10-2006

Two-Instant Reallocation in Two-Echelon Spare Parts Inventory Systems

Huawei SONG

Hoong Chuin LAU
Singapore Management University, hclau@smu.edu.sg

DOI: https://doi.org/10.1109/COASE.2006.326905

Periodic Resource Reallocation in Multi-Echelon Repairable Item Inventory Systems

Huawei SONG
The Logistics Institute – Asia Pacific
National University of Singapore
Singapore 119260
tlish@nus.edu.sg

Hoong Chuin LAU
School of Information Systems
Singapore Management University
Singapore 178902
hclau@smu.edu.sg

***Abstract* – Given stock allocation in an inventory system, it is necessary to do reallocation periodically to address inventory imbalance and maximize availability. In this paper, we allow two opportunities to perform reallocation within a replenishment cycle. We derive a mathematical model to dynamically determine when to reallocate and how to reallocate. Experimental results show that multiple-reallocation achieves better performance compared with single-reallocation approach found in literature. More interestingly, we believe the logic of our model can be used to design optimal periodic resupply policies.**

Keywords – Periodic, Reallocation, Multi-Echelon, Repairable, Inventory

1. INTRODUCTION

We consider an arboreal inventory system in which a central depot serves *n* bases. Military systems such as aircrafts and tanks are deployed at the bases. These systems break down because the underlying components, which are called LRU (line replaceable units), are either worn out over time and/or damaged during usage. Stocks are allocated at the depot and bases to insure continuity of operations. When an LRU fails at a base, a spare replaces it if one is available; otherwise a backorder is incurred. In military practice, due to the limited space at the bases, the failed LRUs are usually sent back to depot for repair. In the mean time, an order is placed by the base to the depot to send a spare. A spare will be sent to the base if one is available; otherwise there is a backorder at the depot. After the failure is repaired, it will be sent to the depot inventory to fulfill future demands.

As demands are stochastic, inventory imbalance will occur and tends to grow with time. This imbalance ultimately reaches a situation where some bases hold excess inventories, while others face critical shortage. To correct the imbalance, stocks need to be *reallocated*. To increase the efficiency and reduce unavailability, we also allow excess inventories at some bases to be reallocated laterally to others with shortage. In this paper, we are concerned with a *two-instant* reallocation scheme within a system replenishment cycle. This work is in response to the open challenge post by Cao and Silver [2] to consider two or more possible reallocations within a cycle. In practice, this problem is also faced by planners who need determine the time *between* periodic reallocations.

Although the (s-1,s) replenishment policy is generally assumed in the literature (i.e. depot will send a spare to the base once a failure occurs) it is almost impossible to supply continuously in practice, especially in a naval environment. The depot has to send spares to offshore bases and bring the failures back for repair *periodically*. Hence, it is important to determine when and how to distribute stocks to the bases periodically in the cycle.

Given that we have two reallocation instances, it is important to determine when each reallocation should occur. If we perform the first reallocation too early, it may prevent early backorders but could lead to growth of backorders before the next reallocation or during the remaining time in the cycle. Conversely, performing reallocation later presents two problems: First, it may cause high levels of early backorders. Second, late reallocation may leave no time to do the second reallocation. Furthermore, the time interval between the first and the second reallocations is also important: if it is too short, failures brought back to depot for repair may not have been completed and consequently the depot has too few spares to perform the second reallocation; if the time interval is too long, it causes higher levels of backorders at bases between reallocations. Therefore, the key issue is how best to synchronize the two reallocations.

Many papers have analyzed resource reallocation, periodic resupply and risk pooling effect. Eppen and Schrage [4] analyze an echelon-structure inventory system considering external lead times and random demands. In their model, the bases place orders directly to the outsider supplier but the supplier sends all stocks to the central depot after the external lead time. The depot will then reallocate the received stocks to bases according to the demands during external lead time. They claim that this method of reallocation yields a risk-pooling effect during lead time. Jackson and Muckstadt [6] propose an optimal allocation procedure for a two-period problem with a two-echelon structure. They formulate the problem using the joint probability distribution of the first period demands at the retailers. However, since they do not permit lateral resupply between retailers, they experience difficulties in trying to ascertain how much stock to allocate to each retailer. Jackson [5] introduce a concept of "pooled-risk period", and used a (s-1, s) policy for replenishment. This is continued until the central stock is depleted. The latest period of allocation is called the "pooled-risk period". In contrast, Jönsson and Silver [7] fix the reallocation time at the beginning of the last period of the system order cycle, and propose a complete reallocation policy at that time allowing transshipments among bases. Complete reallocation means that no inventories are kept at the depot after reallocation. Tsao and Enkawa [13] present a "two-phase push control policy" for considering the optimal reallocation instant in a two-echelon inventory system. Their method predetermines a fixed reallocation instant in all replenishment cycles, independent of the dynamic behavior of the inventories at the retailers. Cao and Silver [2] recently propose a heuristic method to dynamically determine optimal reallocation period in each replenishment cycle and perform reallocation.

