Elasto-plastic Analysis of Plate With and Without Cut-outs

The thesis submitted in partial fulfilment of requirements for the degree of

Master of Technology

in

Civil Engineering

(Specialization: Structural Engineering)

by

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A C K N O W L E D G E M E N T

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CERTIFICATE

This is to certify that the thesis entitled " **ELASTO-PLASTIC ANALYSIS OF PLATE WITH AND WITHOUT CUT-OUTS**" submitted by **KODURU VENKATA SANDEEP** bearing roll number **214CE2065** in partial fulfillment of the requirements of the award ***Master of Technology*** in the Department of Civil Engineering, National Institute of Technology, Rourkela is an authenticate work carried out by him under my supervision and guidance.

To the best of my knowledge, the matter embodied in the thesis has not been submitted to any other university/institute for the award of any Degree or Diploma.

Place: Rourkela

Date: 27/05/2016

Prof. Asha Patel

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ABSTRACT

Plates and shells are important parts of several engineering applications. Therefore analysis and design of these elements are always of interest to the engineering community. Accurate and conservative assessments of the maximum load the structure can carry, along with the equilibrium path followed in elastic and inelastic range are of utmost importance to understand accurate behavior of structures. The elasto-plastic behavior of structural elements can be modelled following mathematical theory of plasticity involving various failure criteria like von Mises , Tresca criteria.

The present study is made to investigate the effect of material nonlinearity on static behavior of plate with and without cutout. A finite element formulation for plate bending problem involving isotropic hardening material following von Mises criteria has been presented. The formulations have incorporated the shear deformation of the plate. The numerical approach has been formulated in incremental form and based on the tangent stiffness concept. The analysis has been carried out following modified Newton Raphson solution technique. The coding based on the formulation has been written in MATLAB environment. A non-layered and layered plate models have been adapted to understand the real elasto-plastic behavior of the plate. The complex nonlinear behavior was graphically traced through load deflection diagrams, plastic flow diagrams at different load factors and first yield and collapse loads.

The same plate problems were analyzed by using commercial software ABAQUS and results were compared and found to be in good agreement. In ABAQUS analysis were performed by taking quadrilateral shell element S8R5 with through the thickness stress integration (with three integration points) and von Mises yield criterion. The effect of shape and size of cutout on the yield, collapse loads and plastic flow patterns have been included in the research.

Keywords: perfectly plastic, isotropic hardening, von Mises yield criterion, yield load, collapse load, plastic flow

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LIST OF ABBREVIATIONS

a, b	:	Length and width of plate
t	:	Thickness of plate
[K]	:	Stiffness matrix
P	:	Applied load vector
d	:	Deflection
δ	:	Deflection vector
ψ (δ)	:	Applied residual load vector
ξ-η	:	Local co-ordinate axes
y_i	:	Co-ordinates in global co-ordinate axes
N	:	Cubic serendipity shape function
U	:	Displacement field vector
θx, θy	:	Average and linear variation in X and Y direction
ϕ_x and ϕ_y	:	Average and linear shear deformation in X and Y direction
w	:	Deflection in Z direction
ϵ	:	Strain
σ	:	Stress
ϵ_f, ϵ_s	:	Strain in flexure and shear respectively
ϵ_x, ϵ_y	:	Strain in X and Y direction
Υ_{xy}, Υ_{xz}, Υ_{yz}	:	Shear strain in X, Y and Z directions
σ_f, σ_s	:	Stress in flexure and shear plane

σ_x, σ_y : stress in X and Y directions

$\tau_{xy} \tau_{yz} \tau_{zx}$: shear stress in Y, Z and X planes

$M_x M_y M_{xy}$: moment in X, Y plane

$Q_x Q_y$: shear force in X and Y directions

$\delta\epsilon_f, \delta\epsilon_s$: incremental strain in flexure and shear

$\delta\epsilon_x, \delta\epsilon_y$: incremental strain in X and Y directions

$\delta\gamma_{xy}, \delta\gamma_{xz}, \delta\gamma_{yz}$: shear strain deformation in X, Y and Z directions

u : nodal displacement

B_f, B_s : strain displacement matrix in flexure and shear

A : hardening parameter

D : rigidity matrix

D_f, D_s : rigidity matrix in flexure and shear

E : Young's modulus

ν : poison's ratio

χ : hardening parameter

J_2' : second deviatoric stress invariants

$\sigma_1 \sigma_2 \sigma_3$: principal stresses

Q : plastic potential

$d\lambda$: Plastic multiplier

CHAPTER-1

INTRODUCTION

1.1 Introduction:

The introduction of geometric nonlinearity arises in situations where it is no longer sufficient to consider the strain displacement relations as being linear. Obviously every structure exhibits a degree of geometric nonlinearity because, even the smallest load modifies the geometry, but in a linear analysis this small change is ignored.

The two basic hypotheses of the linear first order analysis of structures are:

1. Displacements are so small that all computations may be referred to the undeformed configuration,
2. Materials behave according to Hooke's law of linear elasticity.

If one or both hypotheses are not satisfied, nonlinear analysis must be performed. In a vast majority of cases it is entirely appropriate to assume linearity, but as technology seeks greater exploitation of materials and coherent design to achieve, a true account of the likely nonlinear behavior becomes necessary during analysis. An optimum design will always be such that the result of the most accurate assessment and response to a certain environment. In other words, with the success in manufacturing stronger, lighter weight and more flexible materials,safety and economy become the main goals to be achieved with the help of nonlinear analysis.The importance of plate and shell structures and their generic complexities in nonlinear analysis have naturally led to a reliance on the finite element method for the solution to many types of problems.

Nonlinear structural behavior can usually be classified as being caused by:

1. Material nonlinearity: The constitutive equations relating stresses and strains are nonlinear.
2. Geometric nonlinearity: The strain-displacement equations include higher-order terms, resulting in nonlinear relationships.
3. Force nonlinearity: The direction and magnitude of applied forces change with deformations.
4. Kinematic nonlinearity: The specified displacement boundary conditions depend on the deformations of the structure. The contact problems fall into this category

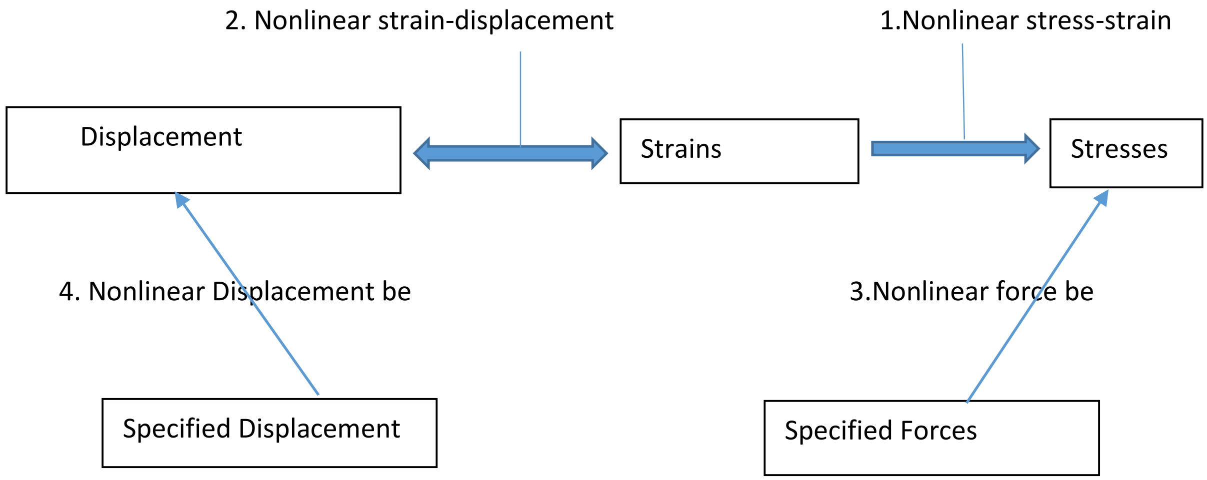


Figure 1: Nonlinear types.

The most general case is, of course, when all four nonlinearities are present in a problem at the same time. However, this may result in a very complex formulation and the cost of computations could be prohibitive. In practical problems, usually only one or two types of nonlinearities are considered at any one time.

Classical approach to solve this problem leads to a set of nonlinear partial differential equations which requires intricate mathematical techniques to achieve a general solution. With the advent of computer alternative numerical approaches to solve the nonlinear problem have been developed, namely the finite difference method and the finite element method. The finite difference method transforms the governing partial differential equations into their nonlinear finite difference equivalents, and the resulting difference equations are solved by iterative procedures. Although the two alternative approaches are readily adaptable to a variety of boundary conditions, the finite element method is considered to be more versatile because of its ability in modelling plates of arbitrary shapes and a to achieve better convergence rate.

Although various researchers have adopted quite a number of different formulation strategies and procedures in the context of geometrically nonlinear plate analysis, the subject is still of considerable interest and practical importance. However elastic behavior of plates has been very closely investigated, whereas inelastic analysis has received less attention from the researchers.

CHAPTER-2

REVIEW OF LITERATURE

2.1 REVIEW OF LITERATURE:

Owen D.R.J.*et al.* (1980) 'Finite Element in Plasticity' Pineridge Press Limited, U.K. demonstrate the use of finite element based methods for the solution of problems involving plasticity. They have given theory and algorithm in detail to solve different problems.

Arthur D.*et al*. (1983) have use triangular element for elasto plastic analysis of plates. This element presents triangular geometry and is the result of a coupling between a plate in bending element and a plane stress element, based on the free formulation (FF).

Owen D.R.J.*et al.* (1983) have given thick shell formulation accounting for shear deformation based on a degenerate three-dimensional continuum element. A numerical model applicable to both thick and thin plates and shells a nine- node heterosis element is introduced. To incorporate anisotropic parameter plasticity Huber-Mises yield criteria has been adopted.

Chung, Wai-cheong (1986) have presented formulation for geometric nonlinear analysis of mindlin Plate adopting higher order finite elements. A number of examples have been carried out in the analyses of square, rectangular, skewed and circular plates as well as shallow shells under different kinds of loading pattern, with a wide range of boundary conditions

Voyiadjis George Z., WoelkePawel (2005) a non-linear finite element analysis is presented, for the elasto-plastic behaviour of thick shells and plates including the effect of large rotations. Shell element based on the refined theory for thick spherical shells is extended here to account for geometric and material non-linearities. The small strain geometric non-linearities are taken into account by means of the updated Lagrangian method. A mathematical representation of multi axial

Bauschinger effect is taken care by means of a quasi-conforming technique, shear and membrane locking are prevented and the tangent stiffness matrix is given explicitly, i.e., no numerical integration is employed which makes the current formulation not only mathematically consistent and accurate.

Murat Yazici (2007) an elasto-plastic theoretical analysis of stresses around a square perforated isotropic plate is studied. The boundary of the plastic stress field around the conformally mapped square holes is obtained by using Savin's complex elastic equations. The elasto-plastic theoretical and FE solutions are compared for isotropic plates, with rounded corner square openings.

Ireneusz Kreja.*et al.* (2007) have given a computational model for a large rotation analysis of elastic laminated shells including a Finite Element Method implementation of the proposed algorithm. The main part of his work deals to examine the relevance of various approximation decisions in the large deformation analysis of plate and shell problems. A number of sample problems of non-linear, large rotation response of composite laminated structures are discussed.

Ki-Du Kim.*et al.* (2008) has taken 4-node quasi shell element for which they had incorporated geometric non linearity and studied the behaviour of FGM plates and shells. The material properties are assumed to be varied in the thickness direction according to a sigmoid function in terms of the volume fraction of the constituents. The series solutions of sigmoid FGM (S-FGM) plates, based on the first-order shear deformation theory and Fourier series expansion are provided as the reference solution for the numerical results.

Hong-xueJia , Xi-la Liu (2014) adopted a force based Large Increment Method (LIM) for the elastoplastic analysis of plates using a force based 4-node quadrilateral plate element which is based on Mindlin–Reissner plate theory. The consistent elastoplastic flexibility matrix of plate element is derived and implemented to solve elastoplastic plate problems. Two simple elastoplastic plate problems are presented to illustrate the accuracy and the computational efficiency of LIM by comparing with the results from the FEM software ABAQUS.

Humberto b coda.*et al.* (2015) proposed a new enhancement strategy which can be applied to the calculated strain field in the analyses of shells by the unconstrained-vector finite element approach, a Solid-Shell-like formulation. This new enhancement is proposed to satisfy the continuity of the shear and normal stresses fields in the transverse direction. The kinematic enhancement is based on the in-plane longitudinal stress equilibrium that is associated with maintaining the elastic strain energy potential in the transverse direction of the shell or plate. Moreover, in contrast to typical elastoplastic procedures, they proposed an alternative plastic flow rule, which includes a new concept of the hardening parameter that depends on the orthotropic directions of the material and a general failure surface that degenerates into the von-Mises or Drucker Prager criteria for isotropic materials.

Rohan Gourav Ray, Patel A (2015) perform analysis of Mindlin plate involving only material nonlinearity incorporating isotropic hardening behavior. The two cases of material behavior, perfectly plastic and linear strain hardening (bilinear) behavior are considered for analysis. The effect of thickness and different boundary conditions on load carrying capacity, load deflection and spread or flow of plastic deformations are studied

CHAPTER-3

NONLINEARITY

3.1 Nonlinearity:

Nonlinearity in the structures means the stiffness of the structure varies as the deflection does not linearly vary with the load. It means that the stiffness is not found by simply dividing the load with the deflection. Generally all our physical structures exhibit non linearity but as an approximation we conveniently transform it to linear methods but it is not possible in every case to convert to the linear analysis. Also structures where the accuracy is of paramount importance we have to go for the nonlinear analysis only.

