View metadata, citation and similar papers at core.ac.uk
brought to you by CORE
provided by Bulgarian Digital Mathematics Library at IMI-BAS

Provided for non-commercial research and educational use.
Not for reproduction, distribution or commercial use.

Serdica
Mathematical Journal
Сердика
Математическо списание

The attached copy is furnished for non-commercial research and education use only.
Authors are permitted to post this version of the article to their personal websites or institutional repositories and to share with other researchers in the form of electronic reprints.
Other uses, including reproduction and distribution, or selling or licensing copies, or posting to third party websites are prohibited.

For further information on
Serdica Mathematical Journal
which is the new series of
Serdica Bulgaricae Mathematicae Publicationes
visit the website of the journal http://www.math.bas.bg/~serdica
or contact: Editorial Office
Serdica Mathematical Journal
Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Telephone: (+359-2)9792818, FAX:(+359-2)971-36-49
e-mail: serdica@math.bas.bg

Serdica Math. J. **38** (2012), 371–394

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

SOME NUMERICAL INVARIANTS OF MULTILINEAR IDENTITIES

Antonio Giambruno, Sergey Mishchenko, Mikhail Zaicev

Communicated by V. Drensky

Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. We consider non-necessarily associative algebras over a field of characteristic zero and their polynomial identities. Here we describe most of the results obtained in recent years on two numerical sequences that can be attached to the multilinear identities satisfied by an algebra: the sequence of codimensions and the sequence of colengths.

1. Introduction. Let A be a non-necessarily associative algebra over a field F of characteristic zero. The purpose of this survey is that of reviewing some of the results obtained in recent years on the growth of the polynomial identities satisfied by A.

Let $F\{X\}$ be the free non-associative algebra on a countable set X over a field of characteristic zero and $\mathrm{Id}(A)$ the T-ideal of polynomial identities of A. One can attach to $\mathrm{Id}(A)$ two numerical sequences $c_n(A)$ and $l_n(A)$,

2010 *Mathematics Subject Classification:* Primary 16R10, 16A30, 16A50, 17B01, 17C05, 17D05, 16P90, 17A, 17D.

Key words: Polynomial identity, codimensions, colengths.

$n = 1, 2, \ldots$, called the sequence of codimensions and the sequence of colengths of A, respectively. We should remark that in characteristic zero both these sequences come as numerical evaluations of characters of the symmetric group. In fact, in order to study the growth of the identities of an algebra, a general strategy is that of studying the space of multilinear polynomials in n fixed variables modulo the identities of the algebra A, through the representation theory of the symmetric group on n symbols. Then one attaches to $\mathrm{Id}(A)$ a sequence of S_n-modules, $n = 1, 2, \ldots$, and studies the corresponding sequence of characters.

When A is an associative PI-algebra, the sequence of codimensions is exponentially bounded and the sequence of colengths is polynomially bounded. Several results have been obtained in this setting showing how the growth of these two sequences are important invariants of the T-ideal $\mathrm{Id}(A)$. Moreover the associative case has inspired research in the more general context of Lie, Jordan and general non-associative algebras. Here we shall report briefly on the main achievements in the associative case, focusing on results in the context of non-associative context.

For general non-associative PI-algebras, the two sequences of codimensions and colengths have a much wilder behavior. Anyway, concerning the sequence of codimensions the class of algebras having such sequence exponentially bounded is quite wide and includes for instance finite dimensional algebras.

Here we shall be mostly concerned with algebras whose sequence of codimensions is exponentially bounded. One section will be devoted to general non-associative algebras. We shall see that the exponential rate of growth of the sequence of codimensions, unlike the associative case, can be any real number greater that one and we shall exhibit algebras having wild behavior of the codimensions and colengths. Another section will be devoted to finite dimensional algebras. One of the main results is that for any such simple algebra one can compute the exponential rate of growth explicitly. In the last section we shall collect results concerning Lie algebras, Jordan algebras, Lie superalgebras.

2. Generalities. Throughout F will be a field of characteristic zero and $F\{X\}$ the absolutely free algebra on a countable set $X = \{x_1, x_2, \ldots\}$. Recall that, given an F-algebra A, a polynomial $f(x_1, \ldots, x_m) \in F\{X\}$ is a polynomial identity of A if $f(a_1, \ldots, a_m) = 0$, for all $a_1, \ldots, a_m \in A$. The identities of A form an ideal $\mathrm{Id}(A)$ stable under all endomorphisms of $F\{X\}$ (T-ideal).

