Relativistic $\langle \sigma v_{\mathrm{rel}} \rangle$ in the calculation of relics abundances: a closer look

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(Dated: November 17, 2013)

In this paper we clarify the relation between the invariant relativistic relative velocity V_{r}, the Møller velocity $\bar{v}$, and the non-relativistic relative velocity v_r. Adopting V_{r} as the true physical relative velocity for pair-collisions in a non-degenerate relativistic gas, we show that in the frame co-moving with the gas (i) the thermally averaged cross section times relative velocity $\langle \sigma v_{\mathrm{rel}} \rangle$ that appears in the density evolution equation for thermal relics is reformulated only in terms of V_{r} and $\mathcal{P}(V_{\mathrm{r}})$ in a manifestly Lorentz invariant form; (ii) the frame-dependent issues of the standard formulation in terms of the Møller velocity, as well as "superluminal" relative velocities, are not present in this formulation. Furthermore, considering the annihilation of dark matter into a particle-antiparticle pair $f\bar{f}$, in the cases $m_f = 0$, $m_f = m$ and $m_f \gg m$, we find that the coefficients of the low velocity expansion of $\langle \sigma V_{\mathrm{r}} \rangle$ admit an exact analytical representation in terms of the Meijer G functions that can be reduced to combinations of modified Bessel functions of the second kind.

PACS numbers: 95.35.+d, 12.60.Jv,03.30.+p,11.80.-m,05.20.-y

I. INTRODUCTION

In the Lee-Weinberg equation [1] for the calculation of the density of relic particles in the expanding Universe

$$\frac{dn}{dt} + 3Hn = \langle \sigma v_{\mathrm{rel}} \rangle (n_{(0)}^2 - n^2), \qquad (1)$$

a fundamental quantity is $\langle \sigma v_{\mathrm{rel}} \rangle$ where σ is the total annihilation cross section and v_{rel} the relative velocity of the annihilating particles. With $n_{(0)}$ we indicate the equilibrium distribution of the number density and H is the Hubble parameter.

If the relics at the freeze-out were non-relativistic, as in the case of the cold dark matter of the standard cosmological model, then the system can be treated as a non-relativistic classical gas. In this case, the meaning of thermal average $\langle ... \rangle$ as well as "who" is v_{rel} is clear. In facts, the interaction rate

$$R = n_1 n_2 \sigma v_r, \qquad (2)$$

contains the product of the cross section times the magnitude of the non-relativistic relative velocity

$$v_r = |\boldsymbol{v}_1 - \boldsymbol{v}_2|. \qquad (3)$$

The reaction rate is then averaged with the Maxwell distribution $f_M(v) = (2/\pi)^{1/2}(m/T)^{3/2}v^2 \exp(-mv^2/(2T))$ for the absolute velocities

$$\langle R \rangle = n_1 n_2 \int d^3\boldsymbol{v}_1 d^3\boldsymbol{v}_2 f_M(v_1) f_M(v_2) \sigma v_r. \qquad (4)$$

By changing variables from the velocities $\boldsymbol{v}_1$, $\boldsymbol{v}_2$ to the velocity of the center of mass $\boldsymbol{v}_c$ and the relative velocity $\boldsymbol{v}_r$, one finds the standard expression for the thermal averaged rate,

$$\langle R \rangle = n_1 n_2 \int_0^\infty dv_r F_{\mathrm{M}}(v_r) \sigma v_r, \qquad (5)$$

where

$$F_M(v_r) = \sqrt{\frac{2}{\pi}} \left(\frac{\mu}{T}\right)^{3/2} v_r^2 \, e^{-\frac{\mu}{T}\frac{v_r^2}{2}} \qquad (6)$$

is the distribution of the relative velocity. Equation (6) has the same form of the Maxwell distribution for the absolute velocity but with the reduced mass $\mu = m_1 m_2/(m_1 + m_2)$ in place of m and v_r in place of v. Considering the gas being composed by particles with mass m such that $\mu = m/2$, the thermally averaged cross section times the relative velocity is thus

$$\langle \sigma v_r \rangle = \int_0^\infty dv_r F_M(v_r) \sigma v_r = \frac{x^{3/2}}{2\sqrt{\pi}} \int_0^\infty dv_r v_r^2 e^{-x\frac{v_r^2}{4}} \sigma v_r, \qquad (7)$$

where we have introduced the standard thermal variable $x = m/T$. In this paper we use natural units with $\hbar = c = k_B = 1$. The non-relativistic average (7) was used in the earlier calculation of the relic density, see for example [2].

On the other hand, the typical freeze-out temperature and masses of weakly interacting massive particles are such that $x \sim m/T \sim 25$. A rough estimate gives that the thermal velocity of the particles is $v \sim \sqrt{3/x} \sim 0.35$, thus relativistic corrections to (7) are expected.

Sredincki, Watkins and Olive [3] found the low-velocity and large x expansion of a relativistic formula based on the definition of $\langle \sigma v_{\mathrm{rel}} \rangle$ given by Bernstein, Brown and Feinberg [4]. In these papers the relative velocity is given by the expression (3). Special relativity enters in the game by replacing the non-relativistic kinetic energy in the Maxwell-Boltzmann distribution with the relativistic $E = \sqrt{\boldsymbol{p}^2 + m^2}$ and using the standard definition of Lorentz invariant cross section.

Gondolo and Gelmini [5] then re-derived the rate equation from the relativistic Boltzmann equation following the book [6] and found that v_{rel} in (1) is not the relative velocity (3), but the so-called Møller velocity

$$\bar{v} = \sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}. \qquad (8)$$

Starting from the general definition of thermal average

$$\langle \sigma \bar{v} \rangle = \frac{\int d^3 \boldsymbol{p}_1 d^3 \boldsymbol{p}_2 e^{-E_1/T} e^{-E_2/T} \sigma \bar{v}}{\int d^3 \boldsymbol{p}_1 e^{-E_1/T} \int d^3 \boldsymbol{p}_2 e^{-E_2/T}}, \qquad (9)$$

they showed that Eq. (9) reduces to the single-integral formula

$$\langle \sigma \bar{v} \rangle = \frac{1}{8m^4 T K_2^2(x)} \int_{4m^2}^{\infty} ds \sqrt{s}(s - 4m^2) K_1\left(\frac{\sqrt{s}}{T}\right) \sigma. \qquad (10)$$

Here s is the Mandelstam variable $s = (p_1 + p_2)^2$, K_i are modified Bessel function of the second kind. Equation (10), and its extension to coahnnilation processes [7], was a step-forward in the precise calculation of the relic density because, ones the annihilation cross section $\sigma(s)$ is known, the single integral can be calculated numerically and does not necessitate any expansion or approximation and is used in public codes for relic density calculation of dark matter.

Conceptually, anyway, formula (10) raises some questions:

(1) The integral on the right-hand side is manifestly Lorentz invariant but the Møller velocity, as much as the product $\sigma \bar{v}$, is not Lorentz invariant. Thus the thermal average of the non-invariant quantity $\sigma \bar{v}$ turns out to be an invariant quantity.