The abovementioned papers deal with consumable items. In a military context, it is important to perform reallocations of spare parts periodically because they are often very expensive and affect system availability greatly ([3], [9]). System availability is usually measured in terms of Expected Backorders (EBO) (e.g. [1], [8], [11]). To our knowledge, there are few works on redistribution of spare parts in multi-echelon systems, except a brief mention of the problem in [11] and a feature within a proprietary commercial tool OPUS [10].

This paper makes technical contributions in the following ways. First, we consider how to perform more than one reallocation within a replenishment cycle, and instead of fixing reallocation instants to certain time points, we propose how to determine the time point instants for reallocation. Second, the replenishment of stocks from depot to bases is periodic, i.e. stocks and failures can be transported only at certain time points in a batch. Third, transshipments among bases are allowed.

The remainder of this paper is organized as follows. Section 2 lists the basic assumptions and notation. The mathematical model and approach are presented in Section 3. Section 4 presents the experimental results. Finally, conclusions and future work are provided in Section 5.

2. PROBLEM DEFINITION

Our resource reallocation problem is based on two important and simplifying assumptions. First we assume negligible internal lead (or transportation) times for moving items from the central depot to each base, and laterally between bases. This is consistent with [2], [5], [13] which assume reallocation can be achieved instantly. The relaxation of this assumption is straightforward. Second, to simplify the problem, we assume all failures are only repairable at the depot which has infinite repair capacities. The repair time is exponentially distributed with mean T.

The system replenishment cycle is H base periods, i.e. every H base periods, the central depot places orders to an outsider supplier. Therefore our reallocation decision time horizon is within this replenishment cycle. Demands over time at the bases are assumed to be independent, normal distributed variables with mean μ_i and standard deviation σ_i at base i during each period.

The main notation used in this paper is as follows.

i: index of site (i = 0 for the depot)
n: number of bases
S_i: initial stock level of LRU at site i
H: system replenishment cycle, in base periods
T: mean repair time of LRU
$y_i(t)$: (random var) demands in a single period t at base i
μ_i: mean value of $y_i(t)$
σ_i: standard deviation of $y_i(t)$
τ: index of the period at the end of which reallocation takes place
$I_i(\tau)$: stock level at site i instantly before reallocation at the end of period τ.
$U_i(\tau)$: stock level at site i instantly after reallocation at the end of period τ
t_1: time point at which the first reallocation is done
t_2: time point at which the second reallocation is done
$EBO_i(t)$: expected backorder at the end of period t at base i (If reallocation takes place at t, it denotes the EBO just before reallocation)
TE_i: total EBO at base i at those time points, namely, just before each allocation and at the end of the cycle.

Given the initial number of stocks at each site, we need to determine the moment series t_1, t_2, at which reallocation takes place as well as the number of stocks at each site after reallocation so that we can keep the total EBO over all bases at those time points just before each allocation and at the end of the cycle as low as possible. Since there are two reallocation instances, we are in fact trying to minimize the total EBO over all bases at *three* points in time (at the end of period t_1 and t_2 just before the reallocations respectively and at the end of H periods). The value of the objective function depends on the times at which the reallocations are carried out and how the reallocations are done. The total EBO at base i (TE_i) is the sum of three components, the first components being the expected backorders just before the first reallocation instant, $EBO_i(t_1)$, when the first reallocation is done at the end of period t_1, the second components being the expected backorders before the second reallocation instant, $EBO_i(t_2)$, when the second reallocation is done at the end of period t_2 and the third component being the expected backorders at the end of the cycle $EBO_i(H)$. Hence,

$$TE_i = EBO_i(t_1) + EBO_i(t_2) + EBO_i(H) \quad (1)$$

Our objective function, the total EBO over all bases at three points in time is $TE = TE(t_1, t_2) = \sum_{i=1}^{n} TE_i$ and our aim is to find t_1, t_2, $U_i(t_1)$, $U_i(t_2)$ such that TE is minimized.