3.1.1. Geometric Nonlinearity:

If a structure experiences large deformations, its changing geometric configuration can cause the structure to respond nonlinearly. An example would be the fishing rod shown in Figure.2 Geometric nonlinearity is characterized by "large" displacements and/or rotations.

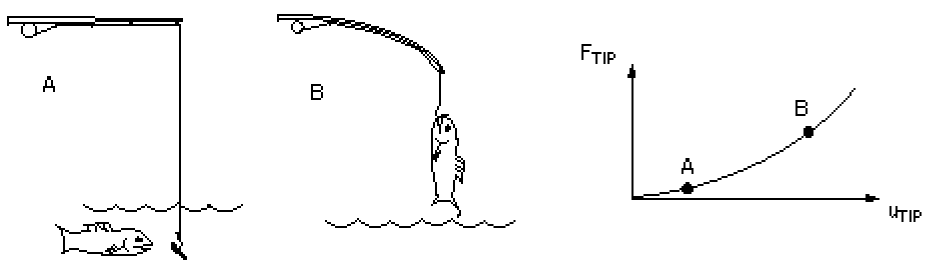


Figure 2: Geometric Nonlinearity

3.1.2. Material Nonlinearity:

Nonlinear stress-strain relationships are a common cause of nonlinear structural behavior. Many factors can influence a material's stress-strain properties, including load history (as in elasto-plastic response), environmental conditions (such as temperature), and the amount of time that a load is applied (as in creep response).

It can be of following ways

a) Perfectly Plastic

An elasto-plastic material model does not account for strain hardening of the material. The stress increases linearly until the yield strength is reached, and then the material offers no further resistance to deformation.

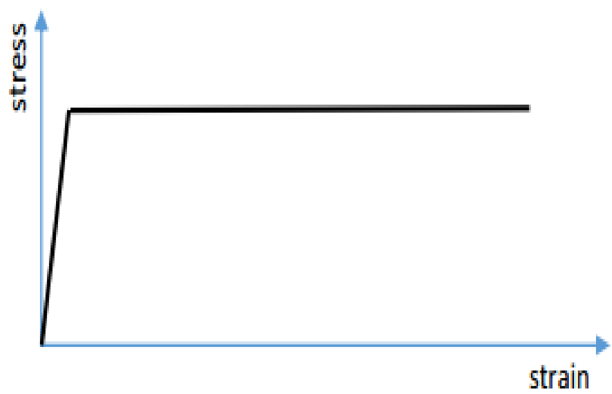


Figure 3: Perfectly plastic

b) Elastic Plastic Strain Hardening:

In some materials after the yield point (or elastic limit) the stress strain curve rises up with increase in the strain values. This is called strain Hardening.

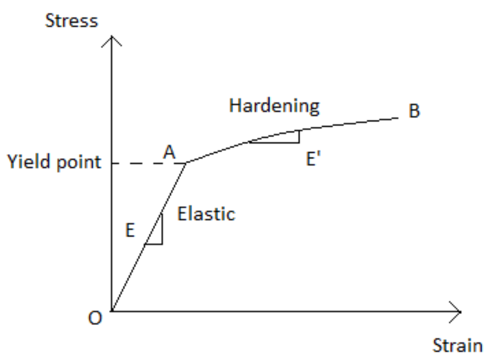


Figure 4: Elasto Plastic Strain Hardening

It is sub divided into two types

1. **Isotropic Hardening:**

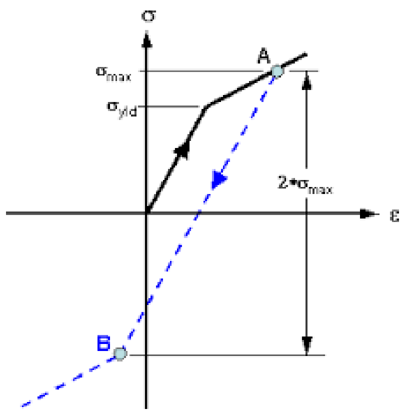


Figure 5: Isotropic Hardening Material Model (Uniaxial)

If the part is taken beyond the yield stress, it begins to deform plastically. If taken to a maximum stress (point A) and the load is released, it unloads along the dashed line. If the part isloaded again, no additional plastic deformation occurs until the stress reaches point A.

If the part is put into compression, it compresses elastically along the dashed line until it reaches point B, and then it yields in compression. With isotropic hardening, the change in stress from point A to point B is twice the maximum stress obtained.

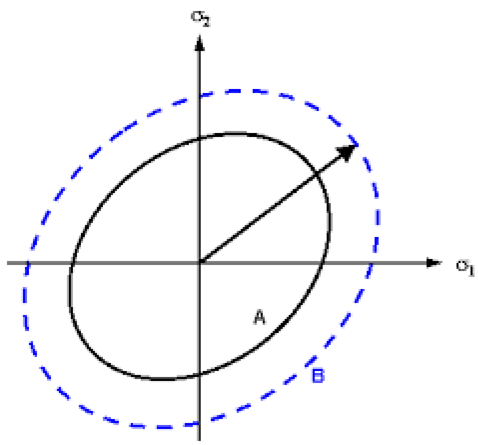


Figure 6: Isotropic Hardening Material Model (Biaxial)

In the biaxial case, any combination of stress inside the initial yield surface (surface A) is in the elastic region. Once the part is taken beyond the initial yield surface, the part experiences plastic deformation.

With isotropic hardening, the center of the yield surface remains fixed but the size of the surface increases. Any stress state inside the new yield surface (surface B) will experience elastic deformation. New plastic deformation occurs when the stress state reaches surface B.

2. Kinematic Hardening:

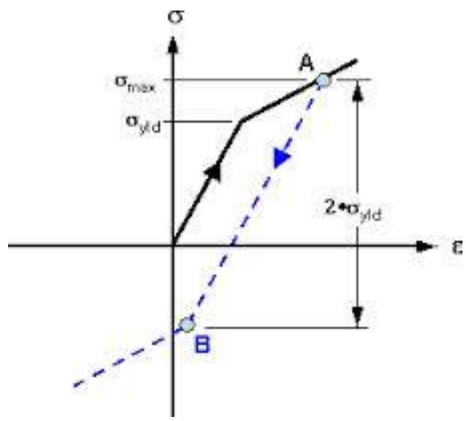


Figure 7: Kinematic Hardening Material Model (Uniaxial)

If the part is taken beyond the yield stress, it begins to deform plastically. If taken to a maximum stress (point A) and the load is released, it unloads along the dashed line. If the part is loaded again, no additional plastic deformation occurs until the stress reaches point A.

If the part is put into compression, it compresses elastically along the dashed line until it reaches point B, and then it yields in compression. With kinematic hardening, the change in stress from point A to point B is twice the yield stress.

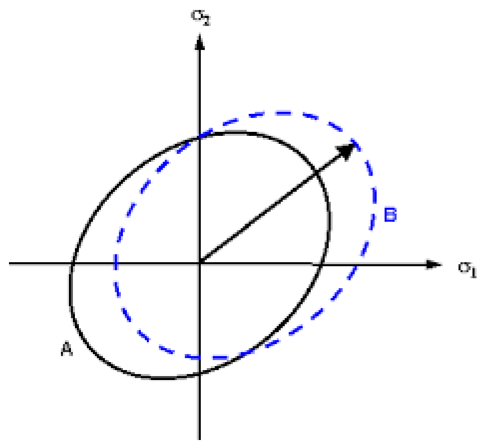


Figure 8: Kinematic Hardening Material Model (Biaxial)

In the biaxial case, any combination of stress inside the initial yield surface (surface A) is in the elastic region. Once the part is taken beyond the initial yield surface, the part experiences plastic deformation.

With kinematic hardening, the center of the yield surface moves but the size of the surface remains constant. Any stress state inside the new yield surface (surface B) will experience elastic deformation. New plastic deformation occurs when the stress state reaches surface B.

CHAPTER-4

AIM

4.1 Aim:

Aim of the proposed work is to study the elasto-plastic analysis of plate using perfectly plastic material. A modified version of the Newton-Raphson method is used to solve the nonlinear equations in the analysis. A non-layered and layered model are used in analysis and results were compared in terms of load deflection and plastic flow diagrams. A non-layered plate with different shapes and percentage of area of concentric cut-outs is analyzed to depict the flow behavior. The analysis is performed for perfectly plastic material. To illustrate the accuracy of numerical model the results are compared with the results obtained from the 'ABAQUS' software.

CHAPTER-5

Formulation for Finite Element Method

5.1 Formulation for Finite Element Method:

Equilibrium equations

$$[\mathbf{K}]\,\boldsymbol{\delta} + \mathbf{P} = \boldsymbol{\psi}(\boldsymbol{\delta}) \neq \mathbf{0} \qquad (1)$$

Where **[K]** is assembled stiffness matrix

P is vector of applied load

δ is vector of basic unknown i.e. defections **d**

ψ(δ) is vector of residual force.

If the coefficients of the matrix **K** depend on the unknowns **δ** or their derivatives, the problem clearly becomes nonlinear. In this case, direct solution of equation system (1) is generally impossible and an iterative scheme must be adopted. For nonlinear situations, in which the stiffness depends on the degree of displacement in some manner, K is equal to the local gradient of the force-displacement relationship of the structure at any point and is termed the tangential stiffness. The analysis of such problems must proceed in an incremental manner since the solution at any stage may not only depend on the current displacements of the structure, but also on the previous loading history. In present study Newton-Raphson technique following the tangential stiffness method is adopted for nonlinear analysis of Mindlin plate.

5.2 Discretization:

The arbitrary shape of the whole plate is mapped into a Master Plate of square region [-1, +1] in the ξ-η plane with the help of the relationship given by

$$x = \sum_{i=1}^{8} N_i(\xi,\eta)x_i \qquad (2)$$

and $$y = \sum_{i=1}^{8} N_i(\xi,\eta)y_i \qquad (3)$$

Where (x_i, y_i) are the coordinates of the i^{th} node on the boundary of the plate in the x-y plane and $N_i\ (\xi, \eta)$ are the corresponding cubic Serendipity shape functions presented below.

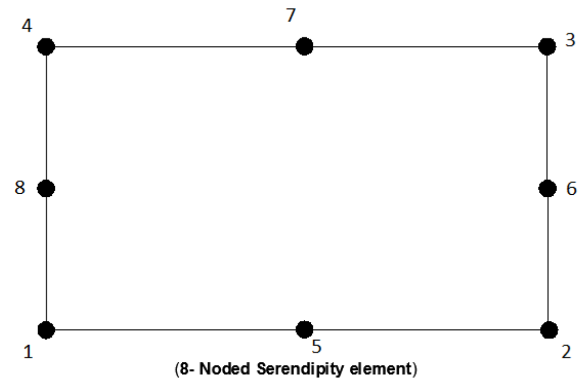


Figure 9: 8 Noded Serendipity element

$$N_1 = {}^{1}/_{4}\ (\eta - 1)\ (1- \xi)\ (\eta + \xi + 1)$$

$$N_2 = {}^{1}/_{2}\ (1 - \eta)\ (1- \xi^{2})$$

$$N_3 = {}^{1}/_{4}\ (\eta - 1)\ (1- \xi)\ (\eta - \xi + 1)$$

$$N_4 = {}^{1}/_{2}\ (1 - \eta^{2})\ (1 + \xi)$$

$$N_5 = {}^{1}/_{4}\ (1 + \eta)\ (1 + \xi)\ (\eta + \xi - 1)$$

$$N_6 = {}^{1}/_{2}\ (1 + \eta)\ (1- \xi^{2})$$

$$N_7 = {}^{1}/_{4}\ (1 + \eta)\ (1 - \xi)\ (\eta - \xi - 1)$$

$$N_8 = {}^{1}/_{2}\ (1 - \eta^{2})\ (1 - \xi)$$

$$[N] = [N_1\ N_2\ N_3\ N_4\ N_5\ N_6\ N_7\ N_8] \quad (4)$$

5.3 Plate element formulation:

The displacement field at any point within the element is given by

$$\{U\} = \begin{bmatrix} u - z\,\theta_x(x, y) \\ u - z\,\theta y(x, y) \\ w(x, y) \end{bmatrix} \tag{5}$$

Owing to the shear deformations, certain warping in the section occurs as shown in Fig.10. However, considering the rotations θx and θy as the average and linear variation along the thickness of the plate, the angles ϕ_x and ϕ_y denoting the average shear deformation in and x-y

