions is important in determining the optimal risk management portfolio. The firm should evaluate financial risk management as an integral part of the firm's overall investment strategy. The next section provides guidelines about optimal portfolio selection.

6 Characteristics of the Optimal Risk Management Portfolio

In this section, we delineate the main drivers of the optimal risk management portfolio and analyze the interplay between financial and operational risk management. In §6.1, we relate the optimal risk management portfolio to firm, industry, technology, product market (demand variability and correlation) and capital market (external financing frictions) characteristics. We then analyze the interaction between operational and financial risk management strategies in §6.2. For this analysis, we proxy the firm size using the level of internal (stage 1) endowment. In particular:

Definition 1 *The firm is defined to be small (large) if the firm borrows (does not borrow) from external markets with flexible technology and full hedging,* $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} \in \Omega_F^2 (\Omega_F^0)$.

The finance literature qualitatively refers to small and large firms according to the degree to which they are affected by external financing frictions. This definition formalizes this concept in the context of our model. We parameterize the internal (stage 1) endowment as $(\lambda\omega_0, \lambda\omega_1)$ and the fixed technology costs as $F_D = F$, $F_F = F + \delta$ with $\delta \geq 0$. For tractability, we impose some parameter restrictions.

Assumption 8 *Let* $\beta = \frac{\omega_0}{\omega_0 + \alpha_0 \omega_1}$, $\gamma_T = 0$, $E \geq \frac{c_T \mathbf{1}' \mathbf{K}_\mathbf{T}^\mathbf{1} (1-ab)}{-(b+1)a}$, $F_T \leq \overline{F}_T = \frac{c_T \mathbf{K}_\mathbf{T}^\mathbf{1} (1+a)}{-(b+1)a}$, *and* $F_T < \underline{F}_T^{-FRM}$.

These assumptions ensure the following: $F_{FRM} \leq \omega_0 + \alpha_0 \omega_1$, so that financial risk management is feasible, and undertaking financial risk management or not can be optimal. If the firm engages in financial risk management, it optimally fully hedges; this rules out cases where the optimal financial risk management level cannot be uniquely characterized. The firm is not constrained by the credit limit, so the effective financing friction is the unit financing cost a. Finally, the optimality of not investing in either technology is ruled out.

6.1 Comparative Statics Results

We define Δ_T as the value of financial risk management (FRM) with technology T:

$$\Delta_T \doteq \mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, \omega_1^{FRM})\right)\right] - \mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right]. \tag{5}$$

To investigate the main drivers of the optimal portfolio choice, we carry out comparative statics analysis on the variable cost thresholds $\bar{c}_F(c_D, \mathbf{H}^*)$ and $\bar{c}_F(c_D, \mathbf{0})$, and on Δ_T. The results below hold locally such that Assumption 8 and the defining regions for small and large firms are not violated.

Proposition 6 *(Technology Choice) With symmetric fixed technology costs* $(F_F = F_D)$, $\bar{c}_F(c_D, \mathbf{H}^*)$ *and* $\bar{c}_F(c_D, \mathbf{0})$ *are invariant to the unit financing cost* (a)*, the fixed costs of both technologies* (F) *and the internal endowment* (λ) *of the firm. With asymmetric fixed costs* $(F_F > F_D)$, $\bar{c}_F(c_D, \mathbf{H}^*)$ *and* $\bar{c}_F(c_D, \mathbf{0})$ *decrease in the fixed costs of both technologies and the unit financing cost, and increase in the internal (stage 1) endowment of the firm.*

With symmetric fixed costs, the technology ordering is independent of financing cost, fixed costs and internal (stage 1) endowment. With asymmetric fixed costs, since flexible technology has a higher investment cost, any increase in costs (fixed cost, financing cost) favors the dedicated technology; a decrease in costs (such as an increase in the internal (stage 1) endowment), favors the flexible technology.

Proposition 7 *(Value of FRM) The value of FRM increases in the external financing cost* (a) *for large firms. For small firms, the value of full hedging increases (decreases) in the external financing cost at low (high) levels of* F_{FRM}*. For large (small) firms, the value of FRM increases (decreases) in the fixed cost of technology* (F) *and the demand variability* (σ)*, and decreases (increases) in the internal (stage 1) endowment* (λ) *and the demand correlation* (ρ)*.*

We now explain the drivers of Proposition 7 by grouping the results that have similar intuition. Since with Assumption 8, the firm optimally fully hedges with financial risk management, we refer to the firm engaging (not engaging) in financial risk management as the hedged (unhedged) firm.

The effect of external financing cost. Financial risk management is valuable since it reduces risk exposure and hence the expected borrowing level. At the same time, it is costly, and there is an opportunity cost for engaging in FRM: the firm may even need to borrow additional funds to finance its operational investments. These two drivers combine to determine how an increase in financing cost impacts the financial risk management decision of the firm. For large firms, the hedged firm – by Definition 1– does not borrow at all, while the unhedged firm is adversely affected from increasing financing costs. Therefore, the value of financial risk management increases in the financing cost. For small firms, this trade-off depends on the fixed cost of financial risk management. For low fixed costs, the value of financial risk management increases in financing costs; at high fixed costs, the opposite occurs.

The effect of fixed technology cost and internal (stage 1) endowment. The proof of the proposition reveals that there is one fundamental driver that explains both comparative statics results: the level of reliance on external financing, as summarized in Table 3. A firm's reliance on external financing increases as the fixed investment cost F increases and the internal (stage 1) endowment level λ decreases. By Definition 1, the large hedged firm does not need to borrow and the large unhedged firm borrows in some budget realizations. Therefore, increasing the reliance on external financing adversely affects the unhedged firm while not affecting the hedged firm. We conclude that for large firms, the value of FRM increases as the need for external financing increases. Since the small hedged firm, by Definition 1, always borrows and the small unhedged firm only borrows in some budget realizations, increasing the reliance on external financing adversely affects the unhedged firm, but it affects the hedged firm even more. We conclude that for small firms, the value of FRM decreases as the need for external financing increases.

Case	Borrowing level	Increasing reliance on external financing
Large unhedged firm	Borrows in some states	Increases the value of FRM since the unhedged firm borrows more in expectation
Large hedged firm	Does not borrow	
Small unhedged firm	Borrows in some states	Decreases the value of FRM since the hedged firm borrows more in expectation
Small hedged firm	Borrows in all states	

Table 3: Increasing the reliance on external financing has the opposite effect on the value of financial risk management for large and small firms. A firm's reliance on external financing increases as the fixed investment cost F increases, and it decreases as the internal (stage 1) endowment level λ increases.

The effect of demand correlation and demand variability. These two factors have an effect on the firm only with flexible technology. The proof of the proposition reveals that there is one fundamental driver that explains these two comparative statics results: the marginal change in the optimal investment level with changes in these factors, as summarized in Table 4. A firm's optimal investment level decreases as the demand variability decreases or the demand correlation increases. The small unhedged firm borrows only in some budget realizations, while the small fully hedged firm always borrows. As a result, the small hedged firm employs a more conservative investment policy (the capacity investment level is lower at each state) than the unhedged firm since its exposure to external financing costs is higher. Consequently, a similar change in variability or correlation alters the small hedged firm's optimal investment policy to a lower extent than the unhedged firm's; its optimal investment level is more robust to changes in these factors. Therefore, while a reduction in the optimal investment level (due to a decrease in variability or an increase in correlation) adversely affects the small hedged firm, it affects the small unhedged firm even more. We conclude that for small firms, the value of FRM increases as the optimal investment level decreases. For large firms, the opposite result holds. This follows from parallel arguments based on the fact that the large unhedged firm needs to borrow in some budget realizations, while the large hedged firm does not.

<table>
<tr><th>Case</th><th>Borrowing level</th><th>Reduction in the optimal investment level at each budget state</th></tr>
<tr><td>Large unhedged firm</td><td>Borrows in some states</td><td rowspan="2">Decreases the value of FRM since the hedged firm's optimal investment is less conservative and less robust</td></tr>
<tr><td>Large hedged firm</td><td>Does not borrow</td></tr>
<tr><td>Small unhedged firm</td><td>Borrows in some states</td><td rowspan="2">Increases the value of FRM since the hedged firm's optimal investment is more conservative and more robust</td></tr>
<tr><td>Small hedged firm</td><td>Borrows in all states</td></tr>
</table>

Table 4: A reduction in the optimal investment level at each state has the opposite effect on the value of financial risk management for large and small firms. A firm's optimal investment level decreases as the demand variability decreases or the demand correlation increases.

Synthesis. Table 5 summarizes the main drivers of each optimal portfolio choice for large and small firms by combining Propositions 4, 6 and 7 for technologies with asymmetric fixed cost ($F_F > F_D$). By definition, if the variable cost thresholds increase in a parameter, flexible technology is preferred under a larger set of conditions as that parameter increases, and we say that "flexible technology is favored." Similarly, if Δ_T increases in a parameter, we say "financial risk management is favored." While not exact, this usage captures the direction of change. For example, high demand variability and low demand correlation favor investing in flexible technology and undertaking financial risk management for large firms. This is how Table 5 is constructed. We note that the capital intensity of an industry can be captured by keeping the internal endowment level constant and altering the fixed technology costs. With a given internal endowment level, a sufficiently high (low) fixed cost implies a small (large) firm according to our definition. Therefore, our results about small and large firms can be interpreted as being relevant for capital intensive and non-capital intensive industries, respectively.

The main message of Table 5 is that the size of the firm is key to optimal portfolio choice. As explained earlier, the three fundamental drivers behind the optimal portfolio choice (opportunity cost of financial risk management, level of reliance on external financing, and robustness of the optimal capacity investment level to variability and correlation) work in opposite directions for small and large firms. Therefore, different size firms may choose the same optimal portfolio for entirely different reasons.

Portfolio Choice	Large Firms	Small Firms
F with FRM	High demand variability Low demand correlation	High internal endowment Low technology fixed costs Low financing costs with low F_{FRM}
D with FRM	Low internal endowment High technology fixed costs High financing costs	Low demand variability High demand correlation High financing costs with low F_{FRM}
F without FRM	High internal endowment Low technology fixed costs Low financing costs	High demand variability Low demand correlation Low financing cost with high F_{FRM}
D without FRM	Low demand variability High demand correlation	Low internal endowment High technology fixed costs High financing costs with high F_{FRM}

Table 5: Main Drivers of the Optimal Risk Management Portfolio with Asymmetric Fixed Technology Costs.

Table 5 is for asymmetric fixed technology costs. With symmetric fixed costs, it follows from Proposition 4 that the technology ordering is independent of changes in any parameter. Therefore, changes in parameter levels only affect the choice between undertaking FRM or not. Consequently, all the conditions in Table 5 that favor flexible or dedicated technology with FRM and without FRM for a given firm size favor using FRM and not using FRM, respectively. We conclude that the technology cost characteristic is also key to the optimal portfolio structure.

We now relate our theoretical findings to the associated empirical literature. The financial risk management literature relates the value of financial risk management to underlying exposure, growth opportunities and size of firms (Allayannis and Weston 1999). Our results demonstrate that the value of financial risk management also depends on the product market and technology characteristics, and that there are subtle differences between large and small firms.

Gay and Nam (1998) say that firms with higher investment opportunities that are exposed to higher external financing frictions and lower levels of cash make greater use of financial derivatives. We show (in the proof of Proposition 7) that the effect of cash ω_0 is the same as the effect of internal (stage 1) endowment: A lower internal (stage 1) endowment increases the value of hedging for small firms, but not for large firms. Therefore, our results support their argument for small firms, but not for large firms.

The financial risk management literature hypothesizes that the value of financial risk management increases as financing frictions increase by invoking the counterbalancing effect of financial risk management with respect to external financing frictions (Mello and Parsons 2000). Our results support this argument for large firms, but not for small firms. The key is how much the firm needs to borrow after undertaking financial risk management.

6.2 The Interaction of Operational and Financial Risk Management

We first investigate whether flexible technology and financial risk management are substitutes or complements in an integrated risk management framework. They are defined to be substitutes if the firm invests in flexible technology when the firm is not allowed to use financial risk management and switches to dedicated technology when the firm engages in financial risk management; they are called complements if the switch is from dedicated to flexible technology.

Proposition 8 *Flexible technology and financial risk management can be complements or substitutes. Small (large) firms tend to substitute (complement) flexible technology with financial risk management.*

The main driver of Proposition 8 is the value of financial risk management with each technology. Flexible technology is more expensive, so it is more exposed to external financing costs. The use of financial risk management allows large firms to secure a budget level sufficient to eliminate borrowing. Thus, large firms complement flexible technology with financial risk management in their integrated risk management portfolio. Small firms need to borrow to invest in flexible technology, even using financial risk management, but may not need to borrow for dedicated technology if they use financial risk management. In other words, the value of financial risk management is higher with dedicated technology. This explains why flexible technology and financial risk management are substitutes for small firms.

Interestingly, the empirical literature also finds mixed results on this question, albeit in other contexts. Geczy et al. (2000) document complementarity between operational (physical storage) and financial means of risk management among natural gas pipeline firms. In a multinational context, Allayannis et al. (2001) find that financial and operational (geographical diversification) risk management tools are substitutes. In a different framework, Chod et al. (2006b) provide another theoretical justification for these mixed empirical results by focusing on the effect of financial risk management on the optimal flexibility level of the firm. They demonstrate that financial risk management is a complement (substitute) to operational flexibility when the optimal flexibility level increases (decreases) with financial hedging.

We next analyze whether the value of operational risk management (defined as the expected (stage 0) equity value difference between flexible and dedicated technologies) is more or less robust to changes in product and capital market conditions when financial risk management is undertaken. Robust strategies are preferable because they perform well under a wider range of parameters, and can be implemented with more confidence.

Proposition 9 *For large (small) firms, the value of operational risk management is less (more) robust to changes in product market conditions* (ρ, σ) *and more (less) robust to changes in capital market conditions* (a) *with financial risk management than without.*

The proof of the proposition reveals that the robustness with respect to product market conditions is linked to the value of FRM with flexible technology. The value of operational risk management is more or less robust with respect to correlation if the value of FRM decreases or decreases in correlation, respectively. This is valid for small and large firms, respectively, as we discussed in §6.1. Robustness with respect to variability follows from a similar argument. Robustness with respect to the unit financing cost is determined by the difference between the value of FRM with flexible and dedicated technologies: The value of operational risk management is more robust to changes in a if the value of FRM with flexible technology increases more rapidly than the value of FRM with dedicated technology in response to an increase in a.

Proposition 9 again illustrates the intertwined nature of operational and financial risk management strategies: Engaging in financial risk management has the opposite impact on the robustness of the value operational risk management with respect to product and capital market conditions.

7 Value and Effect of Integrated Decision Making

Sections 5 and 6 analyzed the properties of the optimal integrated risk management portfolio and its drivers. In practice, firms may not take an integrated approach to these decisions; operational and financial risk management decisions may be taken independently. In this section, we focus on the value and effect of integrated decision making. We relax the restrictions of Assumption 8 and focus on general parameter settings.

If we ignore its effects on operational decisions, financial risk management does not have any value because forward contracts are investments with zero expected return. For this reason, we take no FRM as the non-integrated benchmark. Since the non-integrated benchmark is no FRM, the results of this section can also be interpreted as the effect of engaging in FRM on the firm's performance and optimal decisions. The effect of FRM on the optimal expected capacity investment and external borrowing level is ambiguous:

Proposition 10 *Engaging in financial risk management can increase or decrease the optimal expected capacity investment and the optimal expected borrowing levels.*

Since financing frictions negatively impact the stage 1 capacity investment level at each budget state, and the firm uses FRM to counterbalance the effect of financing frictions, one may expect that with FRM, the firm's expected borrowing level would be lower and the expected capacity investment level would be higher than without. On the other hand, if there is cost associated with engaging in financial risk management ($F_{FRM} > 0$), the firm has less internal endowment to invest in capacity at each budget state, and has to borrow additionally to compensate for F_{FRM}. In the proof of Proposition 10, we illustrate that even if FRM is costless, the optimal expected capacity investment can decrease and the expected borrowing level can decrease. This is a direct consequence of the joint optimization in external borrowing and capacity levels. The fundamental driver of this result is the marginal profit of the capacity investment in the joint optimization problem as we discussed in §4.2.

Proposition 10 shows the dependence of capacity investment on financial risk management. We now analyze the effect of engaging in financial risk management on the technology choice:

Corollary 5 *The firm may make different technology decisions with and without financial risk management.*

In their numerical analysis, Ding et al. (2005) demonstrate that financial risk management can alter more strategic operational decisions (global supply chain structure) than the capacity investment levels. Observation 5 is in line with their conclusion. We analytically prove that the technology choice of the firm may be altered by engaging in FRM. The direction of change in technology choice is determined by the value of FRM with each technology. Proposition 8 is an example for such changes and provides the intuition with some restrictions on the parameter levels.

The analysis above illustrates the effect of integrating risk management decisions on the firm's decisions. We now analyze the value of such integration as a function of firm size. To separate the value of integration from the cost of FRM, we use $F_{FRM} = 0$. Here, our definition of a large firm is the same as Definition 1, but our definition of a small firm is slightly more restrictive. We refer to firms with very limited expected internal endowment value that optimally fully speculate with FRM as small firms. Since under the conditions of Assumption 8, these firms fully hedge with FRM, the new definition is consistent with Definition 1 and corresponds to a subset of small firms in §6 that have a significantly low expected internal endowment value.

Corollary 6 *The value of integration is low for small firms with low cash levels* (ω_0) *and large firms with high cash levels. If the firm uses financial risk management only for hedging purposes, the value of integration is higher for large firms than for small firms.*

The value of integration is equivalent to the value of engaging in FRM. Since large firms with high cash levels are not significantly exposed to external financing frictions without FRM, the value of FRM, and hence the value of integration is low. In the extreme case, a cash level sufficient to finance the budget-unconstrained optimal investment level completely removes the exposure to external financing frictions and FRM has no value. For small firms with low levels of cash, the additional benefit of full speculation ($H_T^* = \frac{\omega_0}{\bar{\alpha}_1}$) over not using FRM ($H_T^* = 0$) is low. In the extreme case, if the small firm does not have any cash ($\omega_0 = 0$), then FRM has no value.

When the firm uses financial risk management only for hedging purposes, it follows from Proposition 5 that small firms optimally do not engage in FRM. In this case, integration has no value. Large firms generally fully hedge with FRM, therefore integration has value for them. In a numerical analysis not reported here, we observe a similar pattern without imposing the hedging constraint.

8 Robustness of Results to Model Assumptions

In this section, we investigate the robustness of our results to the assumptions presented in §3.

Non-identical and exogenous financing costs. We assumed a unique external financing cost structure (a, E). The firm can be exposed to a different external financing cost structure (a_T, E_T) with each technology $T \in \{D, F\}$. All the analytical results of §4 continue to hold by replacing (a, E) with (a_T, E_T) where a lower unit borrowing cost is associated with a higher credit limit. The main insights of the paper do not change except that the technology with lower a_T and higher E_T is favored in the optimal risk management portfolio.

Endogenous financing costs. In this paper, we focus on a partial equilibrium setting where the financing costs are exogenous and identical for each technology. In a general equilibrium setting, the financing cost for each technology is determined by the interaction between the firm and a creditor. In Boyabatlı and Toktay (2006), we derive the equilibrium level of secured loan commitment contracts (a_T^*, E_T^*) for each technology in a creditor-firm Stackelberg game using a similar firm model. We show that the borrowing terms will be independent of technology choice when the creditor has limited information about the firm and the technologies, there is no credible way of information transmission, and the creditor bases its assessment of default probability on the same cash flow distribution of the firm for any technology. These conditions are relevant for bank financing where banks rely on the credit history of the firm for credit risk estimation and do not have operational expertise. All of the results in this paper are valid in the general equilibrium sense under these conditions. We refer the reader to Boyabatlı and Toktay (2006) for a detailed treatment of endogenous financing costs.

Unsecured loan commitment contracts. If the firm uses unsecured loan commitment contracts

($P = 0$), the firm only receives the salvage value of the non-pledgable technology in the default states. The limited liability of the shareholders left-censors the stage 2 equity value distribution at 0. The expected (stage 1) equity value is calculated using conditional expectations with respect to default and non-default events. The probability of default depends on the capacity investment level, external borrowing level and the risk-pooling value of the technology choice. At stage 1, similar to secured lending, the firm optimally borrows so as to finance the optimal capacity investment level. In a single-product price-taking newsvendor setting, Babich et al. (2006) provide conditions under which the expected (stage 1) equity value is unimodal (though not concave) in capacity. With two products and endogenous pricing, the optimal capacity investment level is very hard to solve and becomes intractable for flexible technology because of the dependence on default regions with bivariate product market uncertainty. In our paper, the effect of limited liability is inherent in the financing cost structure (a, E). When the capital market imperfection costs are default-related (e.g. bankruptcy costs), if there were no limited liability then the creditor would be sure to recoup the face value of the loan and default-related costs from the shareholders' personal wealth. With such a riskless loan, the cost of the loan would be the risk-free rate ($a = 0$) and the firm could raise sufficient funds to finance the budget unconstrained capacity level ($E = P$).

If we allow unsecured lending in our setting, we conjecture that the optimal capacity investment level would be lower: The marginal cost of borrowing is less than $1+a$ because of the default, which should induce the firm to borrow more and invest more in capacity. Structural results related to financial risk management are expected to hold. How the technology choice would change is not clear because of the dependence on default regions. The arguments in this section are also relevant for i) partially secured lending (P is positive but not sufficient to finance the budget unconstrained capacity investment), and ii) secured lending with default-related costs deducted from the firm's seized assets by the creditor in the case of default.