(2) Comparing Eq. (10) with Eq. (7), no velocity appears in the integral in the right-hand side of (10). Gondolo and Gelmini derived also a formula for $\langle \sigma \bar{v} \rangle$ that contains explicitly a velocity in the integral:

$$\langle \sigma \bar{v} \rangle = \frac{2x}{K_2^2(x)} \int_0^{\infty} d\varepsilon \sqrt{\varepsilon} \sqrt{1 + 2\varepsilon} K_1(2x\sqrt{1 + \varepsilon}) \sigma \, (v_r)_{\text{lab}}. \qquad (11)$$

Here $\varepsilon = (s - 4m^2)/4m^2$ and v_r are expressed in the lab frame, that is the rest frame of one particle. In other words, the integral that defines the average of $\sigma \bar{v}$ in the form (11), implies that the co-moving frame coincides with the rest frame of one of the colliding particles. As shown in [5], one obtains a different result adopting the center of mass frame of the collision. In this way a velocity has reappeared in the integral but the Lorentz invariance of the integral is lost.

(3) As in the previous papers, the relative velocity (3) is considered as the natural expression also in the relativistic framework. One further problem with v_r and $\bar{v}$ is the fact that both can be larger than c in the center of mass frame.

Given the general relevance of Eq. (1) and Eq. (10), the definition of the relativistic $\langle \sigma v_{\text{rel}} \rangle$ should be free of the exposed conceptual problems and should involve only Lorentz invariant quantities. With this last statement we mean that, given the co-moving frame where the observer sees the gas at rest as a whole both σv_{rel} and the integral that gives average should be independent of frame where the kinematics of the collision is evaluated.

We thus try to answer the following questions:

- Which is the correct relative velocity in special relativity that is Lorentz invariant and has values smaller or equal to c in any inertial frame?

- Which is its probability density function in a relativistic classical gas?

- Is it possible to define rate, flux and cross section in an Lorentz invariant way without using the non-invariant Møller velocity?

- Is it possible to define $\langle \sigma v_{\text{rel}} \rangle$ in the co-moving frame as an integral over the probability distribution of the relative velocity in analogy with the non-relativistic case?

We will see that the answering them we will solve the 3 raised problems.

The plan of the paper is the following. In Section II we first review the concepts of relativistic relative velocity, Møller velocity and their relations with the definition of the invariant reaction rate and cross section. Here we also clarify the paradox of superluminal relative velocities.

In Section III we show that, actually, the velocity both in the left-hand side and right-hand side of (11) is V_{r}, the invariant relativistic relative velocity. The relativistic thermal average in the co-moving frame $\langle \sigma v_{\text{rel}} \rangle$ is $\langle \sigma V_{\text{r}} \rangle$ and the use of $\bar{v}$, v_r and of any other reference frame can be avoided.

After that, in Section IV we re-analyse the low-velocity expansion of $\langle \sigma V_{\text{r}} \rangle$. We find known and new expansions as well as exact relativistic expressions for the coefficients as a function of x in some important cases.

The mathematics behind the results of Section II and Section IV is furnished by the relation between the generalized hypergeometric Meijer G function and the modified Bessel functions. Being G a special function that is not commonly encountered in particle and astro-particle physics, in Appendix A we give a brief introduction and show its use in the calculation of the integrals.

A summary of the main results is given in Section V.

II. RELATIVE VELOCITY, MØLLER VELOCITY AND INVARIANT RATE IN SPECIAL RELATIVITY

In Special Relativity, the relative velocity between two massless particles and between a massless and a massive particle is c in any frame, while the relative velocity between two massive particles is always smaller than c in any frame. These requirements are not satisfied nor by v_r nor by $\bar{v}$. For example in the center of mass frame of two colliding particles with mass m, $|\boldsymbol{v}_1| = |\boldsymbol{v}_2| = v$ and $(\bar{v})_{\text{cm}} = (v_r)_{\text{cm}} = 2v$, thus, for $v > 1/2$, $\bar{v}$ and v_r assume non-physical values larger than c.

On the other hand, a relative velocity compatible with the principles of Special Relativity is well known [8, 9]

and is given by

$$V_{\mathrm{r}} = \frac{\sqrt{(\boldsymbol{v}_1 - \boldsymbol{v}_2)^2 - (\boldsymbol{v}_1 \times \boldsymbol{v}_2)^2}}{1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2}. \tag{12}$$

This expression is symmetric in the two velocities and is valid in any frame. When one particle (or both) is massless, then $|\boldsymbol{v}_i| = 1$ and also $V_{\mathrm{r}} = 1$. Considering the example above, in the center of mass frame we have $(V_{\mathrm{r}})_{\mathrm{cm}} = 2v/(1 + v^2)$, thus, differently from $\bar{v}$ and v_r, V_{r} is always smaller or equal to velocity of light. In the non-relativistic limit V_{r}, as much as $\bar{v}$, reduces to (3). The expression of v_r in terms of the four-momentum $p_{1,2}$ is

$$V_{\mathrm{r}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{p_1 \cdot p_2}, \tag{13}$$

that manifests the Lorentz invariance.

Although $\langle \sigma v_{\mathrm{rel}} \rangle$ is often called simply thermally averaged cross section, actually the quantity is the thermal average of the interaction (or annihilation) rate per unit density. We thus now reformulate the non-relativistic expression (2) in a Lorentz invariant way using the invariant relative velocity V_{r}. The expression of the invariant rate valid in any frame is [8, 9]

$$\mathcal{R} = n_1 n_2 \frac{p_1 \cdot p_2}{E_1 E_2} \sigma V_{\mathrm{r}}. \tag{14}$$

The factor

$$\frac{p_1 \cdot p_2}{E_1 E_2} = \frac{\gamma_{\mathrm{r}}}{\gamma_1 \gamma_2} = 1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2 \tag{15}$$

accounts for the Lorentz contraction of the volumes of number densities in a generic frame and assures the Lorentz invariance of the product $n_1 n_2 p_1 \cdot p_2 / (E_1 E_2)$. In (15) $\gamma_{1,2}$ and γ_{r} are the Lorentz factors $\gamma = 1/\sqrt{1 - v^2}$ associated to the corresponding velocities.

From the definition of the invariant rate it follows the definition of the invariant cross section: $\sigma = \mathcal{R}/F = 1/F \int \overline{|\mathcal{M}|^2} \mathrm{d}\Phi(f)$, where $\overline{|\mathcal{M}|^2}$ is the squared matrix element summed over final spins and averaged over the initial spins, $d\Phi(f)$ is the usual Lorentz invariant phase space for the final state particles, and F is the invariant flux

$$F = n_1 n_2 \frac{p_1 \cdot p_2}{E_1 E_2} V_{\mathrm{r}}. \tag{16}$$

If we normalize the one-particle states to $2E$, that is the number of particles per unit volume are $2E$, the number density is $n = 2E$, then the invariant flux (16) becomes

$$F = 4(p_1 \cdot p_2) V_{\mathrm{r}} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}, \tag{17}$$

and the standard formula for the invariant cross section is obtained.