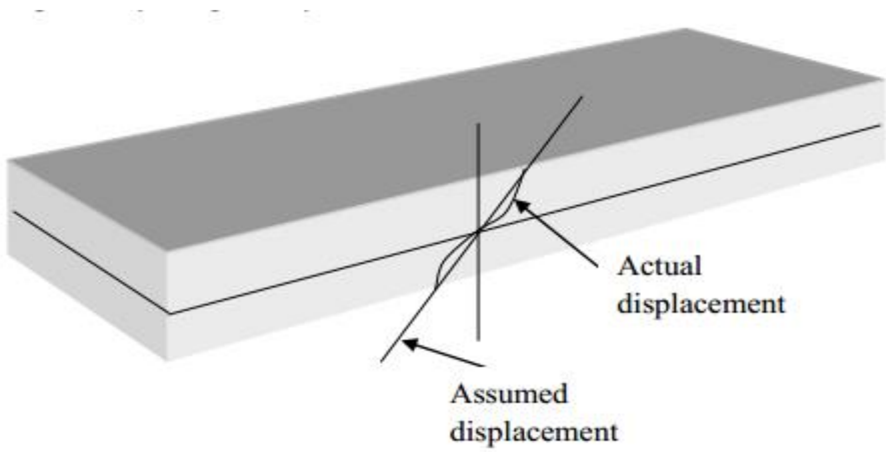


Figure 10: warping in plate section

$$\begin{Bmatrix} \Theta_x \\ \Theta_y \end{Bmatrix} = \begin{Bmatrix} \frac{\partial w}{\partial x} + \varphi_x \\ \frac{\partial w}{\partial y} + \varphi_y \end{Bmatrix} \tag{6}$$

The plate strains are described in terms of middle surface displacements i. e. x-y plane coincides with the middle surface .The strain matrix is given by

$$\{\epsilon\} = \begin{Bmatrix} \epsilon_f \\ \epsilon_s \end{Bmatrix} = \begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \Upsilon_{xy} \\ \Upsilon_{xz} \\ \Upsilon_{yz} \end{Bmatrix} \tag{7}$$

And stress matrix is given by

$$\{\sigma\} = \begin{Bmatrix} \sigma_f \\ \sigma_s \end{Bmatrix} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix}$$

For non-layer approach

We interpret

$$[\sigma_f] = [\, M_x \quad M_y \quad M_{xy}]^{\mathrm{T}} \tag{9}$$

and $$[\sigma_s = [\, Q_x \quad Q_y]^{\mathrm{T}} \tag{10}$$

For layered approach

$$\sigma_f' = \int_{-t/2}^{-t/2} z\sigma_f dz \tag{11}$$

$$\sigma_s' = \int_{-t/2}^{-t/2} z\sigma_s dz \tag{12}$$

Since iterative method is for analysis, the corresponding relations in incremental form can be written as

$$\{\delta\epsilon\} = \begin{Bmatrix} \delta\epsilon_f \\ \delta\epsilon_s \end{Bmatrix} = \begin{Bmatrix} \delta\epsilon_x \\ \delta\epsilon_y \\ \delta\gamma_{xy} \\ \delta\gamma_{xz} \\ \delta\gamma_{yz} \end{Bmatrix} \tag{13}$$

$$\delta\epsilon_f = z\left[-\frac{\partial\delta\theta_x}{\partial x} \quad -\frac{\partial\delta\theta_y}{\partial y} \quad -(\frac{\partial\delta\theta_y}{\partial x} + \frac{\partial\delta\theta_x}{\partial y})\right]^{\mathrm{T}} \tag{14}$$

$$\delta\epsilon_s = \left[\frac{\partial\delta w}{\partial x} - \delta\theta_x, \quad \frac{\partial\delta w}{\partial y} - \delta\theta_y\right]^{\mathrm{T}} \tag{15}$$

5.4 Strain displacement relationship:

For an isotropic material the displacement can be written as

$$U=\sum_{i=1}^{8} N_i(\xi,\eta)u_i \quad (16)$$

Where u_i is nodal displacement vector at i^{th} node may be represented as

$$u_i = [w_i, \theta_{xi}, \theta_{yi}]^T \quad (17)$$

$$U = [w, \theta_x, \theta_y]^T \quad (18)$$

The **flexural strain –displacement equation** in incremental form is given as

$$\delta\epsilon_f = \sum_{i=1}^{8} B_{fi}\delta u_i \quad (19)$$

Where $$B_{fi} = \begin{bmatrix} 0 & -\frac{\partial N_i}{\partial x} & 0 \\ 0 & 0 & -\frac{\partial N_i}{\partial y} \\ 0 & -\frac{\partial N_i}{\partial y} & -\frac{\partial N_i}{\partial x} \end{bmatrix} \quad (20)$$

The incremental **shear strain displacement** equation is

$$\delta\epsilon_s = \sum_{i=1}^{8} B_{si}\delta u_i \quad (21)$$

Where $$B_{si} = \begin{bmatrix} \frac{\partial N_i}{\partial x} & -N_i & 0 \\ \frac{\partial N_i}{\partial y} & 0 & -N_i \end{bmatrix} \quad (22)$$

5.5 Virtual work Equation:

Giving a virtual displacement δu to the system the virtual work statement may be written as

$$\sum_{i=1}^{n}[\delta u_i]^T \left\{\int A \int_{-t/2}^{t/2} [B_{fi}]^T \sigma'_f z + [B_{si}]^T \sigma_s' z - [N_i]^T q\right\} dz\, dA = 0 \quad (23)$$

Or $$\sum_{i=1}^{n} \psi i(u) = 0$$

where ψ_i is residual force vector at i^{th} node.

Since equation (23) must be true for any set of virtual displacements we get (for layered model)

$$\left\{\int A \int_{-t/2}^{t/2} \left[B_{fi}\right]^T \sigma'_f z + [B_{si}]^T \sigma'_s z - [N_i]^T q\right\} dz\, dA = 0 \qquad (24)$$

For layered model

$$\left\{\int A \int_{-t/2}^{t/2} \left[B_{fi}\right]^T \sigma'_f z + [B_{si}]^T \sigma'_s z - [N_i]^T q\right\} d_z d_A = 0 \qquad (25)$$

For nonlayer model

$$\int_A \left[\left[B_{fi}\right]^T \sigma_f + [B_{si}]^T \sigma_s - [N_i]^T q\right] dA = 0 \qquad (26)$$

$$\psi = [\psi_1, \psi_2, \psi_3, \ldots\ldots\ldots \psi_n]^T \qquad (27)$$

Contribution to residual force vector is evaluated at element level and then assembled to for residual force vector ψ.

5.6 Formulation in inelastic region:

In this study material non linearity due to an elasto-plastic material response is considered and isotropic effects are included in the yielding behavior. To model elasto-plastic material behavior in inelastic region two conditions have to be met:

1. A yield criterion representing the stress level at which plastic flow commences must be postulated,

2. A relationship between stress and strain must be developed for post yielding behavior.

Before onset of yielding the relationship between stress and strain is given by

$$\sigma = D^*\varepsilon \qquad (28)$$

D is rigidity matrix

$$D = \begin{bmatrix} D_f \\ D_s \end{bmatrix} \qquad (29)$$

$$D_f = \frac{Et^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix} \quad (30)$$

$$D_s = \frac{Et}{2.4(1+\nu)}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

For the isotropic material the yield criteria adopted is a generalization of the Von Mises law.

5.7 The Von Mises Yield Criterion:

In general form yield criterion is written as

$$F(\sigma, \chi) = f(\sigma) - Y(\chi) = 0 \quad (31)$$

Where f is some function of the deviatoric stress invariants and Y is yield level which is function of hardening parameter χ.

Defining the effective stress **σ** for isotropic Von Mises material as

$$\boldsymbol{\sigma} = \sqrt{3}k \quad (32)$$

$$\text{Where} \quad k = (J_2')^{1/2} \quad (33)$$

and J_2' is the second deviatoric stress invariants

$$J_2' = \frac{1}{6}[(\sigma_1 - \sigma_2)^2 + [(\sigma_2 - \sigma_3)^2 + [(\sigma_3 - \sigma_1)^2] \quad (34)$$

$\sigma_1, \sigma_2, \sigma_3$ are principal stresses

$$= \frac{1}{2}[\sigma_x'^2 + \sigma_y'^2 + \sigma_z'^2] + \tau_{xy}^2 + \tau_{yz}^2 + \tau_{zx}^2 \quad (35)$$

5.8 Elasto-plastic stress strain relation:

After initial yielding the material behavior will be partly elastic and partly plastic. During any increment of stress, the changes of strain are assumed to be divisible into elastic and plastic components, so that

$$d\varepsilon = d\varepsilon^e + d\varepsilon^p \tag{36}$$

The elastic strain increment is given by the incremental form of

$$d\varepsilon^e = [D]^{-1} d\sigma \tag{37}$$

And the plastic strain increment by the flow rule

$$d\varepsilon^p = d\lambda \frac{\partial Q}{\partial \chi} \tag{38}$$

where Q is defined as plastic potential and $d\lambda$ is a proportional constant called plastic multiplier.

The assumption $Q \equiv \boldsymbol{f}$ gives rise to an associated plasticity theory, in which case equation (38) represents the normality condition; since $\frac{\partial f}{\partial \sigma}$ is a vector directed normal to the yield surface in a stress space geometrical interpretation.

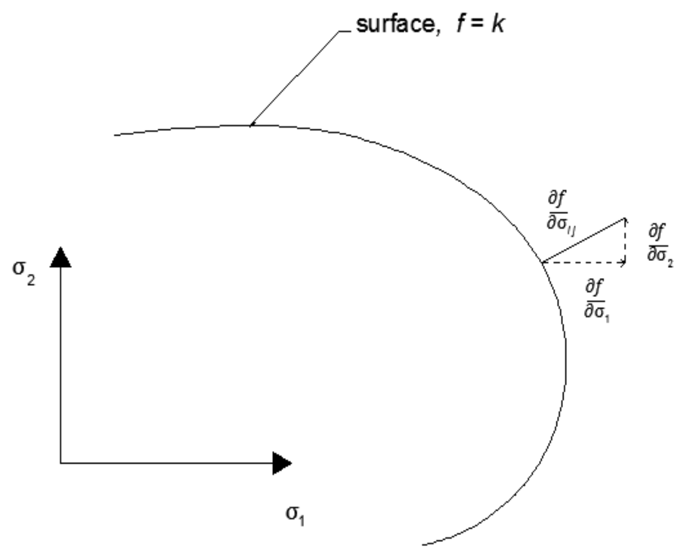


Figure 11: Geometrical representation of the normality rule of associated plasticity

The differential form of eq. (31) is

$$dF = \frac{\partial F}{\partial \sigma} d\sigma + \frac{\partial F}{\partial \chi} d\chi = 0 \tag{39}$$

$$\text{or} \qquad a^T d\sigma - A d\lambda = 0 \tag{40}$$

in which the flow vector $\mathbf{a}^{\mathbf{T}}$ is define as

$$a^T = \frac{\partial F}{\partial \sigma} = \left[\frac{\partial F}{\partial \sigma x}, \frac{\partial F}{\partial \sigma y}, \frac{\partial F}{\partial \tau xy}, \frac{\partial F}{\partial \tau yz}, \frac{\partial F}{\partial \tau zx}\right] \quad (41)$$

Equation (39) & (40) can be reduced to get

$$A = -\frac{1}{d\lambda}\frac{\partial F}{\partial \chi}d\chi \quad (42)$$

Total incremental strain is

$$d\varepsilon = [D]^{-1}d\sigma + d\lambda\, \frac{\partial F}{\partial \chi} \quad (43)$$

Pre-multiplying both sides by a^T D and eliminating $a^T d\sigma$ by using eq. (42), we get $d\lambda$ to be

$$d\lambda = \frac{1}{[A + a^T D\, a]} a^T D^T a\, d\varepsilon \quad (44)$$

Manipulation of equation (36) to equation (44) will give elastoplastic incremental stress strain relationship

$$d\sigma = D_{ep} d\varepsilon \quad (45)$$

Where

$$D_{ep} = D - \frac{D\, a\, a^T D}{[A + a^T D\, a]} \quad (46)$$

The hardening parameter A can be deduced from uniaxial conditions as

$$A = H' = \frac{\partial \sigma}{\partial \epsilon_p} \quad (47)$$

Thus A is obtained to be the local slope of the uniaxial stress/plastic strain curve and can be determined experimentally from Fig.12

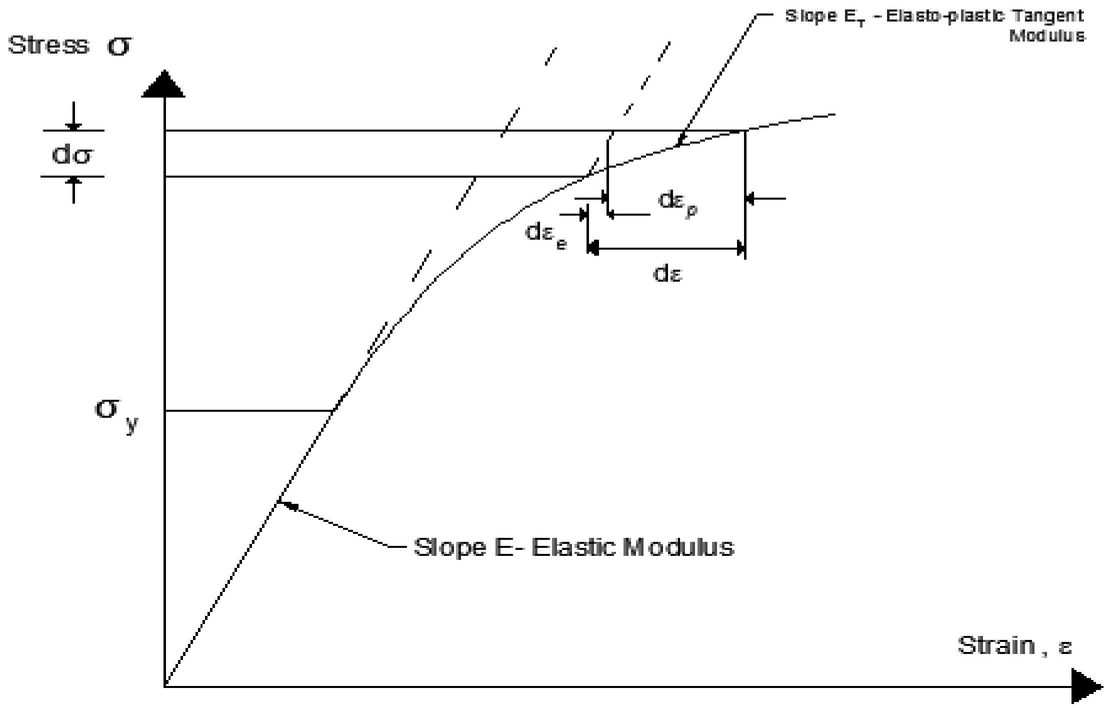


Figure 12: Elasto-plastic strain hardening behavior for the uniaxial case

$$A = H' = \frac{E_T}{1- {E_T}/{E}} \tag{48}$$

The incremental stress-strain resultant relationship is given as

$$\begin{bmatrix} d\sigma_f \\ d\sigma_s \end{bmatrix} = \begin{bmatrix} (D_{ep})_f & 0 \\ 0 & D_s \end{bmatrix} \begin{bmatrix} d\varepsilon_f \\ d\varepsilon_s \end{bmatrix} \tag{49}$$

For the analysis, yield function F is assumed to be function of σ_f, the direct stresses associated with flexure only hence D_s always remain elastic.