Positive production cost at stage 2. Let y denote the unit production cost for both products with either technology. With $y > 0$, the optimal production vector at stage 2 is limited by the cash availability of the firm in addition to the physical capacity constraints. In this case, the literature often uses a clearing-pricing strategy for tractability that fully utilizes the physical capacity (see for example, Chod and Rudi 2005). If we assume a clearing-pricing strategy, the firm optimally borrows so as to fully utilize the physical resource in stage 2 and all the results of our paper continue to hold by replacing c_T with $c_T + y$.

If we focus on the optimal pricing policy with $y > 0$, the optimal production vector with flexible (dedicated) technology is state dependent and has a complex form that is characterized by a two-region (six-region) partitioning of the demand space (ξ_1, ξ_2) with respect to capacity constraints[3].

The optimal capacity level is lower than the $y = 0$ case, and accounts for the state-dependent optimal production vector. With flexible technology, the firm optimally borrows the exact amount required for the full utilization of the physical resource. With dedicated technology, the optimal borrowing level is such that the physical resources are never fully utilized. Financial capacity has a risk-pooling benefit with dedicated technology because the firm can allocate the financial resource to each physical capacity contingent on the demand realization. Because of this additional risk-pooling benefit of dedicated technology, flexible technology is more adversely affected from $y > 0$ compared to $y = 0$. With $y > 0$, the majority of the insights and the structural results obtained with $y = 0$ remain valid. The results concerning the product market characteristics (ρ, σ) are among the few exceptions. Similar to flexible technology, the value of dedicated technology decreases in ρ and increases in σ. This is a direct consequence of the declining risk-pooling value of the financial capacity. The optimal technology choice as a function of product market conditions is not clear in this setting.

Seizable salvage value of technology. We assume that the creditor cannot seize the salvage value of the technology in case of default. If the salvage value of the technology is offered as an additional collateral, then the creditor can seize the technology. With exogenous financing costs, seizable technology does not have any impact on the results of this paper. With endogenous financing costs and immediate liquidation of technology, collateralizing the technology reduces the default risk and hence external financing costs in equilibrium. Different salvage values of the technologies have a significant impact on the technology choice in equilibrium as we discuss in Boyabatlı and Toktay (2006).

Fixed cost of technology is incurred at stage 0. If the firm incurs the fixed cost of technology at the time of commitment (at stage 0), then this fixed cost is deducted from the firm's internal stage 1 endowment (ω_0, ω_1) in the same way as F_{FRM}. With this assumption, the firm always optimally fully hedges with financial risk management; hence Observation 3 and Proposition 5 do not hold. All the other results remain valid. The same conclusions hold in the absence of technology fixed costs ($F_F = F_D = 0$).

9 Conclusions

This paper analyzes the integrated operational and financial risk management portfolio of a firm that determines whether to use flexible or dedicated technology and whether to undertake financial risk management or not. The risk management value of flexible technology is due to its risk pooling benefit under demand uncertainty. The financial risk management motivation comes from the existence of deadweight costs of external financing. Financial risk management has a fixed cost, while technology investment incurs both fixed and variable costs. The firm's limited budget, which depends partly on a tradable asset, can be increased by borrowing from external markets, and its distribution can be altered via financial risk management.

In a parsimonious model, we solve for the optimal risk management portfolio, and the related capacity, production, financial risk management and external borrowing levels, the majority of them in closed form. We characterize the optimal risk management portfolio as a function of firm size, technology and financial risk management costs, product market (demand variability and correlation) and capital market (external financing costs) characteristics.

We find that three fundamental drivers explain the optimal portfolio choice: the robustness of the optimal capacity investment with respect to product market characteristics, the level of reliance on external financing and the opportunity cost of financial risk management. Our results provide managerial insights about the design of integrated operational and financial risk management programs. A firm that operates in highly variable or highly negatively correlated product markets should use flexible technology with financial risk management if the firm has sufficiently high internal endowment (large firm); and without financial risk management if the firm has limited internal endowment (small firm). For large firms with low (high) external financing costs, flexible technology with financial risk management (dedicated technology without financial risk management) is the best risk management portfolio. For small firms, the insights related to technology choice under high and low external financing costs continue to hold but the firm should only use financial risk management if the fixed cost of financial risk management is sufficiently low.

Our analysis clearly shows the intertwined nature of operational and financial risk management strategies and illustrates their subtle interactions. For example, operational and financial risk management can be complements or substitutes depending on the firm size. Flexible technology and financial risk management tend to be substitutes for small firms and complements for large firms. The fundamental driver of this result is the difference in the value of financial risk management with each technology. We also show that the firm's use of financial instruments for speculative reasons can be triggered by choosing the higher cost flexible technology.

Our analysis extends the modelling framework of Froot et al. (1993) by formalizing operational investments and imposing a cost for financial risk management. With our more detailed operational model, some of their findings do not continue to hold. For example, firms can optimally use financial risk management for speculative purposes even if the returns from operational investments are independent from the financially hedgable risk variable. The driver of this result is the fixed cost of technology. In addition, we show that firms may choose not to use financial risk management due to its cost when resources are limited. The effective cost of financial risk management is larger than its fixed cost because of the existence of operational investments: After incurring the fixed cost of financial risk management, the firm may need to borrow additional funds to finance its operational investments, which imposes an opportunity cost on the firm. These results enhance our understanding of the effect of operational factors in risk management and underline the importance of integrated decision making.

This paper brings constructs and assumptions motivated by the finance literature into a classical operations management problem. In turn, we provide theoretical support for some observations made in the empirical finance literature and highlight additional trade-offs in some others. For example, we establish that the value of financial risk management increases in external financing costs only for large firms and not for small firms. This is in contrast to the existing understanding that this is true for any firm. There is evidence that large firms use financial instruments more frequently than small firms. This observation is attributed to the fixed cost of establishing a financial risk management program. Our analysis proposes another explanation that is based on the hedging constraint sometimes imposed in practice: If firms are allowed to use financial instruments for hedging purposes only, it is optimal for small firms to not undertake financial risk management even if it is costless.

Our paper opens new empirical avenues. The existing literature on risk management typically does not capture operational aspects such as characteristics of different technologies and product market characteristics. As demonstrated by our analysis, these can have a significant effect on the risk management portfolio and generally have opposite effects for large and small firms. The distinction we make between large and small firms (or equivalently, between capital intensive and non-capital intensive industries), and our results related to the effect of technology and product market characteristics on the risk management portfolio provide new hypotheses that can be tested empirically. For example, we expect to see that large firms engage in financial risk management less frequently than small firms in highly positively correlated markets. We also expect to see a positive relation between fixed technology costs and the frequency of engaging in financial risk management for large firms and a negative relation for small firms.

In §8, we discussed the implication of relaxing some of our assumptions. Other interesting research directions remain. For example, this paper focuses on a monopolistic firm. In an integrated risk management framework, strategic risk management has not received much attention. Goyal and Netessine (2005) analyze the value of flexible technology under product market competition. It would be interesting to incorporate financial risk management decisions of the firm in this competitive setting. The financially hedged firm may invest in more costly flexible technology whereas the non-hedged competitor may not because of external financing frictions. Financial risk management will certainly have a non-trivial impact on the equilibrium of the game. Dong et al. (2006) take a step in this direction by modeling operational flexibility and financial risk management decisions of a global firm facing a local competitor that can only respond by setting its production quantity.

We assume an exogenous external financing cost structure. Technology characteristics can affect the external financing costs in equilibrium; this occurs if the lender has information about the firm's technology options and the ability to assess their operational and collateral value. In this case and with loan commitment contracts, the financing cost structure would depend on the firm's likelihood of borrowing and the default risk conditional on the borrowing level. Flexible technology has higher costs, and requires more external borrowing than dedicated technology; but the risk-pooling value of flexible technology decreases the default risk. The different collateral values of each technology bring another facet to this interaction. It is interesting to analyze which effect dominates under what conditions. The broader question is whether firms should use flexible versus dedicated technology in imperfect capital markets. We analyze these issues in a companion paper (Boyabatlı and Toktay 2006).

Notes

[1]With the exception of sensitivity results with respect to demand variability and correlation: These results require formalization of demand variability and correlation via specific distributional or structural (using stochastic orderings) assumptions.

[2]To capture the effect of demand correlation and variability, we use different measures that are commonly used in the literature. Throughout the paper, by "an increase in demand variability," we refer to any one of the following cases: i) $\boldsymbol{\xi}$ has a symmetric bivariate lognormal distribution and σ monotonically increases, ii) $\boldsymbol{\xi}$ with independent marginal distributions is replaced with $\boldsymbol{\xi}'$ with independent marginal distributions such that $\overline{\boldsymbol{\xi}} = \overline{\boldsymbol{\xi}'}$ and ξ_i' is stochastically more variable than ξ_i for $i = 1, 2$, or iii) $\boldsymbol{\xi}$ with $\sigma = 0$ is replaced with $\boldsymbol{\xi}'$ with $\sigma \neq 0$. By "an increase in demand correlation," we refer to any one of the following cases: i) $\boldsymbol{\xi}$ has a bivariate lognormal distribution and ρ monotonically increases, ii) $\boldsymbol{\xi}$ is replaced with $\boldsymbol{\xi}'$ which dominates $\boldsymbol{\xi}$ according to the concordance ordering, or iii) $\boldsymbol{\xi}$ with $\rho \neq 1$ is replaced with $\boldsymbol{\xi}'$ with $\rho = 1$. The details of the analysis can be found in the proof.

[3]The proofs for the stage 2 optimal production vector for each technology with $y > 0$ are available upon request.

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A Appendix A

Name	Meaning
(ω_0, ω_1)	cash and asset holdings of the firm, called the firm's endowment
β	proportion of F_{FRM} deducted from cash holdings of the firm
α_0	stage 0 price of tradable asset
(F_T, c_T)	fixed and variable capacity costs of technology T
γ_T	salvage rate of fixed cost of technology T
F_{FRM}	fixed cost of financial risk management (FRM)
B	stage 1 budget
(a, E)	interest rate and credit limit of the loan contract
$r_f(=0)$	risk-free rate
P	value of collateral physical asset
α_1	stage 1 price of tradable asset
$\boldsymbol{\xi} = (\xi_1, \xi_2)$	multiplicative demand intercept in product markets
$\boldsymbol{\Sigma}$	covariance matrix of $\boldsymbol{\xi}$
ρ	coefficient of correlation in $\boldsymbol{\xi}$
σ	standard deviation of ξ_1 and ξ_2)
Γ_T	optimal stage 2 operating profits
Π_T	optimal stage 2 equity value
π_T	optimal expected (stage 1) equity value
Λ^{FRM}	expected (stage 0) equity value of better technology with FRM
Λ^{-FRM}	expected (stage 0) equity value of better technology without FRM
Π^*	optimal expected (stage 0) equity value
Δ_T	Value of financial risk management with technology T

Table 6: Summary of Notation

Proof of Proposition 1: We start by formulating the stage 2 optimization problem. Let $\Gamma_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right)$ denote the optimal stage 2 operating profit as a function of the state vector $(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}})$. Since we assume production is costless, this profit is equal to the maximum sales revenue that can be obtained with the existing capacity.

In stage 1, the firm will have observed the budget realization $\tilde{B}$ and borrowed e_T to invest in capacity level $\mathbf{K_T}$. The remaining cash holdings of $\tilde{B} + e_T - c_T\mathbf{1}'\mathbf{K_T} - F_T$, non-negative by construction, will have been invested into a cash account with return $r_f(=0)$.

Two outcomes are possible in stage 2: If the firm's final cash position (operating profits and cash account holdings) is sufficient to cover the face value of the loan, i.e. $\Gamma_T(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) + (\tilde{B} + e_T - c_T\mathbf{1}'\mathbf{K_T} - F_T) \geq e_T(1 + a)$, then the firm does not default; otherwise, it does. If the firm does not default, it repays the face value of its loan and liquidates the non-pledged technology and the physical assets, generating $\gamma_T F_T$ and P, respectively. If the firm defaults, the cash on hand and the ownership of the collateralized physical asset are transferred to the bank. The firm receives the salvage value of the technology $\gamma_T F_T$ and the cash $R(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}})$ remaining after the face value of the loan is deducted from its seized assets. We write

$$R(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) = P + \Gamma_T(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) + (\tilde{B} + e_T - c_T \mathbf{1}'\mathbf{K_T} - F_T) - e_T(1+a), \tag{6}$$

where we invoke the assumptions that any additional fees in the default state (e.g. bankruptcy fee) are borne by the creditor as out-of-pocket expenditures, and that the loan is fully-collateralized by the physical asset.

Since the shareholders are risk neutral and the risk-free rate is 0, the stage 2 equity value can be written as the sum of the individual components cash flows, regardless of when they are realized:

$$\Pi_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right) = \begin{cases} \Gamma_T(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) + (\tilde{B} + e_T - c_T \mathbf{1}'\mathbf{K_T} - F_T) \\ \quad -e_T(1+a) + \gamma_T F_T + P & \text{if no default} \\ \gamma_T F_T + R(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) & \text{if default} \end{cases} \tag{7}$$

Inspecting (7) reveals that the equity value can simply be written as

$$\Pi_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right) = \Gamma_T(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\xi}) + (\tilde{B} + e_T - c_T \mathbf{1}'\mathbf{K_T} - F_T) + \gamma_T F_T - e_T(1+a) + P \tag{8}$$

regardless of whether the firm defaults or not. Obtaining this unique functional form is essential in preserving tractability and in deriving closed-form expressions for the firm's capacity, technology and financial risk management decisions for a subset of parameter levels.

The production decision only affects the operating profit Γ in (8), so optimizing the stage 2 equity value is equivalent to the following optimization problem:

$$\max_{\mathbf{Q} \in \Theta_T} \mathbf{Q}'\mathbf{p}(\mathbf{Q}; \tilde{\boldsymbol{\xi}}) = \max_{\mathbf{Q} \in \Theta_T} \tilde{\boldsymbol{\xi}}'\mathbf{Q}^{1+\frac{1}{b}}. \tag{9}$$

Here, $\mathbf{p}(\mathbf{Q}; \tilde{\boldsymbol{\xi}})' = \left(p(q_1; \tilde{\xi}_1), p(q_2; \tilde{\xi}_2)\right)$, $\Theta_F \doteq \{\mathbf{Q} : \mathbf{Q} \geq \mathbf{0}, \mathbf{1}'\mathbf{Q} \leq \mathbf{K_F}\}$ and $\Theta_D \doteq \{\mathbf{Q} : \mathbf{Q} \geq \mathbf{0}, \mathbf{Q} \leq \mathbf{K_D}\}$ are the feasibility sets for production quantity levels for each technology T.

Let $f(\mathbf{Q}) \doteq \tilde{\boldsymbol{\xi}}'\mathbf{Q}^{1+\frac{1}{b}}$ and $\mathbf{Q_T^*}$ denote the optimal production vector that solves (9) for technology $T \in \{F, D\}$. It is easy to establish that $f(\mathbf{Q})$ is strictly concave in $\mathbf{Q}' = (q_1, q_2)$. Since the constraints are linear, KKT conditions are necessary and sufficient for optimality and $\mathbf{Q_T^*}$ is unique. Since $\frac{\partial f}{\partial q_i} = (1 + 1/b)\tilde{\xi}_1 q_i^{1/b} > 0$, and with $b \in (\infty, -1)$, $\lim_{q_i \to 0^+} \frac{\partial f}{\partial q_i} \to \infty$, the non-negativity constraints will be non-binding and the capacity constraint will be binding at optimality. With the dedicated technology, this yields $\mathbf{Q_D^*} = \mathbf{K_D}$ and

$$\Gamma_D\left(\mathbf{K_D}, e_D, \tilde{B}, \tilde{\boldsymbol{\xi}}\right) = f(\mathbf{Q_D^*}) = \tilde{\boldsymbol{\xi}}'\mathbf{K_D}^{1+\frac{1}{b}}.$$

With the flexible technology, according to the KKT conditions, $\mathbf{Q_F^*}$ solves $\left.\frac{\partial f}{\partial q_1}\right|_{q_F^1} = \left.\frac{\partial f}{\partial q_2}\right|_{K_F - q_F^1}$. After some algebra, we obtain $\mathbf{Q_F^*} = \frac{K_F}{\tilde{\xi}_1^{-b}+\tilde{\xi}_2^{-b}}\ \tilde{\boldsymbol{\xi}}^{-b}$ and

$$\Gamma_F\left(K_F, e_F, \tilde{B}, \tilde{\boldsymbol{\xi}}\right) = f(\mathbf{Q_F^*}) = \frac{K_F^{1+\frac{1}{b}}}{\left(\tilde{\xi}_1^{-b}+\tilde{\xi}_2^{-b}\right)^{1+\frac{1}{b}}}\left[\tilde{\xi}_1^{-b}+\tilde{\xi}_2^{-b}\right] = \left(\tilde{\xi}_1^{-b}+\tilde{\xi}_2^{-b}\right)^{-\frac{1}{b}} K_F^{1+\frac{1}{b}}.$$

Defining $\mathbf{N_F} \doteq \left(\tilde{\xi}_1^{-b}+\tilde{\xi}_2^{-b}\right)^{-\frac{1}{b}}$ and $\mathbf{N_D} \doteq \tilde{\boldsymbol{\xi}}$ and substituting Γ_T in (8) yields the expression for the optimal equity value Π_T:

$$\Pi_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right) = \mathbf{N_T'}\mathbf{K_T}^{1+\frac{1}{b}} + (\tilde{B} + e_T - c_T\mathbf{1'}\mathbf{K_T} - F_T) + \gamma_T F_T - e_T(1+a) + P \qquad (10)$$

■

Proof of Proposition 2: We start by formulating the stage 1 optimization problem. The optimal expected (stage 1) equity value of the firm, $\pi_T(\tilde{B})$, is given as follows:

$$\pi_T(\tilde{B}) = \begin{cases} \max\left\{\Psi_T(\tilde{B}), \tilde{B} - (1-\gamma_T)F_T + P\right\} & \text{if } \tilde{B} + E > F_T \\ \tilde{B} - (1-\gamma_T)F_T + P & \text{if } \tilde{B} + E \le F_T \end{cases} \qquad (11)$$

where

$$\begin{aligned} \Psi_T(\tilde{B}) \quad = \quad & \max_{\mathbf{K_T},\, e_T} \tilde{B} + e_T - (c_T\mathbf{1'}\mathbf{K_T} + F_T) - (B + e_T - c_T\mathbf{1'}\mathbf{K_T} - F_T) + \mathbb{E}\left[\Pi_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right)\right] \\ \text{s.t.} \qquad & e_T \ge c_T\mathbf{1'}\mathbf{K_T} + F_T - \tilde{B} \\ & e_T \le E_T \qquad (12) \\ & \mathbf{K_T} \ge \mathbf{0}, \quad e_T \ge 0. \end{aligned}$$

We start with explaining the formulation of the optimization problem (12). The firm has available budget $\tilde{B}$ and borrows e_T from the creditor. Out of this sum $\tilde{B} + e_T$, the firm invests $c_T\mathbf{1'}\mathbf{K_T} + F_T$ in capacity and places the remainder $(B + e_T - c_T\mathbf{1'}\mathbf{K_T} - F_T)$ into the cash account. The return from the cash account and the operating profits from the capacity investment are included in the expected value of the equity in stage 2, $\mathbb{E}\left[\Pi_T\left(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}\right)\right]$. Using (8), the objective function can be rewritten as $\tilde{B} + P - (1-\gamma_T)F_T + \Gamma_T(\mathbf{K_T}, e_T, \tilde{B}, \tilde{\boldsymbol{\xi}}) - c_T\mathbf{1'}\mathbf{K_T} - ae_T$. Here, the first three terms are equal to the equity value of the firm if the firm does nothing (does not borrow and does not invest). Note that since the firm has already committed to technology T, the fixed cost F_T is incurred even if $\mathbf{K_T} = \mathbf{0}$. The last three terms are the net profit derived from borrowing and investing in capacity.

The first constraint ensures that the amount of external borrowing is greater than the difference between the cost of the investment and the available budget, otherwise the investment is not feasible.

The second constraint states that the external borrowing is less than the credit limit (E) of the firm.

Equation (11) states the firm will either choose a positive capacity level in stage 1 or do nothing (not borrow and not invest in capacity). The former will be the case when the optimal capacity investment level obtained in (12) is positive, and this solution dominates doing nothing; with $\pi_T(\tilde{B}) = \Psi_T(\tilde{B})$. In the latter case, the equity value of the firm is $\tilde{B} + P - (1-\gamma_T)F_T$, with $\mathbf{K}^*_{\mathbf{T}}(\tilde{B}) = \mathbf{0}$ and $e^*_T(\tilde{B}) = 0$. This is the optimal solution if (i) the budget plus the credit limit is insufficient (or only sufficient) to cover the fixed cost of investment ($\tilde{B} + E \leq F_T$), so the firm liquidates the physical asset and salvages the technology; or if (ii) the budget plus credit limit is sufficient to cover the fixed cost, but the firm optimally chooses not to invest in capacity ($\tilde{B} + P - (1-\gamma_T)F_T > \Psi_T(\tilde{B})$ when $\tilde{B} + E \geq F_T$). Note that if $\mathbf{K_T} = \mathbf{0}$ in the optimal solution of (12), the formulation in (12) forces the firm to (suboptimally) borrow $E - \tilde{B}$, but the optimal objective function value is then dominated by $\tilde{B} + P - (1-\gamma_T)F_T$, the value of doing nothing, so the joint formulation in (5) and (6) yields the correct optimal solution.