An equivalent formulation is obtained by introducing the Møller velocity $\bar{v}$. In facts, using (8), (12) and (13)

we have

$$\bar{v} = (1 - \boldsymbol{v}_1 \cdot \boldsymbol{v}_2) V_{\mathrm{r}} = \frac{p_1 \cdot p_2}{E_1 E_2} V_{\mathrm{r}} = \frac{\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}}{E_1 E_2}. \tag{18}$$

The invariant rate (14) is

$$\mathcal{R} = n_1 n_2 \sigma \bar{v}, \tag{19}$$

and the invariant flux, with same normalization of the densities, $F = 4E_1 E_2 \bar{v} = 4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}$.

The only reason to introduce the Møller velocity is to write the Lorentz invariant rate (14) in the form (19) that is similar to the non relativistic expression (2). Anyway, while in the latter each factor is Galileo invariant, relativistically the products $n_1 n_2 (p_1 \cdot p_2 / E_1 E_2)$ and $n_1 n_2 \bar{v}$ are Lorentz invariant. This redefinition is the reason why the collision term of the relativistic Boltzmann equation has the same form as the one in the non-relativistic equation with v_r replaced by $\bar{v}$ [6, 9].

It should be clear from the previous discussion that neither v_r nor $\bar{v}$ are the relative velocity in special relativity, contrary to what is often claimed in literature and textbooks. In textbooks, when defining the invariant cross section, the flux is usually defined in a frame where the velocities are collinear, say the lab or cm frame,

$$F^{\mathrm{coll}} = n_1 n_2 |\boldsymbol{v}_1 - \boldsymbol{v}_2|, \tag{20}$$

in analogy with the non relativistic case $n_1 n_2 v_r$, and then it is rewritten in the invariant form using $|\boldsymbol{v}_1 - \boldsymbol{v}_2| = \sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}/E_1 E_2$ and $n_1 n_2 = 4E_1 E_2$. It is the formal equivalence of v_r with the quantity $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$ in the definition of the flux that is probably at the origin of the confusion about "who" is the relativistic relative velocity and the paradox that the relative velocity can have values larger than the velocity of the light.

Actually, although $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$ is mathematically equivalent to non-relativistic relative velocity v_r, conceptually the two quantity have nothing to do with each other. In the relativistic framework

$$|\boldsymbol{v}_1 - \boldsymbol{v}_2| = \left(\frac{p_1 \cdot p_2}{E_1 E_2} V_{\mathrm{r}}\right)^{\mathrm{coll}} = (\bar{v})^{\mathrm{coll}}. \tag{21}$$

From this point of view the fact that $|\boldsymbol{v}_1 - \boldsymbol{v}_2|$ assumes in the center of mass frame values larger than c is not a problem because this quantity is not the relative velocity. For example, consider the scattering of two massless particles as seen in the cm frame. The relative velocity is $V_{\mathrm{r}} = 1$, but $p_1 \cdot p_2 / (E_1 E_2) = 2$, thus $|\boldsymbol{v}_1 - \boldsymbol{v}_2| = \bar{v} = 2$: the relative velocity is never larger than 1, $\bar{v}$ can be larger than 1 because is not a physical velocity.

We thus addressed the problem (3) of the Introduction.

III. RELATIVISTIC $\langle \sigma V_{\mathrm{r}} \rangle$

Under the hypothesis that the system can be treated as a non-degenerate relativistic gas in equilibrium [4, 5],

the normalized momentum distribution in the comoving frame is given by the Jüttner [1] distribution

$$f_J(\boldsymbol{p}) = \frac{1}{4\pi m^2 T K_2(x)} e^{-\frac{\sqrt{\boldsymbol{p}^2+m^2}}{T}}, \qquad (22)$$

Averaging the rate (14) with the Jüttner distribution (22), the relativistic analogous of Eq. (2) hence is

$$\langle \mathcal{R} \rangle = n_1 n_2 \int \frac{d^3\boldsymbol{p}_1}{E_1} \frac{d^3\boldsymbol{p}_2}{E_2} f_J(\boldsymbol{p}_1) f_J(\boldsymbol{p}_2) (p_1 \cdot p_2) \sigma v_{\mathbf{r}}. \quad (23)$$

In Ref. [12] we have shown that

$$\int \frac{d^3\boldsymbol{p}_1}{E_1} \frac{d^3\boldsymbol{p}_2}{E_2} p_1 \cdot p_2 f_J(\boldsymbol{p}_1) f_J(\boldsymbol{p}_2) \equiv \int_0^1 dV_{\mathbf{r}} \mathcal{P}_{\mathbf{r}}(V_{\mathbf{r}}) = 1,$$

where $\mathcal{P}_{\mathbf{r}}(V_{\mathbf{r}})$ is the probability density distribution of $V_{\mathbf{r}}$:

$$\mathcal{P}_{\mathbf{r}}(V_{\mathbf{r}}) = \frac{X}{\sqrt{2}\prod_i K_2(x_i)} \frac{\gamma_{\mathbf{r}}^3(\gamma_{\mathbf{r}}^2-1)}{\sqrt{\gamma_{\mathbf{r}}+\varrho}} K_1(\sqrt{2}X\sqrt{\gamma_{\mathbf{r}}+\varrho}). \quad (24)$$

with the abbreviations

$$X = \sqrt{x_1 x_2}, \quad \varrho = \frac{m_1^2+m_2^2}{2m_1 m_2} = \frac{x_1^2+x_2^2}{2x_1 x_2}. \qquad (25)$$

In the following we adopt the symbol $\langle ... \rangle_{\mathcal{P}}$ to indicate that the average of a certain quantity is obtained integrating over $\mathcal{P}(V_{\mathbf{r}})$ in agreement with the general methods of statistical mechanics: $V_{\mathbf{r}}$ is the physical Lorentz invariant quantity, admits a normalized probability density function $\mathcal{P}(V_{\mathbf{r}})$. Given any $f(V_{\mathbf{r}})$ its average is obtained integrating it over $\mathcal{P}(V_{\mathbf{r}})$. This is in complete analogy with the non-relativistic average (7) where the relative velocity is v_r, Eq. (3), and the probability density function is given by the Maxwell distribution $F_M(v_r)$, Eq. (6).

Having clarified the concept of relativistic relative velocity and having found its probability distribution, it is clear that the thermal average of $\sigma V_{\mathbf{r}}$ is

$$\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}} = \int_0^1 dV_{\mathbf{r}} \mathcal{P}_{\mathbf{r}}(V_{\mathbf{r}}) \sigma V_{\mathbf{r}}. \qquad (26)$$

We can write the average in terms of the more practical variables $\gamma_{\mathbf{r}}$ and s:

$$\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}} = \frac{X}{\sqrt{2}\prod_i K_2(x_i)}$$
$$\times \int_1^\infty d\gamma_{\mathbf{r}} \gamma_{\mathbf{r}} \sqrt{\frac{\gamma_{\mathbf{r}}^2-1}{\gamma_{\mathbf{r}}+\varrho}} K_1(\sqrt{2}X\sqrt{\gamma_{\mathbf{r}}+\varrho}) \sigma V_{\mathbf{r}}$$
$$= \frac{1}{4T\prod_i m_i^2 K_2(x_i)}$$
$$\times \int_{M^2}^\infty ds [s-(m_1^2+m_2^2)] p' K_1(\sqrt{s}/T) \sigma V_{\mathbf{r}}, \qquad (27)$$

where p' is

$$p' = \frac{\sqrt{s-(m_1+m_2)^2}\sqrt{s-(m_1-m_2)^2}}{2\sqrt{s}}. \qquad (28)$$

and $M = m_1 + m_2$.