3. MATHEMATICAL MODEL

3.1 Before the First Reallocation

Given the initial stock allocations at all sites, the expected backorders over all bases at time t_1, just before reallocation can be calculated according to the definition of EBO as follows:

$$EBO_1 = \sum_{i=1}^{n} EBO_i(t_1) = \sum_{i=1}^{n} \int_{S_i}^{\infty} (x_i - S_i) f(x_i) dx_i \quad (2)$$

$X_i = \sum_{t=1}^{t_1} y_i(t)$ denotes the demands of LRUs at base i during time interval [0, t_1], which is a normally distributed random variable with mean $E(X_i) = t_1\mu_i$, and variance $\mathrm{Var}(X_i) = t_1\sigma_i^2$. Therefore, after standardization, EBO_1 can be expressed by:

$$EBO_1 = \sum_{i=1}^{n} \sqrt{t_1}\sigma_i G\left(\frac{S_i - t_1\mu_i}{\sqrt{t_1}\sigma_i}\right) \quad (3)$$

where $G(k) = \int_k^{\infty}(z-k)\phi(z)dz$ is the unit normal loss function and $\phi(z)$ is the probability density function of standard normal distribution.

3.2 The First Reallocation

When t = t_1, we will do reallocation. The spares will be distributed from depot to bases and among different bases while failures at all bases will be sent back to depot for repair in the meantime. From our assumption, reallocation will be achieved instantly and repair service will start at the depot just after reallocation. Given the inventory levels at all sites before reallocation $I_i(t_1)$ (i = 0, 1, …, n), our goal is to find the inventory levels at all sites after reallocation $U_i(t_1)$ (i = 0, 1, …, n) such that EBO over all bases by t_2 just before the second reallocation will be minimized. That is, the reallocated spares will be used to last until the next reallocation. Mathematically, EBO over all bases by t_2 can be expressed as:

$$\begin{aligned} EBO_2 &= \min_{U_i(t_1)'s} \sum_{i=1}^{n} EBO_i(t_2) \\ &= \min_{U_i(t_1)'s} \sum_{i=1}^{n} \sqrt{t_2 - t_1}\sigma_i G\left(\frac{U_i(t_1) - (t_2 - t_1)\mu_i}{\sqrt{t_2 - t_1}\sigma_i}\right) \end{aligned} \quad (4)$$

And we have the constraints

$$\begin{aligned} &\sum_{i=1}^{n} U_i(t_1) + U_0(t_1) = \sum_{i=1}^{n} I_i(t_1) + I_0(t_1) \\ &U_i(t_1) \geq 0 (i = 0,1,...,n) \end{aligned} \quad (5)$$

We know $I_0(t_1) = S_0$ and $I_i(t_1) = S_i - D_i(t_1)$ for i = 1, …, n where $D_i(t_1) = \sum_{t=1}^{t_1} y_i(t)$. So Equation (5) can be changed into:

$$\sum_{i=1}^{n} U_i(t_1) + U_0(t_1) = S_0 + \sum_{i=1}^{n} S_i - \sum_{i=1}^{n} D_i(t_1) \quad (6)$$

Using a Lagrange multiplier λ, the optimization can be represented as

$$\min_{U_i(t_1)'s} \sum_{i=1}^{n} \sqrt{t_2 - t_1}\sigma_i G\left(\frac{U_i(t_1) - (t_2 - t_1)\mu_i}{\sqrt{t_2 - t_1}\sigma_i}\right) + \lambda\left(\sum_{i=1}^{n} U_i(t_1) + U_0(t_1) - \sum_{i=1}^{n} I_i(t_1) - I_0(t_1)\right) \quad (7)$$

Differentiating with respect to $U_i(t_1)$ (i = 1, …, n) and setting the result to zero, we obtain

$$-\Psi\left(\frac{U_i(t_1) - (t_2 - t_1)\mu_i}{\sqrt{t_2 - t_1}\sigma_i}\right) + \lambda = 0 \quad (i = 1, \ldots, n) \quad (8)$$

where $\Psi(k) = \int_k^{\infty}\phi(z)dz$ is the right-hand tail area of the standard normal distribution. So according to the property of standard normal distribution,

$$\frac{U_i(t_1) - (t_2 - t_1)\mu_i}{\sqrt{t_2 - t_1}\sigma_i} = c \quad (i = 1, \ldots, n) \quad (9)$$

where c is a constant, independent of i.

