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Incorporating diversification into risk management

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Incorporating Diversification into Risk Management*

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Incorporating Diversification into Risk Management

Abstract

Artzner, Delbaen, Eber and Heath (1999) introduce the concept of a *coherent* risk measure. However, these measures only allow the addition of riskfree capital to reduce portfolio risk. In contrast, through portfolio rebalancing, our proposed measure enables diversification to lower risk. Consequently, the importance of derivative and insurance contracts to risk management is explicitly recognized. Moreover, we derive the price of portfolio insurance, a security whose addition to a portfolio ensures its acceptability to an external regulator. Throughout our analysis, market frictions such as illiquidity and transaction costs may be incorporated into portfolio rebalancing decisions.

1 Introduction

The question of "how a firm should measure its risk?" is of fundamental importance to financial practice. According to Modigliani and Miller (1958), risk management policies are irrelevant. However, market frictions such as bankruptcy costs, managerial incentives, taxes and costly external financing motivate risk management activities (see Stulz (1984), Smith and Stulz (1985) and Froot, Scharfstein and Stein (1993)).

Given the wide-spread usage of risk management tools such as Value-at-Risk (VaR), firms actively manage risk and are concerned with its measurement. Unfortunately, VaR is not derived from fundamental economic principles and its usage may lead to economically sub-optimal decision rules as shown by Shapiro and Basak (2001).

Substantial progress in the academic risk management literature began with Artzner, Delbaen, Eber and Heath (1999), abbreviated ADEH hereafter, who develop an axiomatic framework for risk measurement. Their axioms stem from intuitive economic principles that define a coherent risk measure. The intent of ADEH is to provide a regulator with a methodology for determining the riskfree capital requirements of a firm, conditional on their existing portfolio. Indeed, a coherent risk measure is defined as the minimum amount of riskfree capital a portfolio requires to become acceptable to the regulator.

ADEH spawned an entire literature on coherent risk measures. These extensions include Delbaen (2000) and Roorda, Engwerda and Schumacher (2002) who generalize the framework to an infinite dimensional state space and a multiperiod context respectively. Follmer and Schied (2002) introduce convex risk measures that account for market frictions by allowing risk to increase nonlinearly with portfolio size. After altering an ADEH axiom, Jarrow (2002) enables a put option written on firm value (with zero strike price) to be coherent. Jaschke and Kuchler (2001) study the relationship between coherent risk measures and good-deal bounds on asset prices while Carr, Geman and Madan (2001) introduce acceptable trading opportunities in their study of incomplete markets.

This paper introduces a risk measure that is appropriate for the portfolio selection decisions of firms, while maintaining the concept of acceptable portfolios. To achieve this objective, we define risk on portfolio holdings, a domain conducive to having diversification reduce portfolio risk. We maintain an axiomatic structure and define the risk of a portfolio as its *distance* from the set of acceptable portfolios. More importantly, distance involves as many components as available assets, including but not limited to riskfree capital. After developing our risk measure, we also provide an implementation strategy involving quadratic programming, a technique with prior applications in finance originating from portfolio theory.

In contrast to coherent risk measures which focus on the regulator, this paper operates from the firm's perspective. In particular, we recognize that firms prefer to pursue investment opportunities that are capable of earning excess economic rents. This desire may stem from a perception of having superior information or investment ability. By implication, the ambitions of firms result in portfolios that are not well diversified. Intuitively, firms are unable to demonstrate investment skill by increasing their position in the riskfree asset. Thus, they are adverse to adding riskfree capital to their portfolio for performance considerations, yet are constrained by an external regulator.

Balancing the demands of an external regulator and the performance objectives of firms is accomplished by introducing portfolio theory into the measurement of risk. Specifically, our proposed risk measure offers firms the ability to rebalance their portfolio. During this rebalancing, the addition of riskfree capital remains a potential option, but is not the exclusive means by which a portfolio becomes acceptable. Since every asset portfolio weight may be altered, diversification is capable of reducing portfolio risk. Consequently, as discussed in Merton (1998), instruments with nonlinear payoffs such as derivative and insurance contracts become important tools for risk management. In addition, market frictions may be incorporated into a firm's rebalancing decisions as demonstrated in Section 4.

We also consider the pricing of portfolio insurance, a single contract whose addition to the existing portfolio is capable of ensuring its acceptability. This contract does not reduce positive payoffs but insures against negative outcomes to avoid insolvency. Provided a firm is willing to rebalance their portfolio, only a fraction of this security is required.

The organization of this paper is as follows. Section 2 details the properties of our proposed risk measure while a simple example which illustrates our approach is given in Section 3. Section 4 focuses on the implementation of our risk measure and demonstrates that coherent risk measures are contained in our framework. Section 5 considers the pricing of portfolio insurance while the conclusion of the paper is found in Section 6.

2 Risk Measure with Diversification

Consider the time horizon $[0, T]$ and a finite number N of risky assets denoted x_i for $i = 1, \ldots, N$ with x_0 representing riskfree capital. Let P denote an $M \times (N+1)$ payoff matrix with M rows corresponding to the regulator's set of scenarios and $N+1$ columns corresponding to the available assets. Elements of P are individual asset payoffs in a given scenario. A vector of portfolio holdings $\eta = [\eta_0, \eta_1, \ldots, \eta_N]^\top$ represents the number of units, not dollar amounts or fractions of a portfolio, invested in the various assets. Portfolio values $P\eta$ in the M scenarios determine whether a portfolio complies with the demands of an external regulator by being nonnegative, as in the ADEH literature.

Coherent risk measures evaluate a portfolio's risk according to its value in the worst possible scenario or under the probability measure that produces the largest negative outcome. Mathematically, these risk measures are defined in terms of terminal portfolio values, $X = P\eta$, as

$$\rho(X) = \max_i \frac{E^{P_i}\left[-X \,|\, P_i \in \mathcal{P}\right]}{1+r} \tag{1}$$

with $\mathcal{P}$ representing a set of scenarios and r the riskfree rate of interest. In our framework, $E^{P_i}[-X]$ is replaced by $P\eta_i^-$, the i^{th} row of $P\eta^- = -\min\{0, P\eta\}$ as each row of $P\eta$ corresponds to a regulator's scenario.

It is important to emphasize that coherent risk measures do not account for diversification. Instead, they focus solely on the amount of riskfree capital required to ensure the portfolio has nonnegative terminal values in the scenarios considered relevant by the regulator. This limitation is overcome by our methodology which operates on a different domain. Define $\mathcal{M} \subset \mathcal{R}^{N+1}$ as the space of portfolio holdings with the subset of acceptable portfolios denoted $\mathcal{A}_\eta \subset \mathcal{M}$.

Definition 2.1. *The set of acceptable portfolio holdings $\mathcal{A}_\eta \subset \mathcal{M}$ contains all portfolios that have nonnegative outcomes, $P\eta \geq 0$, in all M scenarios evaluated by the regulator.*

Clearly, the acceptance set $\mathcal{A}_\eta$ depends on the payoff matrix P with the regulator controlling the number of scenarios (rows) in which the firm must remain solvent.