For any $n \geq 1$, denote by P_n the subspace of $F\{X\}$ of multilinear polynomials in $x_1, \ldots, x_n$. Hence $P_n \cap \mathrm{Id}(A)$ is the space of multilinear identities of A of degree n. Since $\mathrm{char} F = 0$, the sequence of spaces $P_n \cap \mathrm{Id}(A)$, $n = 1, 2, \ldots$, completely determines $\mathrm{Id}(A)$ and we define $P_n(A) = \dfrac{P_n}{P_n \cap \mathrm{Id}(A)}$.

In some sense $P_n(A)$ corresponds to the non-identities of A. One of the most useful measures of the identities of an algebra is given by the sequence of codimensions

$$c_n(A) = \dim P_n(A), \ n = 1, 2, \ldots.$$

Clearly if B satisfied all the identities of A, then $\mathrm{Id}(B) \supseteq \mathrm{Id}(A)$ and, so, $c_n(B) \leq c_n(A)$.

In general $c_n(A)$ is a fast growing sequence. For instance, if A is the free associative algebra, then $c_n(A) = n!$. If A is the free Lie algebra, $c_n(A) = (n-1)!$. The fastest growing sequence is obtained when A satisfies no identities. In this case $c_n(A) = \dfrac{1}{n}\dbinom{2n-2}{n-1}n!$ where $\dfrac{1}{n}\dbinom{2n-2}{n-1}$ is the nth Catalan number, i.e., the number of all possible arrangements of brackets on a word of length n. Notice that asymptotically $\dfrac{1}{n}\dbinom{2n-2}{n-1} \simeq 4^n$.

Anyway, as we shall see later, there is a wide class of algebras with exponentially bounded growth of the codimensions, i.e., $0 \leq c_n(A) \leq a^n$, for all n, with a a real number. In this case the sequence $\sqrt[n]{c_n(A)}$, $n = 1, 2, \ldots$, is bounded $0 \leq \sqrt[n]{c_n(A)} \leq a$, and one can consider its lower and upper limit which we call the lower and upper exponent of A

$$\underline{\exp}(A) = \liminf_{n\to\infty} \sqrt[n]{c_n(A)}, \ \overline{\exp}(A) = \limsup_{n\to\infty} \sqrt[n]{c_n(A)}.$$

If the limit of the sequence $\sqrt[n]{c_n(A)}$ exists, then we call it the PI-exponent (or just the exponent) of A,

$$\exp(A) = \underline{\exp}(A) = \overline{\exp}(A).$$

One of the main problems in the theory of codimensions is the existence of the PI-exponent (provided that $c_n(A)$ is exponentially bounded).

There are other numerical invariants of a T-ideal that are closely related to the action of the symmetric group S_n on P_n. Recall that if $f = f(x_1, \ldots, x_n) \in P_n$ and $\sigma \in S_n$, then

$$\sigma f(x_1, \ldots, x_n) = f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

Moreover an identity $f \equiv 0$ implies another identity $g \equiv 0$ if any algebra satisfying f also satisfies g. It is not difficult to see that $f \in P_n$ implies $g \in P_n$ if and only if FS_nf, the S_n-submodule generated by f in P_n, contains g. In particular, two multilinear identities $f \equiv 0$ and $g \equiv 0$ are equivalent if and only if $FS_nf = FS_ng$.

It is well-known that any set theoretical map $X \to X$ can be lifted to an endomorphism of $F\{X\}$. Since $\mathrm{Id}(A)$ is invariant under endomorphisms, it follows that $P_n \cap \mathrm{Id}(A)$ is an S_n-submodule of P_n and $P_n(A) = \dfrac{P_n}{P_n \cap \mathrm{Id}(A)}$ is also an S_n-module. Since $\mathrm{char}F = 0$, $P_n(A)$ is completely reducible. The number of irreducible summands, i.e., the length of the S_n-module $P_n(A)$, is called the nth colength of A, and is denoted $l_n(A)$. Hence a new numerical invariant of $\mathrm{Id}(A)$ is given by the colength sequence

$$l_n(A),\ n = 1, 2, \ldots.$$

Recall that the set of non-equivalent irreducible representations of S_n is in one-to-one correspondence with the partitions λ of n (we refer the reader to [30] for an account of the representation theory of the symmetric group). In order to describe the decomposition of an S_n-module M into irreducibles it is convenient to use the language of characters. Hence if χ_λ denotes the character of an S_n-representation corresponding to the partition $\lambda \vdash n$, we write

$$\chi(M) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda,$$

where $\chi(M)$ is the character of the S_n-module M and m_λ is the multiplicity of χ_λ in $\chi(M)$. In other words M is the direct sum $M = M_1 \oplus \cdots \oplus M_t$ of irreducible components where the number of summands M_i corresponding to λ is equal to the non-negative integer m_λ.