The relative velocity in the integrand can be eliminated using the relations $V_{\mathbf{r}} = \sqrt{\gamma_{\mathbf{r}}^2-1}/\gamma_{\mathbf{r}}$ and, from Eq. (13), $s = (m_1-m_2)^2 + 2m_1 m_2(1+\gamma_{\mathbf{r}})$ that give

$$V_{\mathbf{r}} = \frac{\sqrt{[s-(m_1+m_2)^2][s-(m_1-m_2)^2]}}{s-(m_1^2+m_2^2)}. \qquad (29)$$

The formulas as integrals over the cross section are:

$$\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}} = \frac{X}{\sqrt{2}\prod_i K_2(x_i)}$$
$$\times \int_1^\infty d\gamma_{\mathbf{r}} \frac{\gamma_{\mathbf{r}}^2-1}{\sqrt{\gamma_{\mathbf{r}}+\varrho}} K_1(\sqrt{2}X\sqrt{\gamma_{\mathbf{r}}+\varrho}) \sigma$$
$$= \frac{1}{8T\prod_i m_i^2 K_2(x_i)} \int_{M^2}^\infty ds \frac{[s-(m_1-m_2)^2]}{\sqrt{s}}$$
$$\times [s-(m_1+m_2)^2] K_1(\sqrt{s}/T) \sigma. \qquad (30)$$

We remark that the formulas in (27) and (30) as a function of $\gamma_{\mathbf{r}}$ are more general than the formulas as a function of s. The former are valid also for different temperatures. In facts the temperature enters in the formula only through the scaling variable $x_i = m_i/T_i$, hence even if $m_1 \neq m_2$ and $T_1 \neq T_2$ the expression does not change because X and ϱ maintain the same dependence on x_1 and x_2 given by Eq. (25), as it is easy to verify.

We now compare these formulas with those discussed in the Introduction and find the answer to conceptual questions (1) and (2) raised there.

(1) Setting $m_1 = m_2 = m$ in Eq. (30) we find the integral on the right-hand side of Eq. (10). The integral in the numerator of Eq. (9) is not invariant because both the phase-space elements $d^3\boldsymbol{p}_i$ and $\bar{v}$ are not. However, we have seen that $\bar{v} = (p_1 \cdot p_2)/(E_1 E_2)V_{\mathbf{r}}$, thus ones we explicit this relation, the integral is the same as the one in Eq. (23). Equation (10) gives the correct result because $V_{\mathbf{r}}$ and $\mathcal{P}(V_{\mathbf{r}})$ are present in the integral but are hidden by $\bar{v}$: actually, one is calculating $\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}}$. Put in other words, $\langle \sigma \bar{v} \rangle$ is only a different notation for $\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}}$ as much as Eq. (19) is a different way of writing Eq. (14).

2) Setting $m_1 = m_2 = m$ and using the variable $\varepsilon = (s-4m^2)/4m^2$ in Eq. (27) we get

$$\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}} = \frac{2x}{K_2^2(x)} \int_0^\infty d\varepsilon \sqrt{\varepsilon}\sqrt{1+2\varepsilon} K_1(2x\sqrt{1+\varepsilon}) \sigma V_{\mathbf{r}}.$$

Comparing with Eq. (11), we see that (11) is nothing but $\langle \sigma \bar{v} \rangle_{\mathcal{P}}$ with the integral evaluated in the rest frame of one particle: in facts, only in this frame $(\bar{v})_{\text{lab}} = (v_r)_{\text{lab}} = (V_{\mathbf{r}})_{\text{lab}}$ and $\langle \sigma \bar{v} \rangle_{\mathcal{P}}$ coincides with $\langle \sigma V_{\mathbf{r}} \rangle_{\mathcal{P}}$. Strictly speaking, the averages $\langle \bar{v} \rangle_{\mathcal{P}}$ and $\langle \sigma \bar{v} \rangle_{\mathcal{P}}$ have not a precise physical meaning. To calculate the average $\langle \bar{v} \rangle_{\mathcal{P}} = \int dV_{\mathbf{r}} \mathcal{P}(V_{\mathbf{r}}) \bar{v}$, the co-moving frame is not

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[1] We conform to the practice, common to other fields of Physics where relativistic thermodynamics is used [6, 9, 11], to attribute the distribution (22) to Jüttner that found it in Ref. [10].

enough: one has to specify $\bar{v}$ in the rest frame of one particle or in the center of mass frame. In the former case coincides with $\langle V_{\mathrm{r}} \rangle_{\mathcal{P}}$, but will have a different form and take different values in the center of mass frame.

We stress again that ones it is recognized that V_{r}, Eqs. (12), (13), (29), *is the relative velocity* in the relativistic framework and (17) is the invariant flux, then the invariant cross section follows automatically, a probability density function for V_{r} exists and the average value of the cross section times the relative velocity is obtained integrating over the probability density. The only reference frame is the co-moving frame where the observer measure the velocities of the colliding particles. All the problems connected with the non-invariance of the Møller velocity and the superluminal values in the center of mass frame disappear.

All the conceptual problems are eliminated considering $\bar{v}$ just a short-hand notation for $p_1 \cdot p_2/(E_1 E_2)V_{\mathrm{r}}$ and not as a velocity to work with in formulating relativistic concepts.

IV. LOW VELOCITY EXPANSION

We skip here the subscript $\mathcal{P}$ to lighten the notation. Let us consider the typical annihilation of two dark matter particles with mass m into particle-antiparticle two-body final states with mass m_f, $XX \rightarrow f\bar{f}$. We express the total cross section as the integral of the differential cross section in center of mass frame that we write as:

$$\sigma = \frac{2m^2}{s}\frac{\sqrt{s - 4m_f^2}}{\sqrt{s - 4m^2}}\sigma_0, \quad \sigma_0 = \frac{1}{2m^2}\frac{1}{64\pi^2}\int \overline{|\mathcal{M}|^2}d\Omega_{\mathrm{cm}}.$$

For what follows it is convenient to use the dimensionless variable $y = s/(2m)^2 = (\gamma_{\mathrm{r}} + 1)/2$. Changing variable in (30) with $m_1 = m_2 = m$ we obtain

$$\langle \sigma V_{\mathrm{r}} \rangle = \frac{2x}{K_2^2(x)}\int_{y_0}^{\infty}\frac{dy}{\sqrt{y}}\sqrt{y - 1}\sqrt{y - \rho}K_1(2x\sqrt{y})\sigma_0(y),$$

where, setting $\rho = m_f^2/m^2$, the inferior limit is $y_0 = 1$ if $m \geq m_f$, and $y_0 = \rho$ if $m < m_f$.