Since $a > 0$, the firm optimally does not borrow if it does not invest in capacity ($e_T = 0$ if $\mathbf{K_T} = \mathbf{0}$) and only borrows exactly enough to cover the capacity investment when this investment level is positive ($e_T = \left(c_T \mathbf{1}'\mathbf{K_T} + F_T - \tilde{B}\right)^+$ if $\mathbf{K_T} > \mathbf{0}$). Substituting Π_T from (8) and Γ_T from Proposition 1 in (12), we obtain the equivalent formulation

$$\begin{aligned} \Psi_T(\tilde{B}) = \max_{\mathbf{K_T}} \quad & \tilde{B} - c_T\mathbf{1}'\mathbf{K_T} - (1-\gamma_T)F_T - a\left(c_T\mathbf{1}'\mathbf{K_T} + F_T - \tilde{B}\right)^+ + \mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T}^{1+\frac{1}{b}} + P \\ \text{s.t.} \quad & c_T\mathbf{1}'\mathbf{K_T} + F_T - \tilde{B} \leq E \\ & \mathbf{K_T} \geq \mathbf{0}. \end{aligned} \tag{13}$$

Let $g(\mathbf{K_T})$ denote the objective function in (13) and $\mathbf{K}^{\mathbf{p}}_{\mathbf{T}}(\tilde{B})$ be the optimal solution of (13). The corresponding optimal borrowing $e^p_T(\tilde{B})$ is equal to $\left(c_T\mathbf{1}'\mathbf{K}^{\mathbf{p}}_{\mathbf{T}}(\tilde{B}) + F_T - \tilde{B}\right)^+$. For $\tilde{B} > F_T$, the function $g(\mathbf{K_T})$ has a kink and is not differentiable at $\mathbf{1}'\mathbf{K_T} = \frac{\tilde{B}-F_T}{c_T}$. We rewrite (13) as a combination of two sub-problems $i = 0, 1$ with

$$\Psi_T(\tilde{B}) \quad = \quad \begin{cases} \max_i \Psi^i_T(\tilde{B}) & \text{if} \quad \tilde{B} > F_T \\ \Psi^1_T(\tilde{B}) & \text{if} \quad \tilde{B} \leq F_T \end{cases} \tag{14}$$

such that

$$\begin{aligned} \Psi^i_T(\tilde{B}) = \max_{\mathbf{K_T}} \quad & \tilde{B} - c_T\mathbf{1}'\mathbf{K_T} - (1-\gamma_T)F_T - a^i\left(c_T\mathbf{1}'\mathbf{K_T} + F_T - \tilde{B}\right) + \mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T}^{1+\frac{1}{b}} + P \\ \text{s.t.} \quad & Z^i_L \leq c_T\mathbf{1}'\mathbf{K_T} + F_T - \tilde{B} \leq Z^i_U \\ & \mathbf{K_T} \geq \mathbf{0}, \end{aligned} \tag{15}$$

where $a^0 = 0, a^1 = a$ and $Z_L^0 = -\infty,\ Z_L^1 = 0,\ Z_U^0 = 0,\ Z_U^1 = E$. Subproblem 0 (1) is the restriction of the problem to the no borrowing (borrowing) regions. Let $g^i(\mathbf{K_T})$ denote the objective function and $\mathbf{K_T^{p_i}}(\tilde{B})$ be the optimal solution of sub-problem i. We have

$$g(\mathbf{K_T}) \quad = \quad \begin{cases} g^0(\mathbf{K_T}) & \text{if} \quad c_T \mathbf{1}'\mathbf{K_T} + F_T \leq \tilde{B} \\ g^1(\mathbf{K_T}) & \text{if} \quad c_T \mathbf{1}'\mathbf{K_T} + F_T > \tilde{B}. \end{cases}$$

The remainder of the proof has the following structure:

1. We show that $g^i(\mathbf{K_T})$ is strictly concave and solve each sub-problem i for $\mathbf{K_T^{p_i}}(\tilde{B})$.

2. We show that $g(\mathbf{K_T})$ is strictly concave. It follows that

$$\mathbf{K_T^{p}}(\tilde{B}) = \mathbf{K_T^{p_i}}(\tilde{B}) \text{ where } i \quad = \quad \begin{cases} arg \max_i \Psi_T^i(\tilde{B}) & \text{if} \quad \tilde{B} > F_T \\ 1 & \text{if} \quad \tilde{B} \leq F_T \end{cases}$$

   We derive $\Psi_T(\tilde{B})$ by using $\mathbf{K_T^{p_i}}(\tilde{B})$.

3. We compare $\Psi_T(\tilde{B})$ with $\tilde{B} - (1 - \gamma_T)F_T + P$, the value of not investing in capacity, and derive $\mathbf{K_T^*}(\tilde{B})$ and $e_T^*(\tilde{B})$.

1. Solution for $\mathbf{K_T^{p_i}}(\tilde{B})$

1.a. Flexible Technology:

Let $A \doteq \mathbb{E}\left[\mathbf{N_F}\right] = \mathbb{E}\left[\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}\right]$. The first and second order conditions in (15) are

$$\begin{aligned} \frac{\partial g^i}{\partial K_F} &= -c_F - a^i c_F + (1 + 1/b)\ A\ K_F^{1/b}, \\ \frac{\partial^2 g^i}{\partial K_F^2} &= \frac{1}{b}\,(1 + 1/b)\ A\ K_F^{(1/b-1)}. \end{aligned}$$

Since $b < -1$, we have $\lim_{K_F \to 0^+} \frac{\partial^2 K_F^{(1/b-1)}}{\partial K_F^2} \to \infty$ and $\frac{\partial^2 K_F^{(1/b-1)}}{\partial K_F^2} > 0 \quad \forall K_F \geq 0$. With $b < -1$, it follows that $\frac{\partial^2 g^i}{\partial K_F^2} < 0$ for $K_F \geq 0$ and the function $g^i(K_F)$ is strictly concave for $i = 0, 1$. Since the constraints in (15) are linear, first-order KKT conditions are necessary and sufficient for optimality for each sub-problem i and $K_F^{p_i}(\tilde{B})$ is unique.

From KKT conditions if i has a non-empty feasible region then the optimal solution is either the solution of $\frac{\partial g^i}{\partial K_F} = 0$, $K_F^{p_i}(\tilde{B}) = \left(\frac{A\left(1+\frac{1}{b}\right)}{c_F(1+a^i)}\right)^{-b}$, or is a boundary solution. Since $\tilde{B} > F_F$ for $i = 0$ from (14) and $\tilde{B} > F_F - E$ for $i = 1$ from (11), the non-negativity constraint is never binding in (15). Since $\lim_{K_F \to 0^+} \frac{\partial g^i}{\partial K_F} \to \infty$, $K_F = 0$ is never optimal. If $\frac{Z_L^i + \tilde{B} - F_F}{c_F} > 0$ and $\frac{\partial g^i}{\partial K_F} < 0$ at this point, then $K_F^{p_i}(\tilde{B}) = \frac{Z_L^i + \tilde{B} - F_F}{c_F}$, i.e., the optimal solution occurs at the lower bound of the financing constraint. If $\frac{\partial g^i}{\partial K_F} > 0$ at $K_F = \frac{Z_U^i + \tilde{B} - F_F}{c_F} > 0$, then $K_F^{p_i}(\tilde{B}) = \frac{Z_U^i + \tilde{B} - F_F}{c_F}$, i.e., the optimal solution occurs at the upper bound of the financing constraint. To summarize, $K_F^{p_i}(\tilde{B})$ for $i = 0, 1$ is characterized by

$$\mathbf{K_F^{p_0}}(\tilde{\mathbf{B}}) = \begin{cases} \mathbf{K_F^0} \doteq \left(\frac{A\left(1+\frac{1}{b}\right)}{c_F}\right)^{-b} & \text{if } \; c_F\mathbf{K_F^0} + F_F - \tilde{B} \leq 0 \\ \overline{\mathbf{K}}_\mathbf{F} \doteq \left(\frac{\tilde{B}-F_F}{c_F}\right) & \text{if } \; c_F\mathbf{K_F^0} + F_F - \tilde{B} > 0, \end{cases} \tag{16}$$

$$\mathbf{K_F^{p_1}}(\tilde{\mathbf{B}}) = \begin{cases} \overline{\mathbf{K}}_\mathbf{F} \doteq \left(\frac{\tilde{B}-F_F}{c_F}\right) & \text{if } \; c_F\mathbf{K_F^1} + F_F - \tilde{B} \leq 0 \\ \mathbf{K_F^1} \doteq \left(\frac{A\left(1+\frac{1}{b}\right)}{c_F(1+a)}\right)^{-b} & \text{if } \; 0 < c_F\mathbf{K_F^1} + F_F - \tilde{B} \leq E \\ \overline{\overline{\mathbf{K}}}_\mathbf{F} \doteq \left(\frac{E+\tilde{B}-F_F}{c_F}\right) & \text{if } \; c_F\mathbf{K_F^1} + F_F - \tilde{B} > E. \end{cases}$$

Here, $\mathbf{K_F^0}$ is the budget-unconstrained optimal capacity investment and $\mathbf{K_F^1}$ is the credit-unconstrained optimal capacity investment.

1.b. Dedicated Technology:

We obtain

$$\begin{aligned} \frac{\partial^2 g^i}{\partial (K_D^j)^2} &= \frac{1}{b}\,(1+1/b)\;\bar{\xi}_j\;(K_D^j)^{(1/b-1)} < 0, \\ \frac{\partial^2 g^i}{\partial (K_D^1)^2}\frac{\partial^2 g^i}{\partial (K_D^2)^2} - \left[\frac{\partial^2 g^i}{\partial K_D^1 K_D^2}\right]^2 &= \prod_j \frac{1}{b}\,(1+1/b)\;\bar{\xi}_j\;(K_D^j)^{(1/b-1)} - 0 > 0 \end{aligned}$$

for $i = 0, 1$ and $j = 1, 2$. Therefore, the Hessian matrix $D^2 g^i(\mathbf{K_D})$ is negative definite for $\mathbf{K_D} \geq \mathbf{0}$ and $g^i(\mathbf{K_D})$ is strictly concave. Since the constraints in (15) are linear, first-order KKT conditions are necessary and sufficient for optimality in each sub-problem i and $\mathbf{K_D^{p_i}}(\tilde{\mathbf{B}})$ is unique.

If $\mathbf{K_D^{p_i}}(\tilde{B})$ is an optimal solution to (15), then there exist $\boldsymbol{\lambda}^{i'} = (\lambda_1^i, \lambda_2^i)$ and $\boldsymbol{\mu}^{i'} = (\mu_1^i, \mu_2^i)$ that satisfy

$$\begin{aligned} c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} &\leq Z_U^i, & (17) \\ c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} &\geq Z_L^i, & (18) \\ \mathbf{K_D^{p_i}}(\tilde{B}) &\geq \mathbf{0}, & (19) \\ -(1+a^i)c_D + (1+1/b)\;\bar{\boldsymbol{\xi}}\;\mathbf{K_D^{p_i}}(\tilde{B})^{1/b} - c_D(\lambda_1^i - \lambda_2^i) + \boldsymbol{\mu}^i &= \mathbf{0}, & (20) \\ \lambda_1^i[Z_U^i - c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) - F_D + \tilde{B}] &= 0, & (21) \\ \lambda_2^i[-Z_L^i + c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B}] &= 0, & (22) \\ \boldsymbol{\mu}^i\mathbf{K_D^{p_i}}(\tilde{B}) &= \mathbf{0} & (23) \end{aligned}$$

with $\boldsymbol{\lambda}^i \geq \mathbf{0}$ and $\boldsymbol{\mu}^i \geq \mathbf{0}$ for $i = 0, 1$. Observe that $\lim_{K_D^j \to 0^+} \frac{\partial g^i}{\partial K_D^j} \to \infty$ for $j = 1, 2$, so it is never optimal to invest in only one of the resources. Since we will compare $\Psi_D(\tilde{B})$ with $\tilde{B} - (1-\gamma_D)F_D + P$

(the value of not investing in either resource) in Step 3, we can focus on $\mathbf{K_D^{p_i}}(\tilde{B}) > \mathbf{0}$ here. This implies $\boldsymbol{\mu}^i = \mathbf{0}$ for (23) to be satisfied.

Case 1: $c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} < Z_U^i$ and $c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} > Z_L^i$

In this case $\boldsymbol{\lambda}^i = \mathbf{0}$, and (20) yields

$$\mathbf{K_D^{p_i}}(\tilde{B}) = \mathbf{K_D^i} \doteq \left(\frac{(1+\frac{1}{b})}{c_D(1+a^i)}\right)^{-b} \bar{\boldsymbol{\xi}}^{-b}.$$

For (17), (18) and (19) to be satisfied, and the solution $\mathbf{K_D^{p_i}}(\tilde{B}) = \mathbf{K_D^i}$ to be valid, we need $Z_L^i < c_D\mathbf{1}'\mathbf{K_D^i} + F_D - \tilde{B} < Z_U^i$. Here, $\mathbf{K_D^0}$ is the budget-unconstrained optimal capacity investment and $\mathbf{K_D^1}$ is the credit-unconstrained optimal capacity investment.

Case 2: $c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} = Z_U^i$

In this case (18) holds as a strict inequality, so $\lambda_2^i = 0$ for (22) to be satisfied. Rewriting the equality as $K_D^2 = \frac{Z_U^i + \tilde{B} - F_F - c_D K_D^1}{c_D}$, and combining this with (20) yields

$$\mathbf{K_D^{p_i}}(\tilde{B})' = \left(\left(\frac{Z_U^i + B - F_D}{c_D}\right)\left(\frac{\bar{\xi}_1^{-b}}{\bar{\xi}_1^{-b} + \bar{\xi}_2^{-b}}\right), \left(\frac{Z_U^i + B - F_D}{c_D}\right)\left(\frac{\bar{\xi}_2^{-b}}{\bar{\xi}_1^{-b} + \bar{\xi}_2^{-b}}\right)\right). \tag{24}$$

The condition $\lambda_1^i \geq 0$ should be satisfied at optimality. After some algebra, this condition implies that (24) is optimal if $\tilde{B} \leq c_D\mathbf{1}'\mathbf{K_D^i} + F_D - Z_U^i$.

Case 3: $c_D\mathbf{1}'\mathbf{K_D^{p_i}}(\tilde{B}) + F_D - \tilde{B} = Z_L^i$

This case is only relevant for $i = 1$ since $Z_L^0 = -\infty$. In this case, (17) holds as a strict inequality, so $\lambda_1^1 = 0$ for (21) to be satisfied. Rewriting the equality as $K_D^2 = \frac{Z_L^1 + \tilde{B} - F_F - c_D K_D^1}{c_D}$, and combining with (20) yields

$$\mathbf{K_D^{p_1}}(\tilde{B})' = \left(\left(\frac{Z_L^1 + B - F_D}{c_D}\right)\left(\frac{\bar{\xi}_1^{-b}}{\bar{\xi}_1^{-b} + \bar{\xi}_2^{-b}}\right), \left(\frac{Z_L^1 + B - F_D}{c_D}\right)\left(\frac{\bar{\xi}_2^{-b}}{\bar{\xi}_1^{-b} + \bar{\xi}_2^{-b}}\right)\right). \tag{25}$$

The condition $\lambda_2^1 \geq 0$ should be satisfied at optimality. After some algebra, this condition implies that (25) is optimal if $\tilde{B} \geq c_D\mathbf{1}'\mathbf{K_D^1} + F_D - Z_L^1$.

Combining cases 1, 2 and 3, $\mathbf{K_D^{p_i}}(\tilde{\mathbf{B}})$ for $i = 0, 1$ is characterized by

$$\mathbf{K_D^{p_0}}(\tilde{\mathbf{B}}) = \begin{cases} \mathbf{K_D^0} = \left(\frac{(1+\frac{1}{b})}{c_D}\right)^{-b} \bar{\boldsymbol{\xi}}^{-b} & \text{if } c_D\mathbf{1}'\mathbf{K_D^0} + F_D - \tilde{B} \leq 0 \\ \overline{\mathbf{K}}_\mathbf{D}' = \left(\left(\frac{B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_1^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right), \left(\frac{B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_2^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right)\right) & \text{if } c_D\mathbf{1}'\mathbf{K_D^0} + F_D - \tilde{B} > 0, \end{cases} \tag{26}$$

$$\mathbf{K_D^{p_1}}(\tilde{\mathbf{B}}) = \begin{cases} \overline{\mathbf{K}}_\mathbf{D}' = \left(\left(\frac{B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_1^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right), \left(\frac{B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_2^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right)\right) & \text{if } c_D\mathbf{1}'\mathbf{K_D^1} + F_D - \tilde{B} \leq 0 \\ \mathbf{K_D^1} = \left(\frac{(1+\frac{1}{b})}{c_D(1+a)}\right)^{-b} \bar{\boldsymbol{\xi}}^{-b} & \text{if } 0 < c_D\mathbf{1}'\mathbf{K_D^1} + F_D - \tilde{B} \leq E \\ \overline{\overline{\mathbf{K}}}_\mathbf{D}' = \left(\left(\frac{E+B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_1^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right), \left(\frac{E+B-F_D}{c_D}\right)\left(\frac{\bar{\xi}_2^{-b}}{\bar{\xi}_1^{-b}+\bar{\xi}_2^{-b}}\right)\right) & \text{if } c_D\mathbf{1}'\mathbf{K_D^1} + F_D - \tilde{B} > E. \end{cases}$$

2. Solution for $\mathbf{K_T^p}(\tilde{\mathbf{B}})$ and $\Psi_T(\tilde{B})$:

To show that $g(\mathbf{K_T})$ is strictly concave, we need to show that $\forall\ \mathbf{K_T^I}, \mathbf{K_T^{II}} \geq 0$ and $\lambda \in (0,1)$,

$$g(\lambda \mathbf{K_T^I} + (1-\lambda)\mathbf{K_T^{II}}) - \lambda g(\mathbf{K_T^I}) - (1-\lambda) g(\mathbf{K_T^{II}}) > 0. \tag{27}$$

Since $g^i(\mathbf{K_T})$ is strictly concave, we only need to focus on $\mathbf{K_T^I}, \mathbf{K_T^{II}}$ such that $c_T \mathbf{1}'\mathbf{K_T^I} + F_T \leq \tilde{B}$ and $c_T \mathbf{1}'\mathbf{K_T^{II}} + F_T > \tilde{B}$. We have two cases to consider. First, if $c_T \mathbf{1}' \left(\lambda \mathbf{K_T^I} + (\mathbf{1}-\lambda)\mathbf{K_T^{II}}\right) + F_T \leq \tilde{B}$ then after some algebra, the left-hand side of (27) becomes

$$\mathbb{E}\left[\mathbf{N_T}\right]'\left(\lambda \mathbf{K_T^I} + (\mathbf{1}-\lambda)\mathbf{K_T^{II}}\right)^{1+\frac{1}{b}} - \lambda \mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T^I}^{1+\frac{1}{b}} - (1-\lambda)\mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T^{II}}^{1+\frac{1}{b}} + (1-\lambda) a (c_T \mathbf{1}'\mathbf{K_T^{II}} + F_T - \tilde{B}).$$

Since $x^{1+\frac{1}{b}}$ is strictly concave for $x \geq 0$ and $c_T \mathbf{1}'\mathbf{K_T^{II}} + F_T - \tilde{B}$ is positive by definition, the above equation is strictly greater than 0. Second, if $c_T \mathbf{1}' \left(\lambda \mathbf{K_T^I} + (\mathbf{1}-\lambda)\mathbf{K_T^{II}}\right) + F_T > \tilde{B}$ then after some algebra, the left-hand side of (27) becomes

$$\mathbb{E}\left[\mathbf{N_T}\right]'\left(\lambda \mathbf{K_T^I} + (\mathbf{1}-\lambda)\mathbf{K_T^{II}}\right)^{1+\frac{1}{b}} - \lambda \mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T^I}^{1+\frac{1}{b}} - (1-\lambda)\mathbb{E}\left[\mathbf{N_T}\right]'\mathbf{K_T^{II}}^{1+\frac{1}{b}} - \lambda a_T (c_T \mathbf{1}'\mathbf{K_T^I} + F_T - \tilde{B}).$$

Since $x^{1+\frac{1}{b}}$ is strictly concave for $x \geq 0$ and $c_T \mathbf{1}'\mathbf{K_T^I} + F_T - \tilde{B}$ is negative by definition, the equation above is strictly greater than 0. Since (27) is satisfied for both cases, $g(\mathbf{K_T})$ is strictly concave. It follows that

$$\mathbf{K_T^p}(\tilde{B}) = \mathbf{K_T^{p_i}}(\tilde{B}) \ \text{ where } \ i \quad = \quad \begin{cases} arg \max_i \Psi_T^i(\tilde{B}) & \text{if } \ \tilde{B} > F_T \\ 1 & \text{if } \ \tilde{B} \leq F_T \end{cases}$$

is the unique maximizer of g. Combining (16) and (26), the unique optimal solution to problem (13) and the corresponding optimal amount of borrowing are given by

$$\mathbf{K_T^p}(\tilde{B}) \quad = \quad \begin{cases} \mathbf{K_T^0} & \text{if } \ c_T \mathbf{1}'\mathbf{K_T^0} + F_T \leq \tilde{B} \\ \overline{\mathbf{K}}_\mathbf{T} & \text{if } \ c_T \mathbf{1}'\mathbf{K_T^1} + F_T \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^0} + F_T \\ \mathbf{K_T^1} & \text{if } \ c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^1} + F_T \\ \overline{\overline{\mathbf{K}}}_\mathbf{T} & \text{if } \ \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E, \end{cases} \tag{28}$$

$$e_T^p(\tilde{B}) \quad = \quad \left(c_T \mathbf{1}'\mathbf{K_T^p}(\tilde{B}) + F_T - \tilde{B}\right)^+$$

where

$$
\begin{aligned}
\mathbf{K_D^0}' &= \left(\left(\frac{\overline{\xi}_1\left(1+\frac{1}{b}\right)}{c_D}\right)^{-b}, \left(\frac{\overline{\xi}_2\left(1+\frac{1}{b}\right)}{c_D}\right)^{-b}\right) \\
\overline{\mathbf{K}}_\mathbf{D}' &= \left(\left(\frac{B-F_D}{c_D}\right)\left(\frac{\overline{\xi}_1^{-b}}{\overline{\xi}_1^{-b}+\overline{\xi}_2^{-b}}\right), \left(\frac{B-F_D}{c_D}\right)\left(\frac{\overline{\xi}_2^{-b}}{\overline{\xi}_1^{-b}+\overline{\xi}_2^{-b}}\right)\right) \\
\mathbf{K_D^1}' &= \left(\left(\frac{\overline{\xi}_1\left(1+\frac{1}{b}\right)}{c_D(1+a)}\right)^{-b}, \left(\frac{\overline{\xi}_2\left(1+\frac{1}{b}\right)}{c_D(1+a)}\right)^{-b}\right) \\
\overline{\overline{\mathbf{K}}}_\mathbf{D}' &= \left(\left(\frac{E+B-F_D}{c_D}\right)\left(\frac{\overline{\xi}_1^{-b}}{\overline{\xi}_1^{-b}+\overline{\xi}_2^{-b}}\right), \left(\frac{E+B-F_D}{c_D}\right)\left(\frac{\overline{\xi}_2^{-b}}{\overline{\xi}_1^{-b}+\overline{\xi}_2^{-b}}\right)\right) \\
K_F^0 &= \left(\frac{A\left(1+\frac{1}{b}\right)}{c_F}\right)^{-b} \\
\overline{K}_F &= \left(\frac{B-F_F}{c_F}\right) \\
K_F^1 &= \left(\frac{A\left(1+\frac{1}{b}\right)}{c_F(1+a)}\right)^{-b} \\
\overline{\overline{K}}_F &= \left(\frac{E+B-F_F}{c_F}\right).
\end{aligned}
$$

We substitute (28) in (13) and find

$$
\Psi_T(\tilde{B}) = \begin{cases}
\tilde{B}-(1-\gamma_T)F_T + \frac{c_T\mathbf{1}'\mathbf{K_T^0}}{-(b+1)} + P & \text{if } c_T\mathbf{1}'\mathbf{K_T^0}+F_T \le \tilde{B} \\
M_T\left(\frac{\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} + \gamma_T F_T + P & \text{if } c_T\mathbf{1}'\mathbf{K_T^1}+F_T \le \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^0}+F_T \\
(\tilde{B}-F_T)(1+a) + \frac{c_T\mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)} + \gamma_T F_T + P & \text{if } c_T\mathbf{1}'\mathbf{K_T^1}+F_T-E \le \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^1}+F_T \\
-E(1+a) + M_T\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} + \gamma_T F_T + P & \text{if } \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^1}+F_T-E.
\end{cases} \tag{29}
$$

where $M_F = \mathbb{E}\left[\left(\xi_1^{-b}+\xi_2^{-b}\right)^{-\frac{1}{b}}\right]$ and $M_D = \left(\overline{\xi}_1^{-b}+\overline{\xi}_2^{-b}\right)^{-\frac{1}{b}}$. It follows from (11) that $\Psi_T(\tilde{B})$ is relevant (and is defined) only for $\tilde{B} > F_T - E$.