Now, if A is an F-algebra, we write

$$\chi_n(A) = \chi(P_n(A)) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda. \tag{1}$$

Then clearly

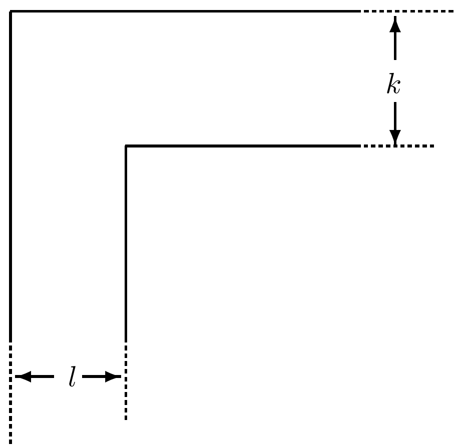
$$l_n(A) = \sum_{\lambda \vdash n} m_\lambda$$

and

$$c_n(A) = \sum_{\lambda \vdash n} m_\lambda d_\lambda,$$

where $d_\lambda =^\circ \chi_\lambda$ is the degree of the irreducible character χ_λ. Recall that $\chi_n(A)$ is called the nth cocharacter of A.

Given integers $k, l \geq 0$, we define the infinite hook $H(k,l)$ as the set of partitions whose corresponding diagram lies in the hook shaped region of the plane:



In other words $H(k,l) = \{\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n \mid \lambda_{k+1} \leq l\}$. In particular, $H(k,0)$ is an infinite strip of height k.

We say that an S_n-character χ_λ (or a partition λ) lies in the hook $H(k,l)$ if $\lambda \in H(k,l)$. More generally, we say that the nth cocharacter $\chi_n(A)$ given in (1) lies in $H(k,l)$, if $m_\lambda \neq 0$ implies that $\chi_\lambda \in H(k,l)$.

3. Associative algebras. In this section we quickly review the main results concerning the above numerical sequences in case of associative algebras. This is not only the starting point for subsequent investigation but also a milestone for comparing the behavior of these sequences for distinct classes of algebras.

We start by recalling the following result proved by Regev in [42] that was the starting point for the theory of codimensions. We remark that this result holds for algebras over a field of any characteristic.

Theorem 1. *Let A be an associative PI-algebra over a field F. Then the sequence of codimensions of A is exponentially bounded, i.e., there exists a constant a such that $c_n(A) \leq a^n$, for all $n \geq 1$.*

Next two major ingredients that allowed to push further the theory were proved in [1] and [5].

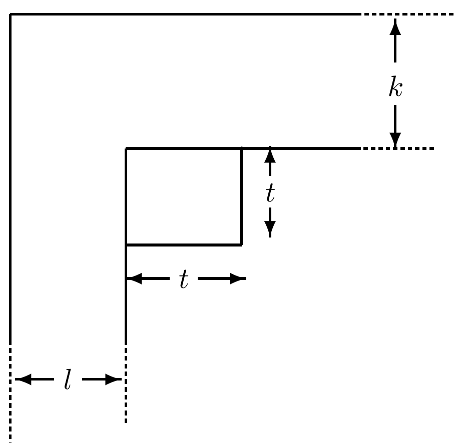
Theorem 2. *Let A be an associative PI-algebra over F. Then*

1. *there exists a hook $H(k, l)$ such that*
$$\chi_n(A) \subseteq H(k, l),$$
for all $n \geq 1$;

2. *the colength sequence $l_n(A)$, $n = 1, 2, \ldots$, is polynomially bounded.*

Recall that an infinite hook $H(k, l)$ is called an essential hook for A if there exists a finite square M (i.e., a partition $M = (t^t)$, for some $t \geq 1$) such that
$$\chi_n(A) \subseteq H(k, l) \cup M,$$
for all $n \geq 1$, but we cannot find squares M' and M'' such that $\chi_n(A) \subseteq H(k-1, l) \cup M'$ or $\chi_n(A) \subseteq H(k, l-1) \cup M''$, for all n.

