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A BASIS FOR THE GRADED IDENTITIES OF THE PAIR $(M_2(K), gl_2(K))$

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Communicated by V. Drensky

Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. Let $M_2(K)$ be the algebra of 2×2 matrices over an infinite integral domain K. In this note we describe a basis for the $\mathbb{Z}_2$-graded identities of the pair $(M_2(K), gl_2(K))$.

1. Introduction. Let K be an associative and commutative unitary ring and let $K\langle X\rangle$ be the free associative algebra over K on a free generating set $X = \{x_1, x_2, \ldots\}$. We say that $f = f(x_1, \ldots, x_n) \in K\langle X\rangle$ is a *polynomial identity* in an associative K-algebra A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. An ideal T in $K\langle X\rangle$ is called a *T-ideal* if $\phi(T) \subseteq T$ for each endomorphism ϕ

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of $K\langle X\rangle$. It can be easily checked that, for a K-algebra A, the set $T(A)$ of all polynomial identities of A is a T-ideal in $K\langle X\rangle$. The converse also holds: every T-ideal is the set of the polynomial identities of a certain K-algebra. A set S of polynomial identities of an algebra A is called a *basis* for the identities of A if it generates $T(A)$ as a T-ideal. We refer to [6, 8] for further terminology and basic results related to polynomial identities.

Let $M_2(K)$ be the algebra of 2×2 matrices over K. One of the most challenging and long standing open problems concerning polynomial identities is the following.

Problem 1. *Let K be an infinite field of characteristic 2. Is there a finite basis for the polynomial identities of $M_2(K)$?*

Let A be an associative K-algebra and let $A^{(-)}$ be its associated Lie algebra (with the Lie multiplication given by $[a,b]=ab-ba$). Let B be a Lie subalgebra of $A^{(-)}$. We say that $f=f(x_1,\dots,x_n)\in K\langle X\rangle$ is an *identity of the pair* (A,B) if $f(b_1,\dots,b_n)=0$ for all $b_1,\dots,b_n\in B$. Let L be the Lie subalgebra of $K\langle X\rangle^{(-)}$ generated by X. It is well known that L is the free Lie algebra freely generated by the set X. An ideal T in $K\langle X\rangle$ is called a *weak T-ideal* if $\psi(T)\subseteq T$ for each endomorphism ψ of $K\langle X\rangle$ such that $\psi(x_i)\in L$ for all i. The set $T(A,B)$ of all identities of the pair (A,B) is a weak T-ideal in $K\langle X\rangle$. A set S of identities of a pair (A,B) is called a *basis* for the identities of (A,B) if it generates $T(A,B)$ as a weak T-ideal.

In order to find an approach to Problem 1 one can study the following.

Problem 2. *Is there a finite basis for the identities of the pair $(M_2(K), gl_2(K))$ if K is an infinite field of characteristic 2?*

It can be easily seen that a basis for the identities of the pair $(M_2(K), gl_2(K))$ is a basis for the polynomial identities of $M_2(K)$ (but in general the converse is not true). Since over an infinite field K of characteristic 2 the Lie algebra $gl_2(K)$ has no finite basis for its identities (Vaughan-Lee [20]), one might expect that it could be easier to solve the latter problem than the former one. However, Problem 2 still remains open as well as Problem 1.

Note that the algebra $M_2(K)$ admits a natural grading and so does the pair $(M_2(K), gl_2(K))$. An algebra A is called *graded* (or $\mathbb{Z}_2$-*graded*) if $A=A_0\oplus A_1$ where A_0, A_1 are submodules of A, and $A_iA_j\subseteq A_{i+j}$ with the sum $i+j$ taken in $\mathbb{Z}_2=\mathbb{Z}/2\mathbb{Z}$. In particular, A_0 is a subalgebra of A. If B is a Lie subalgebra in $A^{(-)}$ such that $B=B_0\oplus B_1$, $B_i=B\cap A_i$, $(i=0,\,1)$ we say that (A,B) is a *graded pair*.

If $A=M_2(K)$ then A_0 is the subalgebra of A consisting of all diagonal matrices and A_1 is spanned by all matrices with 0 on the main diagonal. We refer to the elements of A_0 as even ones and to those in A_1 as odd ones.