5.9 Tangential Stiffness matrix:

From equation (24),the tangential stiffness matrix can be written as

$$K_T = \int_A \left[[B_f]^T (D_{ep})_f \, B_f + [B_s]^T D_s B_s \right] d_A \tag{50}$$

5.10 Equation solving technique for non-layered plate:

1. Begin new load increment, $f = \Delta f$.
2. Set Δf equal to the current load increment vector.
3. Set d^0 equal to 0 for the first increment or equal to the total displacement vector at the end of the last load increment.
4. Set ψ^0 equal to the residual force vector at the end of the last increment or equal to 0 for the first increment.
5. Set $\psi^0 = \psi^0 + \Delta f$.
6. Solve $\Delta d^0 = -[K_T]^{-1}\psi^0$. (Use old or updated value K_T)
7. Set $d^1 = d^0 + \Delta d^0$.
8. Evaluate $\psi^1(d^1)$
9. If solution has converged go to 11; otherwise continue.
10. Iterate until solution has converged.
11. If this is not the last increment go to 1; otherwise stop

5.11 The iteration loop for elasto-plastic non-layered plate:

1. Set iteration number $i = 1$.
2. Solve $\Delta d^i = -[K_T]^{-1}\psi^i$. (Use old or updated value K_T)
3. Set $d^{i+1} = d^i + \Delta d^i$.
4. For each Gauss point, evaluate the increments in strain resultants

$$\Delta\hat{\epsilon}_f^i = B_f \Delta d^i$$

$$\Delta\hat{\epsilon}_s^i = B_s \Delta d^i$$

5. Using the elastic rigidities estimate, at each Gauss point, the increments in stress resultants and hence the total stress resultants

$$\Delta\hat{\sigma}_f^i = \widehat{D}_f \Delta\hat{\epsilon}_f^i \quad \text{Hence} \quad \hat{\sigma}_f^{i+1} = \hat{\sigma}_f^i + \Delta\hat{\sigma}_f^i$$

$$\Delta\hat{\sigma}_s^i = \widehat{D}_s \Delta\hat{\epsilon}_s^i \quad \text{Hence} \quad \hat{\sigma}_s^{i+1} = \hat{\sigma}_s^i + \Delta\hat{\sigma}_s^i$$

6. At each Gauss point, depending on the states of $\hat{\sigma}_f^i$ and $\hat{\sigma}_f^{i+1}$, adjust $\hat{\sigma}_f^{i+1}$ to satisfy the yield criterion and preserve the normality condition.

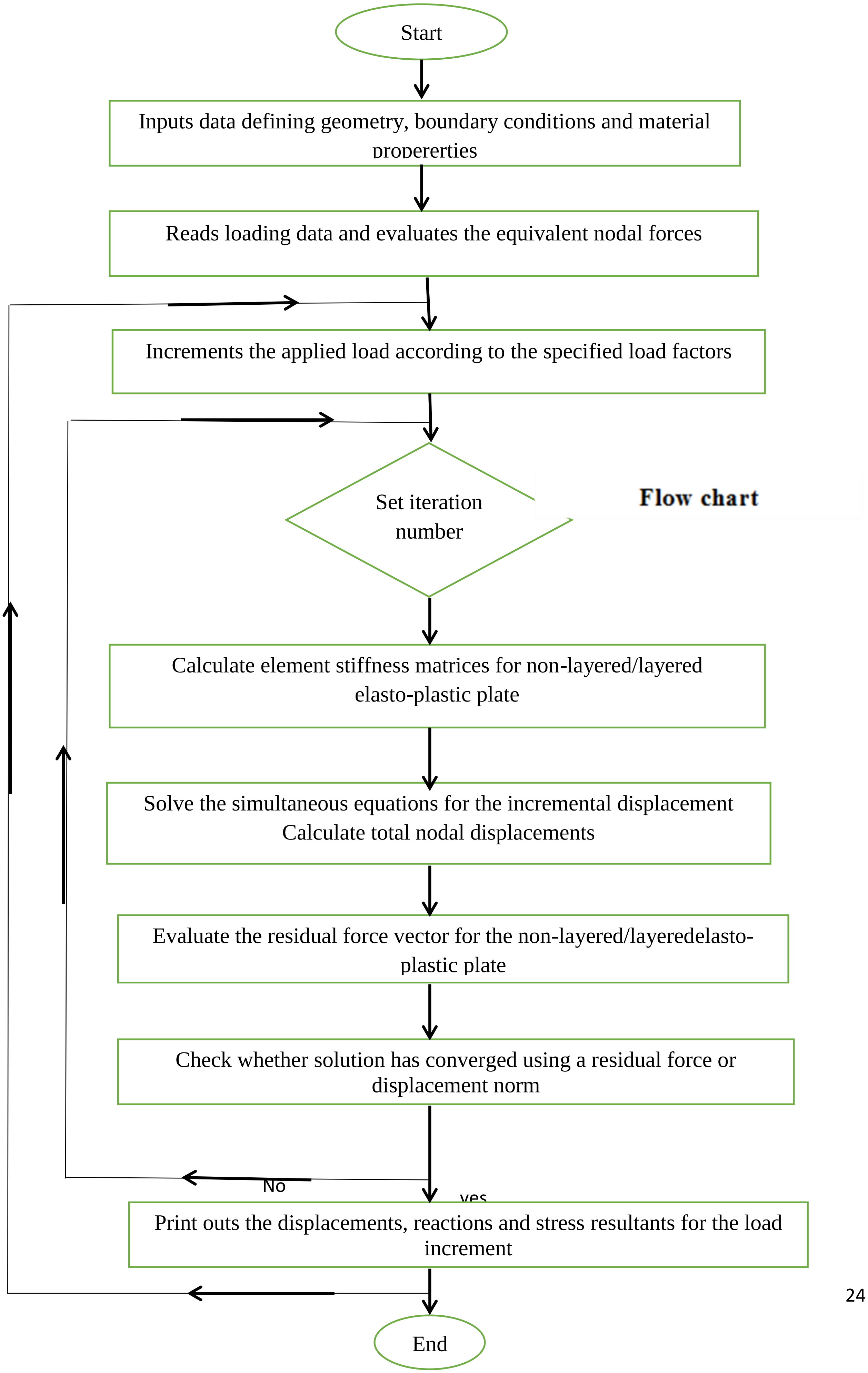
7. Evaluate the residual force vector

$$\psi^{i+1} = \iint \left\{ \left[B_f\right]^T \hat{\sigma}_f + [B_s]^T \hat{\sigma}_s \right\} dx\, dy - f$$

8. If the solution has converged, continue, otherwise set $i = i + 1$ and go to 2.
9. Move to next load increment.

5.12 Plates with cutout:

The same formulation with modifications in element numbers, node numbers etc. Were adopted for the analysis of plate with cutout.


Start
Inputs data defining geometry, boundary conditions and material propererties
Reads loading data and evaluates the equivalent nodal forces
Increments the applied load according to the specified load factors
Set iteration number
Flow chart
Calculate element stiffness matrices for non-layered/layered elasto-plastic plate
Solve the simultaneous equations for the incremental displacement Calculate total nodal displacements
Evaluate the residual force vector for the non-layered/layeredelasto-plastic plate
Check whether solution has converged using a residual force or displacement norm
No
yes
Print outs the displacements, reactions and stress resultants for the load increment
End

CHAPTER-6

MODELING AND ANALYSIS

6.1 Abaqus Modeling and analysis:

In Abaqus modeling and analysis includes following three steps:

1. Preprocessing
2. Simulation
3. Postprocessing

6.2 Preprocessing:

It is the initial step to analyze the physical problem. In this step model of the physical problem is defined and a Abaqus input file (job.inp) in generated. Basic key points are assigned here

1. Planar 3D shell element was taken and geometry will be assigned.
2. Material properties were defined and section was created.
3. Created shell section has to be assigned to the part.
4. Step was created as per the load factor and the iterative method was adopted.
5. Boundary conditions ,load was given and finally meshing will be done with S8R5 element

6.3 Simulation:

The simulation is normally run as a background process. In this step already generated abacus input file solves the numerical problem defined in the model. For example, output from a stress analysis problem includes displacement and stress values which stored in binary files in simulation which are further to be used in postprocessing. The output file is generated as job.odb.

During simulation Abaqus uses Newton Raphson method to solve the non-linear type problems. Unlike linear analysis, load application to the system is incremental in non-linear case. Abaqus breaks the simulation stage into number of *load increments* and at the end of each load increment it finds approximate equilibrium configuration. Sometimes Abaqus/standard takes number iterations to find an acceptable solution for a particular load increment. Finally the cumulative summation of all load incremental responses is the approximate solution to that nonlinear problem. Abaqus uses both incremental and iterative methods to solve the non-linear problems.

There are three phases in simulation stage

a) Analysis step
b) Load increment
c) Iteration

In first phase, steps should be defined which consists of loading option, output request. Output request describes the values of required parameters like displacement, stress, strain, reaction force, bending moment etc.

Second phase is the increment step, in which load increments has to be defined by user and the subsequent increments will be chosen by Abaqus automatically. After each load increment the structure will be in equilibrium and corresponding output request values were written to the output database file.

In iteration step, approximate equilibrium solution in each increment is found out. If the structure is not in equilibrium after iteration, Abaqus tries further iteration till closest possible equilibrium is obtained or the residual force is less than the given tolerance value.

6.4 Postprocessing:

Once the simulation was done and the fundamental variables like stress, displacement, reaction forces were calculated, the results can be evaluated using Visualization module of Abaqus. The visualization module has variety of options to display the results such as animation, color contour plots, deformed shape plots and *X-Y* plots.

Deflection, true and plastic strain, true stress and yield stress values at desired nodes can be found by this module. So all these values can be obtained from the visualization module of Abaqus. From the XY data one has to select Field output for getting the deflection variables at different increments. Once XY data was found, it can import to excel to get load deflection curve.

CHAPTER-7

RESULTS AND DISCUSSION

The finite element formulation of elasto-plastic analysis of plate has been presented in Chaptor-5. A computer program based on the formulation has been written in MATLAB. Examples have been solved to validate the proposed approach. Examples include square plate with and without cutout. Only concentric cutout has been considered. The material type has been perfectly plastic only.

7.1 Problem Statement:

To demonstrate the effectiveness of the present formulation for elasto-plastic plate problem under monotonic loading, simply supported square plate of perfectly plastic material under uniform loading is analyzed. The non-dimensional parameters are as follows

The plate side length L=1,

The thickness t =0.01,

Young's modulus E=10.92,

Poisson's ratio ν=0.3 and

The yield stress σ_y=1600

7.2 Convergence study:

A convergence study for mesh size was performed based on elastic analysis

Convergence study of deflection at mid-point with varying mesh size

Table 1: convergence study

Deflection Value at mid-point ($*10^3$)	**Mesh division**					
	2x2	4x4	6x6	8x8	12x12	15x15
	3.850	4.118	4.091	4.085	4.079	4.079

Hence for analysis 12x12 mesh size has been adopted.

7.3 Plate: Non-layered model:

The non-linear analysis was performed numerically, using finite element method in MATLAB environment

The results obtained was compared to those published by Owen & Hinton (1980), for the non-layered model .The values on center node deflection for two load factors were given in the Table

Table 2: comparison of deflection values

Load factor	Present study	Owen & Hinton (1980)
0.5	2020.89	2020.89
0.856	3501.28	3496.31

The results were further validated by using ABAQUS. The results are compared in terms of load vs. central node displacement and plastic flow and found in good agreement.