In the above definition, the inclusion $\chi_n(A) \subseteq H(k, l) \cup M$, with $M = (t^t)$, means that the diagram of any partition λ with $m_\lambda \neq 0$ in $\chi_n(A)$ lies in the following region of the plane



Essential hooks allow to sharpen the above result. In fact the following result can be found in [22, Theorem 9.4.6].

Theorem 3. *For any PI-algebra A there exists an essential hook $H(k, l)$, for some $k, l \geq 0$.*

We remark that from the proof of the above theorem it follows that for any essential hook $H(k, l)$, we must have $k \geq l \geq 0$.

Concerning the asymptotic behavior of the sequence of codimensions, one of the oldest results was proved by Kemer in [32].

Theorem 4. *For any associative PI-algebra A, either the sequence of codimensions is polynomially bounded or, up to a polynomial factor, $c_n(A) \gtrsim 2^n$, asymptotically.*

Recall that a function $f = f(n)$ of a natural argument n is said to be a function of intermediate growth if

$$\lim_{n\to\infty} \frac{f(n)}{n^k} = \infty,$$

for any integer k, while

$$\lim_{n\to\infty} \frac{f(n)}{a^n} = 0,$$

for any real number $a > 1$.

We remark that from the above theorem in particular it follows that there are no associative algebras with an intermediate codimension growth.

In the 80's Amitsur conjectured that for any associative PI-algebra A, the PI-exponent $\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$ exists and is a non-negative integer. This conjecture was confirmed in [19] and [20].

Theorem 5. *For any associative PI-algebra A, there exist constants $C_1 > 0, C_2, k_1, k_2$ such that*

$$C_1 n^{k_1} d^n \leq c_n(A) \leq C_2 n^{k_2} d^n,$$

where $d \geq 0$ is an integer. As a consequence $\exp(A)$ *exists and is a non-negative integer.*

In [19] it was also shown that the exponent of a finite dimensional algebra over an algebraically closed field equals its dimension if and only if A is simple. In general we have:

Theorem 6. *Let A be a finite dimensional associative algebra over F. Then*

1. $\exp(A) \leq \dim A$*;*
2. $\exp(A) = \dim A$ *if and only if A is central simple over F.*

Actually Regev conjectured that for any PI-algebra A, asymptotically

$$c_n(A) \simeq Cn^t d^n$$

where C is a constant, d is an integer and t is an integer or half integer. This conjecture was first confirmed in some special cases ([43], [28], [21]) and then it was proved in [6], [4] for any associative PI-algebra with 1.

Theorem 7. *Let A be a PI-algebra with 1. Then there exist constants $C \in \mathbb{R}^+$, $t \in \frac{1}{2}\mathbb{Z}$ and $d \in \mathbb{Z}^+$ such that $c_n(A) \simeq Cn^t d^n$.*

As we mentioned above there is no intermediate growth of the codimensions. Concerning PI-algebras with polynomially bounded codimension growth several results have been obtained in recent years (see for instance [33], [10]). The starting point was the following result on the asymptotics proved in [8] and [9].

Theorem 8. *Let A be a PI-algebra with polynomially bounded codimension sequence. Then there exist a rational number $q > 0$ and an integer $k > 0$ such that*

$$c_n(A) = qn^k + O(n^{k-1}) \simeq qn^k.$$

If A is a unitary algebra, then

$$\frac{1}{k!} \le q \le \frac{1}{2!} - \frac{1}{3!} + \cdots + \frac{(-1)^k}{k!} \simeq \frac{1}{e},$$

where e is the base of the natural logarithms.

For any rational number $q > 0$, there exists a (non-unitary) algebra A such that $c_n(A) \simeq qn^k$.

We remark that in [11] the authors constructed for any $k \ge 1$, finite dimensional unitary PI-algebras A with $c_n(A) \simeq qn^k$ realizing the smallest and the largest value of q. They proved that the above lower bound is reached only in case k is even. For k odd the lower bound is given by $(k-1)/k!$.

Finally we remark that one can also compute the exponent when adjoining 1 to the algebra [24].

Theorem 9. *Let A be an associative PI-algebra and let $A^\sharp$ be the algebra obtained from A by adjoining a unit element. Then either $\exp(A^\sharp) = \exp(A)$ or $\exp(A^\sharp) = \exp(A) + 1$.*

4. Non-associative algebras. In the general non-associative case there are many examples of algebras with overexponential codimension growth.