Let $Y = \{y_1, y_2, \ldots\}$ and $Z = \{z_1, z_2, \ldots\}$ and let $X = Y \cup Z$. Recall that $K\langle X\rangle$ is the free associative algebra freely generated by X. The homogeneous degree of a monomial $m \in K\langle X\rangle$, denoted by $w(m)$, equals 0 if its degree with respect to the variables of Z is even; otherwise $w(m) = 1$. Then $K\langle X\rangle$ is graded in a natural way setting $K\langle X\rangle_i$ to be the span of all monomials m such that $w(m) = i$, $i = 0, 1$. A polynomial $f(y_1, \ldots, y_m, z_1, \ldots, z_n) \in K\langle X\rangle$ is called a *graded identity* for a graded algebra $A = A_0 \oplus A_1$ (for a graded pair (A, B)) if $f(u_1, \ldots, u_m, v_1, \ldots, v_n) = 0$ for every $u_i \in A_0$ and $v_i \in A_1$ (for every $u_i \in B_0$ and $v_i \in B_1$).

An ideal I in $K\langle X\rangle$ is called a *T_2-ideal* if $\phi(I) \subseteq I$ for all graded endomorphisms ϕ of $K\langle X\rangle$, that is, endomorphisms ϕ such that $\phi(y_i) \in K\langle X\rangle_0$ and $\phi(z_i) \in K\langle X\rangle_1$ for all i. Recall that L is the Lie subalgebra of $K\langle X\rangle^{(-)}$ generated by X. An ideal I in $K\langle X\rangle$ is called a *weak T_2-ideal* if $\psi(I) \subseteq I$ for all endomorphisms ψ of $K\langle X\rangle$ such that $\psi(y_i) \in L \cap K\langle X\rangle_0$ and $\psi(z_i) \in L \cap K\langle X\rangle_1$ for all i.

The graded identities for a graded algebra A and for a graded pair (A, B) form ideals in $K\langle X\rangle$, denoted by $T_2(A)$ and $T_2(A, B)$ respectively. It can be easily seen that, for a graded algebra A, the ideal $T_2(A)$ is a T_2-ideal and, for a graded pair (A, B), the ideal $T_2(A, B)$ is a weak T_2-ideal in $K\langle X\rangle$. A set $S \subseteq T_2(A)$ is called a *basis* of the graded identities of an algebra A if it generates $T_2(A)$ as a T_2-ideal. In other words, S is a basis of the graded identities of A when $T_2(A)$ is the least T_2-ideal of $K\langle X\rangle$ that contains S. Similarly, a set $S \subseteq T_2(A, B)$ is a *basis* of the graded identities of a pair (A, B) if S generates $T_2(A, B)$ as a weak T_2-ideal.

In order to find an approach to the solution of Problem 2 one can study first its (simpler) graded analog.

Problem 3. *Let K be an infinite field of characteristic 2. Is there a finite basis for the graded identities of the pair $(M_2(K), gl_2(K))$?*

In this paper we solve Problem 3. More precisely we present an explicit finite basis in question. We were able to find such a basis over an arbitrary infinite integral domain K.

Theorem 1. *Let K be an infinite integral domain. The following polynomials form a basis for the graded identities of the pair $(M_2(K), gl_2(K))$:*

$$y_1y_2 - y_2y_1, \quad z_1z_2z_3 - z_3z_2z_1, \quad z_1z_2y - yz_1z_2. \tag{1}$$

Note that for an arbitrary associative K-algebra A, the set of the (graded) identities of the pair $(A, A^{(-)})$ coincides with the set of the (graded) polynomial identities of A. In particular, we have

$$T_2(M_2(K), gl_2(K)) = T_2(M_2(K)).$$

It follows that Theorem 1 is equivalent to the following.

Theorem 2. *Let K be an infinite integral domain. The ideal $T_2(M_2(K))$ is generated as a weak T_2-ideal in $K\langle X\rangle$ by the polynomials* (1).

Remarks. 1. By Theorem 1, over an infinite field K of characteristic 2 the pair $(M_2(K), gl_2(K))$ has a finite basis for its graded identities. On the other hand, over such a field K the graded identities of $gl_2(K)$ admit no finite basis [15]. This gives the first example of a pair of the form $(M_n(K), G)$, where G is a graded Lie algebra with the following properties:

i) the graded identities of G have no finite basis;

ii) the graded identities of the pair $(M_n(K), G)$ have a finite basis.

A pair $(M_2(K), S)$ with similar properties where S is a (multiplicative) semigroup was found in [1]. A pair $(M_4(K), G)$ such that the Lie algebra G has a finite basis for its identities but the pair has no such a basis was constructed in [17] (the field K in the latter example is infinite of characteristic 2).