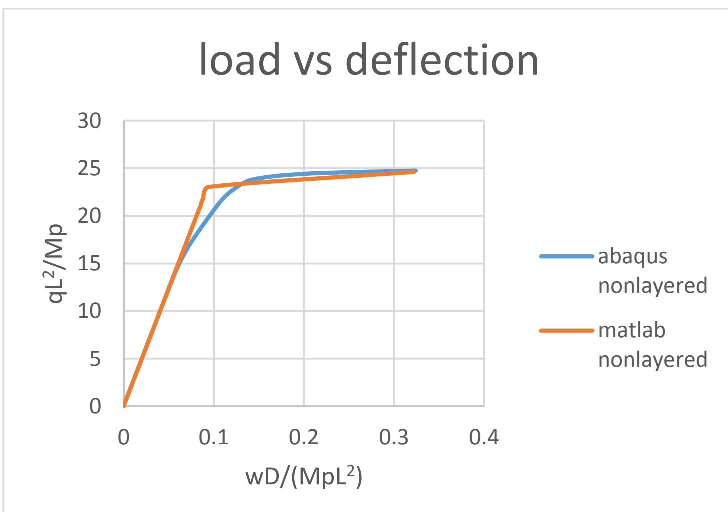
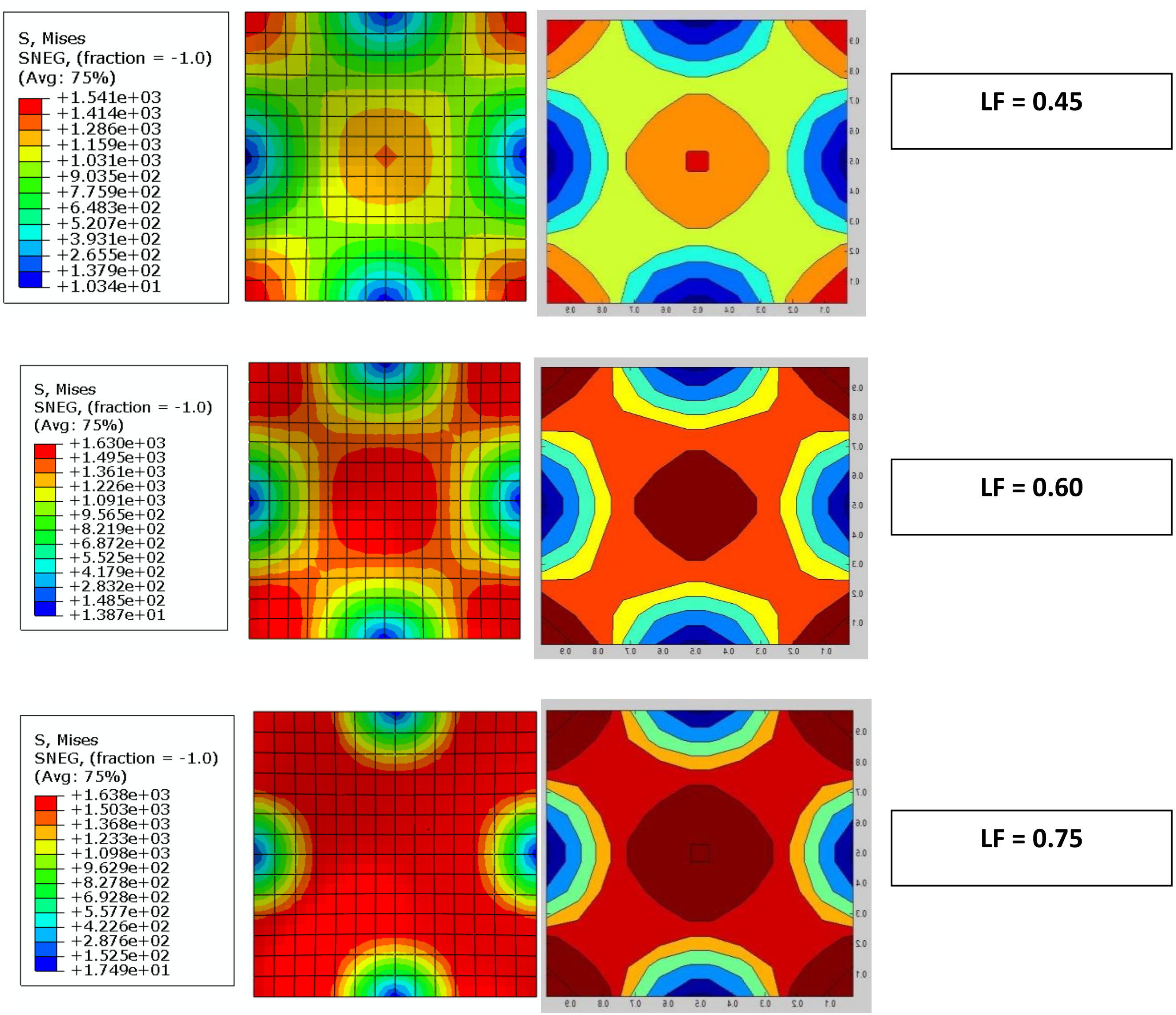


Figure 13: Load Vs. deflection diagram for non-layered model

The deflections are same till the plate is in elastic stage i.e. till yield load. Bifurcation starts at the onset of yielding. The collapse load obtained from ABAQUS is more than that is obtained from MATLAB. Because modeling perfect plastic material in ABAQUS can be done by adopting small gradient and analysis doesn't get aborted when same yield stress is defined against various plastic strain, to avoid this a minimal increase in yield stress has been given against plastic strain values. The slight increase in yield value results in higher value of collapse load.

Plastic Flow:

A study of plastic flow is important to understand the yielding behavior of the element. This can be observed by plotting stress contours. The stress contours were plotted at different load increments. The results obtained from ABAQUS and MATLAB were compared and found to be similar.



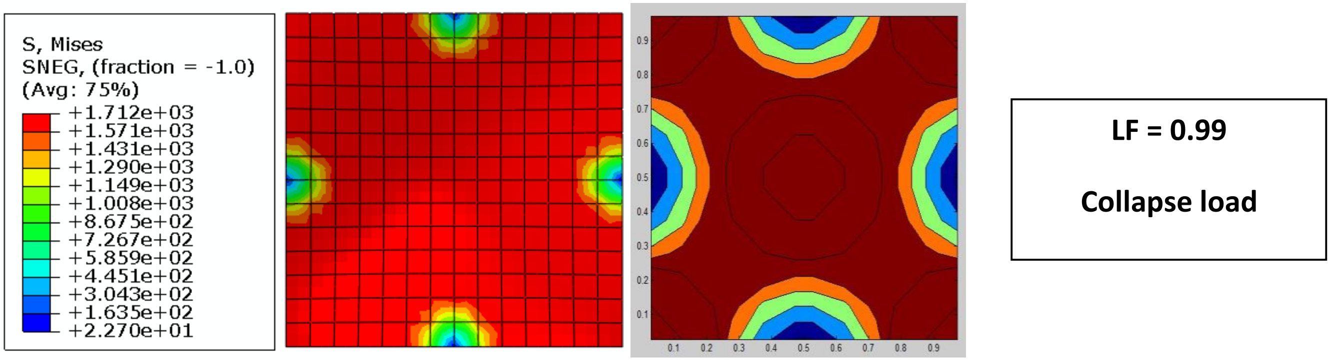


Figure 14: Plastic Flow (Stress Contour) Diagram at different Load Factors

7.4 Plate: Layered model

The non-linear analysis was performed numerically, using finite element method in following the formulation given in chapter 5 for layered plate. The convergence study has been conducted for number of layers and results were found to be converged for eight layers. The results are presented graphically as load deflection diagram and plastic flow diagram along thickness. The pattern of yield zone through thickness were observed at each increment of load and presented in Fig.13 the same model has been analyzed in ABAQUS and load deflection (central node) diagram is presented in Fig.15

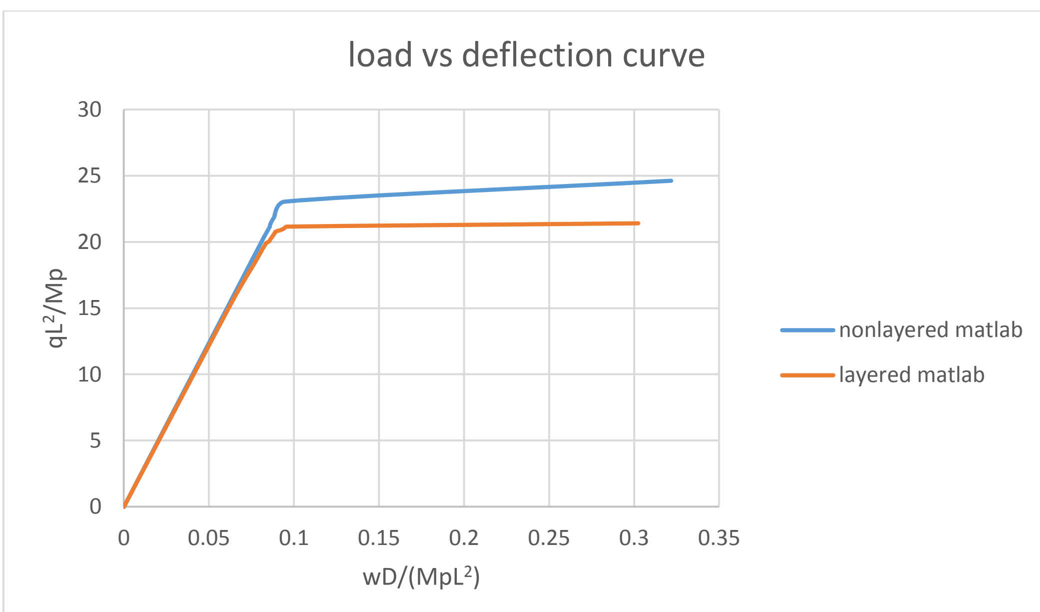


Figure 15: Load vs. deflection diagram for non-layered and layered model

The assumption used in FEM formulation implies that the whole section becomes plastic as soon as the bending moment reaches its yield value i.e. Plastic moment value$M_p = \frac{\sigma_y t^2}{4}$.

The whole cross section is assumed to have yielded when the bending moment at that section has exceeded Mp. In fact at this condition stresses in extreme fibers only have exceeded the actual yield stress value and the whole section has not yielded. Hence, unless steps are taken to improve this problem, the cross section may only be either fully elastic or fully plastic, without any intermediate states

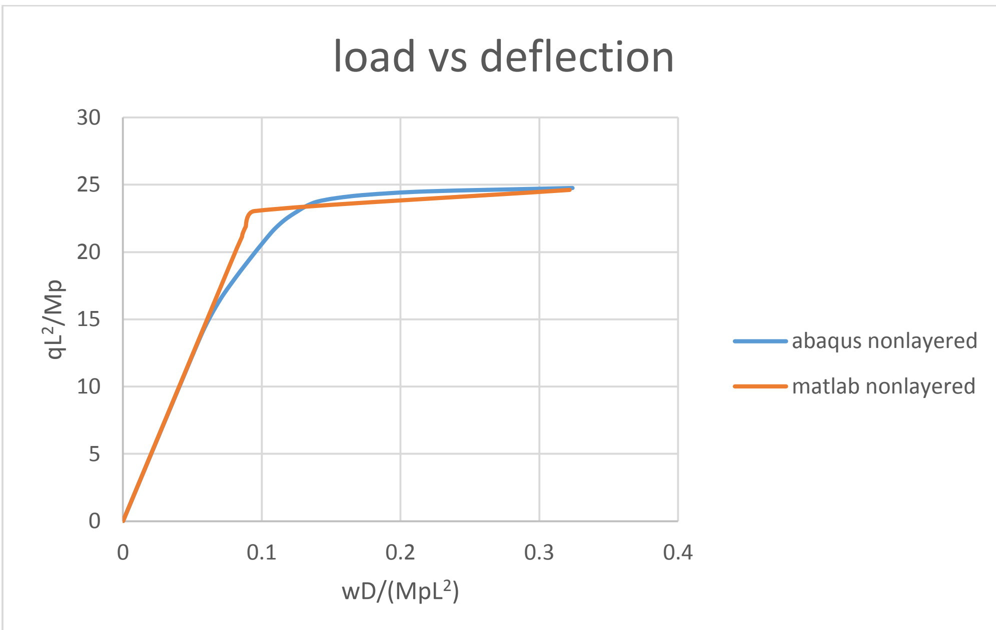
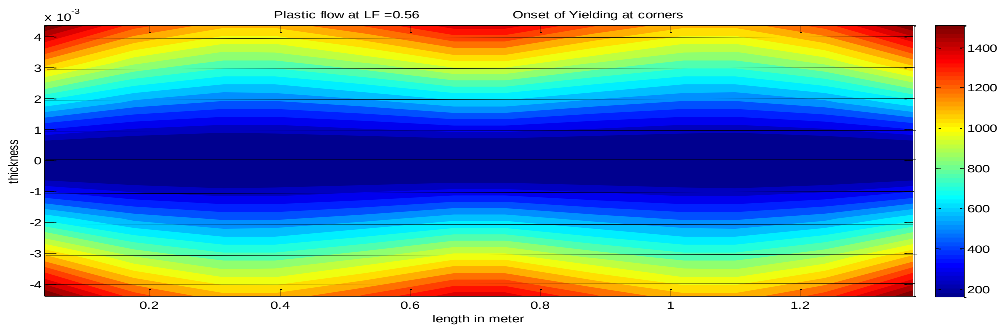


Figure 16: Load vs. deflection (central node) diagram for non-layered model

The deflections are same till the plate is in elastic stage i.e. till yield load. Bifurcation starts at the onset of yielding. The collapse load obtained from ABAQUS is more than that is obtained from MATLAB. Because ABAQUS abort the analysis when same yield stress is defined for various plastic strain, to avoid this a minimal increase in yield stress is input against plastic strain values. The slight increase in yield value results in higher value of collapse load.