Except the case when there are resonant propagators with small width of the exchanged particles, σ_0 is a well behaved smooth function. For cold dark matter $y - 1$ is a small quantity, thus σ_0 can be expanded in powers of $y - 1$,

$$\sigma_0(y) = \sum_{n=0}^{\infty}\frac{1}{n!}\sigma_0^{(n)}(y - 1)^n. \tag{31}$$

where the derivatives $\sigma_0^{(n)}$ are evaluated at $y = 1$. We can write the low-velocity expansion as

$$\langle \sigma V_{\mathrm{r}} \rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\sigma_0^{(n)}\mathcal{K}_n^\rho(x) \tag{32}$$

where the coefficients are given by

$$\mathcal{K}_n^\rho(x) = \frac{2x}{K_2^2(x)}I_n^\rho(x), \tag{33}$$

$$I_n^\rho(x) = \int_{y_0}^{\infty}\frac{dy}{\sqrt{y}}(y - 1)^{(1/2+n)}\sqrt{y - \rho}K_1(2x\sqrt{y}). \tag{34}$$

For $\rho = 0$, $\rho = 1$ and $\rho \gg 1$, the integral (34) can be reduced to the known integral [13] and the properties of the generalized hypergeometric Meijer's G function [13] discussed in the Appendix. We study them separately.

A. $\rho = 0$

When the mass of the annihilation products is much smaller the mass of the dark matter particle we have $\rho \ll 1$ and $\rho = 0$ gives a good approximation. In this case

$$I_n^{\rho=0}(x) = \int_1^{\infty} dy(y - 1)^{1/2+n}K_1(2x\sqrt{y}) \tag{35}$$

corresponds to (A6) with $\lambda = 0$, $\mu = 3/2 + n$ and $\nu = 1$. Hence

$$I_n^{\rho=0}(x) = \frac{1}{2}\Gamma\left(n + \frac{3}{2}\right)G_{1,3}^{3,0}\left(x^2 \left| \begin{matrix} 0 \\ -n - \frac{3}{2}, \frac{1}{2}, -\frac{1}{2} \end{matrix}\right.\right). \tag{36}$$

The coefficients (33) of the low-velocity expansion (32) are

$$\mathcal{K}_n^{\rho=0}(x) = \Gamma(n + \frac{3}{2})\frac{x}{K_2^2(x)}G_{1,3}^{3,0}\left(x^2 \left| \begin{matrix} 0 \\ -\frac{1}{2}, \frac{1}{2}, -n - \frac{3}{2} \end{matrix}\right.\right). \tag{37}$$

The expansion in powers of x^{-1} should give the standard result of Srednicki, Watkins and Olive [3]. To study the large x behavior, we remind that for $x \gg 1$ the Bessel functions are approximated by [13]

$$K_n(z) = e^{-z}\sqrt{\frac{\pi}{2z}}(1 + \frac{4n^2 - 1}{8z} + \frac{16n^4 - 40n^2 + 9}{128z^2} + ...), \tag{38}$$

while the asymptotic expansion of the G function for $x \gg 1$ for some n [13, 14]:

$$G_{1,3}^{3,0}\left(x^2 \left| \begin{matrix} 0 \\ -\frac{1}{2}, \frac{1}{2}, -n - \frac{3}{2} \end{matrix}\right.\right) = \sqrt{\pi}e^{-2x}$$

$$\times \begin{cases} \left(\frac{1}{x^2} + \frac{3}{4}\frac{1}{x^3} - \frac{3}{32}\frac{1}{x^4} + \mathcal{O}(x^{-5})\right), & n = 0 \\ \left(\frac{1}{x^3} + \frac{7}{4}\frac{1}{x^4} + \mathcal{O}(x^{-5})\right), & n = 1 \\ \left(\frac{1}{x^4} + \mathcal{O}(x^{-5})\right), & n = 2 \\ \left(\frac{1}{x^5} + \mathcal{O}(x^{-6})\right) & n = 3. \end{cases} \tag{39}$$

For a given n, the polynomial starts with the power $x^{-(n+2)}$ and continues with increasing powers $x^{-(n+3)}$, $x^{-(n+4)}$ and so on, while

$$\frac{x}{K_2^2(x)} = e^{2x}\frac{2}{\pi}\left(x^2 - \frac{15}{4}x + \frac{285}{32} + \mathcal{O}(x^{-1})\right), \quad (40)$$

starts with x^2 and continues decreasing the exponent by steps of 1. The expansion of the product $x/K_2^2(x)G_{1,3}^{3,0}$, for each n, thus starts with x^{-n}. If we want the thermal averaged cross section up to the order x^{-3}, we have only to consider the terms with n from 0 to 3:

$$\langle\sigma v_{\mathbf{r}}\rangle = \sigma_0^{(0)}\mathcal{K}_0(x)_{(\mathcal{O}(x^{-4}))} + \sigma_0^{(1)}\mathcal{K}_1(x)_{(\mathcal{O}(x^{-4}))}$$
$$+ \frac{1}{2}\sigma_0^{(2)}\mathcal{K}_2(x)_{(\mathcal{O}(x^{-4}))} + \frac{1}{6}\sigma_0^{(3)}\mathcal{K}_3(x)_{(\mathcal{O}(x^{-4}))}. (41)$$

Using in (41) the expansions (39), (40) and the coefficients

$$\Gamma\left(\frac{3}{2}+n\right) = \frac{(2n+1)!!}{2^{n+1}}\sqrt{\pi}, \quad (42)$$

we finally find up to $\mathcal{O}(x^{-4})$,

$$\langle\sigma v_{\mathbf{r}}\rangle = \sigma_0^{(0)} + \left(-3\sigma_0^{(0)} + \frac{3}{2}\sigma_0^{(1)}\right)\frac{1}{x}$$
$$+ \left(6\sigma_0^{(0)} - 3\sigma_0^{(1)} + \frac{15}{8}\sigma_0^{(2)}\right)\frac{1}{x^2}$$
$$- \frac{5}{16}\left(30\sigma_0^{(0)} - 15\sigma_0^{(1)} + 3\sigma_0^{(2)} - 7\sigma_0^{(3)}\right)\frac{1}{x^3}, (43)$$

that is the expected result [3]. This new derivation based on the property of the G function shows clearly why the coefficients of the powers x^{-n} are linear combinations of the derivatives $\sigma_0^{(n)}$ starting from $n = 0$ up the order of the corresponding power of x.