2. If K is an infinite field and $\operatorname{char} K \neq 2$ then the polynomial identities of $M_2(K)$ have a finite basis. Such a basis was found by Razmyslov [18] (see also Drensky [4]) if $\operatorname{char} K = 0$ and by the first named author [12] if $\operatorname{char} K = p > 2$. A (finite) basis for the identities of $gl_2(K)$ was found by Razmyslov [18] if K is a field of characteristic 0 and by Vasilovsky [19] if K is an infinite field of characteristic $p > 2$ (over such a field K the Lie algebras $sl_2(K)$ and $gl_2(K)$ satisfy the same identities). On the other hand, Vaughan-Lee [20] proved that over an infinite field K of characteristic 2 the identities of $gl_2(K)$ admit no finite basis. Over such a field K, $sl_2(K)$ is a nilpotent Lie algebra of dimension 3 so all its identities follow from $[[x_1, x_2], x_3]$.

3. The identities of the pair $(M_2(K), sl_2(K))$ were described by Razmyslov in [18] when $\operatorname{char} K = 0$, and by the first named author when K is an infinite field of characteristic $\neq 2$, see [9]. All of them follow from $[x^2, y]$. Recall that the description of the identities of this pair is an essential step to obtaining a basis of the polynomial identities for the associative algebra $M_2(K)$. The above results admit generalizations, see for example [7, 10, 11]. Over an infinite field K

of characteristic 2 the identities of the pair $(M_2(K), sl_2(K))$ were described by Drensky [5].

4. For an infinite field K, $\operatorname{char} K \neq 2$, the graded identities for $M_2(K)$ were described in [3, 14]. In fact, the proof given in [14] remains valid over an arbitrary infinite integral domain K (see [2, Corollary 2]). A (finite) basis for the graded identities of $sl_2(K)$ (or, equivalently, $gl_2(K)$) over such a field K was found by the first named author [12] (see also [16]). On the other hand, over an infinite field K of characteristic 2 the graded identities of $gl_2(K)$ admit no finite basis [15].

5. In Theorem 2 we prove that $T_2(M_2(K))$ is generated by the polynomials (1) as a *weak* T_2-ideal. As an "ordinary" T_2-ideal it is generated by the first two polynomials only since $z_1z_2y - yz_1z_2$ is contained in the T_2-ideal generated by $y_1y_2 - y_2y_1$.

2. Proof of Theorem 2. Let t_i, u_i, v_i and w_i be commuting variables. Form the polynomial algebra $K[t_i, u_i, v_i, w_i \mid i \geq 1]$. Let $F_2(K)$ be the subalgebra of $M_2(K[t_i, u_i, v_i, w_i])$ generated by the *generic graded* matrices

$$A_i = \begin{pmatrix} t_i & 0 \\ 0 & u_i \end{pmatrix}, \qquad B_i = \begin{pmatrix} 0 & v_i \\ w_i & 0 \end{pmatrix} \qquad (i \geq 1).$$

When K is an infinite integral domain it is easy to check that $F_2(K)$ is isomorphic to the relatively free graded algebra in the variety of graded K-algebras generated by $M_2(K)$, that is,

$$F_2(K) \cong K\langle X\rangle / T_2(M_2(K)).$$

Here the matrices A_i stand for the even variables and B_i for the odd ones. Thus, Theorem 2 follows immediately from the following.

Theorem 3. *Let K be an associative and commutative unitary ring. The ideal $I = T_2(F_2(K))$ is generated as a weak T_2-ideal in $K\langle X\rangle$ by the polynomials* (1)*.*

In order to prove Theorem 3 we will need some auxiliary results.

The following proposition was proved in [3, 14] when K is an infinite field, $\operatorname{char} K \neq 2$. In fact, the proof given in [14] relies on an argument using generic graded matrices. It remains valid for the graded identities of $F_2(K)$, where K is an arbitrary associative and commutative ring with 1.

Proposition 4. *The ideal $I = T_2(F_2(K))$ is generated as a T_2-ideal in $K\langle X\rangle$ by the polynomials*

$$y_1y_2 - y_2y_1, \quad z_1z_2z_3 - z_3z_2z_1. \tag{2}$$

Let $\mathcal{B}$ be the set of the following monomials in $K\langle X\rangle$:

$$y_{a_1}y_{a_2}\cdots y_{a_k},$$
$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}},$$
$$y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}y_{b_1}y_{b_2}\cdots y_{b_l}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}\widehat{z_{d_m}}.$$

Here $a_1\le a_2\le\ldots\le a_k$, $b_1\le b_2\le\ldots\le b_l$, $c_1\le c_2\le\ldots\le c_m$ and $d_1\le d_2\le\ldots\le d_m$, $k\ge 0$, $l>0$, $m>0$. The "hat" over a variable means that it can be missing.