3. Solution for $\mathbf{K_T^*}(\tilde{B})$ and $e_T^*(\tilde{B})$:

To complete the characterization of $\mathbf{K_T^*}(\tilde{B})$ and $e_T^*(\tilde{B})$, we compare $\Psi_T(\tilde{B})$ with $\tilde{B}-(1-\gamma_T)F_T+P$ (the value of the not borrowing and not investing in capacity) for $\tilde{B} > F_T - E$ and establish that the two functions intersect at most once on $\tilde{B} \in (F_T - E, \infty)$; and find $\mathbf{K_T^*}(\tilde{B})$ and $e_T^*(\tilde{B})$.

For $\tilde{B} > F_T - E$, we define $G_T(\tilde{B}) \doteq \Psi_T(\tilde{B}) - (\tilde{B}-(1-\gamma_T)F_T+P)$, the difference between the equity values in (29) and not borrowing and not investing in capacity. It is easy to verify that, $\lim_{\tilde{B}\to\tilde{B}_k^+}\Psi_T(\tilde{B}) = \lim_{\tilde{B}\to\tilde{B}_k^-}\Psi_T(\tilde{B})$ for $\forall\ \tilde{B}_k > F_T - E$ therefore, $\Psi_T(\tilde{B})$ and, in turn, $G_T(\tilde{B})$

are continuous functions of $\tilde{B}$. We have

$$\frac{\partial G_T(\tilde{B})}{\partial \tilde{B}} = \begin{cases} 0 & \text{if } c_T\mathbf{1}'\mathbf{K_T^0} + F_T \le \tilde{B} \\ \frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} - 1 & \text{if } c_T\mathbf{1}'\mathbf{K_T^1} + F_T \le \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^0} + F_T \\ a & \text{if } c_T\mathbf{1}'\mathbf{K_T^1} + F_T - E \le \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^1} + F_T \\ \frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} - 1 & \text{if } \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^1} + F_T - E. \end{cases} \tag{30}$$

For $c_T\mathbf{1}'\mathbf{K_T^1} + F_T \le \tilde{B} < c_T\mathbf{1}'\mathbf{K_T^0} + F_T$,

$$\frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} - 1 > \frac{M_T}{c_T}(1+\frac{1}{b})\ \left(\mathbf{1}'\mathbf{K_T^0}\right)^{\frac{1}{b}} - 1 = 0, \tag{31}$$

and for $\tilde{B} < c_T\mathbf{1}'\mathbf{K_T^1} + F_T - E$,

$$\frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} - 1 > \frac{M_T}{c_T}(1+\frac{1}{b})\ \left(\mathbf{1}'\mathbf{K_T^1}\right)^{\frac{1}{b}} - 1 = a. \tag{32}$$

It follows that $\lim_{\tilde{B}\to\tilde{B}_k^+}\frac{\partial}{\partial\tilde{B}}G_T(\tilde{B}) = \lim_{\tilde{B}\to\tilde{B}_k^-}\frac{\partial}{\partial\tilde{B}}G_T(\tilde{B})$ on the domain of $G_T(.)$. Therefore $G_T(\tilde{B})$ is differentiable for $\tilde{B} > F_T - E$ and $\frac{\partial}{\partial\tilde{B}}G_T(\tilde{B}) \ge 0$ with equality holding only for $\tilde{B} \ge c_T\mathbf{1}'\mathbf{K_T^0} + F_T$. For $c_T\mathbf{1}'\mathbf{K_T^0} + F_T \le \tilde{B}$,

$$G_T(\tilde{B}) = \tilde{B} - (1-\gamma_T)F_T + \frac{c_T\mathbf{1}'\mathbf{K_T^0}}{-(b+1)} + P - (\tilde{B} - (1-\gamma_T)F_T + P) = \frac{c_T\mathbf{1}'\mathbf{K_T^0}}{-(b+1)} > 0. \tag{33}$$

We showed that $G_T(\tilde{B})$ strictly increases for $\tilde{B} \in (F_T - E, c_T\mathbf{1}'\mathbf{K_T^0} + F_T)$ and is positive for $\tilde{B} \in [c_T\mathbf{1}'\mathbf{K_T^0} + F_T, \infty)$. Let $\widehat{B}_T$ denote the budget level at which the two equity value curves intersect, i.e. $G_T(\widehat{B}_T) = 0$. For $F_T \ge E$, we have $\lim_{\tilde{B}\to(F_T-E)^+} G_T(\tilde{B}) = -aE < 0$. Since $G_T(\tilde{B})$ strictly increases in $\tilde{B}$, it follows that for $F_T \ge E$, there exists a unique $\widehat{B}_T > F_T - E$ such that $G_T(\widehat{B}_T) = 0$. For $F_T < E$, the domain of $G_T(\tilde{B})$ is $[0,\infty)$. For notational convenience, we let $\widehat{B}_T \doteq 0$ if the two curves do not intersect on this domain ($G_T(\tilde{B}) > 0$ for $\tilde{B} \ge 0$). Since $G_T(\tilde{B})$ strictly increases in $\tilde{B}$, it follows that for $F_T < E$, $\widehat{B}_T$, if it exists on $[0,\infty)$, is unique. For $\tilde{B} \le \widehat{B}_T$ we have $\mathbf{K_T^*}(\tilde{B}) = \mathbf{0}$ and $e_T^*(\tilde{B}) = 0$. Combining this with (28) gives the desired result. ■

Proof of Corollary 1: The expected (stage 1) equity value of the firm with a given budget level $\tilde{B}$ follows directly from Proposition 2:

$$\pi_T(\tilde{B}) = \begin{cases} \tilde{B} - (1-\gamma_T)F_T + \frac{c_T\mathbf{1}'\mathbf{K_T^0}}{-(b+1)} + P & \text{if } \tilde{B} \in \Omega_T^0 \\ M_T\left(\frac{\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} + \gamma_T F_T + P & \text{if } \tilde{B} \in \Omega_T^1 \\ (\tilde{B} - F_T)(1+a) + \frac{c_T\mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)} + \gamma_T F_T + P & \text{if } \tilde{B} \in \Omega_T^2 \\ -E(1+a) + M_T\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} + \gamma_T F_T + P & \text{if } \tilde{B} \in \Omega_T^3 \\ \tilde{B} - (1-\gamma_T)F_T + P & \text{if } \tilde{B} \in \Omega_T^4 \end{cases} \tag{34}$$

where $M_F = \mathbb{E}\left[\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}\right]$ and $M_D = \left(\overline{\xi}_1^{-b} + \overline{\xi}_2^{-b}\right)^{-\frac{1}{b}}$.
We calculate

$$\left(\frac{\partial \pi_T(\tilde{B})}{\partial \tilde{B}}, \frac{\partial^2 \pi_T(\tilde{B})}{\partial \tilde{B}^2}\right) = \begin{cases} (1,0) & \text{if } \tilde{B} \in \Omega_T^0 \\ \left(\frac{M_T}{c_T}(1+1/b)\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}}, \frac{1}{b}\frac{M_T}{c_T^{(1+1/b)}}(1+1/b)(\tilde{B}-F_T)^{\frac{1}{b}-1}\right) & \text{if } \tilde{B} \in \Omega_T^1 \\ (1+a,0) & \text{if } \tilde{B} \in \Omega_T^2 \\ \left(\frac{M_T}{c_T}(1+1/b)\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}}, \frac{1}{b}\frac{M_T}{c_T^{(1+1/b)}}(1+1/b)(E+\tilde{B}-F_T)^{\frac{1}{b}-1}\right) & \text{if } \tilde{B} \in \Omega_T^3 \\ (1,0) & \text{if } \tilde{B} \in \Omega_T^4 \end{cases}$$

at the points where $\pi_T(\tilde{B})$ is differentiable. It is easy to verify that $\lim_{\tilde{B}\to\tilde{B}_k^+} \frac{\partial}{\partial \tilde{B}}\pi_T(\tilde{B}) = \lim_{\tilde{B}\to\tilde{B}_k^-} \frac{\partial}{\partial \tilde{B}}\pi_T(\tilde{B})$ for $\tilde{B}_k \in \Omega_T^{0123}$, and $\pi_T(\tilde{B})$ is differentiable everywhere in its domain except at $\widehat{B}_T$. Since $\pi_T(\tilde{B})$ is a continuous function of $\tilde{B}$ it follows that $\pi_T(\tilde{B})$ is strictly increasing in $\tilde{B}$. We have $\frac{\partial^2}{\partial \tilde{B}^2}\pi_T(\tilde{B}) \leq 0$ for each Ω_T^i and $\pi_T(\tilde{B})$ is piecewise concave. From (31) we obtain $\frac{\partial}{\partial \tilde{B}}\pi_T(\tilde{B}) > 1$ for $\tilde{B} \in \Omega_T^1$ and from (32) we have $\frac{\partial}{\partial \tilde{B}}\pi_T(\tilde{B}) > 1 + a$ for $\tilde{B} \in \Omega_T^3$. Since $\pi_T(\tilde{B})$ is only kinked at $\widehat{B}_T$ it follows that $\pi_T(\tilde{B})$ is concave in $\tilde{B}$ for $\tilde{B} \geq \widehat{B}_T$, but not globally concave. ∎

Proof of Proposition 3: The optimal risk management level H_T^* is given by

$$H_T^* = argmax_{H_T} \quad \mathbb{E}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right] \qquad \text{s.t.} \quad -\frac{\omega_0^{FRM}}{\overline{\alpha}_1} \leq H_T \leq \omega_1^{FRM} \tag{35}$$

Since $\boldsymbol{\xi}$ and α_1 are independent,

$$\mathbb{E}_{\boldsymbol{\xi},\alpha_1}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right] = \mathbb{E}_{\alpha_1}\left[\mathbb{E}_{\boldsymbol{\xi}}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right]\right] = \mathbb{E}_{\alpha_1}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right].$$

Therefore we can write the expectation in (35) over α_1. Let $r_{\alpha_1}(.)$ and $R_{\alpha_1}(.)$ denote the density and distribution function of α_1, respectively. Since $B_{FRM}(\alpha_1, H_T) = \omega_0^{FRM} + \alpha_1(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T$, for each H_T the unique distribution function of $B_{FRM}(H_T)$ is

$$R_{B_{FRM}(H_T)}(\tilde{B}) = R_{\alpha_1}\left(\frac{\tilde{B} - \omega_0^{FRM} - \overline{\alpha}_1 H_T}{\omega_1^{FRM} - H_T}\right) \qquad \tilde{B} \geq \omega_0^{FRM} + \overline{\alpha}_1 H_T. \tag{36}$$

It follows that H_T determines the range and the probability distribution of the available budget in stage 1. Since we do not impose any specific assumption on the type of the distribution of α_1, we will use general structural properties of the optimization problem (35) to solve for H_T^*. In particular, we will focus on the functional form of $\pi_T(\tilde{B})$ since the expected (stage 0) value of the equity is the expectation of this function with respect to the budget random variable. We first provide the following lemma that we will use throughout the proof. The proof is relegated to Appendix C.

Lemma 1 *There exist unique fixed cost threshold $\overline{F}_T$ such that $\widehat{B}_T = 0$ iff $F_T \leq \overline{F}_T$, and $\widehat{B}_T > 0$ iff $F_T > \overline{F}_T$.*

We now conclude the proof by analyzing each case in Proposition 3.

Case (i)**,** $F_T \leq \overline{F}_T$**:**

It follows from Lemma 1 that $\widehat{B}_T = 0$. Since $\widehat{B}_T = 0$, from Corollary 1 we have that $\pi_T(\tilde{B})$ is concave for $\tilde{B} \geq 0$. From Jensen's inequality,

$$\begin{aligned}\mathbb{E}\left[\pi_T(B_{FRM}(\alpha_1, H_T))\right] \leq \pi_T\left(\mathbb{E}[B_{FRM}(\alpha_1, H_T)]\right) &= \pi_T\left(\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM}\right)\right) \\ &= \pi_T\left(B_{FRM}\left(\alpha_1, \omega_1^{FRM}\right)\right) \end{aligned} \tag{37}$$

for $H_T \in \left[-\frac{\omega_0^{FRM}}{\overline{\alpha}_1}, \omega_1^{FRM}\right]$. This implies that $H_T^* = \omega_1^{FRM}$.

Case (ii)**,** $F_T > \overline{F}_T$**:**

From Lemma 1, we have $\widehat{B_T} > 0$ and we cannot guarantee the concavity of π_T for the whole range of $\tilde{B}$. Therefore Jensen's inequality is not sufficient to find H_T^*. In this case, H_T^* is either a solution to the first order condition $\frac{\partial}{\partial H_T}\mathbb{E}[\pi_T] = 0$, or occurs at a boundary, i.e. $H_T^* \in \{-\frac{\omega_0^{FRM}}{\overline{\alpha}_1}, \omega_1^{FRM}\}$. To write the first-order condition, we utilize the following lemma proven in Appendix C:

Lemma 2 *For any argument κ_T of π_T, the expectation and the derivative operators can be interchanged, i.e. $\frac{\partial}{\partial \kappa_T}\mathbb{E}[\pi_T] = \mathbb{E}\left[\frac{\partial}{\partial \kappa_T}\pi_T\right]$.*

Let

$$\alpha_T^0 \doteq \frac{c_T \mathbf{1'K_T^0} + F_T - \omega_0^{FRM} - H_T\overline{\alpha}_1}{\omega_1^{FRM} - H_T}, \qquad \alpha_T^1 \doteq \frac{c_T \mathbf{1'K_T^1} + F_T - \omega_0^{FRM} - H_T\overline{\alpha}_1}{\omega_1^{FRM} - H_T},$$
$$\alpha_T^2 \doteq \frac{c_T \mathbf{1'K_T^1} + F_T - E - \omega_0^{FRM} - H_T\overline{\alpha}_1}{\omega_1^{FRM} - H_T}, \quad \alpha_T^B \doteq \frac{\widehat{B}_T - \omega_0^{FRM} - H_T\overline{\alpha}_1}{\omega_1^{FRM} - H_T}.$$

From Lemma 2 (letting $\kappa_T = H_T$), we can write the first-order condition $\frac{\partial}{\partial H_T}\mathbb{E}[\pi_T]$ by using the expression for $\pi_T(\tilde{B})$ in (34) of Corollary 1 and the equivalence in (36). The integration ranges correspond to the regions Ω_T^i in (34) of Corollary 1.

$$\begin{aligned}\mathbb{E}\quad&\left[\frac{\partial \pi_T}{\partial H_T}\right] = \int_{\max(\alpha_T^0,0)}^{\infty} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx \\ +\quad& \int_{\max(\alpha_T^1,0)}^{\max(\alpha_T^0,0)} \frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{\omega_0^{FRM} + x(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T - F_T}{c_T}\right)^{\frac{1}{b}} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx \\ +\quad& \int_{\max(\alpha_T^2,0,\alpha_T^B)}^{\max(\alpha_T^1,0)} (\overline{\alpha}_1 - x)(1+a)\, r_{\alpha_1}(x)\, dx \\ +\quad& \int_{\max(0,\alpha_T^B)}^{\max(\alpha_T^2,0,\alpha_T^B)} \frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{\omega_0^{FRM} + x(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T + E - F_T}{c_T}\right)^{\frac{1}{b}} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx \\ +\quad& \int_{0}^{\max(0,\alpha_T^B)} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx \end{aligned} \tag{38}$$

Both the limits of integration and the integrants in (38) are functions of H_T. Since we do not impose any distributional assumptions on α_1 it is not always possible to find a closed-form solution for H_T^*.

We have $\alpha_T^0 > \alpha_T^1 > \alpha_T^2$ by definition. For $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} \leq \widehat{B}_T$, $\alpha_T^B \geq \overline{\alpha}_1$. Therefore, for $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} \leq \widehat{B}_T$, we either have $\alpha_T^0 > \alpha_T^1 > \alpha_t^2 > \alpha_T^B \geq \overline{\alpha}_1 > 0$ or $\alpha_T^0 > \alpha_T^1 > \alpha_T^B \geq \overline{\alpha}_1 > 0 > \alpha_T^2$. Similar to (31) and (32) we establish

$$\int_{\alpha_T^1}^{\alpha_T^0} \frac{M_T}{c_T}(1+\frac{1}{b}) \left(\frac{\omega_0^{FRM} + x(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T - F_T}{c_T} \right)^{\frac{1}{b}} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx < \int_{\alpha_T^1}^{\alpha_T^0} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx,$$

$$\int_{\alpha_T^B}^{\max(\alpha_T^2, \alpha_T^B)} \frac{M_T}{c_T}(1+\frac{1}{b}) \left(\frac{\omega_0^{FRM} + x(\omega_1^{FRM} - H_T) + \overline{\alpha}_1 H_T - F_T}{c_T} \right)^{\frac{1}{b}} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx < \int_{\alpha_T^B}^{\max(\alpha_T^2, \alpha_T^B)} (\overline{\alpha}_1 - x)(1+a)\, r_{\alpha_1}(x)\, dx.$$

It follows that

$$\frac{\partial \pi_T}{\partial H_T} < \int_0^{\infty} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx + a \int_{\alpha_T^B}^{\alpha_T^1} (\overline{\alpha}_1 - x)\, r_{\alpha_1}(x)\, dx. \tag{39}$$

The first term is equal to 0 and the second term is negative, therefore $\frac{\partial \pi_T}{\partial H_T} < 0$ and $H_T^* = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$. This concludes the proof for part (1) of this case.

If $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} > \widehat{B}_T$, then H_T^* either satisfies $\mathbb{E}\left[\frac{\partial \pi_T}{\partial H_T}\right]\Big|_{H_T^*} = 0$ or occurs at a boundary $\{-\frac{\omega_0^{FRM}}{\overline{\alpha}_1}, \omega_1^{FRM}\}$ depending on the distributions of α_1 and $\boldsymbol{\xi}$. From Jensen's inequality, ω_1^{FRM} dominates $H_T \geq \frac{\widehat{B}_T - \omega_0^{FRM}}{\overline{\alpha}_1}$ because by (36) and Corollary 1, $\pi_T(B_{FRM}(\alpha_1, H_T))$ is concave over its domain for $H_T \geq \frac{\widehat{B}_T - \omega_0^{FRM}}{\overline{\alpha}_1}$. It follows that $H_T^* \in \left\{ \left\{ H_T < \frac{\widehat{B}_T - \omega_0^{FRM}}{\overline{\alpha}_1} \right\} \cup \left\{ \omega_1^{FRM} \right\} \right\}$. ■

Proof of Proposition 4: We first prove the existence of $\overline{c}_F(c_D, \mathbf{H}^*)$. Notice from (35) that the optimal financial risk management level H_T^* depends on c_T[4]. For each financial risk management level H_T, the expected (stage 0) equity value $\mathbb{E}\left[\pi_T\left(c_T, B_{FRM}(\alpha_1, H_T)\right)\right]$ is a continuous function of c_T. It follows that the expected (stage 0) equity value at the optimal risk management level $\mathbb{E}\left[\pi_T\left(c_T, B_{FRM}(\alpha_1, H_T^*(c_T))\right)\right]$ is also a continuous function of c_T (because it is the upper envelope of continuous functions). For a finite $c_D > 0$, $\mathbb{E}\left[\pi_D\left(c_D, B_{FRM}(\alpha_1, H_D^*(c_D))\right)\right]$ is also finite. It is easy to prove that

$$\begin{aligned} \lim_{c_F \to \infty} \mathbb{E}\left[\pi_F\left(c_F, B_{FRM}(\alpha_1, H_F^*(c_F))\right)\right] &= \omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} - (1-\gamma_F)F_F + P, \\ \lim_{c_F \to 0} \mathbb{E}\left[\pi_F\left(c_F, B_{FRM}(\alpha_1, H_F^*(c_F))\right)\right] &\to \infty. \end{aligned}$$

[4]Since from Proposition 3 we cannot guarantee the uniqueness of H_T^*, $H_T^*(c_T)$ is a correspondence.