Using Equation (9) to sum over i and using equation (6) leads to

$$U_i(t_1) = (t_2 - t_1)\mu_i + \frac{\sigma_i}{\sum_{j=1}^{n}\sigma_j}[S_0 + \sum_{i=1}^{n} S_i - \sum_{i=1}^{n} D_i(t_1) - U_0(t_1) - (t_2 - t_1)\sum_{j=1}^{n}\mu_j] \quad (10)$$

3.3 The Second Reallocation

Using the same method as above, for the second reallocation, our purpose is to find $U_i(t_2)$ (i = 0, 1, …, n) so that EBO over all bases at the end of the cycle will be minimized. In addition, since there is no opportunity to do reallocation once more, we will completely redistribute all stocks on hand at the depot to bases by t_2. Assuming complete redistribution at the second reallocation instant, we have $U_0(t_2) = 0$.

Our objective function is:

$$\begin{aligned} EBO_3 &= \min_{U_i(t_2)'s} \sum_{i=1}^{n} EBO_i(H) \\ &= \min_{U_i(t_2)'s} \sum_{i=1}^{n} \sqrt{H - t_2}\sigma_i G\left(\frac{U_i(t_2) - (H - t_2)\mu_i}{\sqrt{H - t_2}\sigma_i}\right) \end{aligned} \quad (11)$$

And we have the constraints

$$\sum_{i=1}^{n} U_i(t_2) = \sum_{i=1}^{n} I_i(t_2) + I_0(t_2) \quad (12)$$

We know $I_i(t_2) = U_i(t_1) - D_i(t_2)$ for i = 1, …, n where $D_i(t_2) = \sum_{t=t_1+1}^{t_2} y_i(t)$. But the number of available spares at the depot just before the second reallocation equals to the number of spares left at the depot after the first reallocation $U_0(t_1)$, plus those failures which have been finished repair and sent to depot inventory $R(x, t_2 - t_1)$. So we have $I_0(t_2) = U_0(t_1) + R(x, t_2 - t_1)$. We assume the repair time at the depot follows an exponential distribution with mean T. So the probability that an item has been finished repair and sent to inventory by t_2 is $1 - \exp[-(t_2 - t_1)/T]$. Given the total number of failures x brought back to depot for repair at t_1 after the first reallocation, $R(x, t_2 - t_1) = x\{1 - \exp[-(t_2 - t_1)/T]\}$. We know $E(X) = t_1\sum_{i=1}^{n}\mu_i$ and $Var(X) = t_1\sum_{i=1}^{n}\sigma_i^2$, so $E(R) = t_1\sum_{i=1}^{n}\mu_i(1 - e^{-(t_2 - t_1)/T})$ and $Var(R) = t_1\sum_{i=1}^{n}\sigma_i^2(1 - e^{-(t_2 - t_1)/T})^2$.

Therefore, equation (12) can be

$$\sum_{i=1}^{n} U_i(t_2) = \sum_{i=1}^{n} U_i(t_1) - \sum_{i=1}^{n} D_i(t_2) + U_0(t_1) + R \quad (13)$$

Using equation (6), equation (13) can be further changed into:

$$\sum_{i=1}^{n} U_i(t_2) = S_0 + \sum_{i=1}^{n} S_i - \sum_{i=1}^{n} D_i(t_1) - \sum_{i=1}^{n} D_i(t_2) + R \quad (14)$$

As the above method, using a Lagrange multiplier and differentiating with respect to $U_i(t_2)$ (i = 1, …, n) and setting the result to zero, we obtain:

$$U_i(t_2) = (H - t_2)\mu_i + \frac{\sigma_i}{\sum_{j=1}^{n}\sigma_j}[S_0 + \sum_{i=1}^{n} S_i - Y - (H - t_2)\sum_{j=1}^{n}\mu_j] \quad (15)$$

where $Y = Y_1 + Y_2 - R$, $Y_1 = \sum_{i=1}^{n} D_i(t_1)$, $Y_2 = \sum_{i=1}^{n} D_i(t_2)$.

3.4 Compute Optimal EBO

Using Equation (10) and (15), we can compute the optimal spare allocations of LRU after each reallocation. However, we still have not specified how to compute EBO_2 by Equation (4) and EBO_3 by Equation (11). In addition, $U_0(t_1)$ is still included in Equation (10), i.e. the inventory level at each base after the first reallocation depends on different inventory levels left at the depot.