Proposition 2.1. *The acceptance set $\mathcal{A}_\eta$ has the following two properties:*

1. *Closed under multiplication by $\gamma \geq 0$ and*

2. *Convexity.*

Proof:

1. It must be shown that if $\eta \in \mathcal{A}_\eta$, then $\gamma\eta \in \mathcal{A}_\eta$ for $\gamma \geq 0$. This property follows from $\eta \in \mathcal{A}_\eta$ being equivalent to $P\eta \geq 0$ and the property $P[\gamma\eta] = \gamma P\eta$ which is nonnegative since both γ and $P\eta$ are nonnegative.

2. Convexity is a consequence of the first property. If $\eta_1, \eta_2 \in \mathcal{A}_\eta$, implying $P\eta_1 \geq 0$ and $P\eta_2 \geq 0$, then $\gamma\eta_1 + (1-\gamma)\eta_2 \in \mathcal{A}_\eta$ for $0 \leq \gamma \leq 1$ since

$$P[\gamma\eta_1 + (1-\gamma)\eta_2] = \gamma P\eta_1 + (1-\gamma)P\eta_2 \geq 0\,. \quad \square$$

Unless each element of $P\eta$ is nonnegative, the portfolio η is unacceptable. In this instance, an optimal acceptable η^* is found based on its *proximity* to η as we assume firms prefer to engage in as little portfolio rebalancing as possible given their initial preference for η. Quadratic programming solves for the portfolio η^* in Section 4.

Define a trivial acceptable portfolio η_c consisting of \$1 invested only in riskfree capital, in other words, an $N+1$ vector with one as the first element and zero in the remaining elements. This portfolio has the property $P\eta_c = (1+r)\mathbf{1} > 0$ where $\mathbf{1}$ is an $N+1$ vector of ones. Given the acceptance set $\mathcal{A}_\eta$ in Definition 2.1, portfolio risk is defined in terms of the l_2 norm[1] on $\mathcal{M}$. Our risk function $\rho(\eta)$ maps from the domain of portfolio holdings, $\mathcal{M}$, into the nonnegative real line, $\rho(\eta) : \mathcal{M} \rightarrow \mathcal{R}^1_+$.

Definition 2.2. *Given $\mathcal{A}_\eta$ defined by the payoff matrix P, the risk of a portfolio η equals*

$$\rho(\eta) = \inf\left\{||\eta - \eta'||_2 : \eta' \in \mathcal{A}_\eta\right\}\,.$$

Observe the fundamental difference between our approach and that of ADEH, instead of defining risk on terminal portfolio values, risk is defined on portfolio holdings. Thus, although both measures of risk are derived from a *distance* to the acceptance set, our concept of distance has $N+1$ variables (one for each asset) instead of only one (riskfree capital).

If η already comprises an acceptable portfolio, then its associated risk equals zero. For example, the portfolio η_c has zero risk, $\rho(\eta_c) = 0$. Otherwise, portfolio risk is determined by the amount of rebalancing a portfolio requires to become acceptable. This illustrates a major advantage of our risk measure. A firm may rebalance their portfolio by purchasing derivative instruments, insurance contracts, or simply reducing their exposure to certain risky assets. The important point is that portfolio rebalancing may include, but is not limited to, increasing the amount of riskfree capital.

The lemma below is invoked in subsequent discussions and proofs.

[1]The l_2 norm $||x-y||_2$ equals $\sqrt{\sum_{i=0}^{N}(x_i - y_i)^2}$.

Lemma 2.1. *The acceptance set $\mathcal{A}_\eta$ is closed and compact.*

Proof: The function $\rho(\eta)$ is continuous and equals zero for acceptable portfolios that comprise $\mathcal{A}_\eta$. The inverse image of a closed and compact set $\{0\}$ for a continuous function is itself closed and compact. $\square$

2.1 Properties of Risk Measure

The next proposition summarizes the properties of our risk measure. Interestingly, all but one of ADEH's coherence axioms are preserved. However, removal of the translation invariance axiom results in an important generalization by eliminating the strict dependence on riskfree capital to reduce risk. Note that the operations $\eta_1 \pm \eta_2$ are applied componentwise to signify operations on two vectors representing portfolio holdings.

Proposition 2.2. *The proposed risk measure with diversification has the following properties:*

1. ***Subadditivity*** $\rho(\eta_1 + \eta_2) \leq \rho(\eta_1) + \rho(\eta_2)$
2. ***Monotonicity*** $\rho(\eta_1) \leq \rho(\eta_2)$ *if* $P\eta_1 \geq P\eta_2$
3. ***Positive Homogeneity*** $\rho(\gamma\eta) = \gamma\rho(\eta)$ *for* $\gamma \geq 0$
4. ***Riskfree Capital Monotonicity*** $\rho(\eta + \gamma\eta_c) \leq \rho(\eta)$ *for* $\gamma \geq 0$
5. ***Relevance*** $\rho(\eta) > 0$ *if* $\eta \notin \mathcal{A}_\eta$
6. ***Shortest Path*** *For every* $\eta \notin \mathcal{A}_\eta$ *and for* $0 \leq \gamma \leq ||\eta - \eta^*||_2$,

$$\rho(\eta + \gamma \cdot \tilde{u}) = \rho(\eta) - \gamma$$

where $\tilde{u}$ *is the unit vector in the direction* $\eta^* - \eta$ *defined as* $\frac{\eta^* - \eta}{||\eta^* - \eta||_2}$ *given a portfolio* η^* *that lies on the boundary of* $\mathcal{A}_\eta$ *and minimizes the distance* $||\eta - \eta^*||_2$.

The proof is contained in Appendix A. The shortest path property imposes cardinality on the risk measure with $\tilde{u}$ representing a unit of rebalancing. Observe that *riskier* portfolios are farther from the acceptance set with larger associated risk measures $\rho(\eta)$. Versions of the subadditivity, monotonicity, and positive homogeneity properties found in the original ADEH paper remain with subadditivity responsible for incorporating diversification into our framework. The second and third properties, monotonicity and positive homogeneity, are discussed in ADEH. Monotonicity guarantees that a portfolio whose terminal payoffs are larger than another portfolio in every scenario has lower risk than its counterpart. Positive-homogeneity allows a firm to scale an acceptable portfolio up or down with the resulting portfolio remaining acceptable.[2]

The key distinction arises from ADEH's translation invariance axiom. Our risk measure with diversification employs a weaker concept manifested in the riskfree capital monotonicity and shortest path properties. The relevance property ensures the risk function is positive if there exists a scenario, considered relevant by the regulator, where the terminal value of the portfolio is negative. Consequently, the relevance property ensures unacceptable portfolios have positive risk.[3]

2.2 Economic Motivation

An unacceptable portfolio may initially be chosen by a firm which believes it has superior information or investment skill. Moreover, additional riskfree capital does not permit a firm to exhibit investment ability or skill. Provided firms pursue excess economic rents and fail to maintain well diversified portfolios, a coherent risk measure is shown in Section 4 to overestimate their risk.