Nevertheless the class of algebras with exponentially bounded codimension sequence is quite large. Beside associative PI-algebras, it includes finite dimensional algebras, infinite dimensional simple Lie algebras of Cartan type, affine Kac-Moody algebras, Lie algebras and superalgebras with nilpotent commutator subalgebra and many more.

A general condition giving an exponential bound of the codimensions is given in the following.

Proposition 1. *Let A be an algebra over F and suppose that $\chi_n(A) \subseteq H(k,l)$, for some $k,l \geq 0$. Then $c_n(A)$, $n = 1,2,\ldots$, is exponentially bounded.*

Proof. We fix an arrangement of the brackets on the space of multilinear polynomials in $x_1,\ldots,x_n$. Let $\tilde{P}_n$ be the corresponding subspace of P_n. Notice that the number of such subspaces is $\frac{1}{n}\binom{2n-2}{n-1} \leq 4^n$. Also, as an S_n-module $\tilde{P}_n$ is isomorphic to the group algebra FS_n. Now, if $Q(n)$ is the number of partitions $\lambda \vdash n$ such that $\lambda \in H(k,l)$ we have that $Q(n) \leq n^{k+l}$ is polynomially bounded. On the other hand, $d_\lambda =^\circ \chi_\lambda$ does not exceed $(k+l)^n$ (up to a polynomial factor) as it follows from the hook formula (see [30]). Hence

$$\overline{\exp}(A) \leq 4(k+l),$$

as soon as $\chi_n(A) \subseteq H(k,l)$. $\square$

Even if the codimensions of a non-associative PI-algebra are exponentially bounded, the exponential behavior can be very wild. In fact, by mean of methods pertaining to the combinatorics of infinite words one can construct several examples. Here below we give a general construction that can be used in exhibiting several examples.

Let $K = \{k_i\}_{i\geq 1}$ be a sequence of positive integers. Starting with K we construct an algebra $A(K)$ by giving explicitly a basis and the corresponding multiplication table. A basis of $A(K)$ over F is given by the set

$$\{a,b\} \cup Z_1 \cup Z_2 \cup \cdots$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \leq j \leq k_i\}, \quad i = 1,2,\ldots.$$

The vector a acts on every set Z_i by right multiplication according to the following rule

$$z_1^{(i)}a = z_2^{(i)},\ \ldots,\ z_{k_i-1}^{(i)}a = z_{k_i}^{(i)},\ z_{k_i}^{(i)}a = 0, \quad i = 1,2\ldots,$$

if $k_i \geq 2$, while we set $z_1^{(i)}a = 0$ if $k_i = 1$. The element b acts on each Z_i by right multiplication as follows

$$z_{k_i}^{(i)}b = z_1^{(i+1)}, \quad i = 1, 2, \ldots$$

and all other products are equal to zero.

Of special interest are sequences constructed from infinite words as follows. Let $w = w_1w_2\cdots$ be an infinite word in the alphabet $\{0, 1\}$. Given an integer $m \geq 2$, we let $K_{m,w} = \{k_i\}_{i\geq 1}$ be the sequence defined by

$$k_i = \begin{cases} m, & \text{if } w_i = 0, \\ m+1, & \text{if } w_i = 1 \end{cases}$$

and we write $A(m, w) = A(K_{m,w})$.

Among infinite words Sturmian and periodic words play an important role (see [34]). Recall that w is a Sturmian word if $\mathrm{Comp}_w(n) = n + 1$, where $\mathrm{Comp}_w : \mathbb{N} \to \mathbb{N}$ is the complexity function. Among the remarkable properties of such words we recall that if $w = w_1w_2\cdots$ is a Sturmian word, then the limit

$$\alpha = \pi(w) = \lim_{n\to\infty} \frac{w_1 + \cdots + w_n}{n} \tag{2}$$

exists and is an irrational number called the slope of w. Moreover for any irrational number α, $0 < \alpha < 1$, there exists a Sturmian word whose slope is α. On the other hand, if w is a periodic word, the limit in (2) also exists and is a rational number. Accordingly, for any rational number α, $0 < \alpha < 1$, there exists a periodic word w with $\pi(w) = \alpha$.

Next three theorems can be found in [15].