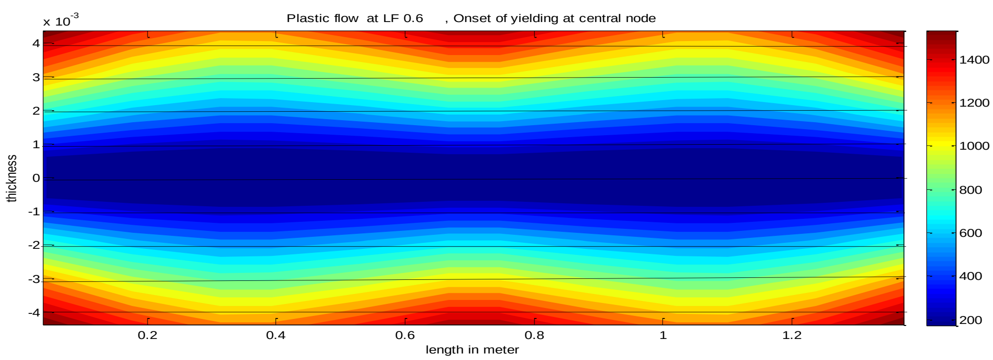
Plastic Flow:

In a layered model development of plastic deformations can be tracked directly, since stresses are calculated at several different layers in the model and the plastic bending moment is calculated for a fully plastic cross section. The layered model provides a good approximation of plastic strain growing gradually from the outer layers to middle layers. This transition is depicted as smooth load deflection curved as shown in Fig.17



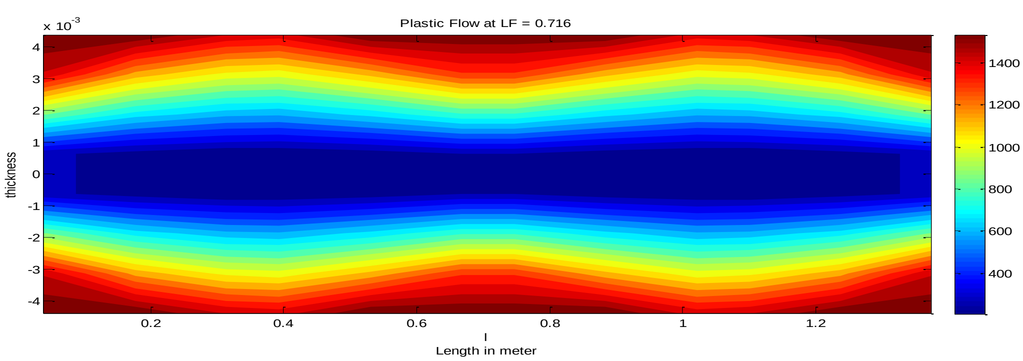
LF = 0.56

Onset of first yielding at corners top layers



LF = 0.6

Onset of yielding at center top layers



LF = 0.716

inward spreading of yielding at corners and centre

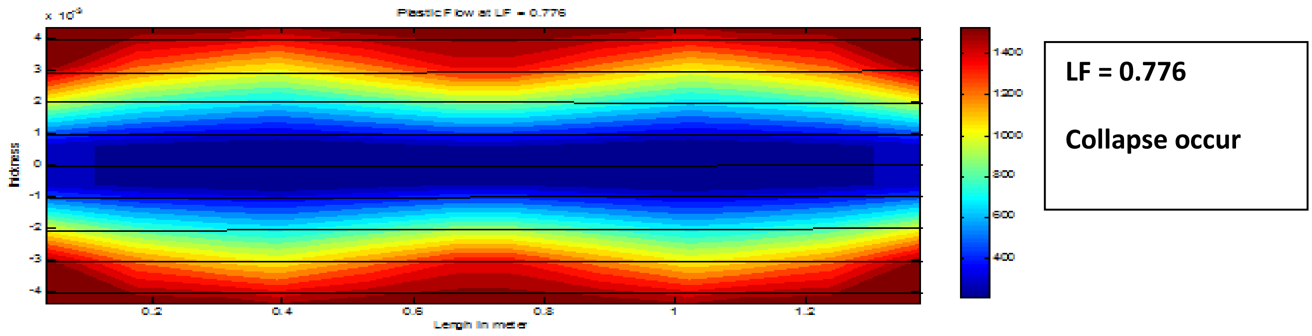


Figure 17: Plastic flow through the thickness

7.5 Comparison between Layered and Non-layered Models:

The load deflection diagram obtained from Abaqus and Matlab are shown in fig. Compared with non-layered model, the layered model exhibit more deflection at a particular load increment**.**

Table 3: Yield Load and Collapse Load for non-layered and layered s/s plate

Thickness(m) 0.01	ABAQUS		MATLAB	
	Yield load	Collapse load	Yield load	Collapse load
Non-layered	0.57	0.99	0.54	0.94
Layered	0.59	0.99	0.55	0.866

The yield load in non-layered model has been found more than layered model whereas the collapse load was less for layered model. This is because in a layered model stress resultants are calculated layer wise which allow for redistribution of stresses. Therefore a smooth curve have been obtained for layered model.

To proof this plates with different boundary conditions have been analysed in ABAQUS for both non-layered and layered models. The summary of yield and collapse loads are given in Table 3. and respective load deflection graphs are shown in Fig.18-Fig.22

Table 4: Yield Load and Collapse Load for plate with different boundary conditions

S.NO:	Boundary Condition	First yield load	Collapse Load
1.	Fixed	0.76	1.88
2.	Simply Supported	0.57	0.99
3.	Three Sides Fixed And One Free	0.40	1.12
4.	Two Opposite Sides Fixed And Other Free	0.61	0.80
5.	Two Opposite Sides Simply Supported And Other Free	0.22	0.35

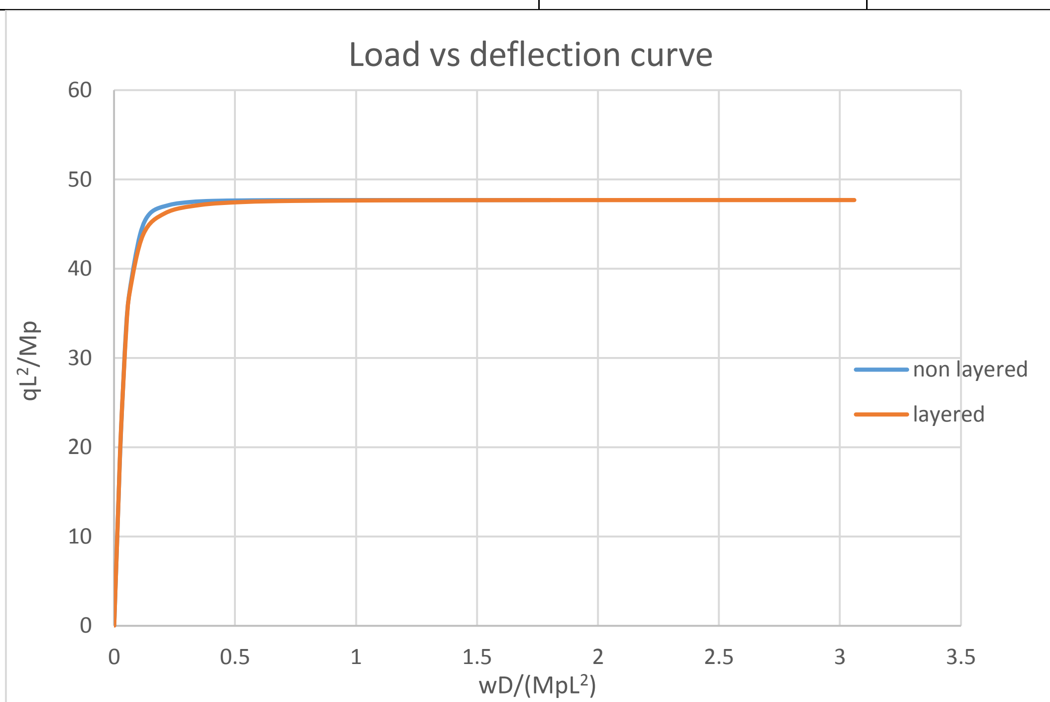


Figure 18: Load vs. deflection (central node) diagram for Fixed Supports

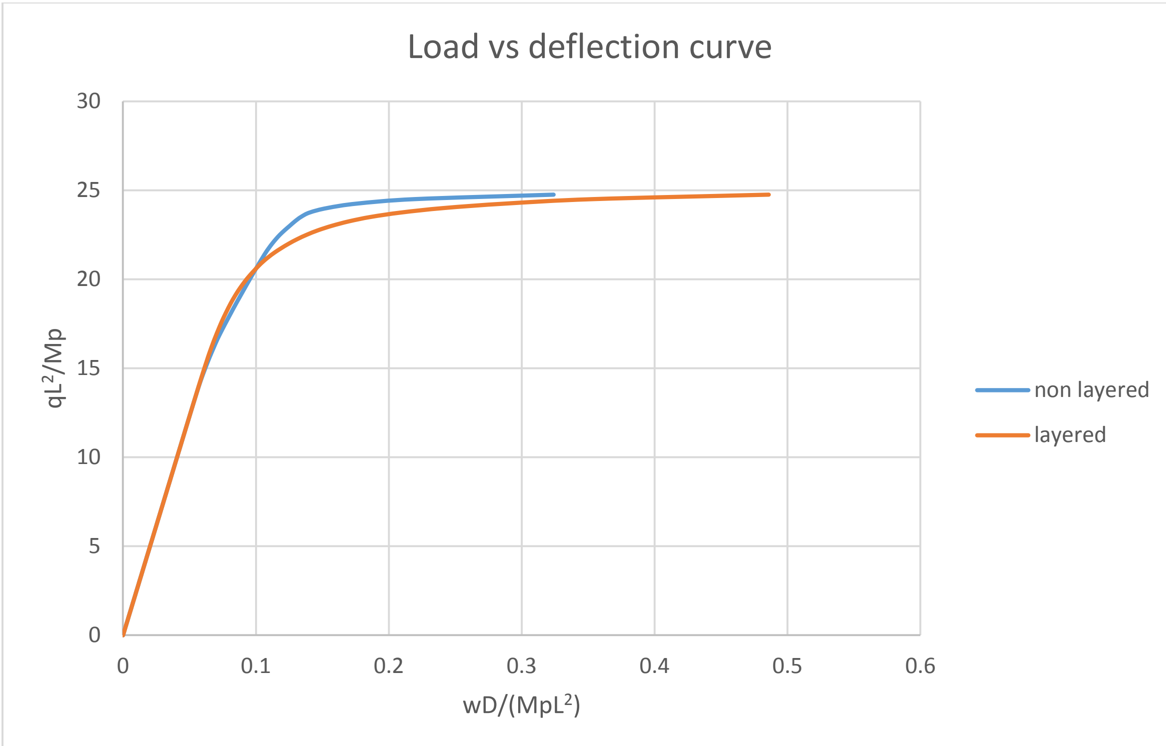


Figure 19: Load vs. deflection (central node) diagram for all sides simply supported.