To enhance the motivation behind our risk measure, we introduce a nonnegative metric $R(\eta) \geq 0$ which determines the aggregate desirability of a portfolio. Since the selection

[2]To account for market frictions, Follmer and Schied (2002) replace positive homogeneity and subadditivity with a convexity axiom. In our framework, market frictions influence the solution for η^* as demonstrated in Section 4.

[3]When $\rho(\eta) = 0$, an amount γ^* of riskfree capital may be removed from the portfolio according to $\sup_{\gamma^*} \rho(\eta - \gamma^* \eta_c) = 0$, which is unique by the monotonicity of riskfree capital property. Since $\mathcal{A}_\eta$ is closed, there exists a boundary point which minimizes the required amount of riskfree capital. Although quadratic programming is capable of solving for γ^*, this issue is not elaborated on further as our focus concerns unacceptable η portfolios.

criteria and perceived desirability of individual assets are highly variable across firms, very little structure is imposed on $R(\eta)$. For illustration, we merely assume this function equals

$$R(\eta) \quad = \quad \frac{\sum_{i=0}^{N} \eta_i \cdot c_i}{\sum_{i=0}^{N} \eta_i} \tag{2}$$

where c_i implicitly denotes a ranking of the assets. For example, c_i may represent numerical weightings associated with *strong outperform*, *weak outperform*, or *hold* among other possibilities. Equation (2) allows several variables, including expected returns and variances, to influence a portfolio's desirability. However, covariances are not considered in equation (2) as diversification is reserved for our subsequent discussion of the proposed risk measure.

Regardless of the exact functional form for $R(\eta)$, the c_i elements may be derived from an infinite number of scenarios, not only those evaluated by the regulator. Indeed, the regulator is primarily concerned with a small subset of *extreme* scenarios. In contrast, the firm's investment criteria is comprised of more frequently occuring scenarios. This disparity reflects the diverging interests of the regulator and firm which our proposed risk measure attempts to bridge.

In the absence of portfolio rebalancing, define the amount of additional riskfree asset required to ensure the portfolio η becomes acceptable as $\alpha \geq 0$. This quantity equals

$$\begin{aligned} \alpha &= \inf\{\gamma : \eta + \gamma\eta_c \in \mathcal{A}_\eta\} \\ &= \min\{0, P\eta\}\,, \end{aligned} \tag{3}$$

and depends on η but is written as α rather than $\alpha(\eta)$ for notational simplicity. Overall, for $\eta \notin \mathcal{A}_\eta$, diversification is beneficial from the firm's perspective whenever there exists an $\eta' \in \mathcal{M}$ (not necessarily acceptable) such that

$$\text{Condition 1:} \qquad \eta + \eta' \in \mathcal{A}_\eta \tag{4}$$

$$\text{Condition 2:} \qquad R(\eta + \eta') \geq R(\eta + \alpha\eta_c)\,. \tag{5}$$

The first condition ensures that η', when added to η, is capable of constituting an acceptable portfolio. A solution for η' that satisfies the first condition is provided in Section 4. The second condition states that portfolio rebalancing is preferred to the addition of riskfree capital when complying with the regulator.[4] The existence of a portfolio η' is motivated by the inability of η_c to generate excess economic rents.[5]

In practice, the regulator may impose a fine denoted f on firms that continue to hold unacceptable portfolios. Thus, the second condition expressed in equation (5) may be extended to

$$R(\eta + \eta') \quad \geq \quad \max\left\{R(\eta + \alpha\eta_c), R(\eta) - f 1_{\{\eta \notin \mathcal{A}_\eta\}}\right\} . \tag{6}$$

Assuming the fine is large enough to satisfy both

1. $f \geq R(\eta) - R(\eta + \alpha\eta_c)$
2. $f \geq R(\eta) - R(\eta + \eta')$,

firms strive to be in compliance with the regulator. Indeed, the firm is better off rebalancing the portfolio than adding riskfree capital or paying the fine and maintaining their original portfolio. Since $R(\eta + \eta') \geq R(\eta + \alpha\eta_c)$, the two requirements above reduce to the first statement,

$$f \geq R(\eta) - R(\eta + \alpha\eta_c) .$$

Hence, the required fine is a function of both η and the firm's aversion to adding riskfree capital expressed via $R(\eta)$. Intuitively, firms which are less adverse to holding riskfree capital require smaller fines to induce compliance.

Observe that the addition of riskfree capital increases a portfolio's payoffs in all scenarios, even those for which the original portfolio already has nonnegative values. Indeed, the portfolio payoff increases in scenarios that are not even considered by the regulator. Therefore, the addition of riskfree capital is a very conservative approach to risk management, one suitable from the perspective of a regulator but not firms. Section 5 investigates the pricing of portfolio insurance, a security which only increases payoffs in scenarios that prevent the portfolio from being acceptable. Note that firms are able to evaluate scenarios beyond those

[4]Observe that setting $\eta' = \alpha\eta_c$ results in equality for the second condition.

[5]Although many other functions besides equation (2) are possible, the property $c_i \geq c_0$ for $i \geq 1$ guarantees the second condition is satisfied.

considered by the regulator if their internal risk management procedures are designed to be more stringent.

The next section considers a simple example to differentiate our risk measure from coherent risk measures.

3 Numerical Example

Consider an economy with two risky assets and riskfree capital. Uncertainty in the economy is captured by a coin toss. For the first risky asset, the payoff is \$4 if heads and -\$2 if tails, while their counterparts are \$0 and \$2 respectively for the second risky asset. The rate of interest is assumed to be zero ($r = 0$) implying riskfree capital is worth \$1 at time T.

Asset	Heads	Tails
Riskfree Capital	1	1
Risky Asset #1	4	-2
Risky Asset #2	0	2

Table 1: Asset payoffs at time T in both scenarios.

The two risky assets are negatively correlated. Indeed, the second risky asset resembles a "put" option on the first risky asset. The space of portfolio holdings whose terminal values are nonnegative in both scenarios is characterized as follows:

$$1\,\eta_0 + 4\,\eta_1 + 0\,\eta_2 \;\geq\; 0 \quad \text{Heads} \tag{7}$$

$$1\,\eta_0 - 2\,\eta_1 + 2\,\eta_2 \;\geq\; 0 \quad \text{Tails} \tag{8}$$

Consider the portfolio $\eta = [1, 1, 0]^{\top}$ consisting of one unit of riskfree capital, one unit of the first risky asset and none of the second. The portfolio η is not acceptable since the payoff is negative if the coin toss results in tails.

Portfolio	Heads	Tails
$\eta = [1, 1, 0]^\top$	5	-1

Table 2: Payoffs at time T in both scenarios for the unacceptable portfolio η.

In the coherent risk measure framework, η requires an additional unit of riskfree capital resulting in $\eta^*_{ADEH} = [2, 1, 0]^\top$.