Since the equity value is continuous in c_F, if $\mathbb{E}\left[\pi_D\left(c_D, B_{FRM}(\alpha_1, H_D^*(c_D))\right)\right] > \omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} - (1-\gamma_F)F_F + P$, then there exists a c_F such that the equity values with both technologies coincide. If $\mathbb{E}\left[\pi_D\left(c_D, B_{FRM}(\alpha_1, H_D^*(c_D))\right)\right] \leq \omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} - (1-\gamma_F)F_F + P$ then the threshold does not exist and the flexible technology is always preferred over the dedicated technology. This concludes the proof for existence of $\overline{c}_F(c_D, \mathbf{H}^*)$. The existence of $\overline{c}_F(c_D, \mathbf{0})$ can be proven in the same manner by substituting $B_{FRM}(.)$ with $B_{-FRM}(.)$ and $H_T^*(c_T)$ with 0.

To prove the uniqueness of $\overline{c}_F(c_D, \mathbf{H}^*)$ and $\overline{c}_F(c_D, \mathbf{0})$ we first provide the following lemma and relegate the proof to Appendix C:

Lemma 3 *In the optimal set of financial risk management levels, for a fixed level of H, the expected (stage 0) value of the equity with technology T strictly decreases in the unit capacity investment cost ($\frac{\partial}{\partial c_T}\mathbb{E}[\pi_T(c_T, B_{FRM}(\alpha_1, H))] < 0$).*

From Lemma 3 it follows that the expected (stage 0) equity value with flexible technology is strictly decreasing in c_F for any (relevant) financial risk management level H_F. This implies the uniqueness of $\overline{c}_F(c_D, \mathbf{H}^*)$. The uniqueness of $\overline{c}_F(c_D, \mathbf{0})$ follows from Lemma 3 using the identity $B_{-FRM}(\alpha_1) = B_{FRM}(\alpha_1, H)$ for $H = 0$ and $F_{FRM} = 0$. For the comparative statics results with respect to demand variability and correlation we first provide the following two lemmas and relegate their proofs to Appendix C. Recall from Corollary 1 that $M_F(\boldsymbol{\xi}) = \mathbb{E}\left[\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}\right]$.

Lemma 4 *$M_F(\boldsymbol{\xi}) \leq M_F(\boldsymbol{\xi}')$ for $\boldsymbol{\xi}'$ that is obtained from $\boldsymbol{\xi}$ with an increase in σ in one of the following ways:*

i) $\boldsymbol{\xi}'$ is obtained by an increase in σ where $\boldsymbol{\xi}$ has a symmetric bivariate lognormal distribution,

ii) $\boldsymbol{\xi}'$ and $\boldsymbol{\xi}$ have independent marginal distributions, equal means ($\overline{\boldsymbol{\xi}} = \overline{\boldsymbol{\xi}'}$), and $\xi_i' \succeq_v \xi_i$ (ξ_i' is stochastically more variable than ξ_i) for $i = 1, 2$ or the variability ordering holds for only one of the marginals and the other marginal is identical,

iii) $\boldsymbol{\xi}'$ is random ($\sigma \neq 0$) while $\boldsymbol{\xi}$ is deterministic ($\sigma = 0$).

Lemma 5 *$M_F(\boldsymbol{\xi}) \geq M_F(\boldsymbol{\xi}')$ for $\boldsymbol{\xi}'$ that is obtained from $\boldsymbol{\xi}$ with an increase in ρ in one of the following ways:*

i) $\boldsymbol{\xi}'$ is obtained by an increase in ρ where $\boldsymbol{\xi}$ has a symmetric bivariate lognormal distribution,

ii) $\boldsymbol{\xi}'$ dominates $\boldsymbol{\xi}$ according to the concordance ordering ($\boldsymbol{\xi}' \succeq_c \boldsymbol{\xi}$),

iii) $\boldsymbol{\xi}'$ is perfectly positively correlated ($\rho = 1$) and $\boldsymbol{\xi}$ is less than perfectly positively correlated ($\rho < 1$).

In Lemma 4 and Lemma 5, case i imposes distributional assumptions on $\boldsymbol{\xi}$ to analyze the effect of σ and ρ, respectively. Case ii of each lemma analyzes different stochastic orderings to capture the effect of product market conditions. Variability ordering is often used in the literature to analyze the effect of increasing variability. Concordance ordering $\boldsymbol{\xi}' \succeq_c \boldsymbol{\xi}$, as stated in Corbett and Rajaram (2005, p. 13), essentially means that (ξ_1', ξ_2') move together more closely than (ξ_1, ξ_2). Case iii focuses on limiting cases.

To establish the comparative statics results, we provide the following lemma and relegate the proof to Appendix C:

Lemma 6 *In the optimal set of financial risk management levels, for a fixed level of H, the expected (stage 0) value of the equity with technology T*

i) strictly decreases in the fixed cost of technology and strictly increases in the salvage rate ($\frac{\partial}{\partial F_T}\mathbb{E}[\pi_T(F_T, B_{FRM}(\alpha_1, H))] < 0$ and $\frac{\partial}{\partial \gamma_T}\mathbb{E}[\pi_T(\gamma_T, B_{FRM}(\alpha_1, H))] > 0$),

ii) decreases in unit financing cost ($\frac{\partial}{\partial a}\mathbb{E}[\pi_T(a, B_{FRM}(\alpha_1, H))] \leq 0$), and the equality only holds for H such that $\omega_0^{FRM} + \overline{\alpha}_1 H \geq c_T \mathbf{1}'\mathbf{K_T^1} + F_T$,

iii) increases in credit limit ($\frac{\partial}{\partial E}\mathbb{E}[\pi_T(E, B_{FRM}(\alpha_1, H))] \geq 0$), and the equality only holds for H such that $\omega_0^{FRM} + \overline{\alpha}_1 H \geq c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E$,

iv) increases in demand variability ($\frac{\partial}{\partial \sigma}\mathbb{E}[\pi_T(E, B_{FRM}(\alpha_1, H))] \geq 0$),

v) decreases in demand correlation ($\frac{\partial}{\partial \rho}\mathbb{E}[\pi_T(E, B_{FRM}(\alpha_1, H))] \leq 0$).

Since the expected (stage 0) equity value is a continuous function of parameters $a, E, F_T, \gamma_T, \rho, \sigma$ for a given financial risk management level H, the expected (stage 0) equity value at the optimal risk management level (which also depends on these parameters) is also continuous in these parameters. Therefore the monotonic relations stated in Lemma 6 are also satisfied in the weak sense (not strict inequality) at the optimal financial risk management level without assuming differentiability (because the expected (stage 0) equity value might not be differentiable at the points where the optimal financial risk management level changes). The comparative static results for $\overline{c}_F(c_D, \mathbf{H}^*)$ follow from Lemma 6. The comparative static results for $\overline{c}_F(c_D, \mathbf{0})$ also follow from Lemma 6 using the identity $B_{-FRM}(\alpha_1) = B_{FRM}(\alpha_1, H)$ for $H = 0$ and $F_{FRM} = 0$.

With symmetric fixed costs and salvage rates, we establish the functional form of $\overline{c}_F^S(c_D)$ with the following Lemma and relegate the proof to Appendix C:

Lemma 7 *When the fixed costs and the salvage rates of the two technologies are symmetric, at $c_F = \overline{c}_F^S(c_D)$ expected (stage 1) equity values, expected (stage 0) equity values at an arbitrary financial* *risk management level H and the optimal financial risk management actions are the same for both technologies, i.e.* $\pi_F(c_F, \tilde{B})\Big|_{c_F=\bar{c}_F^S(c_D)} = \pi_D(c_D, \tilde{B})$ *for* $\tilde{B} \geq 0$, $\mathbb{E}[\pi_F(\bar{c}_F^S(c_D), B_{FRM}(\alpha_1, H))] = \mathbb{E}[\pi_D(c_D, B_{FRM}(\alpha_1, H))]$ *and* $H_F^*(\bar{c}_F^S(c_D)) = H_D^*(c_D)$.

It follows from Lemma 7 that $\bar{c}_F^S(c_D)$ is the unique threshold with financial risk management in the symmetric case ($\bar{c}_F(c_D, \mathbf{H}^*) = \bar{c}_F^S(c_D)$). Using the identity $B_{-FRM}(\alpha_1) = B_{FRM}(\alpha_1, H)$ for $H = 0$ and $F_{FRM} = 0$, it follows from Lemma 7 that $\bar{c}_F^S(c_D)$ is also the unique threshold without financial risk management in the symmetric case ($\bar{c}_F(c_D, \mathbf{0}) = \bar{c}_F^S(c_D)$). We now prove the relation $\bar{c}_F^S(c_D) \geq c_D$. It is sufficient to show

$$\mathbb{E}^{-b}\left[\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}\right] \geq \mathbb{E}^{-b}[\xi_1] + \mathbb{E}^{-b}[\xi_2].$$

From Hardy et al. (1988, p.133,146) if $d \in (0, 1)$ and X and Y are non-negative random variables then the following is true:

$$E^{1/d}\left[(X+Y)^d\right] \geq \mathbb{E}^{1/d}[X^d] + \mathbb{E}^{1/d}[Y^d] \tag{40}$$

where the equality only holds when X and Y are effectively proportional, i.e. $X = \lambda Y$. In the expression for $\bar{c}_F^S(c_D)$ we have $d = -\frac{1}{b} \in (0, 1)$ and $\boldsymbol{\xi} > \mathbf{0}$ therefore we can use this inequality. Replacing X with ξ_1^{-b} and Y with ξ_2^{-b} gives the desired result. Notice that $\bar{c}_F^S(c_D) = c_D$ only if $\xi_1 = k\xi_2$ for $k > 0$. This is only possible if either $\boldsymbol{\xi}$ is deterministic or it is perfectly positively correlated and has a proportional bivariate distribution. ■

Proof of Corollary 2: If the capital markets are perfect we have $E = P \geq c_T \mathbf{1}'\mathbf{K_T^0} + F_T$ and $a = 0$ (as we discussed in Assumption 6). Since we have $\Omega_T^{1234} = \emptyset$, it follows from Proposition 2 that the firm invests in the budget-unconstrained capacity investment level for any budget realization, $\mathbf{K_T^*}(\tilde{B}) = \mathbf{K_T^0}$, and borrows to finance this capacity level, $e_T^*(\tilde{B}) = [c_T \mathbf{1}'\mathbf{K_T^0} + F_T - B]^+$. We obtain

$$\mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right] = \mathbb{E}\left[\pi_T(B_{FRM}(\alpha_1, H))\right]|_{F_{FRM}=0} = \omega_0 + \overline{\alpha}_1\omega_1 - (1-\gamma_T)F_T + \frac{c_T \mathbf{1}'\mathbf{K_T^0}}{-(b+1)} + P,$$

and it follows from Proposition 12 that $\overline{F}_{FRM}^T = 0$ for $T \in \{D, F\}$. If the product markets are perfect ($\mathbf{\Sigma} = \mathbf{0}$), then with symmetric fixed costs and salvage rates, it follows from Proposition 4 that $\bar{c}_F(c_D, \mathbf{H}^*) = \bar{c}_F(c_D, \mathbf{0}) = c_D$. ■

Proof of Corollary 3: The proof of the first argument follows from Proposition 3. For the second argument, we provide a numerical example where the firm optimally fully speculates with flexible technology and fully hedges with dedicated technology. We focus on the case with $F_{FRM} = 0$ such that financial risk management is costless. The horizontal line in Figure 1 denotes the value of not investing any technology; hence the firm optimally chooses flexible technology with full speculation in this example. ■

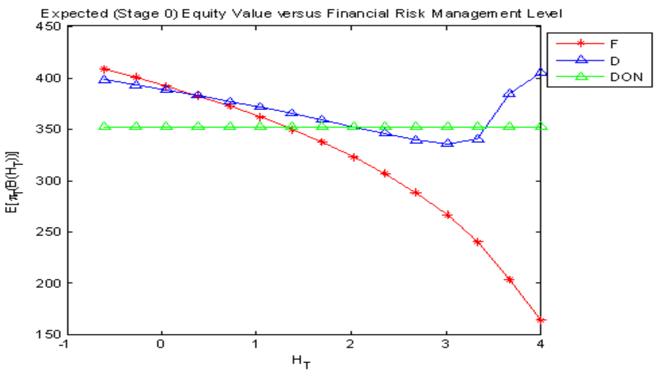

Figure 1: Optimal Speculation is triggered by flexible technology investment: Dedicated technology with full hedging ($H_D^* = \omega_1 = 4$) is dominated by flexible technology with full speculation ($H_F^* = -\frac{w_0}{\alpha_1} = -0.61$.

Proof of Proposition 5: With a hedging constraint, the range of forward contracts is $[0, \omega_1^{FRM}]$ in (35). Substituting $F_{FRM} = 0$ in (38) of Proposition 3, similar to (39), we obtain $\frac{\partial \pi_T}{\partial H_T} < 0$. It follows that $H_T^* = 0$. ■

Proof of Corollary 4: It follows from Proposition 4 that for symmetric fixed costs and salvage rates of technologies and for $F_{FRM} = 0$, the optimal risk management portfolio is flexible (dedicated) technology with financial risk management if $c_F < \overline{c}_F^S(c_D)$ ($c_F > \overline{c}_F^S(c_D)$). From the proof of Proposition 12, for $\beta = \frac{\omega_0}{\omega_0 + \alpha_0 \omega_1}$ we can have a sufficiently large feasible F_{FRM} such that engaging in financial risk management is not profitable. In this case, the optimal risk management portfolio is flexible (dedicated) technology with financial risk management if $c_F < \overline{c}_F^S(c_D)$ ($c_F > \overline{c}_F^S(c_D)$). ■

Proof of Proposition 6: The invariance of $\overline{c}_F(c_D, \mathbf{H}^*)$ and $\overline{c}_F(c_D, \mathbf{0})$ to the unit financing cost, the fixed cost of both technologies and the internal endowment of the firm follows from the definition of $c_F^S(c_D)$ in Proposition 4. For $F = F_D < F_F = F + \delta$ with $\delta > 0$, we obtain $\overline{c}_F(c_D, \mathbf{H}^*) < \overline{c}_F^S(c_D)$ and $\overline{c}_F(c_D, \mathbf{0}) < \overline{c}_F^S(c_D)$ from Proposition 4. We first provide the proof of the results with respect to technology fixed costs. Comparative statics with respect to the internal endowment follow from a similar argument. We define

$$S^{-FRM}(c_F) \doteq \mathbb{E}[\pi_F(c_F, F + \delta, B_{-FRM}(\alpha_1))] - \mathbb{E}[\pi_D(c_D, F, B_{-FRM}(\alpha_1))] \text{ where } S^{-FRM}(\overline{c}_F(c_D, \mathbf{0})) = 0. \quad (41)$$

From the implicit function theorem we have $\frac{\partial}{\partial F}\overline{c}_F(c_D, \mathbf{0}) = -\frac{\partial}{\partial F} S^{-FRM} \left(\frac{\partial}{\partial c_F} S^{-FRM}\right)^{-1}\Big|_{\overline{c}_F(c_D, \mathbf{0})}$.
From Lemma 2, we can interchange derivative and expectation operators, and using Lemma 3 with $B_{-FRM}(\alpha_1) = B_{FRM}(\alpha_1, H)$ for $H = 0$ and $F_{FRM} = 0$, we obtain

$$\frac{\partial S^{-FRM}}{\partial c_F}\Big|_{\overline{c}_F(c_D, \mathbf{0})} = \mathbb{E}\left[\frac{\partial \pi_F(B_{-FRM}(\alpha_1))}{\partial c_F}\right]\Big|_{\overline{c}_F(c_D, \mathbf{0})} < 0.$$

Similarly we have

$$\left.\frac{\partial S^{-FRM}}{\partial F}\right|_{\overline{c}_F(c_D,\mathbf{0})} = \left.\left[\mathbb{E}\left[\frac{\partial \pi_F(B_{-FRM}(\alpha_1))}{\partial F}\right] - \mathbb{E}\left[\frac{\partial \pi_D(B_{-FRM}(\alpha_1)))}{\partial F}\right]\right]\right|_{\overline{c}_F(c_D,\mathbf{0})}.$$

Since $\overline{c}_F(c_D,\mathbf{0}) < \overline{c}_F^S$ and $\delta > 0$ it follows that $c_F\mathbf{K_F^i}|_{\overline{c}_F(c_D,\mathbf{0})} + F + \delta > c_D\mathbf{1}'\mathbf{K_D^i} + F$ for $i = 0, 1$. This implies that $\Omega_F^0 \subset \Omega_D^0$ and $\Omega_F^2 \supset \Omega_D^2$. We obtain

$$\begin{aligned}
\left.\frac{\partial S^{-FRM}}{\partial F}\right|_{\overline{c}_F(c_D,\mathbf{0})} &= \int_{\Omega_F^0 \bigcap \Omega_D^0} (-1+1) \;\; dR_{B_{-FRM}}(\tilde{B}) \\
&+ \int_{\Omega_F^1 \bigcap \Omega_D^0} \left(-\frac{M_F\left(1+\frac{1}{b}\right)}{\overline{c}_F(c_D,\mathbf{0})}\left(\frac{\tilde{B}-F-\delta}{\overline{c}_F(c_D,\mathbf{0})}\right)^{\frac{1}{b}} + 1\right) \;\; dR_{B_{-FRM}}(\tilde{B}) \\
&+ \int_{\Omega_F^1 \bigcap \Omega_D^1} \left(1+\frac{1}{b}\right)\left[-\frac{M_F}{(\overline{c}_F(c_D,\mathbf{0}))^{1+\frac{1}{b}}}\left(\tilde{B}-F-\delta\right)^{\frac{1}{b}} + \frac{M_D}{c_D^{1+\frac{1}{b}}}\left(\tilde{B}-F\right)^{\frac{1}{b}}\right] \;\; dR_{B_{-FRM}}(\tilde{B}) \\
&+ \int_{\Omega_F^2 \bigcap \Omega_D^0} (-(1+a)+1) \;\; dR_{B_{-FRM}}(\tilde{B}) \\
&+ \int_{\Omega_F^2 \bigcap \Omega_D^1} \left[-(1+a) + (1+\frac{1}{b})\frac{M_D}{c_D^{1+\frac{1}{b}}}\left(\tilde{B}-F\right)^{\frac{1}{b}}\right] \;\; dR_{B_{-FRM}}(\tilde{B}) \\
&+ \int_{\Omega_F^2 \bigcap \Omega_D^2} (-(1+a)+(1+a)) \;\; dR_{B_{-FRM}}(\tilde{B}).
\end{aligned} \tag{42}$$

From (31), we have $\frac{\partial}{\partial F}S^{-FRM}|_{\overline{c}_F(c_D,\mathbf{0})} < 0$ for $\tilde{B} \in \Omega_F^1 \bigcap \Omega_D^0$ and $\tilde{B} \in \Omega_F^2 \bigcap \Omega_D^1$. From Lemma 7, we have $\frac{M_D}{c_D^{1+\frac{1}{b}}} = \frac{M_F}{(\overline{c}_F^S(c_D))^{1+\frac{1}{b}}}$. Since $\overline{c}_F(c_D,\mathbf{0}) < \overline{c}_F^S(c_D)$ and $\delta > 0$, we obtain $\frac{\partial}{\partial F}S^{-FRM}|_{\overline{c}_F(c_D,\mathbf{0})} < 0$ for $\tilde{B} \in \Omega_F^1 \bigcap \Omega_D^1$. In conclusion, we have $\left.\frac{\partial}{\partial c_F}S^{-FRM}\right|_{\overline{c}_F(c_D,\mathbf{0})} < 0$ and $\frac{\partial}{\partial F}S^{-FRM}|_{\overline{c}_F(c_D,\mathbf{0})} \le 0$. It follows from the implicit function theorem that $\frac{\partial}{\partial F}\overline{c}_F(c_D,\mathbf{0}) \le 0$ where the equality holds only for $\omega_0 > c_F\mathbf{K_F^0}|_{\overline{c}_F(c_D,\mathbf{0})} + F + \delta$.

To prove the result for $\overline{c}_F(c_D,\mathbf{H}^*)$, we define $S^{FRM}(c_F)$, the counterpart of (41) by replacing $B_{-FRM}(\alpha_1)$ with $B_{FRM}(\alpha_1, H_T^*)$. We have $H_F^*(c_F) = H_D^* = \omega_1^{FRM}$ for $c_F = \overline{c}_F(c_D,\mathbf{H}^*)$. We establish $\left.\frac{\partial}{\partial c_F}S^{FRM}\right|_{\overline{c}_F(c_D,\mathbf{H}^*)} < 0$ using $\left.\frac{\partial}{\partial c_F}H_F^*\right|_{\overline{c}_F(c_D,\mathbf{H}^*)} = 0$. The rest of the proof follows in a similar manner using the facts that $\frac{\partial}{\partial F}H_T^*|_{\overline{c}_F(c_D,\mathbf{H}^*)} = 0$ and that with full-hedging $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM}$ is realized in only one of the regions in (42). In conclusion, it follows from the implicit function theorem that $\frac{\partial}{\partial F}\overline{c}_F(c_D,\mathbf{H}^*) \le 0$ where the equality holds only for $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_F^0 \bigcap \Omega_D^0$ or $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_F^2 \bigcap \Omega_D^2$.