From Equation (15), we know no matter how many spares are left at the depot after the first reallocation, under the complete redistribution assumption for the second reallocation, the term $U_0(t_1)$ will disappear, i.e. the inventory level is independent of $U_0(t_1)$. Therefore, in order to reduce EBO just before the second reallocation at t_2, $U_0(t_1)$ should be set to zero according to Equation (10). That is, we also adopt complete redistribution for the first reallocation. Thus,

$$U_i(t_1) = (t_2 - t_1)\mu_i + \frac{\sigma_i}{\sum_{j=1}^{n}\sigma_j}[S_0 + \sum_{i=1}^{n} S_i - Y_1 - (t_2 - t_1)\sum_{j=1}^{n}\mu_j] \tag{16}$$

where $Y_1 = \sum_{i=1}^{n} D_i(t_1) = \sum_{t=1}^{t_1}\sum_{i=1}^{n} y_i(t)$.

Substituting $U_i(t_1)$ in Equation (4) by Equation (16), we can compute the EBO_2 just before the second reallocation for a given value of Y_1. However, Y_1 is a normally distributed random variable with mean $E(Y_1) = t_1\sum_{i=1}^{n}\mu_i$ and variance $Var(Y_1) = t_1\sum_{i=1}^{n}\sigma_i^2$. Thus, weighting EBO_2 for a given value of Y_1 by the density of Y_1 and integrating over Y_1, we have

$$EBO_2 = \int_{-\infty}^{\infty}\sum_{i=1}^{n}\sqrt{t_2 - t_1}\sigma_i G\left(\frac{U_i(t_1) - (t_2 - t_1)\mu_i}{\sqrt{t_2 - t_1}\sigma_i}\right) f(y_1)dy_1 \tag{17}$$

Substituting

$$\xi = \frac{Y_1 - t_1\sum_{i=1}^{n}\mu_i}{\sqrt{t_1\sum_{i=1}^{n}\sigma_i^2}}$$

and using Equation (16), Equation (17) can be rewritten as

$$EBO_2 = \int_{-\infty}^{\infty}\sum_{i=1}^{n}\sqrt{t_2 - t_1}\sigma_i G(a\xi + b)\phi(\xi)d\xi \tag{18}$$

where

$$a = \frac{-\sqrt{t_1\sum_{i=1}^{n}\sigma_i^2}}{\sqrt{t_2 - t_1}\sum_{i=1}^{n}\sigma_i}$$

and

$$b = \frac{S_0 + \sum_{i=1}^{n} S_i - t_2\sum_{i=1}^{n}\mu_i}{\sqrt{t_2 - t_1}\sum_{i=1}^{n}\sigma_i}$$

Using the following result in [2] and [12]

$$\int_{-\infty}^{\infty} G(ax + b)\phi(x)dx = \sqrt{1 + a^2}\, G\left(\frac{b}{\sqrt{1 + a^2}}\right)$$

we obtain

$$EBO_2 = \sqrt{(t_2 - t_1)(\sum_{i=1}^{n}\sigma_i)^2 + t_1\sum_{i=1}^{n}\sigma_i^2} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - t_2\sum_{i=1}^{n}\mu_i}{\sqrt{(t_2 - t_1)(\sum_{i=1}^{n}\sigma_i)^2 + t_1\sum_{i=1}^{n}\sigma_i^2}}\right) \tag{19}$$

Respectively, using Equation (11) and (15), we can also construct the formula of EBO_3 for a given value of Y, in which Y is a normally distributed random variable. $Y = Y_1 + Y_2 - R$, $Y_1 = \sum_{i=1}^{n} D_i(t_1) = \sum_{t=1}^{t_1}\sum_{i=1}^{n} y_i(t)$, and $Y_2 = \sum_{i=1}^{n} D_i(t_2) = \sum_{t=t_1+1}^{t_2}\sum_{i=1}^{n} y_i(t)$. Thus, it has mean

$$E(Y) = E(Y_1) + E(Y_2) - E(R)$$
$$= t_1\sum_{i=1}^{n}\mu_i + (t_2 - t_1)\sum_{i=1}^{n}\mu_i - t_1\sum_{i=1}^{n}\mu_i[1 - e^{-(t_2 - t_1)/T}]$$
$$= t_2\sum_{i=1}^{n}\mu_i - t_1\sum_{i=1}^{n}\mu_i(1 - e^{-(t_2 - t_1)/T})$$

and variance

$$Var(Y) = Var(Y_1) + Var(Y_2) + Var(R)$$
$$= t_1\sum_{i=1}^{n}\sigma_i^2 + (t_2 - t_1)\sum_{i=1}^{n}\sigma_i^2 + t_1\sum_{i=1}^{n}\sigma_i^2(1 - e^{-(t_2 - t_1)/T})^2$$
$$= t_2\sum_{i=1}^{n}\sigma_i^2 + t_1\sum_{i=1}^{n}\sigma_i^2(1 - e^{-(t_2 - t_1)/T})^2$$