Acceptable Portfolio - ADEH	Heads	Tails
$\eta^*_{ADEH} = [2, 1, 0]^\top$	6	0

Table 3: Payoffs at time T in both scenarios for the η^*_{ADEH} portfolio.

Solving for our optimal portfolio η^* involves minimizing the distance between $\eta = [1, 1, 0]^\top$ and $\eta^* \in \mathcal{A}_\eta$ under the l_2 norm using quadratic programming (QP). The portfolio η^* equals[6] $[1.11, 0.78, 0.22]^\top$ with details pertaining to its solution found in the next section.

Optimal Portfolio - QP	Heads	Tails
$\eta^* = [1.11, 0.78, 0.22]^\top$	4.22	0

Table 4: Payoffs at time T in both scenarios for the η^* portfolio.

As demonstrated above, a coherent risk measure evaluates the risk of η as 1 due to the negative payoff when the coin toss is tails. However, the portfolio $[1.11, 0.78, 0.22]^\top \in \mathcal{A}_\eta$ implies the risk of η in our framework is

$$||\eta^* - \eta||_2 = \sqrt{(1.11 - 1)^2 + (0.78 - 1)^2 + (0.22 - 0)^2} = 0.33.$$

Thus, our proposed risk measure evaluates the risk of η at one third that of a coherent risk measure. However, the rebalanced portfolio has nonnegative payoffs in both scenarios and therefore satisfies the regulator.

Finding more general solutions for η^* that incorporate market frictions into the rebalancing decision is addressed in Section 4. Note that our risk measure reduced the positive portfolio payoff in the heads scenario. Section 5 computes the value of a portfolio insurance contract that eliminates negative terminal values without reducing positive terminal values.

[6] MATLAB code which solves for η^* is available from the authors.

To summarize, the example illustrates that firms may comply with the demands of a regulator while holding less riskfree capital. Indeed, regulators may adopt our risk measure without compromising their original role of preventing insolvency in each scenario.

4 Implementation

If $P\eta$ has any negative elements, then the regulator deems the portfolio to be unacceptable. This section is concerned with implementing our risk measure by solving for the portfolio $\eta^* \in \mathcal{A}_\eta$ such that $P\eta^* \geq 0$ and η^* is "as close as possible" to the firm's original portfolio η.

Definition 4.1. *Allowing g to represent the l_2 norm, the portfolio $\eta^* \in \mathcal{A}_\eta$ is the solution to the optimization problem:*

$$\begin{aligned} &min_{\eta^* \in \mathcal{R}^{N+1}} \qquad g\left(\eta^* - \eta\right) \\ &\textit{subject to} \qquad P\eta^* \geq 0\,. \end{aligned} \tag{9}$$

In a financial context, quadratic programming, implied by the l_2 norm, is equivalent to the mean-variance analysis underlying much of portfolio theory. Since the objective function g is twice differentiable and strictly convex and the feasible region is also convex, the Kuhn Tucker conditions imply a unique solution. Although this problem cannot be solved analytically, very efficient numerical solutions are available. In particular, the problem is well suited for a pivoting scheme described in Luenberger (1990).

Proposition 4.1. *Let $y = P\eta^*$ and $g(\eta^*) = \frac{1}{2}(\eta^* - \eta)^\top(\eta^* - \eta)$. The optimal solution to equation (9) is given by $\eta^* = \eta + P^\top\lambda$ where λ solves the linear complementarity conditions*

$$\begin{cases} y - PP^\top\lambda = P\eta \\ y \geq 0, \quad \lambda \geq 0, \quad \lambda^\top y = 0\,. \end{cases} \tag{10}$$

Proof: The Kuhn Tucker conditions are

$$\begin{cases} \eta^* - P^\top\lambda = \eta \\ P\eta^* \geq 0, \quad \lambda \geq 0, \quad \lambda^\top P\eta^* = 0 \end{cases}$$

since the gradient of the objective function, $g(\eta^*) - (P\eta^*)^\top \lambda$, equals $\eta^* - \eta - P^\top \lambda$. Hence, with $y = P\eta^*$, the above conditions become

$$\begin{cases} y - PP^\top \lambda = P\eta \\ y \geq 0, \quad \lambda \geq 0, \quad \lambda^\top y = 0 \end{cases}$$

which completes the proof. □

Hence, the optimization problem in equation (9) is reduced to solving the linear complementary conditions in (10). Furthermore, the optimal portfolio η^* is a linear function of the vector λ which satisfies these linear complementary conditions. However, there may exist multiple solutions to (10), raising the question whether all possible solutions yield the same optimal portfolio η^* in Definition 4.1. This issue is addressed in the following proposition whose proof is found in appendix B.

Proposition 4.2. *All solutions to the linear complementary conditions in* (10) *yield the same optimal portfolio* η^* *in Definition 4.1.*

The λ parameters have interesting interpretations as each element corresponds to a specific regulator scenario. If the constraint $P\eta \geq 0$ is not binding in scenario i with $(P\eta)_i \geq 0$, then the corresponding λ_i equals 0. Otherwise, the optimal λ_i is a positive number representing the *cost* of preventing insolvency.

If $P\eta \geq 0$, then (10) has an obvious solution; $\lambda = 0$ and $y = P\eta$, implying η is optimal. Otherwise, the general pivoting approach transforms (10) to optimality. After finitely many pivots, bounded above by the number of rows (scenarios), the vector $P\eta^*$ is nonnegative. In terms of computational complexity, a total of M linear equations are solved for each pivot operation. The algorithm stops when $P\eta^* \geq 0$, providing the optimal solution to (10).

4.1 Incorporating Market Frictions

In general, the objective function g may be defined with respect to a positive definite matrix A as in $(\eta^* - \eta)^\top A\ (\eta^* - \eta)$. Consider a diagonal matrix of positive elements a_i

$$A = \begin{bmatrix} a_0 & & & \\ & a_1 & & \\ & & \ddots & \\ & & & a_N \end{bmatrix}$$

representing the associated market friction (illiquidity and transaction costs) of the i^{th} asset as well as the firm's unwillingness to alter their position in this asset. Larger a_i values correspond to larger *penalties* for altering that element of the portfolio. Even if riskfree capital has the smallest corresponding penalty, the addition of riskfree capital may still be sub-optimal. Indeed, a portfolio may require a large amount of additional riskfree capital to become acceptable, but only minor modifications to positions with larger a_i penalties. This issue is re-examined in the next section when pricing portfolio insurance.

Also note that we do not incorporate the c_i elements of the $R(\eta)$ function from equation (2) into A. Indeed, solving for the optimal acceptable portfolio that maximizes $R(\eta)$ is well beyond the scope of this paper and would require far greater structure on firm preferences, information and beliefs. Since η represents the firm's optimal portfolio in the absence of the regulator, we merely assume any deviation from η is disliked by the firm.

Proposition 4.1 has an immediate corollary when the positive definite matrix A is inserted into the objective function which alters the values of both η^* and λ.