To prove the results with respect to the unit financing cost for $\overline{c}_F(c_D,\mathbf{0})$, we follow the same steps by replacing F with a in $S^{-FRM}(c_F)$. We obtain

$$\left.\frac{\partial S^{-FRM}}{\partial a}\right|_{\overline{c}_F(c_D,\mathbf{0})} = \int_{\Omega_F^2\setminus\Omega_D^2}\left(\tilde{B}-(c_F\mathbf{K_F^1}+F_F)\big|_{\overline{c}_F(c_D,\mathbf{0})}\right)\ dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_F^2\bigcap\Omega_D^2}\left(c_D\mathbf{1}'\mathbf{K_D^1}+F_D-(c_F\mathbf{K_F^1}+F_F)\big|_{\overline{c}_F(c_D,\mathbf{0})}\right)\ dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_F^3\setminus\Omega_D^{23}}-E\ dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_F^3\bigcap\Omega_D^2}\left(-\tilde{B}+(c_F\mathbf{K_F^1}+F_F)\big|_{\overline{c}_F(c_D,\mathbf{0})}-E\right)\ dR_{B_{-FRM}}(\tilde{B}). \tag{43}$$

The first term and the last integrands are negative by the definition of the regions. From above (comparative static with respect to fixed cost) we have $c_D\mathbf{1}'\mathbf{K_D^1}+F_D < c_F\mathbf{K_F^1}\big|_{\overline{c}_F(c_D,\mathbf{0})}+F_F$. This implies $\frac{\partial}{\partial a}S^{-FRM}\big|_{\overline{c}_F(c_D,\mathbf{0})}\leq 0$. We conclude $\frac{\partial}{\partial a}\overline{c}_F(c_D,\mathbf{0})\geq 0$ where the equality holds for $\omega_0 > c_F\mathbf{K_F^1}\big|_{\overline{c}_F(c_D,\mathbf{0})}+F_F$.
The result for $\overline{c}_F(c_D,\mathbf{H}^*)$ can be proven in a similar fashion. It follows that $\frac{\partial}{\partial a}\overline{c}_F(c_D,\mathbf{H}^*)\geq 0$ where the equality holds if $\omega_0^{FRM}+\overline{\alpha}_1\omega_1^{FRM}\in\Omega_F^{01}\bigcup\left(\Omega_F^3\bigcap\Omega_D^3\right)$. ■

Proof of Proposition 7: We only prove the results for small firms. Results related to large firms follow from similar arguments. We define

$$\Upsilon^{\varphi}\doteq\frac{\partial\Delta_T}{\partial\varphi}=\frac{\partial\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1,\omega_1^{FRM})\right)\right]}{\partial\varphi}-\frac{\partial\mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right]}{\partial\varphi} \tag{44}$$

as the derivative of the value of full hedging with respect to the argument φ. For small firms, we have $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1,\omega_1^{FRM})\right)\right]=(\omega_0^{FRM}+\overline{\alpha}_1\omega_1^{FRM}-F_T)(1+a)+\frac{c_T\mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)}+\gamma_TF_T+P$. We analyze each comparative static result separately.

Fixed cost of technology. We obtain

$$\Upsilon^{F_T}=-(1+a)-\int_{\Omega_T^0}-1\ dR_{B_{-FRM}}(\tilde{B}) - \int_{\Omega_T^1}(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}}dR_{B_{-FRM}}(\tilde{B}) - \int_{\Omega_T^2}-(1+a)\ dR_{B_{-FRM}}(\tilde{B}).$$

It is easy to show that $(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}}<1+a$ for $\tilde{B}\in\Omega_T^1$. It follows that $\Upsilon^{F_T}<0$.

Initial endowment. After parameterizing the initial endowment, we obtain $\beta^{\lambda}=\frac{\lambda\omega_0}{\lambda\omega_0+\alpha_0\lambda\omega_1}=\frac{\omega_0}{\omega_0+\alpha_0\omega_1}=\beta$. We have $(\omega_0^{FRM},\omega_1^{FRM})\doteq(\lambda\omega_0-\beta F_{FRM},\lambda\omega_1-\frac{1-\beta}{\alpha_0}F_{FRM})$ and for small firms, it follows that $\frac{\partial\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1,\omega_1^{FRM})\right)\right]}{\partial\lambda}=(\omega_0+\overline{\alpha}_1\omega_1)(1+a)$. After parameterizing the initial endowment, we define $\alpha_T^0 \doteq \frac{\frac{c_T \mathbf{1'K_T^0}+F_T}{\lambda}-\omega_0}{\omega_1}$, $\alpha_T^1 \doteq \frac{\frac{c_T \mathbf{1'K_T^1}+F_T}{\lambda}-\omega_0}{\omega_1}$. We obtain

$$\begin{aligned}\Upsilon^\lambda &= (\omega_0+\overline{\alpha}_1\omega_1)(1+a) - \int_{\max(\alpha_T^0,0)}^{\infty} (\omega_0 + x\omega_1)\, r_{\alpha_1}(x)\, dx \\ &- \int_{\max(\alpha_T^1,0)}^{\max(\alpha_T^0,0)} \frac{M_T(1+1/b)}{c_T}\left(\frac{\lambda(\omega_0+x\omega_1)-F_T}{c_T}\right)^{\frac{1}{b}} (\omega_0+x\omega_1)\, r_{\alpha_1}(x)\, dx \\ &- \int_0^{\max(\alpha_T^1,0)} (\omega_0+x\omega_1)(1+a)\, r_{\alpha_1}(x)\, dx\end{aligned}$$

Notice that negative terms above are the expected value of the following function

$$f(\alpha_1) = \begin{cases} \omega_0+\alpha_1\omega_1 & \text{if } \alpha_1 \geq \alpha_T^0 \\ \frac{M_T(1+1/b)}{c_T}\left(\frac{\lambda(\omega_0+\alpha_1\omega_1)-F_T}{c_T}\right)^{\frac{1}{b}} (\omega_0+\alpha_1\omega_1) & \text{if } \alpha_T^0 > \alpha_1 \geq \alpha_T^1 \\ (\omega_0+\alpha_1\omega_1)(1+a) & \text{if } \alpha_1 < \alpha_T^1 \end{cases}$$

with respect to the asset price distribution α_1. It is easy to prove that $(\omega_0+\alpha_1\omega_1)(1+a) \geq f(\alpha_1)$ for $\alpha_1 \geq 0$ with strict inequality for some α_1. It follows that $\mathbb{E}[(\omega_0+\alpha_1\omega_1)(1+a)] = (\omega_0+\overline{\alpha}_1\omega_1)(1+a) > \mathbb{E}[f(\alpha_1)]$ and we obtain $\Upsilon^\lambda > 0$.

To analyze the effect of cash holdings (ω_0) on the value of financial risk management, we only parameterize the cash holdings as $(\lambda'\omega_0, \omega_1)$ and set $\beta = 0$ such that F_{FRM} is only deducted from the value of asset holdings ω_1. It follows that $\omega_0^{FRM} = \lambda'\omega_0$ and $\omega_1^{FRM} = \omega_1 - \frac{F_{FRM}}{\alpha_0}$. $\Upsilon^{\lambda'} > 0$ follows from the similar lines with $\Upsilon^\lambda > 0$.

Demand variability and correlation. We only provide the proof for demand variability. The proof for demand correlation is along the similar lines. It is sufficient to focus on flexible technology because dedicated technology is not affected from changes in σ and ρ. We obtain

$$\begin{aligned}\Upsilon^\sigma &= \frac{\partial M_F}{\partial\sigma}\left(\frac{c_F M_F(1+\frac{1}{b})}{1+a}\right)^{-b-1} - \int_{\Omega_F^0} \frac{\partial M_F}{\partial\sigma}\left(c_F M_F(1+\frac{1}{b})\right)^{-b-1} dR_{B_{-FRM}}(\tilde{B}) \\ &- \int_{\Omega_F^1} \frac{\partial M_F}{\partial\sigma}\left(\frac{\tilde{B}-F_F}{c_F}\right)^{1+\frac{1}{b}} dR_{B_{-FRM}}(\tilde{B}) - \int_{\Omega_F^2} \frac{\partial M_F}{\partial\sigma}\left(\frac{c_F M_F(1+\frac{1}{b})}{1+a}\right)^{-b-1} dR_{B_{-FRM}}(\tilde{B}).\end{aligned}$$

It is easy to show $\left(\frac{\tilde{B}-F_F}{c_F}\right)^{1+\frac{1}{b}} > \left(\frac{c_F M_F(1+\frac{1}{b})}{1+a}\right)^{-b-1}$ for $\tilde{B} \in \Omega_F^1$. From Lemma 4, we have $\frac{\partial}{\partial\sigma}M_F \geq 0$ and it follows that $\Upsilon^\sigma \leq 0$.

Unit financing cost. We obtain

$$\Upsilon^a = \omega_0 + \overline{\alpha}_1\omega_1 - (\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0})F_{FRM} - c_T\mathbf{1'K_T^1} - F_T - \int_{\Omega_T^2} (\tilde{B} - c_T\mathbf{1'K_T^1} - F_T)\ dR_{B_{-FRM}}(\tilde{B}).$$

It follows that for $\omega_0 > c_T\mathbf{1'K_T^1} + F_T$, when the non-hedged firm does not borrow at all, we have $\Upsilon^a < 0$. We focus on the case where the firm borrows at some budget states without financial risk

management ($\omega_0 < c_T \mathbf{1}'\mathbf{K_T^1} + F_T$).
For $F_{FRM} = 0$, we have $\Upsilon^a = \int_{\Omega_T^2} (c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \tilde{B})\ dR_{B\_FRM}(\tilde{B}) - (c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \overline{B})$ where $\overline{B} = \omega_0 + \overline{\alpha}_1 \omega_1$. Notice that the first term is the expected value of the function

$$f(\tilde{B}) = \begin{cases} c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \overline{B} & \text{if } \ \tilde{B} \leq c_T \mathbf{1}'\mathbf{K_T^1} + F_T \\ 0 & \text{if } \ \tilde{B} > c_T \mathbf{1}'\mathbf{K_T^1} + F_T \end{cases}$$

with respect to the budget distribution. Since $f(\tilde{B})$ is a convex function, $\Upsilon^a \geq 0$ for $F_{FRM} = 0$ follows from Jensen's inequality.
For $F_{FRM} > 0$, we have

$$\Upsilon^a = \int_{\Omega_T^2} (c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \tilde{B})\ dR_{B\_FRM}(\tilde{B}) - (c_T \mathbf{1}'\mathbf{K_T^1} + F_T + (\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}) F_{FRM} - \overline{B}).$$

We observe that the first term is strictly less than $c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \omega_0$. For $F_{FRM} \geq F^0_{FRM} = \frac{\alpha_0 \omega_1}{(1-\beta) + \frac{\beta}{\overline{\alpha}_1}}$, we obtain $c_T \mathbf{1}'\mathbf{K_T^1} + F_T + (\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}) F_{FRM} - \overline{B} \geq c_T \mathbf{1}'\mathbf{K_T^1} + F_T - \omega_0$ and it follows that $\Upsilon^a < 0$. Notice that $F_{FRM} \leq \frac{\alpha_0 \omega_1}{(1-\beta)}$ is the feasiblity condition; hence such F_{FRM} exists. We calculate $\frac{\partial}{\partial F_{FRM}} \Upsilon^a = -(\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}) < 0$. Since Υ^a strictly decreases in F_{FRM}, $\Upsilon^a \geq 0$ for $F_{FRM} = 0$ and $\Upsilon^a < 0$ for F^0_{FRM}, we conclude that there exists a unique $\widehat{F}_{FRM}$ such that $\Upsilon^a < 0$ for $F_{FRM} > \widehat{F}_{FRM}$ and $\Upsilon^a \geq 0$ for $F_{FRM} \leq \widehat{F}_{FRM}$. ■

Proof of Proposition 8: We focus on the case where it is profitable for the firm to engage in financial risk management. To prove the proposition, we use the ordering between $\overline{c}_F(c_D, \mathbf{H}^*)$ and $\overline{c}_F(c_D, \mathbf{0})$. If $\overline{c}_F(c_D, \mathbf{0}) < \overline{c}_F(c_D, \mathbf{H}^*)$ ($\overline{c}_F(c_D, \mathbf{0}) > \overline{c}_F(c_D, \mathbf{H}^*)$) then flexible technology and financial risk management are complements (substitutes) because engaging in financial risk management enables the firm to invest in flexible (dedicated) technology at some technology cost levels where dedicated (flexible) technology was more profitable without financial risk management. From Proposition 4, we obtain $\overline{c}_F(c_D, \mathbf{H}^*) < \overline{c}^S_F(c_D)$ and $\overline{c}_F(c_D, \mathbf{0}) < \overline{c}^S_F(c_D)$. From Assumption 8, we have $H^*_D(c_D) = H^*_F(\overline{c}_F(c_D, \mathbf{H}^*)) = \omega_1^{FRM}$. From Lemma 3, it follows that $\overline{c}_F(c_D, \mathbf{H}^*) \gtreqless \overline{c}_F(c_D, \mathbf{0})$ if and only if

$$\mathbb{E}\left[\pi_D(B_{FRM}(\alpha_1, \omega_1^{FRM}))\right] \lesseqgtr \mathbb{E}\left[\pi_F(\overline{c}_F(c_D, \mathbf{0}), B_{FRM}(\alpha_1, \omega_1^{FRM})))\right]. \tag{45}$$

Recall that $\triangle_T(c_T, F_T)$ is the value of financial risk management with technology $T \in \{D, F\}$ at given cost parameters (c_T, F_T) as defined in (5). Inequality (45) holds if and only if $\triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_F) \gtreqless \triangle_D(c_D, F_D)$. We will use the relation between $\triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_F)$ and $\triangle_D(c_D, F_D)$ to prove the proposition. We provide the following lemma and relegate the proof to Appendix C.

Lemma 8 *For $F_T < \overline{F}_T$ and $E > c_T \mathbf{1}'\mathbf{K_T^1} + F_T$,*

(i) If $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} \in \Omega_T^0$ ($\in \Omega_T^2$) then $\frac{\partial}{\partial F_T} \triangle_T \geq 0$ ($\frac{\partial}{\partial F_T} \triangle_T < 0$);

(ii) If $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} \in \Omega_T^0$ ($\in \Omega_T^2$) then $\frac{\partial}{\partial c_T} \triangle_T \leq 0$ ($\frac{\partial}{\partial c_T} \triangle_T > 0$).

For large firms ($\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_F^0$), we obtain from Lemma 8, $\overline{c}_F^S(c_D) > \overline{c}_F(c_D, \mathbf{0})$ and $F_F \geq F_D$ that

$$\triangle_D(c_D, F_D) = \triangle_F(\overline{c}_F^S(c_D), F_D) \leq \triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_D) \leq \triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_F).$$

From the proof of Lemma 8, the inequalities above are strict for sufficiently low ω_0. We conclude that $\overline{c}_F(c_D, \mathbf{H}^*) \geq \overline{c}_F(c_D, \mathbf{0})$ and large firms tend to use flexible technology and financial risk management as complements.

For small firms ($\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_F^2$), we obtain

$$\triangle_D(c_D, F_D) = \triangle_F(\overline{c}_F^S(c_D), F_D) > \triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_D) > \triangle_F(\overline{c}_F(c_D, \mathbf{0}), F_F).$$

We conclude that $\overline{c}_F(c_D, \mathbf{H}^*) < \overline{c}_F(c_D, \mathbf{0})$ and small firms tend to substitute flexible technology with financial risk management. ■

Proof of Proposition 9: We only prove the results for small firms. Results related to large firms follow from similar arguments. Recall that in the proof of Proposition 6 we defined

$$\begin{aligned} S^{FRM} &= \mathbb{E}\left[\pi_F\left(B_{FRM}(\alpha_1, \omega_1^{FRM})\right)\right] - \mathbb{E}\left[\pi_D\left(B_{FRM}(\alpha_1, \omega_1^{FRM})\right)\right] \\ S^{-FRM} &= \mathbb{E}\left[\pi_F\left(B_{-FRM}(\alpha_1)\right)\right] - \mathbb{E}\left[\pi_D\left(B_{-FRM}(\alpha_1)\right)\right] \end{aligned}$$

as the value of operational risk management with and without financial risk management respectively. The value of operational risk management is more robust to a change in $\varphi \in \{a, \rho, \sigma\}$ with financial risk management then without if

$$\left|\frac{\partial\, S^{FRM}}{\partial\varphi}\right| < \left|\frac{\partial\, S^{-FRM}}{\partial\varphi}\right|.$$

To analyze the robustness of the value of operational risk management, we focus on the cases where operational risk management has a value, i.e. flexible technology is preferred over dedicated technology with and without financial risk management. Recall from the proof of Proposition 6 that we have $\overline{c}_F(c_D, \mathbf{H}^*) < \overline{c}_F^S(c_D)$ and $\overline{c}_F(c_D, \mathbf{0}) < \overline{c}_F^S(c_D)$ in this setting. Therefore, for any relevant unit investment cost pair (c_F, c_D) we have $c_F < \overline{c}_F^S(c_D)$. We now analyze each market condition separately.

Robustness with respect to capital market condition (a). Since $c_F < \overline{c}_F^S(c_D)$, it follows from (43) that $\frac{\partial}{\partial a}S^{FRM} \leq 0$ and $\frac{\partial}{\partial a}S^{-FRM} \leq 0$. Therefore, it is sufficient to show $\frac{\partial}{\partial a}S^{FRM} \leq \frac{\partial}{\partial a}S^{FRM}$ to prove the result of lower robustness. It follows from (44) that this condition is equivalent to $\frac{\partial}{\partial a}\Delta_F \leq \frac{\partial}{\partial a}\Delta_D$. We obtain

$$\begin{aligned} \frac{\partial\Delta_F}{\partial a} - \frac{\partial\Delta_D}{\partial a} &= \omega_0 + \overline{\alpha}_1\omega_1 - s(\overline{\alpha}_1)F_{FRM} - c_F\mathbf{K_F^1} - F_F - \int_{\Omega_F^2}(\tilde{B} - c_F\mathbf{K_F^1} - F_F)\ dR_{B_{-FRM}}(\tilde{B}) \\ - &\quad [\omega_0 + \overline{\alpha}_1\omega_1 - s(\overline{\alpha}_1)F_{FRM} - c_D\mathbf{1}'\mathbf{K_D^1} - F_D]\chi(\overline{B} \in \Omega_D^2) + \int_{\Omega_D^2}(\tilde{B} - c_D\mathbf{1}'\mathbf{K_D^1} - F_D)\ dR_{B_{-FRM}}(\tilde{B}). \end{aligned}$$

where $s(\overline{\alpha}_1) = \beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}$, $\overline{B} = \omega_0 + \overline{\alpha}_1\omega_1 - s(\overline{\alpha}_1)F_{FRM}$ and $\chi(.)$ is the indicator function. We have the indicator function because a small firm (that always borrows with financial risk management with flexible technology) need not to borrow with financial risk management with dedicated technology. We now show that $\frac{\partial}{\partial a}\Delta_F \leq \frac{\partial}{\partial a}\Delta_D$ by focusing on two cases.

Case i : $(\overline{B} \in \Omega_D^2)$ We obtain

$$\begin{aligned}\frac{\partial \Delta_F}{\partial a} - \frac{\partial \Delta_D}{\partial a} &= \int_{\Omega_F^2 \backslash \Omega_D^2} (c_F \mathbf{K_F^1} + F_F - \tilde{B}) \ dR_{B_{-FRM}}(\tilde{B}) \\ &+ \int_{\Omega_F^2 \bigcap \Omega_D^2} (c_F \mathbf{K_F^1} + F_F - c_D \mathbf{1}'\mathbf{K_D^1} - F_D) \ dR_{B_{-FRM}}(\tilde{B}) - (c_F \mathbf{K_F^1} + F_F - c_D \mathbf{1}'\mathbf{K_D^1} - F_D).\end{aligned}$$

Since for $\tilde{B} \in \Omega_F^2 \backslash \Omega_D^2$ we have $\tilde{B} \geq c_D \mathbf{1}'\mathbf{K_D^1} + F_D$, it follows that $\frac{\partial}{\partial a}\Delta_F < \frac{\partial}{\partial a}\Delta_D$.

Case ii : $(\overline{B} \in \Omega_D^{01})$ We obtain

$$\frac{\partial \Delta_F}{\partial a} - \frac{\partial \Delta_D}{\partial a} = \int_{\Omega_F^2 \backslash \Omega_D^2} (c_F \mathbf{K_F^1} + F_F - \tilde{B}) \ dR_{B_{-FRM}}(\tilde{B})$$

$$+ \int_{\Omega_F^2 \bigcap \Omega_D^2} (c_F \mathbf{K_F^1} + F_F - c_D \mathbf{1}'\mathbf{K_D^1} - F_D) \ dR_{B_{-FRM}}(\tilde{B}) - (c_F \mathbf{K_F^1} + F_F - \omega_0 - \overline{\alpha}_1\omega_1 + s(\overline{\alpha}_1)F_{FRM}).$$

Since we have $\omega_0 + \overline{\alpha}_1\omega_1 - s(\overline{\alpha}_1)F_{FRM} > c_D \mathbf{1}'\mathbf{K_D^1} + F_D$, it follows that $\frac{\partial}{\partial a}\Delta_F < \frac{\partial}{\partial a}\Delta_D$. This concludes the proof for the robustness result with respect to capital market condition.