The following fact can be proved exactly in the same way as Proposition 5 in [14] (see also [2, Proposition 3]).

Proposition 5. *Let K be an associative and commutative ring with 1. Then the relatively free graded algebra $K\langle X\rangle/I$ is a free K-module with a basis*

$$\{g+I \mid g\in\mathcal{B}\}$$

over K.

The linear independence of the above monomials in $K\langle X\rangle/I$ was proved in [14] by substituting the variables by generic graded matrices (that is, by identifying the graded algebras $K\langle X\rangle/I$ and $F_2(K)$), and computing the entries of the matrices thus obtained.

We write $[a,b]=ab-ba$, $[a,b,c]=[[a,b],c]$.

Lemma 6. *The following polynomials generate I as a weak T_2-ideal in $K\langle X\rangle$:*

$$[y_1,y_2],\qquad v_0=z_1z_2z_3-z_3z_2z_1,$$

$$u_k=[z_1y_1y_2\ldots y_kz_2,y_0]\qquad (k=0,1,\ldots),$$

$$v_k=z_1y_1y_2\ldots y_kz_2z_3-z_3y_1y_2\ldots y_kz_2z_1\qquad (k=1,2,\ldots).$$

P r o o f. Let J be the weak T_2-ideal in $K\langle X\rangle$ generated by $[y_1,y_2]$ together with u_k and v_k ($k=0,1,2,\ldots$). Then $J\subseteq I$. Indeed, the polynomials u_k ($k\ge 0$) belong to the ("strong") T_2-ideal generated by $[y_1,y_2]$ and the polynomials v_k ($k\ge 1$) belong to the T_2-ideal generated by $z_1z_2z_3-z_3z_2z_1$. Since, by Proposition 4, $[y_1,y_2]$ and $z_1z_2z_3-z_3z_2z_1$ belong to I, so do u_k ($k\ge 0$) and v_k ($k\ge 1$).

To complete the proof of Lemma 6 we need the following.

Lemma 7. *Let* h *be a monomial in* $K\langle X\rangle$. *Then there exists* $h' \in \mathcal{B}$ *such that*

$$h = h' \pmod{J}.$$

Proof. Let h be an arbitrary monomial in $K\langle X\rangle$,

$$h = Y_1 z_{i_1} Y_2 z_{i_2} \ldots Y_s z_{i_s} Y_{s+1}$$

where $s \geq 0$ and, for all m, Y_m are monomials in y_1, y_2, ... Note that $y_i y_j = y_j y_i \pmod{J}$ for all i and j. It follows that if $s = 0$ then

$$h = y_{a_1} y_{a_2} \ldots y_{a_k} \pmod{J}, \qquad a_1 \leq a_2 \leq \ldots \leq a_k$$

and if $s = 1$ then

$$h = y_{a_1} y_{a_2} \ldots y_{a_k} z_{c_1} y_{b_1} y_{b_2} \ldots y_{b_l} \pmod{J},$$

where $a_1 \leq a_2 \leq \ldots \leq a_k$ and $b_1 \leq b_2 \leq \ldots \leq b_l$.

Suppose that $s \geq 2$. Since $u_k \in J$ $(k \geq 0)$, for all $i_1, i_2, j_0, j_1, \ldots, j_k$ and all $f, g \in K\langle X\rangle$ we have

$$f\ [z_{i_1} y_{j_1} \ldots y_{j_k} z_{i_2}, y_{j_0}]\ g \in J,$$

that is,

$$f\ z_{i_1} y_{j_1} \ldots y_{j_k} z_{i_2} y_{j_0}\ g = f\ y_{j_0} z_{i_1} y_{j_1} \ldots y_{j_k} z_{i_2}\ g \pmod{J}.$$

It follows that

$$h = y_{a_1} y_{a_2} \ldots y_{a_k} z_{c_1} y_{b_1} y_{b_2} \ldots y_{b_l} z_{d_1} z_{c_2} z_{d_2} \ldots z_{c_m} \widehat{z_{d_m}} \pmod{J} \tag{3}$$

where $a_1 \leq a_2 \leq \ldots \leq a_k$, $b_1 \leq b_2 \leq \ldots \leq b_l$. Since

$$z_{i_1} y_{j_1} y_{j_2} \ldots y_{j_k} z_{i_2} z_{i_3} - z_{i_3} y_{j_1} y_{j_2} \ldots y_{j_k} z_{i_2} z_{i_1} \in J$$

for all $k \geq 0$ and all i_s and j_r, we can permute, modulo J, the elements z_{c_1}, ..., z_{c_m} and z_{d_1}, ..., z_{d_m} in (3) in order to get the conditions $c_1 \leq c_2 \leq \ldots \leq c_m$ and $d_1 \leq d_2 \leq \ldots \leq d_m$ satisfied.