Corollary 4.1. *Let $y = P\eta^*$ and $g(\eta^*) = \frac{1}{2}(\eta^* - \eta)^\top A(\eta^* - \eta)$ where A is a positive definite matrix. The optimal portfolio η^* equals $\eta + A^{-1}P^\top\lambda$, where λ satisfies the modified linear complementarity conditions*

$$\begin{cases} y - PA^{-1}P^\top\lambda = P\eta \\ y \geq 0, \quad \lambda \geq 0, \quad \lambda^\top y = 0\,. \end{cases}$$

Given Corollary 4.1 above, we now reconsider the example in Section 3 for different A matrices and their corresponding optimal acceptable portfolios.

4.2 Continuation of Example

Once again, the original unacceptable portfolio $\eta = [1, 1, 0]^\top$ is considered. Suppose a firm is extremely[7] adverse to adding riskfree capital to their portfolio. This preference is expressed through the matrix

$$A_1 = \begin{bmatrix} \infty & & \\ & 1 & \\ & & 1 \end{bmatrix}$$

which implies η_1^* equals $[1, 0.75, 0.25]^\top$. The portfolio η_1^* is acceptable with $P\eta_1^*$ being non-negative in both scenarios. Therefore, our proposed risk measure generates an acceptable portfolio without any additional riskfree capital by reducing the firm's exposure to the first risky asset and purchasing a portion of the second risky asset as a hedge.

Interestingly, one may begin with the portfolio $\bar{\eta} = \eta - \eta_c = [0, 1, 0]^\top$ and find $\bar{\eta}_1^*$, with the prevailing A_1 matrix, without utilizing any additional riskfree capital. Indeed, $[0, 1, 1]^\top$ consists entirely of risky assets and is acceptable.

Furthermore, suppose the firm also has a strong desire to maintain their position in the first risky asset. Returning to the original η portfolio, the A matrix

$$A_2 = \begin{bmatrix} \infty & & \\ & \infty & \\ & & 1 \end{bmatrix}$$

generates an optimal portfolio $\eta_2^* = [1, 1, 0.50]^\top$. As expected, only the position in the second risky asset is modified.

Finally, we examine an A matrix capable of replicating the optimal ADEH portfolio

$$A_{ADEH} = \begin{bmatrix} 1 & & \\ & \infty & \\ & & \infty \end{bmatrix}$$

[7] For implementation, ∞ is replaced with a large number, 1000 in the context of the numerical examples presented in this paper.

which implies $\eta^*_{ADEH} = [2, 1, 0]^\top$. In this situation, only additional riskfree capital is chosen. Overall, by eliminating the possibility of rebalancing the risky assets, the ADEH risk measure implicitly has $a_{j\neq 0} = \infty$.

The above examples illustrate the ability of our methodology to find optimal acceptable portfolios that reflect market frictions, as well as an aversion to additional riskfree capital or altering positions in specific risky assets. In summary, implementing our framework reduces to solving a quadratic programming problem, a situation encountered in many financial applications involving portfolio theory.

5 Pricing Portfolio Insurance

This section determines the price of portfolio insurance, a single contract whose combination with the original portfolio satisfies the regulator.

Let IC denote the nonnegative price of the contract in circumstances where $P\eta$ contains at least one negative value. Denote $X^+ = \max\{0, X\}$ and $X^- = -\min\{0, X\}$. To become acceptable, the firm requires a contract with a payoff profile equal to $(P\eta)^-$. In addition, we ensure the portfolio, when combined with the insurance contract, continues to provide $(P\eta)^+$ in scenarios with positive values. Thus, the insurance contract does not reduce positive terminal values, only increases negative terminal values to zero. Hence, in contrast to riskfree capital, portfolio insurance only provides a positive payoff in scenarios where it is necessary.

We endogenously determine the value of portfolio insurance by equating the dollar value of the optimal portfolios at time zero with and without this contract. This indifference stems from portfolio insurance being redundant since an acceptable portfolio may be obtained via rebalancing. Indeed, portfolio insurance provides an economically intuitive *short-cut* to acceptability.

5.1 Insurance without Rebalancing

Let q denote the price vector of the $N + 1$ assets at time zero which is assumed to be free of arbitrage. The proposition below solves for the price of portfolio insurance under the assumption that no additional rebalancing is conducted after its introduction.

Proposition 5.1. *The price of the portfolio insurance, without additional portfolio rebalancing, equals*

$$IC_{wo} = q^\top P^\top \lambda_{wo}$$

where λ_{wo} *is determined by the resulting linear complementary conditions.*

Proof: Consider the alternative to purchasing an insurance contract. The firm must rebalance their portfolio to obtain η^* which satisfies $P\eta^* \geq (P\eta)^+$. The optimization problem which solves for η^* is

$$\begin{aligned} \min_{\eta^*} \quad & g(\eta^* - \eta) \\ \text{subject to} \quad & P\eta^* \geq (P\eta)^+ . \end{aligned} \tag{11}$$

The Kuhn Tucker conditions imply that the optimal solution is given by the solution to the following linear complementarity conditions[8]

$$\begin{cases} y - PP^\top \lambda = -(P\eta)^- \\ y \geq 0, \quad \lambda \geq 0, \quad \lambda^\top y = 0 \end{cases} \tag{12}$$

where $y = P\eta^* - (P\eta)^+$. Denote the solution to (12) by $(\eta_{wo}, \lambda_{wo})$ where η_{wo} represents the optimal portfolio without the insurance contract. The following linear relationship between η_{wo} and the original portfolio η holds

$$\eta_{wo} = \eta + P^\top \lambda_{wo} . \tag{13}$$

With firms indifferent between buying the contract or rebalancing their portfolio, the dollar values of the two acceptable portfolios at time zero are equated. Thus, the price of the insurance contract equals $IC_{wo} + q^\top \eta = q^\top \eta_{wo}$ implying

$$IC_{wo} = q^\top (\eta_{wo} - \eta) = q^\top P^\top \lambda_{wo} , \tag{14}$$

which completes the proof. □

The value of IC_{wo} is positive since the payoff $(P\eta)^-$ is nonnegative in each scenario and strictly positive in at least one scenario.[9]

[8] Using the property $P\eta = -(P\eta)^- + (P\eta)^+$, $y - PP^\top \lambda = -(P\eta)^-$ in equation (12) is equivalent to $P\eta^* - P\eta - PP^\top \lambda = 0$ in Proposition 4.1.

[9] $P\eta \geq 0$ with strict inequality in at least one scenario implies the initial cost of the portfolio $q^\top \eta$ is positive.

5.2 Insurance with Rebalancing

The following analysis has firms willing to engage in additional rebalancing to exploit the diversification benefit offered by the availability of portfolio insurance. Let the insurance contract be the $N + 2^{nd}$ security resulting in an additional column being appended to P to form $Q = [P \ \ (P\eta)^-]$. This column increases negative terminal values in scenarios that previously implied insolvency. In addition, enhanced portfolios with and without portfolio insurance are defined as

$$\delta_1 = \begin{bmatrix} \eta \\ 1 \end{bmatrix} \text{ and } \delta_0 = \begin{bmatrix} \eta \\ 0 \end{bmatrix}.$$

While δ_0 is not acceptable, δ_1 is acceptable since $Q\delta_1 = P\eta + (P\eta)^- = (P\eta)^+ \geq 0$. However, we later prove that δ_1 is not optimal when there are fewer scenarios than available assets.