Robustness with respect to product market conditions (ρ, σ)***.*** We only provide the proof for ρ. From Lemma 6, we have $\frac{\partial}{\partial \rho}S^{FRM} \leq 0$ and $\frac{\partial}{\partial \rho}S^{-FRM} \leq 0$. Therefore, it is sufficient to show $\frac{\partial}{\partial \rho}S^{FRM} \geq \frac{\partial}{\partial \rho}S^{FRM}$ to prove the result of higher robustness for small firms. It follows from (44) that this condition is equivalent to $\frac{\partial}{\partial a}\Delta_F \geq 0$. The result follows from Proposition 7. ■

Proof of Proposition 10: To demonstrate the ambiguous effect of financial risk management on expected (stage 0) capacity investment level, it is sufficient to provide examples for each case of $\mathbb{E}[\mathbf{1}'\mathbf{K}^*_{\mathbf{T}^*}(B_{-FRM}(\alpha_1))] \gtreqless \mathbb{E}[\mathbf{1}'\mathbf{K}^*_{\mathbf{T}^*}(B_{FRM}(\alpha_1, H^*_{T^*}))]$. We consider $F_F = F_D = 0$ which implies from Proposition 3 that the firm optimally fully hedges with both technologies $(\Omega_T^4 = \emptyset)$. Let $F_{FRM} = 0$ such that financial risk management is costless. Without loss of generality we consider $c_F < \overline{c}_F$ which implies from Proposition 4 that $T^* = F$ with or without financial risk management. Let E be sufficiently large ($E \geq \frac{c_T \mathbf{1}'\mathbf{K_T^1}(1-ab)}{-(b+1)a}$ is sufficient as follows from Lemma 9 in Appendix B) such that the firm does not borrow up to the credit limit $(\Omega_F^3 = \emptyset)$. With these parameter restrictions, we obtain

$$\begin{aligned}\mathbb{E}[\mathbf{K_F^*}(B_{-FRM}(\alpha_1))] &= \int_{\Omega_F^0} \mathbf{K_F^0} \ dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_F^1} \overline{\mathbf{K}}_\mathbf{F} \ dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_T^2} \mathbf{K_F^1} \ dR_{B_{-FRM}}(\tilde{B}), \\ \mathbb{E}[\mathbf{K_F^*}(B_{FRM}(\alpha_1, \omega_1))] &= \begin{cases} \mathbf{K_F^0} & \text{if} \ \ \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^0 \\ \overline{\mathbf{K}}_\mathbf{F} & \text{if} \ \ \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^1 \\ \mathbf{K_F^1} & \text{if} \ \ \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^2. \end{cases}\end{aligned}$$

We have $\mathbf{K_F^0} > \mathbf{K_F^1}$, and $\mathbf{K_F^0} > \overline{\mathbf{K}}_\mathbf{F} \geq \mathbf{K_F^1}$ for $\tilde{B} \in \Omega_F^1$ with equality only holding for the lower bound of the region Ω_F^1. For $\omega_0 \in \Omega_F^0$ (and hence $\omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^0$), $\mathbb{E}[\mathbf{K_F^*}(B_{-FRM}(\alpha_1))] = \mathbb{E}[\mathbf{K_F^*}(B_{FRM}(\alpha_1,\omega_1))]$. For $\omega_0 \in \Omega_F^2$ and $\omega_0+\overline{\alpha}_1\omega_1 \in \Omega_F^0$, $\mathbb{E}[\mathbf{K_F^*}(B_{-FRM}(\alpha_1))] < \mathbb{E}[\mathbf{K_F^*}(B_{FRM}(\alpha_1,\omega_1))]$. For $\omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^2$ (and hence $\omega_0 \in \Omega_F^2$), $\mathbb{E}[\mathbf{K_F^*}(B_{-FRM}(\alpha_1))] > \mathbb{E}[\mathbf{K_F^*}(B_{FRM}(\alpha_1,\omega_1))]$.

If we relax our assumption on E, we obtain

$$\mathbb{E}[e_F^*(B_{-FRM}(\alpha_1))] = \int_{\Omega_F^2} [\mathbf{c_F K_F^1} - \tilde{B}] \; dR_{B_{-FRM}}(\tilde{B}) + \int_{\Omega_F^3} E \; dR_{B_{-FRM}}(\tilde{B}),$$

$$\mathbb{E}[e_F^*(B_{FRM}(\alpha_1,\omega_1))] = \begin{cases} 0 & \text{if } \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^{01} \\ \mathbf{c_F K_F^1} - \omega_0 - \overline{\alpha}_1\omega_1 & \text{if } \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^2 \\ E & \text{if } \omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^3. \end{cases}$$

It follows that for $\omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^3$ we have $\mathbb{E}[e_F^*(B_{-FRM}(\alpha_1))] < \mathbb{E}[e_F^*(B_{FRM}(\alpha_1,\omega_1))]$ and for $\omega_0 + \overline{\alpha}_1\omega_1 \in \Omega_F^0$ we have $\mathbb{E}[e_F^*(B_{-FRM}(\alpha_1))] > \mathbb{E}[e_F^*(B_{FRM}(\alpha_1,\omega_1))]$. ■

Proof of Corollary 5: The proof follows from Proposition 8. ■

Proof of Corollary 6: From Proposition 3, it follows that small firms, as we define in §7, optimally fully speculates $H_T^* = -\frac{\omega_0}{\overline{\alpha}_1}$. For $\omega_0 = 0$, the firm optimally does not engage in financial risk management. The low value of integration follows from a continuity argument and the bounded derivative of expected (stage 0) equity value with respect to ω_0. For large firms, financial risk management does not have any value if $\omega_0 \geq \mathbf{c_T 1' K_T^0} + F_T$, i.e. the cash level is sufficient to finance the budget-unconstrained optimal capacity investment level. Low value of financial risk management at high cash levels follow from similar arguments with small firms. When the firm uses financial risk management only for hedging purposes, it follows from Proposition 5 that small firms optimally do not engage in financial risk management. Therefore, the value of integration is zero for small firms. Large firms tend to use financial risk management for full-hedging purposes. For $\omega_0 < \mathbf{c_T 1' K_T^0} + F_T$, financial risk management has positive value; hence the value of integration is higher for large firms than small firms. This concludes the proof. ■

Proposition 11 *If the firm does not engage in financial risk management, there exists a unique technology fixed cost threshold* $\underline{F}_T^{-FRM} < \frac{c_T \mathbf{1' K_T^0}}{-(b+1)(1-\gamma_T)}$ *for technology* $T \in \{D, F\}$ *such that when* $F_T < \underline{F}_T^{-FRM}$*, investing in technology* T *without financial risk management is more profitable than not investing in technology.*

If the firm engages in financial risk management, only one of the following cases holds, depending on the level of the fixed cost F_{FRM}*:*

i) There exists a unique technology fixed cost threshold $\underline{F}_T^{FRM} \leq \frac{\frac{c_T \mathbf{1' K_T^0}}{-(b+1)} - \left(\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}\right) F_{FRM}}{1-\gamma_T}$ *for technology* $T \in \{D, F\}$ *such that when* $F_T < \underline{F}_T^{FRM}$*, investing in technology* T *is more* *profitable than not investing in technology; this case occurs at sufficiently low levels of F_{FRM}.*

ii) Not investing in technology is more profitable for $F_T \geq 0$.

Proof of Proposition 11: We first prove the first part of the proposition. From Lemma 6 in the proof of Proposition 4 (using $H = 0$ and $F_{FRM} = 0$), $\mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1), F_T)\right]$ is strictly decreasing in F_T. We define $L_T(\tilde{B}) \doteq \pi_T(\tilde{B}) - (\tilde{B} + P)$, the difference between the equity values of investing in technology T and not investing in technology at each state $\tilde{B}$. It is easy to verify that for $F_T^0 \doteq 0$, $\pi_T(\tilde{B}) > \tilde{B} + P$ for $\tilde{B} \geq 0$. It follows that $\mathbb{E}\left[\pi_T(F_T^0, B_{-FRM}(\alpha_1))\right] > \omega_0 + \overline{\alpha}_1\omega_1 + P$. For $F_T^1 > \frac{c_T \mathbf{1'K_T^0}}{-(b+1)(1-\gamma_T)}$ we have $L_T(\tilde{B}) < 0$ for $\tilde{B} \geq 0$. It follows that $\mathbb{E}\left[\pi_T(F_T^1, B_{-FRM}(\alpha_1))\right] < \omega_0 + \overline{\alpha}_1\omega_1 + P$. Since $\mathbb{E}\left[\pi_T(F_T, B_{-FRM}(\alpha_1))\right]$ is strictly decreasing in F_T, there exists a unique $\underline{F}_T^{-FRM} < \frac{c_T \mathbf{1'K_T^0}}{-(b+1)(1-\gamma_T)}$.

The second part of the proposition follows from a similar argument. We obtain $\mathbb{E}\left[\pi_T(B_{FRM}(\alpha_1, H_T^*))\right] \leq \omega_0 + \overline{\alpha}_1\omega_1 - \left(\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}\right) F_{FRM} - (1-\gamma_T)F_T + \frac{c_T \mathbf{1'K_T^0}}{-(b+1)} + P$, where the latter is the expected (stage 0) equity value with budget-unconstrained optimal capacity investment. It follows that for $F_T > F_T^1 = \frac{\frac{c_T \mathbf{1'K_T^0}}{-(b+1)} - \left(\beta + \frac{(1-\beta)\overline{\alpha}_1}{\alpha_0}\right) F_{FRM}}{1-\gamma_T}$, not investing in technology is more profitable. Two cases may arise with respect to the level of F_{FRM}. When F_{FRM} is sufficiently low, for $F_T^0 = 0$ we have $\mathbb{E}\left[\pi_T(F_T^0, B_{FRM}(\alpha_1, H_T^*))\right] > \omega_0 + \overline{\alpha}_1\omega_1 + P$. In this case (case i), a unique $\underline{F}_T^{FRM} < F_T^1$ exists since $\mathbb{E}\left[\pi_T(F_T, B_{FRM}(\alpha_1, H_T^*))\right]$ is strictly decreasing in F_T. For a sufficiently high level of F_{FRM} and appropriate allocation scheme β (that makes such a F_{FRM} feasible), not investing in technology is more profitable for $F_T = 0$. In this case (case ii), $\underline{F}_T^{FRM}$ does not exist and not investing in technology is more profitable for $F_T \geq 0$. ■

Proposition 12 *Only one of the following cases holds for technology T:*

i) There exists a unique financial risk management fixed cost threshold $\underline{F}_{FRM}^T$ such that when $F_{FRM} < \underline{F}_{FRM}^T$, it is more profitable to engage in financial risk management than not;

ii) For any feasible F_{FRM}, engaging in financial risk management is more profitable than not.

Proof of Proposition 12: The proof follows from showing that $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right]$ strictly decreases in F_{FRM}. From Lemma 2, we can interchange the derivative and expectation operators and using the Leibniz' rule we obtain

$$
\begin{aligned}
\mathbb{E}\left[\frac{\partial \pi_T(B_{FRM}(\alpha_1,H))}{\partial F_{FRM}}\right] &= \int_{\max(\alpha_T^0,0)}^{\infty}(-\beta-\frac{1-\beta}{\alpha_0}x)\; r_{\alpha_1}(x)\; dx \\
&+ \int_{\max(\alpha_T^1,0)}^{\max(\alpha_T^0,0)} \frac{M_T(1+1/b)}{c_T}\left(\frac{U(x)}{c_T}\right)^{\frac{1}{b}}(-\beta-\frac{1-\beta}{\alpha_0}x)\; r_{\alpha_1}(x)\; dx \\
&+ \int_{\max(\alpha_T^2,0,\alpha_T^B)}^{\max(\alpha_T^1,0)}(-\beta-\frac{1-\beta}{\alpha_0}x)(1+a)\; r_{\alpha_1}(x)\; dx \\
&+ \int_{\max(0,\alpha_T^B)}^{\max(\alpha_T^2,0,\alpha_T^B)} \frac{M_T(1+1/b)}{c_T}\left(\frac{U(x)+E}{c_T}\right)^{\frac{1}{b}}(-\beta-\frac{1-\beta}{\alpha_0}x)\; r_{\alpha_1}(x)\; dx \\
&+ \int_{0}^{\max(0,\alpha_T^B)}(-\beta-\frac{1-\beta}{\alpha_0}x)\; r_{\alpha_1}(x)\; dx
\end{aligned} \tag{46}
$$

for any feasible H, where $U(x) = \omega_0^{FRM} + x(\omega_1^{FRM} - H) + \overline{\alpha}_1 H - F_T$. Since all terms are negative, it follows that $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right]$ is strictly decreasing in F_{FRM}. For $F_{FRM} = 0$, we have $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right] \geq \mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right]$ from the optimality of H_T^*. The existence of $\underline{F}_{FRM}^T \leq \min\left(\frac{\omega_0}{\beta}, \frac{\alpha_0\omega_1}{1-\beta}\right)$ depends on the allocation scheme β. If β is such that a sufficiently large level of F_{FRM} is feasible, then since $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right]$ is strictly decreasing in F_{FRM}, there exists a unique $\underline{F}_{FRM}^T$ (case i). Otherwise, since $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right]$ is preferred for $F_{FRM} = 0$, case ii holds.

We show that $\exists\, \beta$ such that case i holds. Let $\beta = \frac{\omega_0}{\omega_0+\alpha_0\omega_1}$. It follows that the condition $F_{FRM} \leq \min\left(\frac{\omega_0}{\beta}, \frac{\alpha_0\omega_1}{1-\beta}\right)$ is equivalent to $F_{FRM} \leq \omega_0+\alpha_0\omega_1$. We obtain $\lim_{F_{FRM}\to\omega_0+\alpha_0\omega_1} B_{FRM}(\alpha_1, H) = 0$; therefore $\mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, H_T^*)\right)\right]|_{F_{FRM}\to\omega_0+\alpha_0\omega_1} < \mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right]$. It follows that a unique $\underline{F}_{FRM}^T$ exists. ∎

Proposition 13 *For technology $T \in \{D, F\}$ there exists a unique variable cost threshold $\overline{c}_T(c_{-T}, H_T^*, 0)$ such that investing in technology T with financial risk management is more profitable than investing in the other technology $(-T)$ without financial risk management.*

Proof of Proposition 13: The proof follows as in Proposition 4, and is omitted. ∎

B Appendix B. Characterization of $\widehat{B}_T$

Recall from Proposition 2 that $\widehat{B}_T$ is the budget threshold below which the firm does not borrow or invest. From the proof of Proposition 2, for $F_T \geq E$, $\widehat{B}_T > F_T - E$ is the unique solution to $G_T(\widehat{B}_T) = 0$ where $G_T(\tilde{B}) \doteq \Psi_T(\tilde{B}) - (\tilde{B} - (1-\gamma_T)F_T + P)$, the difference between the equity values in (29) and not borrowing and not investing in capacity. For $F_T < E$, $\widehat{B}_T$, if it exists on $[0,\infty)$, is unique. For notational convenience, we let $\widehat{B}_T \doteq 0$ if the two curves do not intersect on the domain of $G_T(.)$ for $F_T < E$. From (29) for $\tilde{B} \geq F_T$ we obtain $\lim_{\mathbf{K_T}\to\mathbf{0}^+} \nabla_{\mathbf{K_T}} \Psi_T \to \infty$. It follows that the firm always optimally invests in capacity if internal budget $\tilde{B}$ is sufficient to cover the fixed cost of the technology. We conclude that $F_T - E < \widehat{B}_T < F_T$. Since $\Psi_T(\tilde{B})$ can take four different forms we have four different cases to analyze.

Case 1: $c_T \mathbf{1}'\mathbf{K_T^0} + F_T \leq \tilde{B}$

From (33), $G_T(\tilde{B}) > 0$ in this range, so it is not possible to have $c_T \mathbf{1}'\mathbf{K_T^0} + F_T \leq \widehat{B}_T$.

Case 2: $c_T \mathbf{1}'\mathbf{K_T^1} + F_T \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^0} + F_T$

$$\begin{aligned} G_T(\tilde{B}) &= M_T \frac{(\tilde{B} - F_T)}{c_T} \left(\frac{c_T}{\tilde{B} - F_T}\right)^{-\frac{1}{b}} + \gamma_T F_T + P - (\tilde{B} - (1-\gamma_T)F_T + P) \\ &\geq M_T \frac{(\tilde{B} - F_T)}{c_T} \left(\frac{1}{\mathbf{1}'\mathbf{K_T^0}}\right)^{-\frac{1}{b}} + F_T - \tilde{B} = \frac{-1}{b+1}(\tilde{B} - F_T) > 0. \end{aligned}$$

Therefore, it is not possible to have $c_T \mathbf{1}'\mathbf{K_T^1} + F_T \leq \widehat{B}_T < c_T \mathbf{1}'\mathbf{K_T^0} + F_T$.

Case 3: $c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E \leq \tilde{B} < c_T \mathbf{1}'\mathbf{K_T^1} + F_T$

$$\begin{aligned} G_T(\widehat{B}_T) &= (\widehat{B}_T - F_T)(1+a) + \frac{c_T \mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)} + \gamma_T F_T + P - (\widehat{B}_T - (1-\gamma_T)F_T + P) = 0 \\ &\Rightarrow \widehat{B}_T = F_T - \frac{c_T \mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)a}. \end{aligned}$$

For $\widehat{B}_T$ to be feasible in Case 3, $\widehat{B}_T \geq 0$ and $\widehat{B}_T \geq c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E$ should hold. Therefore, if $F_T \geq \frac{c_T \mathbf{1}'\mathbf{K_T^1}(1+a)}{-(b+1)a}$ and $E \geq \frac{c_T \mathbf{1}'\mathbf{K_T^1}(1-ab)}{-(b+1)a}$ then $\widehat{B}_T$ is feasible. Otherwise, it is not possible to have $c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E \leq \widehat{B}_T < c_T \mathbf{1}'\mathbf{K_T^1} + F_T$.

Case 4: $c_T \mathbf{1}'\mathbf{K_T^1} + F_T - E > \tilde{B}$

In this case, we can derive a sufficient condition for non-existence of intersection. We obtain

$$\begin{aligned} G_T(\tilde{B}) &= -E(1+a) + M_T \frac{(E + \tilde{B} - F_T)}{c_T} \left(\frac{c_T}{E + \tilde{B} - F_T}\right)^{-\frac{1}{b}} + \gamma_T F_T + P - (\tilde{B} - (1-\gamma_T)F_T + P) \\ &\geq -E(1+a) + M_T \frac{(E + \tilde{B} - F_T)}{c_T} \left(\frac{1}{\mathbf{1}'\mathbf{K_T^1}}\right)^{-\frac{1}{b}} + F_T - \tilde{B} \\ &\geq \frac{E(1+a)}{-(b+1)} + \frac{1-ab}{-(b+1)}(\tilde{B} - F_T). \end{aligned}$$

Therefore if $F_T < \frac{E(1+a)}{1-ab}$, then $G_T(\tilde{B}) > 0$ and it is not possible to have $c_T\mathbf{1'K^1_T} + F_T - E > \widehat{B}_T$. Otherwise, $\widehat{B}_T$ is a solution of a non-integer polynomial of degree $\frac{b}{b+1}$ and it is not possible to find closed-form expression in the whole range of parameters. The following lemma summarizes the analysis and provides a closed-form expression for $\widehat{B}_T$ for a subset of parameter levels.

Lemma 9 *Let E be such that $E \geq \frac{c_T\mathbf{1'K^1_T}(1-ab)}{-(b+1)a}$.*
If $F_T \leq \frac{c_T\mathbf{K^1_T}(1+a)}{-(b+1)a}$ then $\widehat{B}_T = 0$ and $\Omega^{34}_T = \emptyset$.
If $F_T > \frac{c_T\mathbf{1'K^1_T}(1+a)}{-(b+1)a}$ then $\widehat{B}_T = F_T - \frac{c_T\mathbf{1'K^1_T}(1+a)}{-(b+1)a}$ and $\Omega^3_T = \emptyset$.

We also provide the following lemma which we will occasionally use in the comparative statics analysis throughout the paper.