The proof of Lemma 7 is complete. □

Now we are in a position to prove Lemma 6. We take $f \in I$, then $f + J = \sum \alpha_i g_i + J$ for some $\alpha_i \in K$ and $g_i \in \mathcal{B}$ because, by Lemma 7, the image of $\mathcal{B}$ generates $K\langle X\rangle/J$ as a K-module. Since $J \subseteq I$, we have $f + I = \sum \alpha_i g_i + I$. On the other hand, $f \in I$ so $f + I = I$. It follows that $\alpha_i = 0$ for all i because $g_i \in \mathcal{B}$ and the set $\{g + I \mid g \in \mathcal{B}\}$ is a basis of $K\langle X\rangle/I$ over K.

Thus, if $f \in I$ then $f + J = \sum \alpha_i g_i + J = J$, that is, $f \in J$. Since $J \subseteq I$, it follows that $J = I$, as required.

The proof of Lemma 6 is complete. □

Lemma 8. *The polynomial v_k $(k \geq 1)$ is contained in the weak T_2-ideal generated by v_{k-1} and u_{k-1}.*

Proof. We have

$$v_k(y_1, \ldots, y_k, z_1, z_2, z_3) = z_1 y_1 \ldots y_k z_2 z_3 - z_3 y_1 \ldots y_k z_2 z_1$$

$$= z_1 y_1 \ldots y_{k-1} z_2 y_k z_3 + z_1 y_1 \ldots y_{k-1} [y_k, z_2] z_3$$

$$- z_3 y_1 \ldots y_{k-1} z_2 y_k z_1 - z_3 y_1 \ldots y_{k-1} [y_k, z_2] z_1$$

$$= y_k z_1 y_1 \ldots y_{k-1} z_2 z_3 + [z_1 y_1 \ldots y_{k-1} z_2, y_k] z_3$$

$$- y_k z_3 y_1 \ldots y_{k-1} z_2 z_1 - [z_3 y_1 \ldots y_{k-1} z_2, y_k] z_1$$

$$+ v_{k-1}(y_1, \ldots, y_{k-1}, z_1, [y_k, z_2], z_3)$$

$$= y_k v_{k-1}(y_1, \ldots, y_{k-1}, z_1, z_2, z_3) + v_{k-1}(y_1, \ldots, y_{k-1}, z_1, [y_k, z_2], z_3)$$

$$+ u_{k-1}(y_k, y_1, \ldots, y_{k-1}, z_1, z_2) z_3 - u_{k-1}(y_k, y_1, \ldots, y_{k-1}, z_3, z_2) z_1.$$

The result follows. □

Lemma 9. *The polynomial u_k $(k \geq 1)$ is contained in the weak T_2-ideal generated by u_{k-1} and $[y_1, y_2]$.*

Proof. We have

$$u_k(y_0, y_1, \ldots, y_k, z_1, z_2) = [z_1 y_1 y_2 \ldots y_k z_2, y_0]$$

$$= [z_1 y_1 \ldots y_{k-1} z_2 y_k, y_0] + [z_1 y_1 \ldots y_{k-1} [y_k, z_2], y_0]$$

$$= z_1 y_1 \ldots y_{k-1} z_2 [y_k, y_0] + [z_1 y_1 \ldots y_{k-1} z_2, y_0] y_k + [z_1 y_1 \ldots y_{k-1} [y_k, z_2], y_0]$$

$$= z_1 y_1 \ldots y_{k-1} z_2 [y_k, y_0] + u_{k-1}(y_0, y_1, \ldots, y_{k-1}, z_1, z_2) y_k$$

$$+ u_{k-1}(y_0, y_1, \ldots, y_{k-1}, z_1, [y_k, z_2]).$$

The result follows. □

P r o o f o f T h e o r e m 3. The theorem follows immediately from the above Lemmas 6, 8, and 9. □

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