Proposition 5.2. *The price of portfolio insurance, with additional portfolio rebalancing, equals*

$$IC_w = \frac{q^\top P^\top (\lambda_{wo} - \lambda_w)}{((P\eta)^-)^\top \lambda_w}$$

with λ_{wo} previously determined in Proposition 5.1 and λ_w by the resulting linear complementary conditions.

Proof: Denote $\delta^* = \begin{bmatrix} \eta_w \\ x_w \end{bmatrix}$. The optimal solution defined over the $N + 2$ assets is given by

$$\begin{aligned} \min_{\delta^* \in \mathcal{R}^{N+2}} \quad & g(\delta^* - \delta_0) \\ \text{subject to} \quad & Q\delta^* \geq (P\eta)^+ \end{aligned} \tag{15}$$

with linear complementarity conditions

$$\begin{cases} y - QQ^\top \lambda = -(P\eta)^- \\ y \geq 0, \quad \lambda \geq 0, \quad \lambda^\top y = 0 \end{cases} \tag{16}$$

The condition $y \geq 0$ in (12) implies $P\eta_{wo} - (P\eta)^+ \geq 0$ which implies that $P(\eta_{wo} - \eta) \geq 0$ with strict inequality in at least one scenario provided $(P\eta)^- \neq 0$. Therefore, no arbitrage implies $IC_{wo} = q^\top(\eta_{wo} - \eta) > 0$.

for $y = Q\delta^* - (P\eta)^+$. Denote the optimal solution to (16) by (η_w, x_w, λ_w) which implies

$$\begin{bmatrix} \eta_w \\ x_w \end{bmatrix} = \begin{bmatrix} \eta \\ 0 \end{bmatrix} + \begin{bmatrix} P^\top \\ ((P\eta)^-)^\top \end{bmatrix} \lambda_w \,. \tag{17}$$

Therefore, the second equation of (17) implies the optimal amount of insurance to purchase equals

$$x_w = ((P\eta)^-)^\top \lambda_w \geq 0 \,. \tag{18}$$

Hence, conditional on additional rebalancing from η to η_w, the price of the insurance contract is $IC_w \cdot x_w + q^\top \eta_w = q^\top \eta_{wo}$ which is equivalent to

$$IC_w = \frac{q^\top P^\top (\lambda_{wo} - \lambda_w)}{((P\eta)^-)^\top \lambda_w} \tag{19}$$

by equation (18) and the relationship $\eta_{wo} - \eta_w = \eta + P^\top \lambda_{wo} - \eta - P^\top \lambda_w = P^\top(\lambda_{wo} - \lambda_w)$. □

The magnitude of x_w in equation (18) quantifies the importance of diversification. Additional portfolio rebalancing reduces the required amount of portfolio insurance contract from 1 to x_w when P is of full row rank as proved in the following corollary.

Corollary 5.1. *The optimal amount of portfolio insurance to purchase, x_w, is strictly less than one unit if P is of full row rank.*

Proof: The inequality $x_w \leq 1$ follows from $\lambda_w^\top Q Q^\top \lambda_w = \lambda_w^\top (P\eta)^-$ by (16), which is equivalent to $\lambda_w^\top P P^\top \lambda_w + (\lambda_w^\top (P\eta)^-)^2 = \lambda_w^\top (P\eta)^-$. When P is of full row rank, $PP^\top$ is positive definite implying $\lambda_w^\top P P^\top \lambda_w \geq 0$ which yields $(\lambda_w^\top (P\eta)^-)^2 \leq \lambda_w^\top (P\eta)^-$ and proves that

$$x_w = \lambda_w^\top (P\eta)^- \leq 1 \,. \tag{20}$$

Thus, the optimal amount of insurance to purchase is strictly less than one unit. □

The strict inequality in the above corollary reinforces the importance of diversification. Specifically, we are able to diversify risk more effectively once the insurance contract becomes available. With little loss of generality, we may assume the number of assets $N + 1$ exceeds the number of scenarios M.[10] Therefore, for the remainder of this paper, it is assumed that P is of full row rank.

[10]For example, consider the total number of futures contracts and options ranging across time-to-maturities and strike prices for scenarios involving the underlying asset.

To summarize, it is not necessary for firms to purchase the entire insurance contract provided they engage in subsequent portfolio rebalancing. As indicated in the next corollary, fewer dollars are also required to be spent on portfolio insurance, a result that is later reinforced by Proposition 5.3.

Corollary 5.2. *The dollar value of required insurance is less with portfolio rebalancing,* $x_w IC_w < IC_{wo}$.

Proof: This result follows from equations (18) and (19),

$$x_w IC_w = IC_{wo} - q^\top P^\top \lambda_w$$

and the fact that the last term $q^\top P^\top \lambda_w = q^\top(\eta_w - \eta)$ is positive. Indeed, $q^\top(\eta_w - \eta) > 0$ is a consequence of the condition $y = Q\delta^* - (P\eta)^+ = P\eta_w + (P\eta)^- x_w - (P\eta)^+ \geq 0$ from (16) which implies $P\eta_w + (P\eta)^- - (P\eta)^+ \geq 0$ since $1 > x_w \geq 0$. Therefore, $P\eta_w - P\eta \geq 0$ with strict inequality in at least one scenario and by the assumption of no arbitrage, $q^\top(\eta_w - \eta) > 0$. □

In addition, equation (17) implies that neither δ_1 nor η_{wo} are optimal in the presence of the insurance contract. These statements are formalized in the following corollary.

Corollary 5.3. *If η is an unacceptable portfolio and P is of full row rank, then neither*

$$\delta_{wo} = \begin{bmatrix} \eta_{wo} \\ 0 \end{bmatrix} \quad nor \quad \delta_1 = \begin{bmatrix} \eta \\ 1 \end{bmatrix}$$

are optimal in the presence of the insurance contract.

Proof: If δ_{wo} is acceptable, then it is also acceptable in the presence of the insurance contract. But if δ_{wo} is optimal, then (13) and (17) jointly imply that

$$P^\top(\lambda_{wo} - \lambda_w) = 0 \quad \text{and} \quad ((P\eta)^-)^\top \lambda_w = 0\,.$$

Hence, with the payoff matrix P being of full row rank, it follows that $\lambda_{wo} = \lambda_w$ with (12) implying $\lambda_{wo}^\top P P^\top \lambda_{wo} = 0$ which contradicts $PP^\top$ being positive definite since $(P\eta)^-$ is strictly greater than 0 in at least one scenario. Hence δ_{wo} is not optimal. A similar contradiction is obtained if one assumes δ_1 is optimal. □

The next proposition states that the two portfolio insurance prices, IC_{wo} and IC_w, are identical when the market is arbitrage free.