B. $\rho = 1$

The integral (34) is

$$I_n^{\rho=1}(x) = \int_1^\infty dy\, y^{-1/2}(y-1)^{1+n}K_1(2x\sqrt{y}), \quad (44)$$

that reduces to (A6) with $\lambda = -1/2$, $\mu = 2+n$ and $\nu = 1$:

$$I_n^{\rho=1}(x) = \Gamma(2+n)\frac{x}{4}G_{1,3}^{3,0}\left(x^2\left|\begin{array}{c}0\\-n-2,0,-1\end{array}\right.\right)$$
$$= \Gamma(2+n)\frac{1}{x^{(2+n)}}K_{n+1}(2x). \quad (45)$$

The last equality is proved in Appendix A 1. The coefficients (33) of the low-velocity expansion (32) are:

$$\mathcal{K}_n^{\rho=1}(x) = \Gamma(n+2)\frac{2}{x^{n+1}}\frac{K_{n+1}(2x)}{K_2^2(x)}. \quad (46)$$

Using (38) and (40) we find the large x expansion:

$$\langle\sigma v_{\mathbf{r}}\rangle_{\rho=1} = \frac{2x^{-1/2}}{\sqrt{\pi}}[\sigma_0^{(0)} + (-\frac{57}{16}\sigma_0^{(0)} + 2\sigma_0^{(1)})\frac{1}{x}$$
$$+ 3(\frac{1395}{512}\sigma_0^{(0)} - \frac{960}{512}\sigma_0^{(1)} + \sigma_0^{(2)})\frac{1}{x^2} + ...].(47)$$

This expansion never appeared before.

C. $\rho \gg 1$

Neglecting the unity in $(y-1)^{n+1/2}$ and with ρ as lower limit of integration we find after changing variable to y/ρ,

$$I_n^{(\rho\gg1)}(x) = \int_\rho^\infty dy\, y^n\sqrt{y-\rho}K_1(2x\sqrt{y})$$
$$= \rho^{3/2+n}\int_1^\infty dy\, y^n\sqrt{y-1}K_1(2x\sqrt{\rho}\sqrt{y}), \quad (48)$$

that reduces to (A6) with $\lambda = n$, $\mu = 3/2$ and $\nu = 1$:

$$I_n^{(\rho\gg1)}(x) = \frac{\sqrt{\pi}}{4}\frac{\rho^{3/2}}{x^{2n}}G_{1,3}^{3,0}\left(\rho x^2\left|\begin{array}{c}0\\-\frac{3}{2},\frac{1}{2}+n,-\frac{1}{2}+n\end{array}\right.\right). \quad (49)$$

The coefficients (33) of the low-velocity expansion (32) are:

$$\mathcal{K}_n^{\rho\gg1}(x) = \frac{\sqrt{\pi}\rho^{3/2}x}{2x^{2n}K_2^2(x)}G_{1,3}^{3,0}\left(\rho x^2\left|\begin{array}{c}0\\-\frac{3}{2},\frac{1}{2}+n,-\frac{1}{2}+n\end{array}\right.\right). \quad (50)$$

For the expansion at large x we find

$$\langle\sigma v_{\mathbf{r}}\rangle_{\rho\gg1} \approx e^{-2x(\sqrt{\rho}-1)}[\sqrt{\rho}(\sigma_0^{(0)} + \rho\sigma_0^{(1)}) + \mathcal{O}(x^{-1})], \quad (51)$$

that show the well-known suppression for heavy masses [2, 5]. Expansions (46) and (50) correspond to the cases "at threshold" and "above threshold", respectively first discussed in Refs. [2, 5] in the non-relativistic case.

D. Constant cross section, s and p wave scattering

In many cases the cross section in the low-velocity limit goes as $1/v_r$ and the product σv_r can be considered constant and factorize out the thermal integral. Both for non-relativistic and relativistic averages we have $\langle\sigma v_{\text{rel}}\rangle = \sigma v_r$ because of the normalization of the probability distributions $\int_0^\infty dv_r F_M(v_r) = \int_0^1 dv_{\mathbf{r}}\mathcal{P}(v_{\mathbf{r}}) = 1$.

If the cross section is velocity independent, $\sigma = k$, then

$$\langle\sigma v_{\mathbf{r}}\rangle_{\sigma=k} = \sigma\langle v_{\mathbf{r}}\rangle = \sigma\frac{4}{x}\frac{K_3(2x)}{K_2^2(x)}. \quad (52)$$
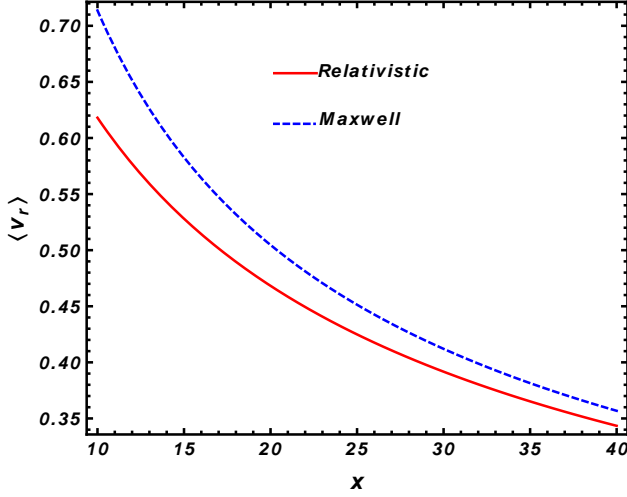
Figure 1. Mean relative velocity as a function of $x = m/T$. The red line is the relativistic value in Eq. (52), see also (A10). The blue-dashed line is the non-relativistic Maxwell value $4/\sqrt{\pi x}$.

The exact expression for the mean value of the relativistic relative velocity was found in Ref. [12], see also Appendix A1. In the range of x between 20 and 40 that is typical for masses and freeze-out temperatures of weakly interacting massive particles, the mean relativistic relative velocity is always smaller than the Maxwell's value $4/\sqrt{\pi x}$ as can be seen in Fig. 1.

We consider now the case $n = 0$ as the dominant contribution to the cross section, the so-called s-wave scattering. The coefficients for $n = 0$ of the expansions (37), (46), (50) take a simple form in terms of the modified Bessel function:

$$\mathcal{K}_{n=0}^{\rho=0}(x) = \frac{K_1^2(x)}{K_2^2(x)}, \tag{53}$$

$$\mathcal{K}_{n=0}^{\rho=1}(x) = \frac{2}{x}\frac{K_1(2x)}{K_2^2(x)}, \tag{54}$$

$$\mathcal{K}_{n=0}^{\rho\gg1}(x) = \rho\frac{K_1^2(\sqrt{\rho}x)}{K_2^2(x)}, \tag{55}$$

The coefficients for the cases $\rho = 0$ and $\rho \gg 1$ follows from Eq. (A5) given in the Appendix.