Weighting EBO_3 for a given value of Y by the density of Y and integrating over Y, we have

$$EBO_3 = \int_{-\infty}^{\infty}\sum_{i=1}^{n}\sqrt{H - t_2}\sigma_i G\left(\frac{U_i(t_2) - (H - t_2)\mu_i}{\sqrt{H - t_2}\sigma_i}\right) f(y)dy \tag{20}$$

And using the same method as EBO_2, we obtain

$$EBO_3 = \sqrt{(H - t_2)(\sum_{i=1}^{n}\sigma_i)^2 + t_2\sum_{i=1}^{n}\sigma_i^2 + t_1\sum_{i=1}^{n}\sigma_i^2(1 - e^{-(t_2 - t_1)/T})^2} \times G\left(\frac{S_0 + \sum_{i=1}^{n} S_i - H\sum_{i=1}^{n}\mu_i + t_1\sum_{i=1}^{n}\mu_i(1 - e^{-(t_2 - t_1)/T})}{\sqrt{(H - t_2)(\sum_{i=1}^{n}\sigma_i)^2 + t_2\sum_{i=1}^{n}\sigma_i^2 + t_1\sum_{i=1}^{n}\sigma_i^2(1 - e^{-(t_2 - t_1)/T})^2}}\right) \tag{21}$$

3.5 Dynamic Reallocation Approach

Using Equation (3), (19) and (21), we can calculate our objective function TE, the expected backorders over all bases at three time points for a given reallocation instant pair t_1 and t_2.

$$TE = TE(t_1, t_2) = EBO_1 + EBO_2 + EBO_3 \tag{22}$$

Our purpose is to find such t_1 and t_2 ($0 \le t_1 < t_2 \le H$) that TE is minimized. Given a replenishment cycle of H periods, the total number of different combinations of pair (t_1, t_2) is C_{H+1}^2 , noticing that the first reallocation can take place at the beginning of the cycle. Hence, we can compute TE for C_{H+1}^2 possible pairs of (t_1, t_2) to determine the minimal TE and corresponding (t_1, t_2).

However, if the second reallocation instant is the end of the cycle, we do not make good use of two reallocation chances. TE will be the same as that in [2] with only one reallocation because our objective function is the expected backorders just before reallocation. [2] proves that TE will decrease first and then increase as t_2 increases for a given

t_1, so our dynamic heuristic approach can be stated as follows:

```
for (t1 = 1; t1 < H – 1; t1++) {
    for (t2 = t1+1; t2 < H; t2++) {
        if (TE(t1, t2) > TE(t1, t2+1))
        // postpone the 2nd reallocation until at least the end of the next period.
            continue;
        if (TE(t1, t2) ≤ TE(t1, t2+1)), reallocate for the 2nd time now. Compute TE(t1, t2) and break
    }
}
Compare TE(t1, t2) for different t1 and choose the minimal one with corresponding t1 and t2.
```

4. Experimental Result

We use the test case provided in [2] and compare the results with those allowing single reallocation, as seen in [2]. In our experiment, one depot supports 5 bases (n=5). The length of the replenishment cycle is 30 periods (H=30). For convenience in experimentation, the coefficient of variation (CV = σ/μ) of demand is assumed to be the same at all bases. First we focus on the identical independent demand distributions at all bases. We set $\mu_i=\mu=50$ and CV= 0.4, i.e. $\sigma_i=\sigma=20$ for all i (i = 1, …, n). The stock level at each base i is set to $S_i=H\mu_i=1500$. Also, using the method and test case in [2], we determine the stock level at depot to be $S_0 = 2.33\sqrt{H\sum_{i=1}^{n}\sigma_i^2} = 570$, where 2.33 represents the probability of 1% chances that the total system demands in a cycle H exceeds the total stocks at the depot and all bases. We set the mean repair time at the depot to be T=10 (this is not needed in [2] where all demands are consumable).