Theorem 10. *Let w be an infinite Sturmian or periodic word with slope α, $0 < \alpha < 1$, and let $A = A(m, w)$ where $m \geq 2$. Then the PI-exponent of A exists and* $\exp(A) = \Phi(\beta)$ *where* $\beta = \dfrac{1}{m + \alpha}$ *and* $\Phi(x) = \dfrac{1}{x^x(1 - x)^{1-x}}$.

Corollary 1. *For any real number α, $1 < \alpha < 2$, there exists an algebra A such that* $\exp(A) = \alpha$.

A slight modification of the above construction allows to build finite dimensional algebras. In fact, if w is a periodic word of period T, define $B(w, m)$ as the algebra with basis

$$\{a, b\} \cup Z_1 \cup Z_2 \cup \cdots \cup Z_T$$

where

$$Z_i = \{z_j^{(i)} \mid 1 \le j \le k_i\}, \quad i = 1, 2, \ldots, T,$$

and multiplication table is given by

$$z_1^{(i)} a = z_2^{(i)}, \ \ldots, \ z_{k_i-1}^{(i)} a = z_{k_i}^{(i)}, \ z_{k_i}^{(i)} a = 0, \quad i = 1, 2 \ldots,$$

$$z_{k_i}^{(i)} b = z_1^{(i+1)}, \quad i = 1, 2, \ldots, (T-1)$$

and

$$z_{k_i}^{(T)} b = z_1^{(1)}.$$

All remaining products are zero.

Theorem 11. *The algebras $A(m, w)$ and $B(m, w)$ satisfy the same identities.*

Corollary 2. *For any rational number $\beta, 0 < \beta \le \frac{1}{2}$, there exists a finite dimensional algebra B such that $\exp(B) = \Phi(\beta)$.*

In order to construct algebras whose PI-exponent is greater that 2, we define

$$A(K, d) = A(K) \oplus B \oplus W,$$

the direct sum of vector spaces $A(K), B$ and W where $K = \{k_i\}_{i \ge 1}$, $W = \operatorname{span}\{w_{i,j}^{(t)} \mid 1 \le i \le d, j \ge 1, t \ge 1\}$ and $B = B(d)$ is an algebra with basis $\{u_1, \ldots u_d, s_1, \ldots, s_d\}$ and multiplication table given by

$$s_1 u_1 = u_2, \ldots, s_{d-1} u_{d-1} = u_d, s_d u_d = u_1,$$

and all other products are zero.

The multiplication in $A(K, d)$ is induced by the multiplication in $A(K), B$ and the rule $u_s z_j^i = w_{si}$, $1 \le s \le d$, $1 \le k_i$, $i \ge 1$. All other products are zero. With these ingredients one can prove the following [15].

Theorem 12. *Let $m \ge 2$ and let w be a periodic or Sturmian word. Then the PI-exponent of the algebra $A(K_{m,w}, d)$ exists and equals $d + \delta$ where $\delta = \exp(A(K_{m,w}))$.*

Corollary 3. *For any real number $\alpha \geq 1$ there exists an algebra A such that* $\exp(A) = \alpha$.

Corollary 4. *For any $1 \leq \alpha < \beta$ there exists a finite dimensional algebra B such that $\alpha < \exp(B) < \beta$.*

Concerning the sequence of colengths, as we mentioned above, if A is an associative algebra, whose codimension growth is polynomial, i.e., $c_n(A) \simeq Cn^t$, then t is an integer. For non-associative algebras, t can be a rational non-integer number. In fact by a suitable choice of the sequence K one can prove the following result [53].

Theorem 13. *There exists a non-associative algebra A such that $c_n(A) \simeq n^{7/2}$.*

As a further application of the algebras given above one can construct a non-associative algebra with intermediate growth of the codimensions of the type n^{n^β}, $0 < \beta < 1$ [13].

Theorem 14. *For any real number β, $0 < \beta < 1$, let $K = \{k_i\}_{i \geq 1}$ be defined by the relation $k_1 + \cdots + k_t = [t^{1/\beta}]$, for all $t \geq 1$, where $[x]$ is the integral part of $x \in \mathbb{R}$. Then, if $A = A(K)$, the sequence of codimensions of A satisfies*

$$\lim_{n \to \infty} \log_n \log_n c_n(A) = \beta.$$

Hence, $c_n(A)$ asymptotically equals n^{n^β}.