In the case of p-wave scattering also the term with $n = 1$ is important. The coefficients have the following expressions in terms of Bessel functions:

$$\mathcal{K}_{n=1}^{\rho=0}(x) = \frac{1}{2}(1 - \frac{K_1^2(x)}{K_2^2(x)}), \tag{56}$$

$$\mathcal{K}_{n=1}^{\rho=1}(x) = \frac{4}{x^2}\frac{K_2(2x)}{K_2^2(x)}, \tag{57}$$

$$\mathcal{K}_{n=1}^{\rho\gg1}(x) = \frac{\rho^2}{2}\frac{K_1^2(\sqrt{\rho}x) + K_2^2(\sqrt{\rho}x)}{K_2^2(x)}. \tag{58}$$

The coefficient for the $\rho = 0$ follows from Eq. (A15) and that for $\rho \gg 1$ from Eq. (A17). These exact coefficients
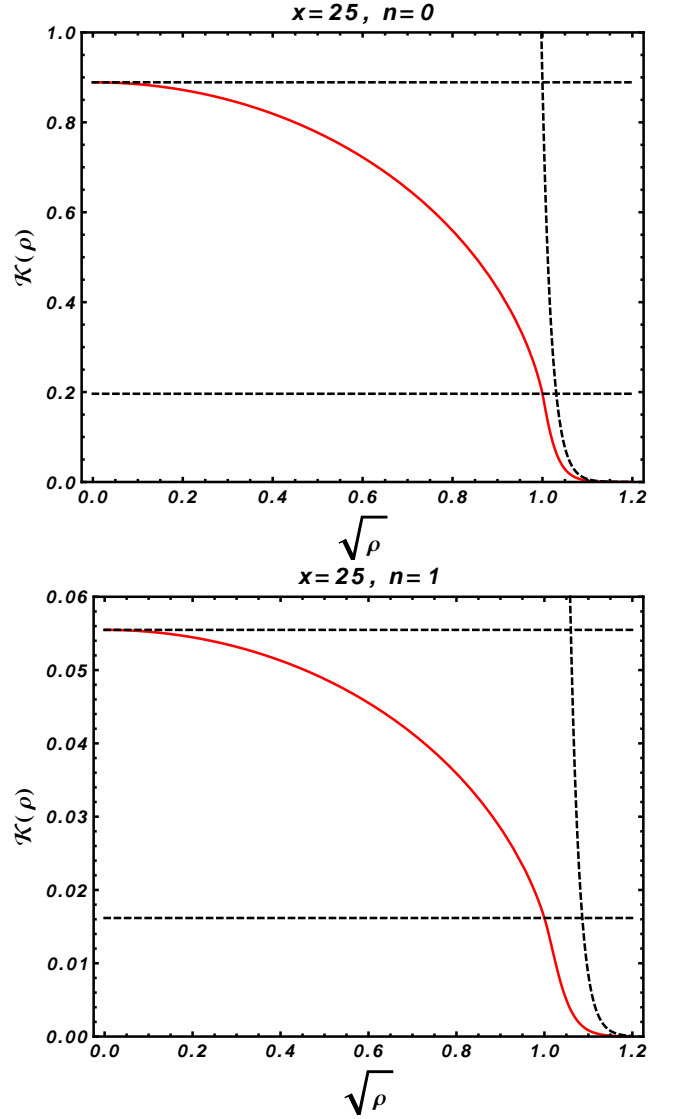

Figure 2. Coefficients of the expansion (32) as a function of the threshold parameter $\sqrt{\rho} = m_f/m$. s wave scattering, top panel, and p wave scattering, bottom panel, at $x = 25$. The red lines are the exact relativistic value obtained by numerical integration of (34). The black-dashed lines correspond to the three particular values $\rho = 0$, $\rho = 1$, $\rho \gg 1$ given by Eqs. (53), (54), (55) for $n = 0$ and Eqs. (56), (57), (58) for $n = 1$.

that we found in the case of a constant cross section, s and p wave scattering can be useful for a rapid estimation of the thermal average. In Fig. 2 we plot the coefficients of the expansion (32) as a function of $\sqrt{\rho} = m_f/m$ at $x = 25$ for the cases $n = 0$, top panel, and $n = 1$, bottom panel. The red line is exact and obtained with numerical integration of (33) while the dashed lines are the constant values given by the formulas (53), (54), (55) for $n = 0$, and (56), (57), (58) for $n = 1$. The agreement show the correctness of the above formulas.

8

V. SUMMARY AND FINAL REMARKS

In this paper we have clarified the meaning of $\langle \sigma v_{\rm rel} \rangle$ in the relativistic framework. The velocity $v_{\rm rel}$ is actually the invariant relative velocity $V_{\rm r}$ defined by (12), (13), (29). The thermal average of $\sigma V_{\rm r}$ is given by the integral over probability distribution of $V_{\rm r}$, Eq. (24), in the same way as the non-relativistic average is determined by the non-relativistic relative velocity v_r and the Maxwell distribution.

We have remarked that the Møller velocity is not a fundamental physical velocity and it is at the origin of the conceptual issues discussed in the paper. Its use as a physical quantity should be avoided in favor of the true invariant relative velocity $V_{\rm r}$.

We have found that the coefficients of the low-velocity expansion of $\langle \sigma V_{\rm r} \rangle$ admit exact analytical representation in the cases that masses of the final state particles are $m_f = 0$, $m_f = m$ and $m_f \gg m$. The coefficients are given by the generalized hypergeometric Meijer G functions and can be reduced to expressions involving combinations of modified Bessel functions K_ν.

ACKNOWLEDGMENTS

Appendix A: G functions and integrals of K_ν

In this Appendix we show how some integrals involving the products of powers and the modified Bessel function K_1 can be evaluated in terms of combination of other Bessel functions K_n by using the generalized hypergeometric Meijer's G function $G_{p,q}^{m,n}(x|\{a_1,...,a_n,...,a_p\},\{b_1,...,b_m,...,b_q\})$. Detailed definition and relations with other special and usual functions can be found for example in [13]. The same kind of integrals were also found in Ref. [12].

The G function is usually written in tabular representation as

$$G_{p,q}^{m,n}\left(z \left| \begin{array}{c} a_1,...,a_n,a_{n+1},...,a_p \\ b_1,...,b_m,b_{m+1}...,b_q \end{array}\right.\right). \quad (A1)$$

Note that the top-left index m counts the first m bottom-left parameters, the bottom-right index p counts the top-right parameters and so on. The function is invariant for permutations of the top a_i parameters or permutations of the b_j's.

The G functions can be simplified thanks to the property

$$G_{p,q}^{m,n}\left(z \left| \begin{array}{c} a_1-\alpha,...,a_p-\alpha \\ b_1-\alpha,...,b_q-\alpha \end{array}\right.\right) = \frac{1}{z^\alpha}G_{p,q}^{m,n}\left(z \left| \begin{array}{c} a_1,...,a_p \\ b_1,...,b_q \end{array}\right.\right), \quad (A2)$$

and when one of the upper index is equal to one of the lower index the function is reduced, for example if $a_p = b_q = c$

$$G_{p,q}^{m,n}\left(z \left| \begin{array}{c} a_1,...,c \\ b_1,...,c \end{array}\right.\right) = G_{p-1,q-1}^{m-1,n}\left(z \left| \begin{array}{c} a_1,...,a_{p-1} \\ b_1,...,b_{q-1} \end{array}\right.\right). \quad (A3)$$