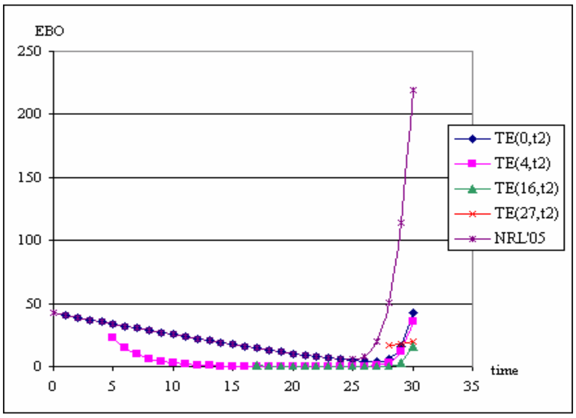

Fig. 1: Comparison of EBO vs. time for two-reallocation and single-reallocation

We implement both our method and that in [2] so that we can compare them with each other to see the effect of multiple reallocations during a cycle. The results are shown in Fig. 1. From Fig. 1, we can see firstly that given the first reallocation instant, the total EBO TE decreases and then increases as the second reallocation instant t_2 increase. This is consistent with what we mentioned in the above section. Therefore, the time interval between reallocations can be neither too short because of fewer repaired failures at the depot nor too long because of more failures at bases. Secondly, we can see from Fig. 1 that TE decreases and then increases as the first reallocation instant t_1 increases. This is indicated by comparing $TE(0, t_2)$, $TE(4, t_2)$ and $TE(27, t_2)$. The curve of $TE(4, t_2)$ is below that of $TE(0, t_2)$, indicating it better to do the first reallocation at t_1=4 than t_1=0 while the curve of $TE(27, t_2)$ is above that of $TE(4, t_2)$, indicating it worse to do the first reallocation at t_1=27 than t_1=4. Comparing all combinations of (t_1, t_2), we find the optimal reallocation instants pair is (16, 23) with TE = 7e-10. Thirdly, we can see from Fig. 1 that multiple reallocations can reduce the total EBO compared with single reallocation. From Fig. 1, the optimal reallocation instant in [2] (i.e. NRL'05 curve) is at t=25 with TE = 5.4874. We can also see that our first reallocation instant should be before t=25. This is because if we reallocate at later than t=25, there will be a large number of backorders at bases according to NRL'05. That is why the curve of $TE(27, t_2)$ is above the line EBO = 5.4874. More interestingly, we compare NRL'05 with those whose first reallocation takes place at t_1=25. The results are shown in Table 1.

Table 1: Total EBO when the first reallocation instant is 25

(t_1, t_2)	TE
(25, 26)	1.0806
(25, 27)	1.0021
(25, 28)	1.0897
(25, 29)	1.2866
(25, 30)	5.4874

From Table 1, we can see that when the first reallocation instant is t=25, if we do the second reallocation immediately after the first one, TE can also be improved because more failures can be repaired as more failures are brought back the depot under the assumption of infinite repair capacities. However, if we do the second reallocation at the end of the cycle, it is equivalent to reallocate only once (recall our objective function is that just before reallocation). Hence, TE should be the same as that of NRL'05, which is proved to be 5.4874 in Table 1.

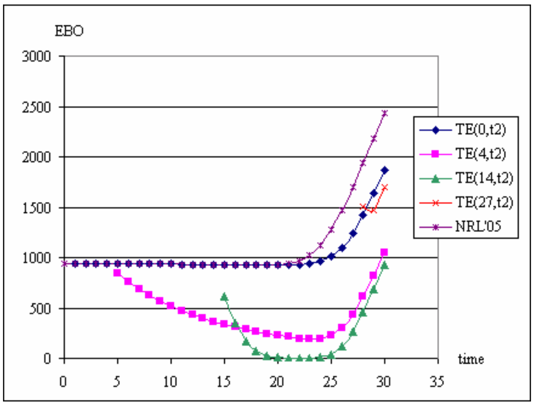

Fig. 2: Comparison of EBO vs. time for two-reallocation and single-reallocation with fewer stocks

Secondly, we set fewer stock levels at bases to see what will happen if more EBO is expected. The results are shown in Fig. 2, where we multiply the stock level at each base in the previous case by 0.8. Cao and Silver [2] claim that because generally it is more costly to delay allocation beyond the best time than to do it somewhat early, it tends to hedge against the higher penalties by committing to an earlier allocation time. Fig. 2 shows firstly that the optimal reallocation instant for NRL'05 is t=19 with TE=931.26. Our optimal reallocation instant pair is (14, 22) with TE=0.3887 correspondingly, both the first reallocation instant and the second one being brought forward. This is consistent with the claim in [2]. Furthermore, Fig. 2 shows secondly that under the same reallocation instant for the first time, the second reallocation instant is also brought forward. In the previous case, when t_1=0, the optimal t_2 is t_2=27 while in the current case the optimal t_2 is t_2=20. Similarly, when t_1=4, t_2=24 in the previous case while t_2=23 in the current one. Thirdly, Fig. 2 shows that the curve of $TE(4, t_2)$ intersects with that of $TE(14, t_2)$. This indicates that it should not always delay the first reallocation and do the second one in a hurry. Reallocation at 4 and 15 is better than reallocation at 14 and 15. In fact, there is also an intersection between the curve of $TE(0, t_2)$ and the curve of $TE(27, t_2)$ although not obvious.