Concerning the colength behavior for general non-associative PI-algebras, one can get quite different results. Even for Lie algebras the colength behavior can be overpolynomial or overexponential. We state the next result [38] in the language of varieties of algebras. We denote by $\ln^{(k)}(x)$ the multiple logarithm

$$\ln^{(k)}(x) = \underbrace{\ln \cdots \ln}_{k}(x).$$

Theorem 15. *Let $\mathcal{V} = \mathcal{N}_{s_q}\mathcal{N}_{s_{q-1}} \cdots \mathcal{N}_{s_1}$ be a product of nilpotent varieties of Lie algebras.*

1) *If $q \geq 3$, then* $l_n(\mathcal{V}) \geq \dfrac{\sqrt{n!}}{(\ln^{(n-2)} n)^{n/s_1}}$.

2) *If $q = 2$ and $a \geq 3$, then* $\ln(\mathcal{N}_b\mathcal{N}_a) \geq b^{n/a}(n!)^{\frac{a-2}{2a}}$.

3) *If* $b \geq 2$, *the variety* $\mathcal{N}_b\mathcal{N}_2$ *has exponential colength growth* $l_n(\mathcal{N}_b\mathcal{N}_2) \simeq (\sqrt{b})^n$.

Even for the variety $\mathcal{A}^3$ of solvable Lie algebras of length 3, it was shown that $l_n(\mathcal{A}^3) \geq \dfrac{\sqrt{n!}}{(\ln n)^n}$.

We recall that earlier in [12] it was shown that $l_n(\mathcal{A}\mathcal{N}_2) \simeq e^t$ where $t = \pi\sqrt{\dfrac{2n}{3}}$, i.e., the colength of $\mathcal{A}\mathcal{N}_2$ has intermediate growth.

We shall see below that for a Lie algebra L the colength sequence is polynomially bounded as soon as the cocharacter $\chi_n(L)$ lies in some hook $H(k,l)$. Nevertheless for general non-associative algebras, even if the cocharacter $\chi_n(A)$ lies in a strip, the colength sequence may have exponential growth. In fact by constructing a quite involved sequence K one can prove the following [51].

Theorem 16. *There exists a sequence* $K = \{k_i\}_{i\geq 1}$ *such that the colength sequence of the algebra* $A(K)$ *satisfies the inequalities*

$$\frac{1}{2n(n-1)}2^n \leq l_n(A(K)) \leq n2^n.$$

Moreover the cocharacter sequence $\chi_n(A(K))$, $n = 1, 2, \ldots$, *lies in a strip of height* 3.

5. Finite dimensional algebras.

In the study of the growth of identities an important topic is that of (non-necessarily associative) finite dimensional algebras. A main motivation is given by the property that such algebras have exponentially bounded codimensions. In fact, if A is a finite dimensional algebra, it satisfies a Capelli polynomial and by standard arguments its cocharacter lies in strip. But then by Proposition 1 the sequence of codimensions of A is exponentially bounded.

The best upper bound was given in [2] (see also [25]).

Theorem 17. *Let* A *be a finite dimensional algebra,* $\dim A = d$. *Then* $c_n(A) \leq d^{n+1}$, *for all* $n \geq 1$.

Two more important properties of finite dimensional algebras are: the absence of intermediate growth of the codimensions and the polynomial bound of the colength sequence [13].

Theorem 18. *Let* A *be a finite dimensional algebra,* $\dim A = d$. *Then*

$$l_n(A) \leq d(n+1)^{d^2+d}.$$

Theorem 19. *Let A be a finite dimensional algebra of overpolynomial codimension growth and let* $\dim A = d$. *Then*

$$c_n(A) > \frac{1}{n^2} 2^{\frac{n}{3d^3}},$$

for all n large enough.

The lower bound for the PI-exponent of a finite dimensional algebra deduced from Theorem 19 is not sharp in general. Anyway, as in the associative case, algebras of dimension 2 and 3, cannot have exponent less than 2. In fact we have.

Theorem 20 ([14])**.** *Let A be a two-dimensional algebra. Then either* $c_n(A) \leq n+1$ *or* $\exp(A) = 2$.

Theorem 21 ([16], [49])**.** *Let A be a three-dimensional algebra. Then either $c_n(A)$ is polynomially bounded or* $\overline{\exp}(A) \geq 2$. *If A is unitary, then* $\exp(A)$ *exists and is a non-negative integer not greater than* 3. *Moreover if A is simple,* $\exp(A) = 3$.