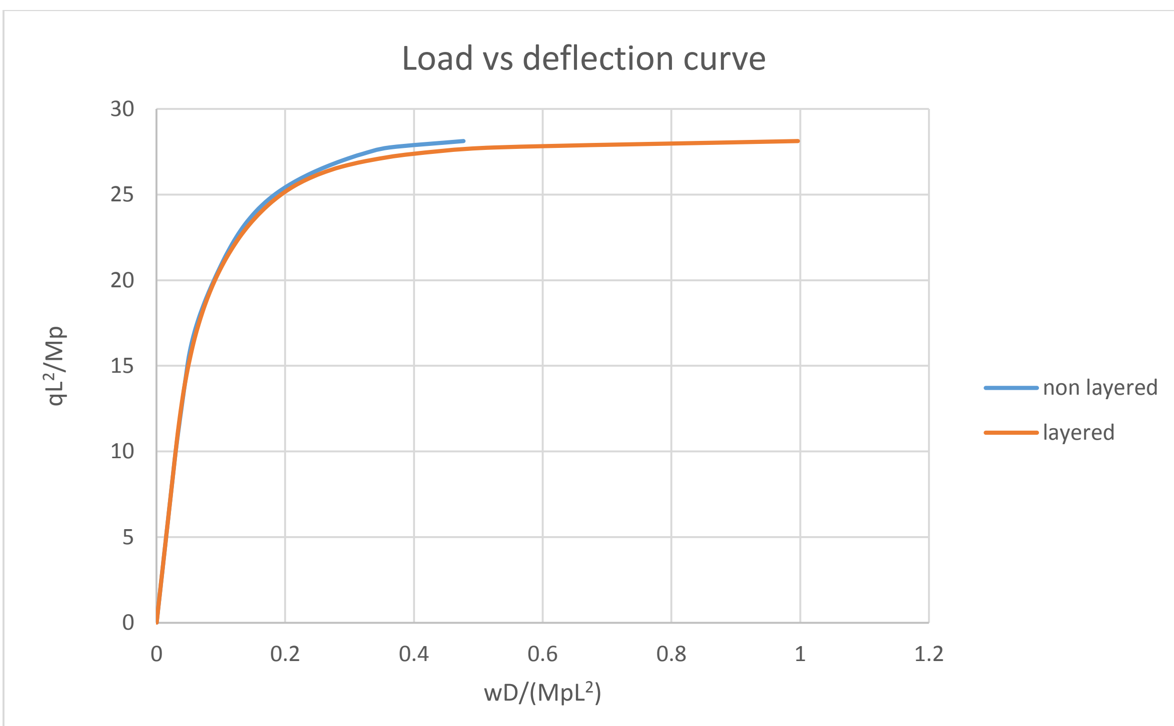


Figure 20: Load vs. deflection (central node) diagram for three sides fixed and one free

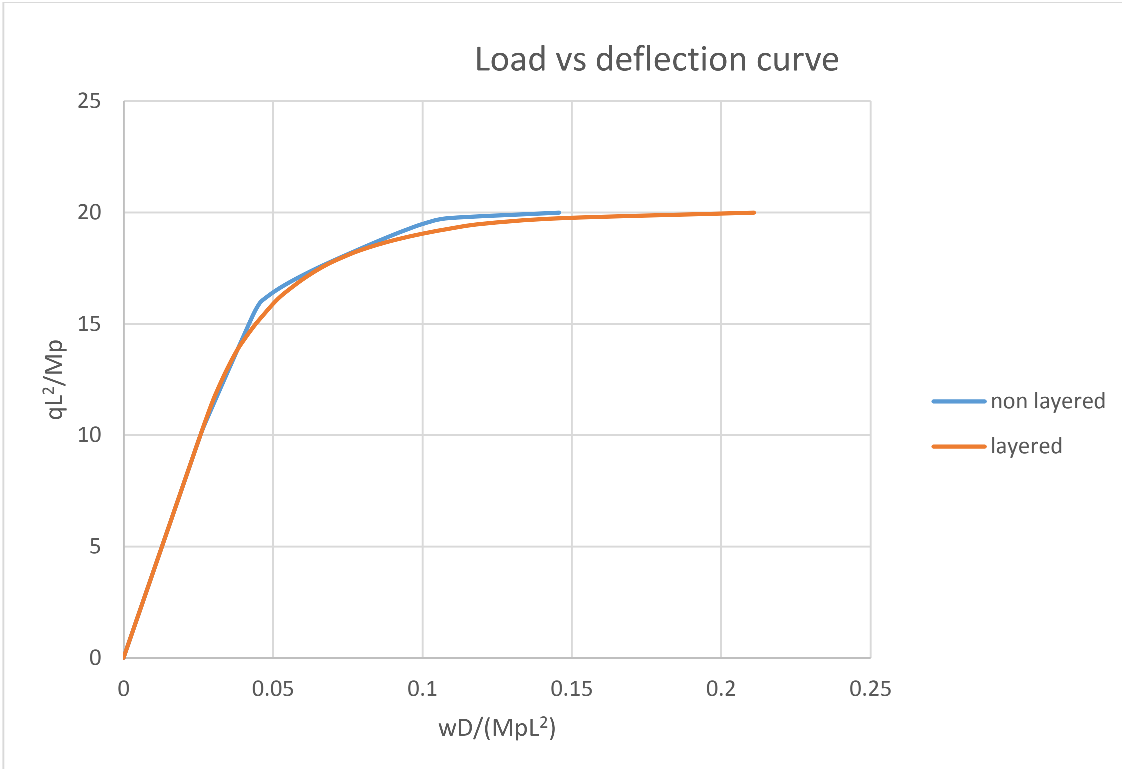


Figure 21: Load vs. deflection (central node) diagram for two opposite sides fixed and others free

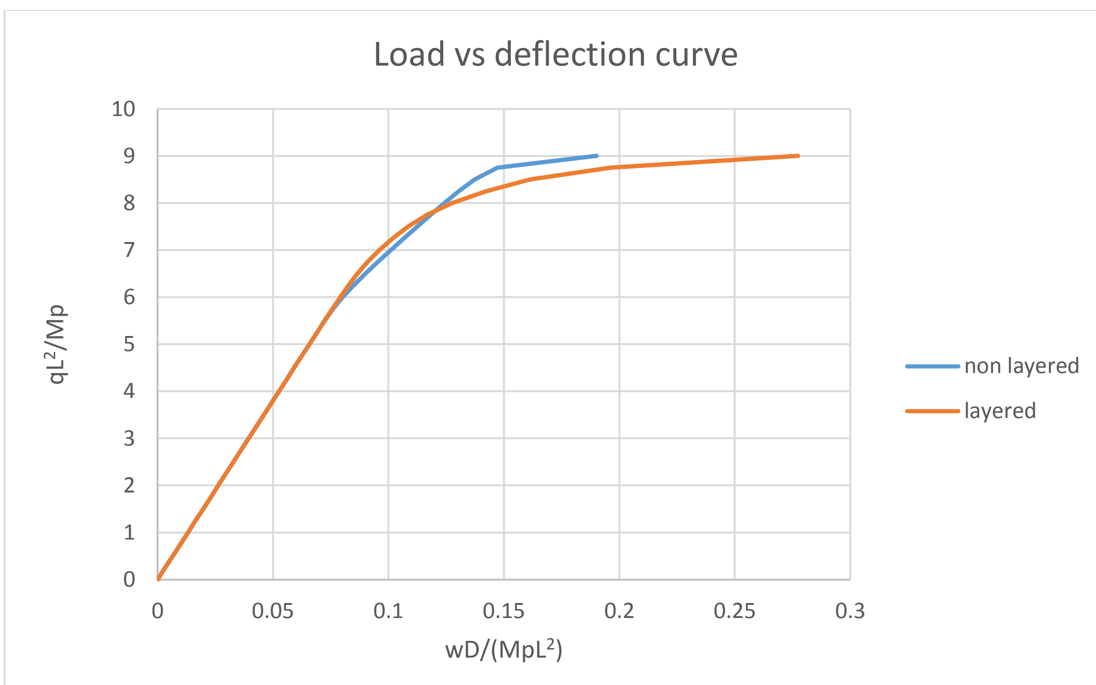


Figure 22: Load vs. deflection (central node) diagram for two opposite sides simply supported and others free

7.5.1 Load Vs. Plastic strain curves for layered and non-layered s/s plate

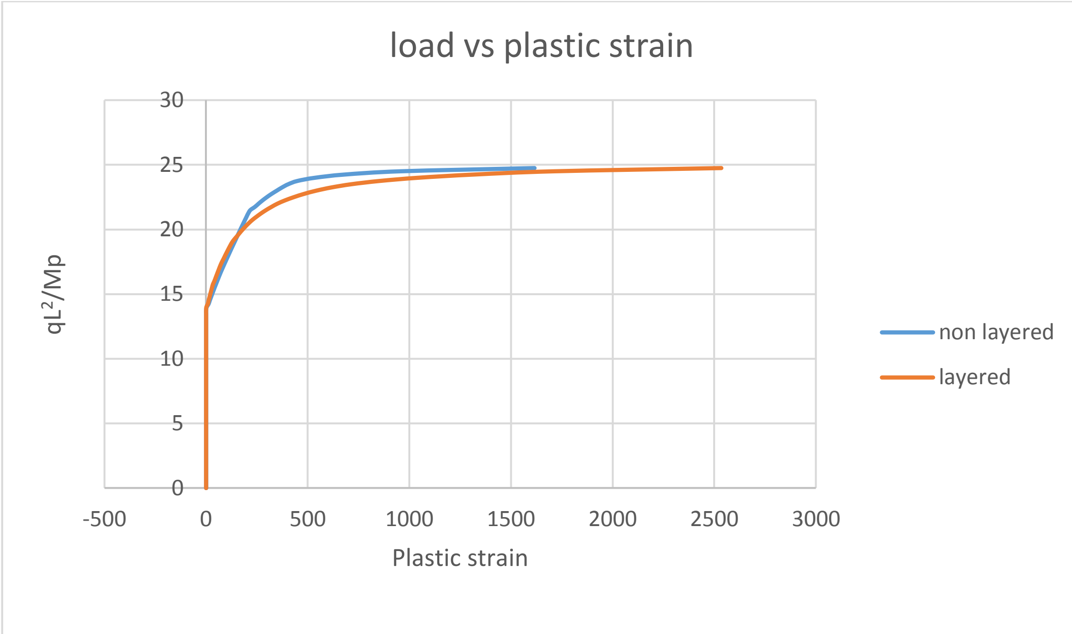


Figure23: Load vs. Plastic strain diagram for layered and non-layered s/s plate

7.6 Comparison between total strain and plastic strain for a s/s non-layered model

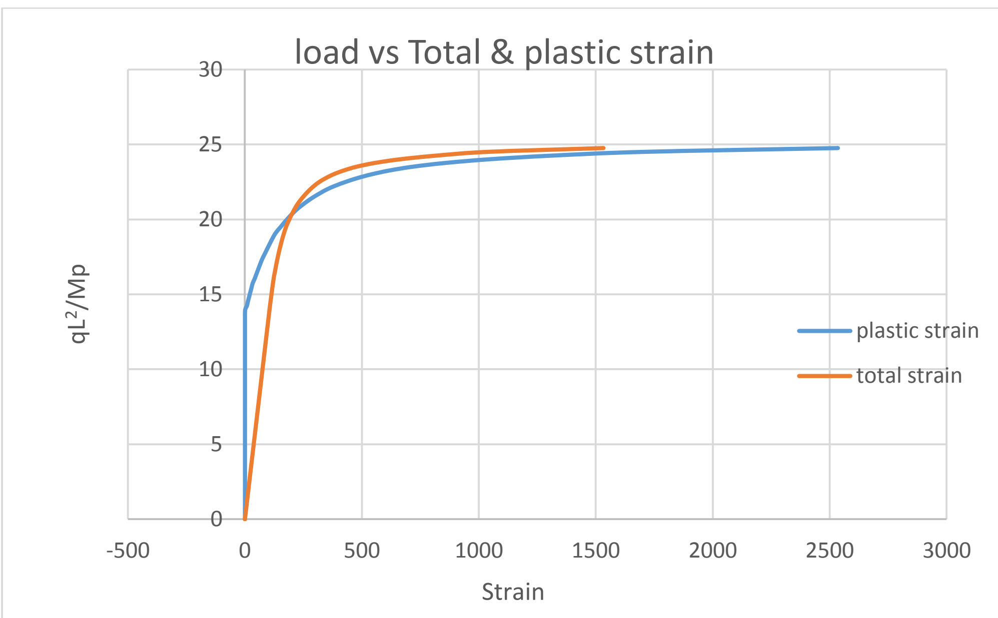


Figure 24: Load vs. total strain and plastic strain for a simply supported non-layered model

7.7 Plate with concentric cutout:

Another problem a plate with concentric rectangular cutout has been analyzed. The computer program based on the presented finite element formulation has been modified to incorporate the hole in the plate. The cutout area is 6.25% of plate area. A non-layered model has been adopted for the analysis. The nonlinear behavior was observed through load deflection diagram, plastic flow, first yield and collapse loads. For validation of results the same problem has been analyzed in ABAQUS and results were found in good agreement.

Load deflection diagrams

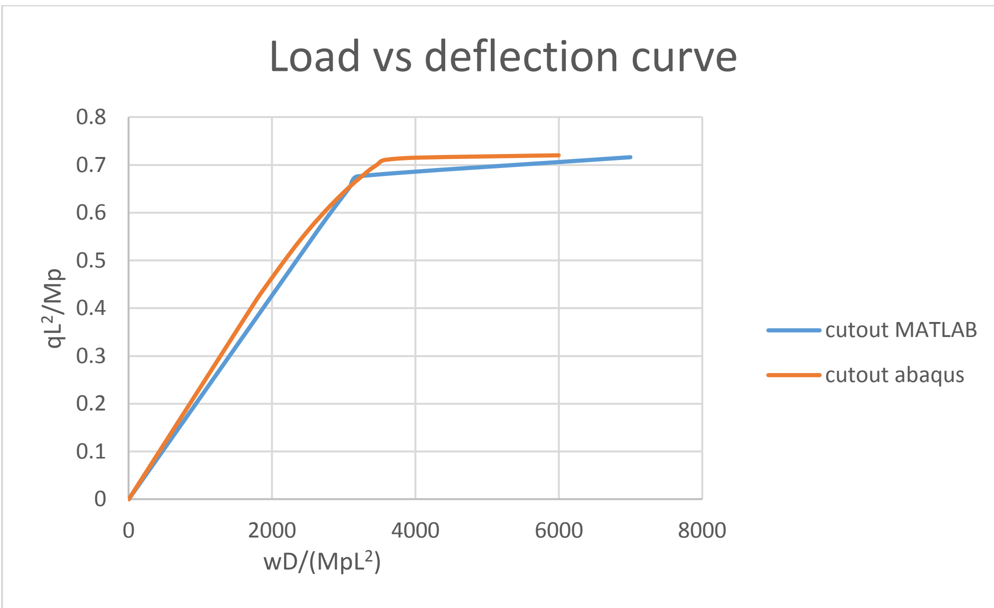


Figure 25: Load vs. deflection diagram at corner of the cut out

The first yielding has been observed at corner nodes of the cutout and second yielding were observed at corner ends of the plate as anticipated. The first yielding has been observed at load factor 0.54 in numerical analysis and collapse load at load factor =0.706.