Proposition 5.3. *If η is an unacceptable portfolio, then the prices IC_{wo} and IC_w are equal.*

Proof: The binding property of the constraints in equations (11) and (15) imply

$$\begin{cases} P\eta_{wo} = P\eta + (P\eta)^- \\ P\eta_w + (P\eta)^- x_w = P\eta + (P\eta)^-. \end{cases}$$

It follows that η plus the insurance contract, η_{wo} and $\delta^* = \begin{bmatrix} \eta_w \\ x_w \end{bmatrix}$ all have the same payoff, $P\eta + (P\eta)^-$. By no arbitrage, their values at time zero are also equal with

$$\begin{cases} q^\top \eta_{wo} = q^\top \eta + IC_{wo} \\ q^\top \eta_w + IC_w \cdot x_w = q^\top \eta + IC_w \end{cases}$$

implying $IC_{wo} - IC_w = q^\top \eta_{wo} - q^\top \eta_w - IC_w \cdot x_w = 0$ which completes the proof. □

In summary, prices for portfolio insurance without portfolio rebalancing and with portfolio rebalancing are given by Propositions 5.1 and 5.2 respectively. Additional portfolio rebalancing exploits the diversification benefit offered by the introduction of the insurance contract. As a result, the firm is able to purchase strictly less than one unit of the contract. However, with or without portfolio rebalancing, the price for one unit of portfolio insurance is identical according to Proposition 5.3. More intuition behind Proposition 5.3 is given in the next subsection.

5.3 Insurance and Dollar-Denominated Risk

We now demonstrate that although the risk metric $\rho(\eta)$ is defined on portfolio weights, our results may be interpreted in terms of a dollar-denominated quantity. Furthermore, the dollar-denominated amount of rebalancing equals the price of portfolio insurance.

Specifically, the difference between η^* and η equals $\eta^* - \eta = P^T\lambda$ implying the dollar-denominated amount of risk is

$$\begin{aligned} q^T(\eta^* - \eta) &= q^T P^T \lambda \\ &= IC_{wo}\,. \end{aligned} \tag{21}$$

Therefore, although risk is defined in terms of the l^2 norm on portfolio weights, it may be converted into the more traditional dollar-based domain and coincides with the price of portfolio insurance (with or without rebalancing).

As a consequence of equation (21), minimizing the distance in portfolio weights between η and the acceptance set is equivalent to minimizing the dollar-denominated amount of rebalancing. Therefore, the price of portfolio insurance equals the amount of rebalancing, in dollars, required to ensure the portfolio η becomes acceptable.

5.4 Example Revisited

Returning to the example in Section 3, let the price vector equal $q = [1, 1.3, 0.9]^\top$. Existing specifications imply $(P\eta)^- = [0, 1]^\top$, and $\eta = [1, 1, 0]^\top$ along with the payoff matrix P illustrates the results in Propositions 5.1 and 5.2. The vector λ_{wo} equals $[0.0673, 0.1635]^\top$ implying a price for portfolio insurance of $IC_{wo} = q^\top P^\top \lambda_{wo}$ which equals \$0.45. The λ_{wo} parameters are associated with two restrictions; preventing negative terminal values and not reducing positive terminal values.

The second optimization in equation (15) based on δ_0 and Q yields $\lambda_w = [0.0579, 0.1405]^\top$. According to Proposition 5.2, the price IC_w equals \$0.45, in accordance with Proposition 5.3.

However, the optimal amount of portfolio insurance to purchase is $x_w = \lambda_w^\top (P\eta)^- = (\lambda_w)_2 = 0.1405$, a quantity strictly less than one since P is of full row rank. Thus, with additional portfolio rebalancing, the dollar-denominated reduction in the amount of portfolio insurance that is required equals $IC_{wo} - x_w IC_w = (1 - 0.1405) \times \$0.45 = \$0.39$.

6 Conclusion

A risk measure defined on the space of portfolio holdings is proposed which enables diversification to reduce portfolio. Consequently, derivative and insurance contracts have important roles in risk management. Through portfolio rebalancing, our risk measure offers firms greater flexibility than coherent risk measures when complying with an external regulator. Indeed, our approach allows every asset in the portfolio, including riskfree capital, to be adjusted. Thus, as in the existing literature, risk is defined as the *distance* to an acceptance set. However, to incorporate diversification, the concept of distance is extended to include the risky assets as well as riskfree capital.

Our analysis incorporates market frictions such as illiquidity and transaction costs into the portfolio rebalancing decision. The price of portfolio insurance is also derived. When combined with the original portfolio, this contract ensures nonnegative portfolio values in every scenario considered by the regulator. Furthermore, the amount of required portfolio insurance is determined by the firm's willingness to rebalance their portfolio once this contract is available.

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Appendices

A Proof of Proposition 2.2

Recall the properties of Proposition 2.1 regarding the acceptance set $\mathcal{A}_\eta$.

1. Consider two portfolios η_1 and η_2 and let η_1^* be the closest portfolio on the acceptance set $\mathcal{A}_\eta$. In other words, $\eta_1^* = \eta'$ such that $\inf\{||\eta_1 - \eta'||_2 : \eta' \in \mathcal{A}_\eta\}$. Similarly define η_2^* as the equivalent quantity for η_2. Therefore, by definition

$$
\begin{aligned}
\rho(\eta_1) &= ||\eta_1 - \eta_1^*||_2 \\
\rho(\eta_2) &= ||\eta_2 - \eta_2^*||_2
\end{aligned}
$$

and the following holds by the triangle inequality property of norms

$$||\eta_1 + \eta_2 - \eta_1^* - \eta_2^*||_2 \leq ||\eta_1 - \eta_1^*||_2 + ||\eta_2 - \eta_2^*||_2 = \rho(\eta_1) + \rho(\eta_2).$$

However, the quantity $\eta_1^* + \eta_2^*$ is also in the acceptance set since $\mathcal{A}_\eta$ is convex and closed under multiplication by $\gamma \geq 0$. For two portfolios $\eta_1^*, \eta_2^* \in \mathcal{A}_\eta$ convexity implies $\eta^* = \frac{1}{2}\eta_1^* + \frac{1}{2}\eta_2^* \in \mathcal{A}_\eta$ while $2\eta^* = \eta_1^* + \eta_2^* \in \mathcal{A}_\eta$ as a consequence of $\mathcal{A}_\eta$ being closed under multiplication of positive scalars. Therefore

$$\rho(\eta_1 + \eta_2) \leq ||\eta_1 + \eta_2 - \eta_1^* - \eta_2^*||_2 = ||(\eta_1 + \eta_2) - (\eta_1^* + \eta_2^*)||_2$$

since $\eta_1^* + \eta_2^*$ is an element of $\mathcal{A}_\eta$ but need not be optimal. Hence, $\rho(\eta_1 + \eta_2) \leq \rho(\eta_1) + \rho(\eta_2)$ and subadditivity is proved.