Next, we run a test case by relaxing the assumption of identical demand distributions at bases while the "independent" assumption is still kept. For nonidentical demand distributions at bases, we use the same method as in [2], setting the mean demand of each base i by $\mu_i = 2i\mu / (n+1)$ (i = 1, …, n). The coefficient of variation (CV) of demands is still assumed to be the same at all bases. The results are shown in Table 2.

Table 2: TE for identical demand distributions vs. nonidentical demand distributions

Case	NRL'05-1	Ours-1	NRL'05-0.8	Ours-0.8
Identical	5.4874	7e-10	931.26	0.3887
Nonidentical	4.6391	0	863.22	0.3368

From Table 2 we observe that the total backorders for nonidentical demand distributions are less than those for identical distributions, which is the same as the case in [2]. This is caused by the assumption of a common CV leading to a higher value of the depot stock level from 570 to 627.

5. Conclusion and Further Research

In this paper, we consider reallocation in a multi-echelon inventory system. During the replenishment cycle, we have two opportunities to reallocate the spares by redistributing the depot stocks to the bases and by lateral transshipment. We develop a mathematical model and use Lagrange multiplier to determine how to reallocate the spares to achieve a minimized total expected backorders under a given reallocation instant pair. Then we derive the dynamic reallocation approach to determine when to do the first and second reallocation respectively. Experimental results show that multiple-reallocation is better than single-reallocation. More interestingly, the logic of our approach is appealing and the calculations are easy to carry out.

In our model, we assume negligible internal lead time, and reallocation can be achieved instantly. A natural extension is to incorporate nonidentical, nonzero lead time in the problem. Another work is to extend our model to multiple (more than 2) reallocations, where the challenge is to determine the time interval between reallocations assuming the intervals are the same for a given time horizon. This would pave the way for the design of optimal periodic resupply policies. It is also interesting to study the balance between the costs of reallocations and shortages. Finally, it is interesting to consider nonstationary demand distributions, i.e. demands at each period are not identical with time-varying mean and standard deviation.

References

[1] P. Alfredsson, "Optimization of multi-echelon repairable item inventory systems with simultaneous location of repair facilities", *European Journal of Operational Research*, v99, 584-595, 1997.

[2] D.B. Cao, E.A. Silver, "Dynamic Allocation Heuristic for Centralized Safety Stock", *Naval Research Logistics*, 2005

[3] A. Díaz, M.C. Fu, "Models for multi-echelon repairable item inventory systems with limited repair capacity", *European Journal of Operational Research*, v97, 480-492, 1997.

[4] G.D. Eppen and L. Schrage, "Centralized ordering policies in a multi-warehouse system with lead times and random demand", TIMS Studies in the Management Sciences, v16, p51-67, 1981

[5] P.L. Jackson, "Stock allocation in a two-echelon distribution system or what to do until your ship comes in", *Management Science*, v34, p880-895, 1988

[6] P.L. Jackson, J.A. Muchstadt, "Risk pooling in a two-period, two-echelon inventory stocking and allocation problem", *Naval Research Logistics*, v36, p1-26, 1989

[7] H. Jönsson, E.A. Silver, "Analysis of a two-echelon inventory control system with complete redistribution", *Management Science*, v33, p215-227, 1987

[8] H.C. Lau, H. Song, C.T. See, S.Y. Cheng, "Evaluation of Time-Varying Availability in Multi-Echelon Spare Parts Systems with Passivation", *European Journal of Operational Research*, in press.

[9] H.C. Lau, H. Song, "Two-Echelon Repairable Item Inventory System with Limited Repair Capacity under Nonstationary Demands", In Proc. *Decision Science Institute Annual Meeting*, p1901-1908, 2004

[10]OPUS10 User's Reference – Logistics Support and Spares Optimization version 3, Systecon AB, May 1998.

[11]C.C. Sherbrooke, "Optimal Inventory Modeling of System: Multi-Echelon Techniques", John Wiley & Sons, New York, 1992.

[12]E.A. Silver, D.F. Pyke, R. Peterson, "Inventory management and production planning and scheduling", 3rd edition, 1998

[13]D.B. Tsao, T. Enkawa, "Optimal second replenishment policy in two-phased push control system", *Journal of Operational Research Society Japan*, v35, 273-289, 1992