The relations between the modified Bessel functions and the Meijer function are

$$G_{0,2}^{2,0}\left(\frac{z^2}{4} \left| \begin{array}{c} \\ \frac{\delta+\nu}{2},\frac{\delta-\nu}{2} \end{array}\right.\right) = \frac{z^\delta}{2^{\delta-1}}K_\nu(z), \quad (A4)$$

$$G_{1,3}^{3,0}\left(x^2 \left| \begin{array}{c} 0 \\ -\frac{1}{2},\nu+\frac{1}{2},\nu-\frac{1}{2} \end{array}\right.\right) = \frac{2}{\sqrt{\pi}}\frac{K_\nu^2(x)}{x}. \quad (A5)$$

$$\int_1^\infty dz z^\lambda (z-1)^{\mu-1}K_\nu(a\sqrt{z}) =$$
$$\Gamma(\mu)2^{2\lambda-1}a^{-2\lambda}G_{1,3}^{3,0}\left(\frac{a^2}{4} \left| \begin{array}{c} 0 \\ -\mu,\frac{\nu}{2}+\lambda,-\frac{\nu}{2}+\lambda \end{array}\right.\right). \quad (A6)$$

1. Mean relative velocity

In this sub-section we explicitly evaluate the mean value of the relative velocity (A10). For the derivation of the expression with $m_1 \neq m_2$ see Ref. [12]. We evaluate the integral $\int_0^\infty d\gamma_{\rm r} \mathcal{P}(\gamma_{\rm r})v_{\rm r}$ with $V_{\rm r} = \sqrt{\gamma_{\rm r}-1}\sqrt{\gamma_{\rm r}+1}/\gamma_{\rm r}$. Changing variable to $z = (\gamma_{\rm r}+1)/2$ we find

$$\langle V_{\rm r} \rangle = \frac{4x}{K_2^2(x)}\int_1^\infty dz z^{1/2}(z-1)K_1(2x\sqrt{z}). \quad (A7)$$

The integral corresponds to Eq. (A6) with $\lambda = 1/2$, $\mu = 2$, $\nu = 1$,

$$\mathcal{I} = \frac{1}{2x}G_{1,3}^{3,0}\left(x^2 \left| \begin{array}{c} 0 \\ -2,1,0 \end{array}\right.\right) = \frac{1}{2x}G_{0,2}^{2,0}\left(\frac{(2x)^2}{4} \left| \begin{array}{c} \\ -2,1 \end{array}\right.\right), \quad (A8)$$

where we used the property (A3) for the second equality. Now we use Eq. (A4) with $\delta = -1$ and $\nu = 3$ that gives

$$\mathcal{I} = \frac{1}{2x}\frac{4}{2x}K_3(2x). \quad (A9)$$

Finally, multiplying by $4x/K_2^2(x)$,

$$\langle V_{\rm r} \rangle = \frac{4}{x}\frac{K_3(2x)}{K_2^2(x)}. \quad (A10)$$

2. G functions for $\rho = 1$

In the case $\rho = 1$ the coefficients of the expansion involve a reducible G function:

$$G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-n-2,-1,0\end{array}\right.\right) = G^{2,0}_{0,2}\left(\frac{(2x)^2}{4}\left|\begin{array}{c}\\-n-2,-1\end{array}\right.\right)$$
$$= \frac{2}{x^{3+n}}K_{n+1}(2x) \qquad (A11)$$

where we have used Eq. (A3) and then Eq. (A4) with $\delta = -3 - n$ and $\nu = -n - 1$.

3. G functions for $n = 1$ and $\rho = 0$, $\rho \gg 1$

Changing variable to $y = (1 + \gamma_{\mathbf{r}})/2$, the normalization condition of the probability distribution in the diagonal case $x_1 = x_2 = x$ defines two integrals

$$\frac{2x}{K_2^2(x)}\int_1^\infty dy(2y-1)\sqrt{y-1}K_1(2x\sqrt{y})$$
$$= \frac{2x}{K_2^2(x)}(2I_1 - I_0) = 1. \qquad (A12)$$

The integral

$$I_0 = \frac{\sqrt{\pi}}{4}G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-\frac{3}{2},-\frac{1}{2},\frac{1}{2}\end{array}\right.\right) = \frac{K_1^2(x)}{2x}, \qquad (A13)$$

is evaluated thanks to the basic integral (A6) with $\lambda = 0$, $\mu = 3/2$, $\nu = 1$ and then the second equality follows from Eq. (A5). The integral (A6) with $\lambda = 1$, $\mu = 3/2$, $\nu = 1$ reads

$$I_1 = \frac{\sqrt{\pi}}{4x^2}G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-\frac{3}{2},\frac{1}{2},\frac{3}{2}\end{array}\right.\right). \qquad (A14)$$

Combining Eq. (A12) with Eqs. (A13) and (A14) we find

$$G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-\frac{3}{2},\frac{1}{2},\frac{3}{2}\end{array}\right.\right) = \frac{x}{\sqrt{\pi}}(K_1^2(x) + K_2^2(x)). \qquad (A15)$$

Eq. (A6) with $\mu = 5/2$, $\lambda = 0$, $\nu = 1$ gives the difference

$$I_1 - I_0 = \frac{3}{8}\sqrt{\pi}G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-\frac{5}{2},-\frac{1}{2},\frac{1}{2}\end{array}\right.\right). \qquad (A16)$$

Combining (A13), (A14) and (A15) we then find

$$G^{3,0}_{1,3}\left(x^2\left|\begin{array}{c}0\\-\frac{5}{2},-\frac{1}{2},\frac{1}{2}\end{array}\right.\right) = \frac{2}{3}\frac{K_2^2(x) - K_1^2(x)}{\sqrt{\pi}x}. \qquad (A17)$$

[1] B. W. Lee and S. Weinberg, Phys. Rev. Lett. **39**, 165 (1977).
[2] K. Griest and D. Seckel, Phys. Rev. D **43**, 3191 (1991).
[3] M. Srednicki, R. Watkins and K. A. Olive, Nucl. Phys. B **310**, 693 (1988).
[4] J. Bernstein, L. S. Brown and G. Feinberg, Phys. Rev. D **32**, 3261 (1985).
[5] P. Gondolo and G. Gelmini, Nucl. Phys. B **360**, 145 (1991).
[6] S. R. De Groot, W. A. Van Leeuwen and C. G. Van Weert, *Relativistic Kinetic Theory. Principles and Applications*, Amsterdam, Netherlands (1980).
[7] J. Edsjo and P. Gondolo, Phys. Rev. D **56**, 1879 (1997).
[8] L. D. Landau and E. M. Lifschits, *The Classical Theory of Fields: Course of Theoretical Physics, Vol. 2*, (Pergamon Press, New York, 1975).
[9] C. Cercignani and G. M. Kremer, *The relativistic Boltzmann equation: theory and applications*, Birkhäuser Verlag, (2002)
[10] F. Jüttner, Ann. Phys. **339**, 856 (1911).
[11] L. P. Csernai, *Introduction to relativistic heavy ion collisions*, (John Wiley & Sons, 1994).
[12] M. Cannoni, arXiv:1311.4494. The material has published in a merged version with the present manuscript in Phys. Rev. D **89**, 103533 (2014).
[13] I.S. Gradshteyn and I.M. Ryzhik, *Table of Integrals, Series, and Products, 7th Edition*, Academic Press, (2007).
[14] http://www.wolfram.com/mathematica/
[15] http://functions.wolfram.comHypergeometricFunctions/ MeijerG.