2. Consider two portfolios η_1 and η_2 and let $P\eta_1 \geq P\eta_2$ a.s. The proof for monotonicity follows by recognizing that $\eta_1 = \eta_1 - \eta_2 + \eta_2$ and $\rho(\eta_1 - \eta_2) = 0$ since the portfolio $\eta_1 - \eta_2$ always generates a nonnegative payoff implying $\eta_1 - \eta_2 \in \mathcal{A}_\eta$. Applying subadditivity, $\rho(\eta_1) = \rho(\eta_1 - \eta_2 + \eta_2) \leq \rho(\eta_2)$, demonstrates that $\rho(\eta_1) \leq \rho(\eta_2)$ and monotonicity is proved.

3. Consider a portfolio η and a scalar $\gamma \geq 0$. Define η^* as in the proof of subadditivity. The function $\rho(\eta)$ is defined as $||\eta - \eta^*||_2$ which implies that

$$\gamma\rho(\eta) = \gamma\,||\eta - \eta^*||_2 = ||\gamma\eta - \gamma\eta^*||_2 \geq ||\gamma\eta - (\gamma\eta)^*||_2 = \rho(\gamma\eta)$$

since $\gamma\eta^*$ is in the acceptance set but need not be optimal in terms of minimizing the distance to the acceptable set. The reverse direction is proved by defining $\rho(\gamma\eta)$ as $||\gamma\eta - (\gamma\eta)^*||_2 = \gamma \left|\left|\eta - \frac{(\gamma\eta)^*}{\gamma}\right|\right|_2 \geq \gamma\rho(\eta)$ since $\frac{1}{\gamma}(\gamma\eta)^*$ is an element of $\mathcal{A}_\eta$ but need not be optimal. Thus, $\rho(\gamma\eta)$ and $\gamma\rho(\eta)$ are equal and positive homogeneity is proved.

4. Consider two portfolios η_1 and η_2 that differ only in terms of the riskfree asset with $\eta_{2,0} > \eta_{1,0}$. It suffices to show that $\rho(\eta_2) \leq \rho(\eta_1)$. Consider a portfolio that is a combination of η_1 and another portfolio $\gamma\eta_c$ for $\gamma \in [0,1]$ that consists entirely of an amount $\eta_{2,0} - \eta_{1,0}$ in riskfree capital. This new portfolio is equivalent to η_2 and implies that

$$\eta_2 = \eta_1 + \gamma\eta_c \Rightarrow \rho(\eta_2) = \rho(\eta_1 + \gamma\eta_c) \leq \rho(\eta_1) + \rho(\gamma\eta_c)$$

using subadditivity. However, $\rho(\gamma\eta_c)$ equals zero since this portfolio is accepted by the regulator, $\gamma\eta_c \in \mathcal{A}_\eta$. Hence, $\rho(\eta_2) \leq \rho(\eta_1)$ and the monotonicity of riskfree capital is proved.

5. Consider a portfolio $\eta \notin \mathcal{A}_\eta$ such that $P\eta_i^- < 0$ for some i. It must be proved that $\rho(\eta) > 0$. Proceed by contradiction by supposing that $\rho(\eta) = 0$ which implies that $\eta \in \mathcal{A}_\eta$ by Definition 2.2. However, Definition 2.1 requires that $P\eta \geq 0$ for $\eta \in \mathcal{A}_\eta$, contradicting $P\eta_i^- < 0$ for any i. Hence, relevance is proved.

6. Consider a portfolio η that does not belong to the acceptance set. Recall that $\mathcal{A}_\eta$ is a closed and convex set according to Lemma 2.1 and Proposition 2.1. Since η is a point outside this set, by the separating hyperplane theorem, there exists a point η^* on the boundary of $\mathcal{A}_\eta$ such that $||\eta - \eta^*||_2$ is the unique minimum distance of η from set $\mathcal{A}_\eta$. Now consider any scaler γ and let $\tilde{u}$ be the unit directional vector in the direction $\eta^* - \eta$. The vector $\eta + \gamma \cdot \tilde{u}$ is a point along the path of minimum distance and proves the shortest path property.

$$\begin{aligned}
\rho\left(\eta + \gamma \cdot \tilde{u}\right) &= ||\eta + \gamma \cdot \tilde{u} - \eta^*||_2 \\
&= \left|\left| \eta + \gamma \cdot \left(\frac{\eta^* - \eta}{||\eta - \eta^*||_2} \right) - \eta^* \right|\right|_2 \\
&= \left|\left| \eta^* - \eta - \gamma \cdot \left(\frac{\eta^* - \eta}{||\eta - \eta^*||_2} \right) \right|\right|_2 \\
&= \left(1 - \frac{\gamma}{||\eta - \eta^*||_2} \right) ||\eta - \eta^*||_2 \\
&= ||\eta - \eta^*||_2 - \gamma \\
&= \rho(\eta) - \gamma\,.
\end{aligned}$$

□

B Proof of Proposition 4.2

It is sufficient to prove that any two solutions to the linear complementary conditions in (10) yield the same optimal portfolio η^*. Therefore, our procedure is optimal. Let (y_1, λ_1) and

(y_2, λ_2) denote two solutions to (10) with the following conditions

$$\begin{cases} y_1 - PP^\top \lambda_1 = P\eta \\ y_2 - PP^\top \lambda_2 = P\eta \\ \lambda_1 \geq 0, \lambda_2 \geq 0, y_1 \geq 0, y_2 \geq 0 \\ \lambda_1^\top y_1 = 0, \lambda_2^\top y_2 = 0\,. \end{cases} \tag{22}$$

We proceed to show

$$P^\top \lambda_1 = P^\top \lambda_2$$

with both solutions generating the same optimal portfolio $\eta^* = \eta + P^\top \lambda_i$ for $i = 1, 2$. From (22),

$$\begin{cases} \lambda_1^\top P\eta = -\lambda_1^\top PP^\top \lambda_1 \\ \lambda_2^\top P\eta = -\lambda_2^\top PP^\top \lambda_2\,. \end{cases}$$

Therefore,

$$\begin{aligned} (\lambda_1 + \lambda_2)^\top (y_1 + y_2) &= \lambda_1^\top y_2 + \lambda_2^\top y_1 \\ &= \lambda_1^\top PP^\top \lambda_2 + \lambda_2^\top PP^\top \lambda_1 + \lambda_1^\top P\eta + \lambda_2^\top P\eta \\ &= -(\lambda_1 - \lambda_2)^\top PP^\top (\lambda_1 - \lambda_2) \\ &\leq 0\,. \end{aligned}$$

Since $\lambda_1 \geq 0$, $\lambda_1 \geq 0$, $y_1 \geq 0$, and $y_2 \geq 0$, it follows that

$$(\lambda_1 - \lambda_2)^\top PP^\top (\lambda_1 - \lambda_2) = 0$$

which implies

$$P^\top (\lambda_1 - \lambda_2) = 0\,.$$

Therefore, the optimal solution to equation (9) is

$$\eta^* = \eta + P^\top \lambda_1 = \eta + P^\top \lambda_2\,,$$

which completes the proof. □