Lemma 10 *The budget threshold $\widehat{B}_T$ is increasing in c_T, F_T, a and decreasing in E.*

Proof We only provide the proof for the result related to a. The other results can be shown in a similar fashion. Let $\widehat{B}_T(a^i),\ \ i = 0, 1$ define the threshold levels for an arbitrary $a^0 < a^1$. We want to show that $\widehat{B}_T(a^0) \leq \widehat{B}_T(a^1)$. Notice that not only the functional form of $G_T(\tilde{B})$ in any region but also the budget levels defining the regions in (29) depend on a. We obtain

$$\frac{\partial G_T(\tilde{B})}{\partial a} = \begin{cases} 0 & \text{if } \ c_T\mathbf{1'K^0_T} + F_T \leq \tilde{B} \\ 0 & \text{if } \ c_T\mathbf{1'K^1_T} + F_T \leq \tilde{B} < c_T\mathbf{1'K^0_T} + F_T \\ \tilde{B} - c_T\mathbf{1'K^1_T} - F_T & \text{if } \ c_T\mathbf{1'K^1_T} + F_T - E \leq \tilde{B} < c_T\mathbf{1'K^1_T} + F_T \\ -E & \text{if } \ F_T - E < \tilde{B} < c_T\mathbf{1'K^1_T} + F_T - E. \end{cases}$$

at the points where $G_T(\tilde{B})$ is differentiable in a. It follows that $\frac{\partial}{\partial a}G_T(\tilde{B}) \leq 0$ for any $\tilde{B}$ where the function is differentiable. Since $G_T(\tilde{B})$ is a continuous function of $\tilde{B}$ for any a, we conclude that $G_T(\tilde{B})$ is decreasing in a. This implies $G_T(\tilde{B}, a^0) \geq G_T(\tilde{B}, a^1)$ for $\tilde{B} > F_T - E$. At this point, two different cases may arise regarding the definition of $\widehat{B}_T(a^0)$. If $\widehat{B}_T(a^0)$ is the solution of $G_T(\tilde{B}, a^0) = 0$, then we have $G_T(\widehat{B}_T(a^0), a^1) \leq G_T(\widehat{B}_T(a^0), a^0) = 0$. Since $G_T(\tilde{B})$ is increasing in $\tilde{B}$ from (30), it follows that $\widehat{B}_T(a^1) \geq \widehat{B}_T(a^0)$. If $\widehat{B}_T(a^0) = 0$ because $G_T(\tilde{B}, a^0) > 0$ for $\tilde{B} \geq 0$, then from (30) either we have $\widehat{B_T}(a^1) = 0$, ($G_T(\tilde{B}, a^1) > 0$ for $\tilde{B} \geq 0$) or $\widehat{B}_T(a^1)$ is a solution to $G_T(\tilde{B}, a^1) = 0$. In either case, we have $\widehat{B}_T(a^1) \geq \widehat{B}_T(a^0)$. ■

C Appendix C. Proofs of Supporting Lemmas

Proof of Lemma 1: From Appendix B, we calculate

$$\frac{\partial G_T(\tilde{B})}{\partial F_T} = \begin{cases} 0 & \text{if } c_T \mathbf{1}'\mathbf{K}^{\mathbf{0}}_{\mathbf{T}} + F_T \le \tilde{B} \\ -\frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} + 1 & \text{if } c_T \mathbf{1}'\mathbf{K}^{\mathbf{1}}_{\mathbf{T}} + F_T \le \tilde{B} < c_T \mathbf{1}'\mathbf{K}^{\mathbf{0}}_{\mathbf{T}} + F_T \\ -a & \text{if } c_T \mathbf{1}'\mathbf{K}^{\mathbf{1}}_{\mathbf{T}} + F_T - E \le \tilde{B} < c_T \mathbf{1}'\mathbf{K}^{\mathbf{1}}_{\mathbf{T}} + F_T \\ -\frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{\frac{1}{b}} + 1 & \text{if } F_T - E < \tilde{B} < c_T \mathbf{1}'\mathbf{K}^{\mathbf{1}}_{\mathbf{T}} + F_T - E. \end{cases}$$

From (31), (32) and the continuity of $G_T(\tilde{B})$, it follows that $G_T(\tilde{B})$ strictly decreases in F_T for $\tilde{B} < c_T \mathbf{1}'\mathbf{K}^{\mathbf{0}}_{\mathbf{T}} + F_T$. Recall from Proposition 2 (or Appendix B) that either $\widehat{B}_T$ is a solution to $G_T(\tilde{B}) = 0$ or $\widehat{B}_T = 0$ (if $G_T(\tilde{B}) > 0$ for $\tilde{B} \ge 0$).

We first prove the necessity of the second argument. Let $\overline{F}_T$ be the fixed cost that satisfies $G_T(\widehat{B}_T(\overline{F}_T), \overline{F}_T) = 0$ with $\widehat{B}_T(\overline{F}_T) = 0$. In other words, $\overline{F}_T$ is the fixed cost of technology T that makes the two equity values intersect at $\tilde{B} = 0$. From Appendix B, it follows that for $F_T = 0$, $G_T(\tilde{B}) > 0$ for $\tilde{B} \ge 0$. For $F_T \ge E$, we have $\lim_{\tilde{B} \to (F_T - E)^+} G_T(\tilde{B}) < 0$, and two curves intersect at $\widehat{B}_T > F_T - E$. Since $G_T(\tilde{B})$ is strictly decreasing in F_T, such an $\overline{F}_T < E$ always exists. Let $F^0_T \le \overline{F}_T$ be an arbitrary fixed cost. We have $G_T(\widehat{B}_T(\overline{F}_T), F^0_T) < G_T(\widehat{B}_T(\overline{F}_T), \underline{F}_T) = 0$ since $\widehat{B}_T < c_T \mathbf{1}'\mathbf{K}^{\mathbf{0}}_{\mathbf{T}} + F_T$ (follows from Appendix B) and G_T strictly decreases in F_T. From (30) we have $G_T(\tilde{B})$ is strictly increasing in $\tilde{B}$ so it follows that $\widehat{B}_T(F^0_T) > \widehat{B}_T(\overline{F}_T) = 0$.

We now prove the necessity of the first argument. Let $F^1_T < \overline{F}_T$ be an arbitrary fixed cost. Since $\widehat{B}_T(\overline{F}_T) = 0$ and $G_T(\tilde{B})$ strictly decreases in F_T, we have $G_T(\tilde{B}, F^1_T) > 0$ for $\tilde{B} \ge 0$. This implies that $\widehat{B}_T(F_T) = 0$ for $F_T < \overline{F}_T$. The uniqueness of $\overline{F}_T$ follows from the fact that $G_T(\tilde{B})$ is strictly decreasing in F_T and the uniqueness of $\widehat{B}_T$.

The proof for sufficiency follows easily using a contrapositive argument. ■

Proof of Lemma 2: The expectation and differentiation operators can be interchanged if the function under expectation is integrable and satisfies the Lipschitz condition of order one (Glasserman 1994, p.245). The function $\pi_T(\tilde{\alpha}_1)$ satisfies the Lipschitz condition of order one if

$$\frac{|\pi_T(\tilde{\alpha}_1') - \pi_T(\tilde{\alpha}_1'')|}{|\tilde{\alpha}_1' - \tilde{\alpha}_1''|} \le Y_{\pi_T} \;\; \forall\, (\tilde{\alpha}_1', \tilde{\alpha}_1'') > 0 \text{ for some } Y_{\pi_T} \text{ with } \mathbb{E}[Y_{\pi_T}] < \infty. \tag{47}$$

Clearly, condition (47) is satisfied if $\left|\frac{\partial \pi_T}{\partial \tilde{\alpha}_1}\right|$ is bounded. Note that $\frac{\partial}{\partial \tilde{\alpha}_1}\pi_T = \left(\frac{\partial}{\partial \tilde{B}}\pi_T\right)\left(\frac{\partial}{\partial \tilde{\alpha}_1}\tilde{B}\right) = \left(\frac{\partial}{\partial \tilde{B}}\pi_T\right)(\omega_1 - H_T)$. From Corollary 1, we know that π_T is differentiable in $\tilde{\alpha}_1$ everywhere except at α^B_T as defined in (38). If $\tilde{B} \in \Omega^1_T$ we have

$$\frac{\partial \pi_T}{\partial \tilde{B}} \le \frac{M_T}{c_T}(1 + \frac{1}{b})\left(\mathbf{1}'\mathbf{K}^{\mathbf{1}}_{\mathbf{T}}\right)^{\frac{1}{b}} \le (1+a),$$

and for $\tilde{B} \in \Omega_T^3$ since $\widehat{B_T} \geq 0$ and $E_T > F_T$ (from (11)) we have

$$\frac{\partial \pi_T}{\partial \tilde{B}} \leq \frac{M_T}{c_T}(1+\frac{1}{b})\left(\frac{E-F_T}{c_T}\right)^{\frac{1}{b}} \leq Y_T$$

where $1 + a_T < Y_T < \infty$. It follows that $\left|\frac{\partial \pi_T}{\partial \tilde{\alpha}_1}\right| \leq Y_T(\omega_1 - H_T) < \infty$ for $\alpha_1 \geq 0$ except α_T^B. Since π_T is continuous in α_1 and the first derivative is bounded at the differentiable points of π_T, the non-differentiability at α_T^B does not violate (47). Since $\pi_T(\tilde{\alpha}_1)$ is integrable, the interchange of the derivative and expectation is justified. ■

Proof of Lemma 3: From Lemma 2, we can interchange the derivative and the expectation operators and using the Leibniz' rule we obtain

$$\begin{aligned} \frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))]}{\partial c_T} &= \int_{\Omega_T^0} -\mathbf{1'K_T^0}\ dR_{B_{FRM}(H)}(\tilde{B}) \\ &+ \int_{\Omega_T^1} -(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} dR_{B_{FRM}(H)}(\tilde{B}) \\ &+ \int_{\Omega_T^2} -\mathbf{1'K_T^1}(1+a)\ dR_{B_{FRM}(H)}(\tilde{B}) \\ &+ \int_{\Omega_T^3} -(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{E+\tilde{B}-F_T}{c_T}\right)^{1+\frac{1}{b}} dR_{B_{FRM}(H)}(\tilde{B}). \end{aligned} \quad (48)$$

It follows that $\frac{\partial}{\partial c_T}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] \leq 0$ with equality holding only for $H = \omega_1^{FRM}$ and $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_T^4$ (i.e. $\Omega_T^{0123} = \emptyset$). From Proposition 3, we know that in this case $H_T^* = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$, so we can ignore $H = \omega_1^{FRM}$. In other words, in the relevant set of $B_{FRM}(\alpha_1, H)$ we have $\frac{\partial}{\partial c_T}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] < 0$. ■

Proof of Lemma 4:

Case *i*): The proof follows from Lemma 3 of Chod et al. (2006) by substituting $\tau = 1$ and noting that ρ and σ in that paper correspond to parameters of the underlying bivariate normal distribution ($\ln \boldsymbol{\xi}$) of $\boldsymbol{\xi}$. In our paper, ρ and σ are the parameters of $\boldsymbol{\xi}$ in the covariance matrix $\boldsymbol{\Sigma}$.

Case *ii*): We only prove the more general case where both of the marginal distributions of $\boldsymbol{\xi}'$ are pairwise stochastically more variable than the marginal distributions of $\boldsymbol{\xi}$. The proof for the case where one of the marginals is identical for ξ_i' and ξ_i is a special case of this proof. For $\xi_i \geq 0$, $\xi_i' \geq 0$ and $\overline{\xi}_i = \overline{\xi'}_i$ it follows from Ross (1983, p.271) that $\xi_i' \succeq_v \xi_i$ if and only if $\mathbb{E}[h(\xi_i)] \leq \mathbb{E}[h(\xi_i')]$ for all convex functions $h(.)$. With independent marginal distributions of $\boldsymbol{\xi}$ we have

$$\mathbb{E}\left[\left(\xi_1^{-b}+\xi_2^{-b}\right)^{-\frac{1}{b}}\right] = \int_0^\infty \int_0^\infty \left(x_1^{-b}+x_2^{-b}\right)^{-\frac{1}{b}} f_1(x_1)f_2(x_2)dx_1dx_2 = \int_0^\infty g(x_1;x_2)f_1(x_1)dx_1$$

where $f_i(.)$ is the marginal distribution of ξ_i and $g(k;x_2) = \int_0^\infty \left(k^{-b}+x_2^{-b}\right)^{-\frac{1}{b}} f_2(x_2)dx_2$ for $k \geq 0$. To conclude the proof, we need to show that $g(k;x_2)$ is convex in k and $\left(k^{-b}+x_2^{-b}\right)^{-\frac{1}{b}}$ is convex in x_2. To prove both of the desired convexity results, it is sufficient to show that $g^{'}(k, x_2)$ is convex in k. We obtain

$$\frac{\partial^2 g^{'}}{\partial k} = (-b-1)\left(k^{-b} + x_2^{-b}\right)^{-\frac{1}{b}-1} k^{-b-2} \frac{x_2^{-b}}{\left(k^{-b} + x_2^{-b}\right)} \geq 0$$

for $k \geq 0$ and $x_2 \geq 0$. This concludes the proof.

Case iii): Follows from (40) in the proof of Proposition 4. ■

Proof of Lemma 5:

Case i): The proof follows from Lemma 4 of Chod et al. (2006) by substituting $\tau = 1$ and noting that ρ and σ in that paper correspond to parameters of the underlying bivariate normal distribution ($\ln \boldsymbol{\xi}$) of $\boldsymbol{\xi}$. In our paper, ρ and σ are the parameters of $\boldsymbol{\xi}$ in the covariance matrix $\boldsymbol{\Sigma}$.

Case ii): The proof of this case is adapted from Corbett and Rajaram (2005). If $\boldsymbol{\xi}^{'} \succeq_c \boldsymbol{\xi}$, it follows from Muller and Scarsini (2000, p.110) that $\boldsymbol{\xi}^{'} \succeq_{sm} \boldsymbol{\xi}$ ($\boldsymbol{\xi}^{'}$ dominates $\boldsymbol{\xi}$ in the sense of supermodular order). From the definition of supermodular stochastic ordering, it is sufficient to show that $g(\xi_1, \xi_2) = -\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}}$ is supermodular. From Muller and Scarcini (2003), it follows that g is supermodular if and only if all mixed derivatives are non-negative, i.e. $\frac{\partial^2}{\partial \xi_1 \partial \xi_2} g \geq 0$ for $\boldsymbol{\xi} \geq \mathbf{0}$. We obtain

$$\frac{\partial^2 g}{\partial \xi_1 \partial \xi_2} = (-b-1)\left(\xi_1^{-b} + \xi_2^{-b}\right)^{-\frac{1}{b}-2} (\xi_1 \xi_2)^{-b-2} \geq 0.$$

This concludes the proof.

Case iii): Follows from (40) in the proof of Proposition 4. ■

Proof of Lemma 6:

Case i): As in Lemma 3, we obtain

$$\begin{aligned}
\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))]}{\partial F_T} &= \int_{\Omega_T^0} -(1-\gamma_T)\ dR_{B_{FRM}(H)}(\tilde{B}) \\
&+ \int_{\Omega_T^1} -\left[(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B} - F_T}{c_T}\right)^{\frac{1}{b}} - \gamma_T\right] dR_{B_{FRM}(H)}(\tilde{B}) \\
&+ \int_{\Omega_T^2} -(1-\gamma_T + a)\ dR_{B_{FRM}(H)}(\tilde{B}) \\
&+ \int_{\Omega_T^3} -\left[(1+\frac{1}{b})\frac{M_T}{c_T}\left(\frac{E + \tilde{B} - F_T}{c_T}\right)^{\frac{1}{b}} - \gamma_T\right] dR_{B_{FRM}(H)}(\tilde{B}) \\
&+ \int_{\Omega_T^4} -(1-\gamma_T)\ dR_{B_{FRM}(H)}(\tilde{B}).
\end{aligned} \tag{49}$$

Since $\gamma_T < 1$ by definition, it follows from (31) and (32) that the second and the fourth terms are negative. This implies that $\frac{\partial}{\partial F_T}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] < 0$. We have $\frac{\partial}{\partial \gamma_T}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] = F_T$

and it follows that the expected (stage 0) equity value is strictly increasing in the salvage rate for $F_T > 0$.

Case ii): We obtain

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))]}{\partial E} = \int_{\Omega_T^3} \left(-(1+a) + \frac{M_T\left(1+\frac{1}{b}\right)}{c_T} \left(\frac{E + \tilde{B} - F_T}{c_T} \right)^{\frac{1}{b}} \right) \; dR_{B_{FRM}(H)}(\tilde{B}).$$

It follows that $\frac{\partial}{\partial E}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] \geq 0$ with equality holding for H such that $\omega_0^{FRM} + \overline{\alpha}_1 H \geq c_T \mathbf{1'K_T^1} + F_T - E$; or $H = \omega_1^{FRM}$ and $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} < \widehat{B}_T$. From Proposition 3 we know that in the latter case $H_T^* = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$ and we can ignore this case in the relevant set of financial risk management levels.

Case iii): We obtain

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))]}{\partial a} = \int_{\Omega_T^2} \left(\tilde{B} - c_T \mathbf{1'K_T^1} - F_T \right) \; dR_{B_{FRM}(H)}(\tilde{B}) - \int_{\Omega_T^3} E_T \; dR_{B_{FRM}(H)}(\tilde{B}).$$

It follows that $\frac{\partial}{\partial a}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] \leq 0$ with equality holding for $\omega_0^{FRM} + \overline{\alpha}_1 H \geq c_T \mathbf{1'K_T^1} + F_T$; or $H = \omega_1^{FRM}$ and $\omega_0^{FRM} + \overline{\alpha}_1 \omega_1^{FRM} < \widehat{B}_T$. From Proposition 3 we know that in the latter case $H_T^* = -\frac{\omega_0^{FRM}}{\overline{\alpha}_1}$ and we can ignore this case in the relevant set of financial risk management levels.

Case iv): The expected (stage 0) equity value with dedicated technology is independent of σ. Therefore, we focus only on flexible technology. We obtain

$$\begin{aligned} \frac{\partial \mathbb{E}[\pi_F(B_{FRM}(\alpha_1, H))]}{\partial \sigma} &= \int_{\Omega_F^0} \frac{\partial M_F}{\partial \sigma} \left(c_F M_F (1 + \frac{1}{b}) \right)^{-b-1} dR_{B_{FRM}(H)}(\tilde{B}) \qquad (50) \\ &+ \int_{\Omega_F^1} \frac{\partial M_F}{\partial \sigma} \left(\frac{\tilde{B} - F_F}{c_F} \right)^{1+\frac{1}{b}} dR_{B_{FRM}(H)}(\tilde{B}) \\ &+ \int_{\Omega_F^2} \frac{\partial M_F}{\partial \sigma} \left(\frac{c_F M_F (1 + \frac{1}{b})}{1+a} \right)^{-b-1} dR_{B_{FRM}(H)}(\tilde{B}) \\ &+ \int_{\Omega_F^3} \frac{\partial M_F}{\partial \sigma} \left(\frac{E + \tilde{B} - F_F}{c_F} \right)^{1+\frac{1}{b}} dR_{B_{FRM}(H)}(\tilde{B}). \end{aligned}$$

From Lemma 4, we have $\frac{\partial}{\partial \sigma} M_F \geq 0$ with respect to our definitions of demand variability. It follows that $\frac{\partial}{\partial \sigma}\mathbb{E}[\pi_T(B_{FRM}(\alpha_1, H))] \geq 0$.

Case v): The proof of the comparative static result with respect to ρ is similar to σ and is omitted.

■

Proof of Lemma 7: It is easy to verify that we have $c_F \mathbf{K_F^j}\big|_{\overline{c}_F^S(c_D)} = c_D \mathbf{1'K_D^j}$ for $j = 0, 1$. Since $F_F = F_D$ and $\gamma_F = \gamma_D$ from (29) we have $\Psi_F(\tilde{B}) = \Psi_D(\tilde{B})$ which implies $\widehat{B}_F = \widehat{B}_D$. It follows that the regions in (1) overlap, i.e. $\Omega_F^i \equiv \Omega_D^i$ for $i = 0, .., 4$. Since the budget distribution

$B_{FRM}(H)$ is independent of cost parameters, the expected (stage 0) equity values are the same at the threshold level. Moreover, from (35), it follows that $H_F^*(\bar{c}_F^S(c_D)) = H_D^*(c_D)$ because both of them are solutions to the same optimization problem. ■

Proof of Lemma 8: Recall from the proof of Proposition 7 we have

$$\Upsilon^\varphi \doteq \frac{\partial \Delta_T}{\partial \varphi} = \frac{\partial \mathbb{E}\left[\pi_T\left(B_{FRM}(\alpha_1, \omega_1^{FRM})\right)\right]}{\partial \varphi} - \frac{\partial \mathbb{E}\left[\pi_T(B_{-FRM}(\alpha_1))\right]}{\partial \varphi}$$

For $\varphi = c_T$ ($\varphi = F_T$), we calculate the derivative from Lemma 3 (Lemma 6) by letting $\Omega_T^{34} = \emptyset$ (because of our assumptions on F_T and E).

In (48) of Lemma 6, for $\tilde{B} \in \Omega_T^1$ we have

$$1 - \gamma_T < \left[(1 + \frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B} - F_T}{c_T}\right)^{\frac{1}{b}} - \gamma_T\right] < 1 + a - \gamma_T.$$

For $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_T^0$, it follows that

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, \omega_1^{FRM}))]}{\partial F_T} = -(1 - \gamma_T) \geq \frac{\partial \mathbb{E}[\pi_T(B_{-FRM}(\alpha_1))]}{\partial F_T},$$

and we obtain $\Upsilon^{F_T} \geq 0$ where the equality holds for $\omega_0 > c_T\mathbf{1}'\mathbf{K_T^0} + F_T$.

For $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_T^2$,

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, \omega_1^{FRM}))]}{\partial F_T} = -(1 + a - \gamma_T) < \frac{\partial \mathbb{E}[\pi_T(B_{-FRM}(\alpha_1))]}{\partial F_T},$$

and we obtain $\Upsilon^{F_T} < 0$. This concludes the proof for part (i).

Similarly, in (49) of Lemma 6, for $\tilde{B} \in \Omega_T^1$ we have

$$\left|\mathbf{1}'\mathbf{K_T^0}\right| > \left|(1 + \frac{1}{b})\frac{M_T}{c_T}\left(\frac{\tilde{B} - F_T}{c_T}\right)^{1+\frac{1}{b}}\right| > \left|\mathbf{1}'\mathbf{K_T^1}(1 + a)\right|.$$

For $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_T^0$, it follows that

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, \omega_1^{FRM}))]}{\partial c_T} = -\mathbf{1}'\mathbf{K_T^0} \leq \frac{\partial \mathbb{E}[\pi_T(B_{-FRM}(\alpha_1))]}{\partial c_T}$$

and we obtain $\Upsilon^{c_T} \leq 0$ where the equality holds for $\omega_0 > c_T\mathbf{1}'\mathbf{K_T^0} + F_T$.

For $\omega_0^{FRM} + \overline{\alpha}_1\omega_1^{FRM} \in \Omega_T^2$,

$$\frac{\partial \mathbb{E}[\pi_T(B_{FRM}(\alpha_1, \omega_1^{FRM}))]}{\partial c_T} = -\mathbf{1}'\mathbf{K_T^1}(1 + a) > \frac{\partial \mathbb{E}[\pi_T(B_{-FRM}(\alpha_1))]}{\partial c_T}$$

and we obtain $\Upsilon^{c_T} > 0$. This concludes the proof for part (ii). ■

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