Plastic flow:

The plastic flow has started at corner points of cutout due to high stress concentration at these points and the from corner ends of the simply supported ends of the plate.

The flow occurred along the diagonals of the plate. All these nonlinear behavior are clearly portrayed in the plastic flow diagrams shown in Fig.26

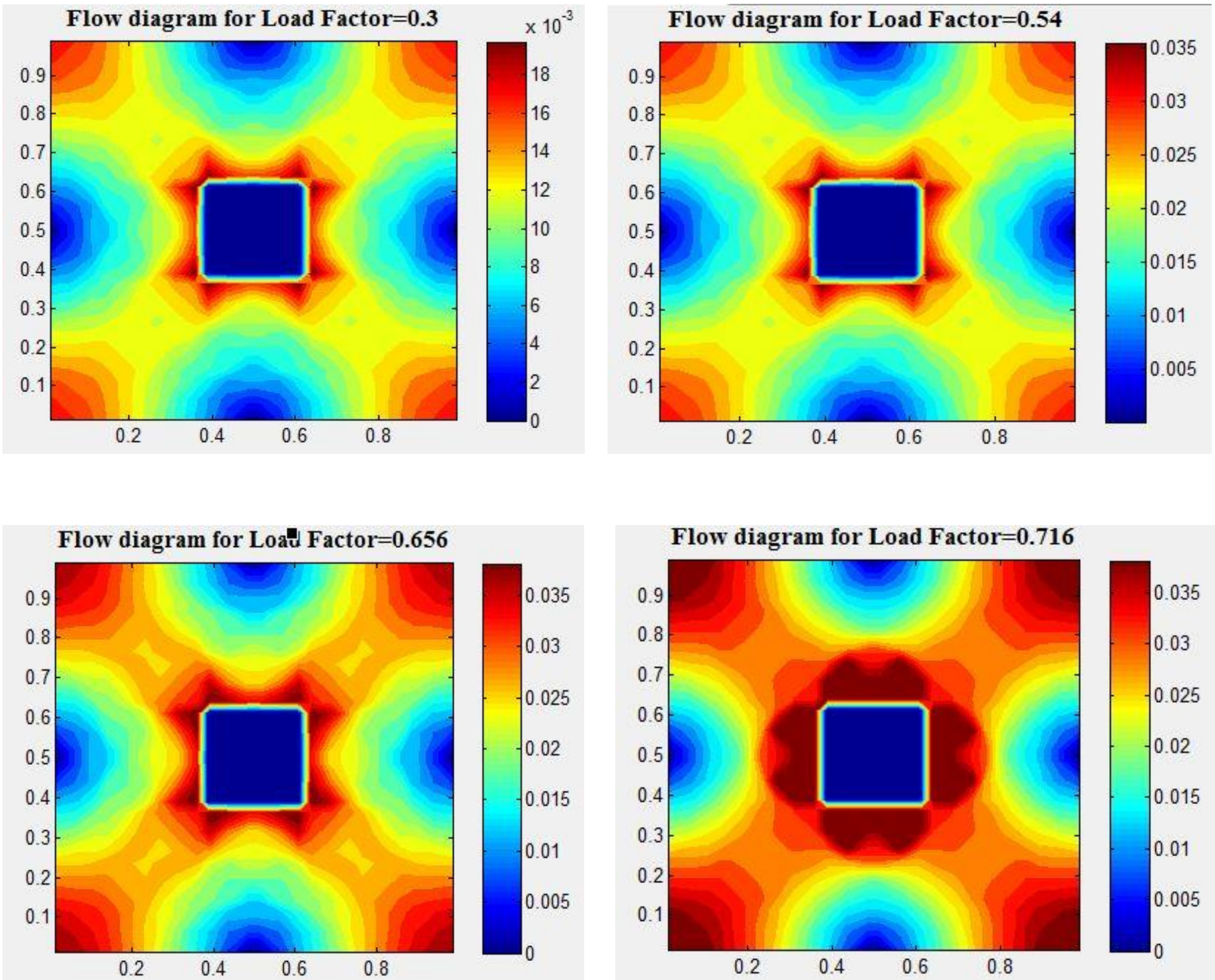


Figure 26: Plastic Flow (Stress Contour) Diagram at different Load Factors

To study the effect of shape on elasto-plastic behavior, three different cutouts of same area have been analyzed in ABAQUS. The first yield load and collapse load were observed to be minimum for square cutout, then for elliptical and maximum for circular cutout as represented in the Table. This is because of smoothening of the edges of the cutout which reduced the stress concentration. The plastic flow diagrams also upkeep these outcomes.

Table **5:** Load at first yield and collapse load for a plate of 0.01m thickness for different cutouts

Sl. No.	Type of cutout	LOAD AT FIRST YIELD	COLLAPSE LOAD
1	Plate without cutout	0.57	0.99
2	Square	0.37	0.90
3	Circular	0.51	0.94
4	Ellipse	0.50	0.93

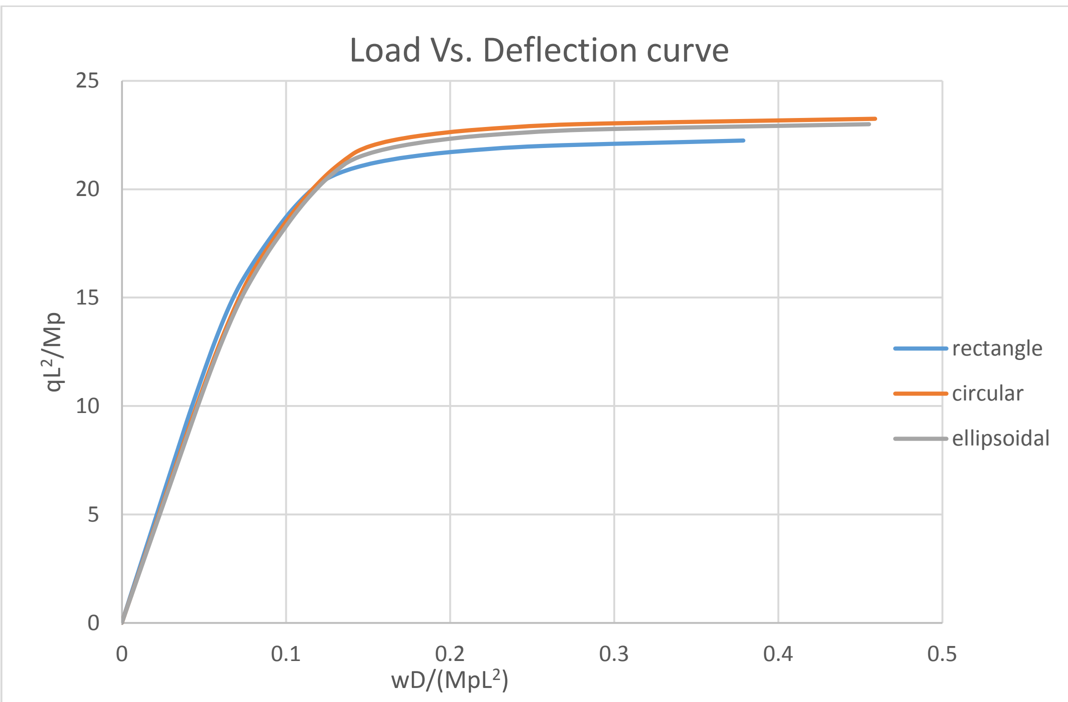


Figure 27: Load vs. deflection curve at corner of the cut out for various shapes of cut out

To see the effect of area of cutout on elasto-plastic behavior of plate analysis were conducted on plates with square cutout of areas 6.25 %, 10%, 15 %, 20% and 30 %. The load deflection diagrams were drawn for corner points of cutout and are shown in Fig. The observation described that with increase in area of central cut out collapse load increase. The observations were shown in the Fig.28 This maybe because when cutout area was increased the deflection at cutout corner point was decreased hence causing less stresses there. Therefore more load is required to attain the first yield load and collapse load.

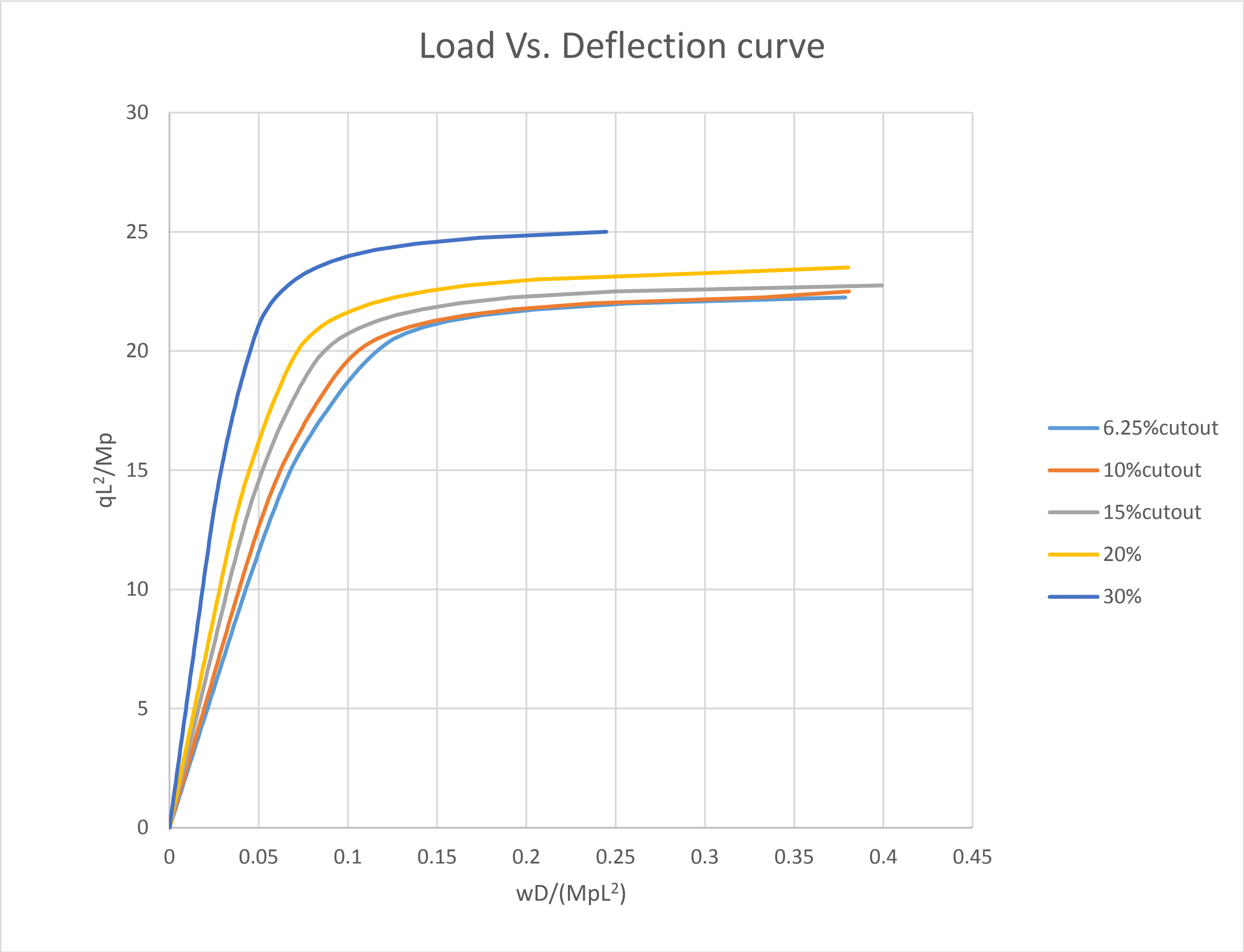


Figure 28: Load vs. deflection curve at corner of the cut out for different percentage of cut outs

CHAPTER-8
CONCLUSIONS

8.1 Conclusions:

1. In non-layered model the assumption used in formulation implies that whole cross section becomes plastic as soon as the moment reaches its yield value i.e. plastic moment value which is unrealistic whereas layered model captured the correct load at which first yield occurred and plastic flow begins.
2. Therefore layered model exhibited less yield load and collapse loads compared with non-layered model.
3. The layered model approach provides information of plastic flow along the thickness a hence realistic model but requires smaller increments for convergence of solution and thereby more computational time.
4. The plate with cutout exhibited beginning of yielding at corner points due to high stress concentration there.
5. The rectangular cutout reduced the first yield load and collapse load to maximum while the circular cutout exhibited less reduction, whereas elliptical cutout showed reduction in between.
6. The observation described that with increase in area of central hole first yield load and collapse load increase owing to less deflection at corner points of hole.
7. It is found that the presented FEM approach can be used to trace the complex nonlinear behavior like progressive yielding, collapse etc. effectively for plate bending problem.
8. In problems dealing material non-linearity convergence are required to satisfy the equilibrium condition and the stress conditions , since the method follow step by step incremental approach with repetitive computations, this may lead to the error accumulations . Therefore the procedure is very sensitive and results depend on adopted incremental load (time step) and tolerance value.

9. The other source of limitation of this FEM formulation is inaccuracy in calculation of stress. The stresses are calculated using the differentiation from the displacement, which may cause numerical errors in stress predictions and thereby in converging the solution.

CHAPTER-9

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