For higher dimension in [16] the authors constructed a 5-dimensional algebra A such that

$$\exp(A) = \frac{3}{\sqrt[3]{4}} \simeq 1.89.$$

The above algebra was further studied in [49] and the following result was proved:

Theorem 22. *Let A be the* 5*-dimensional algebra constructed above with* $\exp(A) = \dfrac{3}{\sqrt[3]{4}}$. *If $A^\sharp$ is the* 6*-dimensional algebra obtained from A by adjoining a unit element, then* $\exp(A^\sharp) = \dfrac{3}{\sqrt[3]{4}} + 1 = \exp(A) + 1$.

We remark that in [40] the author announced the construction of a 4-dimensional algebra whose exponent is strictly in between 2 and 3.

A general question still left unanswered is the following: does the exponent of a finite dimensional algebra exist? Moreover, in case the exponent exists, can one decide when is equal to the dimension of the algebra?

For simple algebras there are two remarkable results that answer the above two questions and we shall describe them here.

Given an algebra A, denote by $\alpha(x, y)$ a fixed linear combination of linear transformations of A of the type $T_u T'_v, T_{uv}$ where $\{u, v\} = \{x, y\}$ and T_w, T'_w are, respectively, the right and the left multiplication by w in A. When A is finite dimensional we denote by $\langle x, y\rangle = \mathrm{tr}(\alpha(x, y))$ the bilinear form determined by α, where tr is the usual trace.

Theorem 23 ([18]). *Let A be a finite dimensional simple algebra over an algebraically closed field,* $\dim A = d$. *Suppose that for some α, the form $\langle x, y\rangle = tr(\alpha(x, y))$ is non-degenerate. Then there exist constants $C > 0$, t such that for any $n \geq 1$,*

$$Cn^t d^n \leq c_n(A) \leq d^{n+1}.$$

Hence the PI-exponent of A exists and $\exp(A) = \dim A = d$.

A more general result has been recently proved in [26].

Theorem 24. *Let A be a finite dimensional simple algebra. Then the PI-exponent of A exists and* $\exp(A) \leq \dim A$.

In the next section we shall see that $\exp(A)$ can be less that $\dim A$ even for finite dimensional simple algebras.

We close this section with a result proved in [49] concerning algebras with 1 clarifying the asymptotic behavior of the codimensions.

Theorem 25. *Let A be a finite dimensional unitary algebra. Then either $c_n(A)$ is polynomially bounded or* $\underline{\exp}(A) \geq 2$.

6. Special classes of algebras. In this section we study the asymptotic behavior of the sequence of codimensions and colengths of special classes of algebras such as Lie algebras, Jordan algebras, alternative algebras, Lie superalgebras.

We start by analyzing Lie algebras. Some results were already mentioned in the previous sections. We start by stating a result that resembles the associative case, that is, for a Lie algebra no intermediate growth of the codimensions is allowed.

Theorem 26 ([36]). *For any Lie algebra L, either $c_n(L)$ is polynomially bounded or $c_n(L) \gtrsim 2^n$ (up to a polynomial factor).*

Recall that in general a Lie algebra L with non-trivial identity can have overexponential growth of the codimensions (see [46]). Nevertheless, there is a wide class of Lie algebras with exponentially bounded codimension growth. This class contains finite dimensional algebras, infinite dimensional simple algebras of Cartan type, affine Kac-Moody algebras, Virasoro algebra, Lie algebras with nilpotent commutator subalgebra, finitely generated Lie algebras solvable of length 3, Lie algebras L whose cocharacter sequence lies in a hook $\chi_n(L) \subseteq H(k,l)$.

For many algebras whose codimensions are exponentially bounded, the PI-exponent exists and is an integer. In particular, for finite dimensional algebras the answer is similar to the associative case.

Theorem 27 ([48]). *Let L be a finite dimensional Lie algebra. Then* $\exp(L)$ *exists and is an integer.*

Theorem 28 ([17]). *Let L be a finite dimensional Lie algebra over an algebraically closed field of characteristic zero. Then* $\exp(L) = \dim L$ *if and only if L is simple.*

We remark that from Theorem 27 and [47], one can prove the existence and integrality of the PI-exponent of any affine Kac-Moody algebra.

Recall that by Proposition 1, the codimensions of an algebra are exponentially bounded provided the corresponding cocharacter lies in a hook. Another property of such algebras is that the colenghts are polynomially bounded.

Theorem 29 ([50]). *Let L be a Lie algebra such that $\chi_n(L) \subseteq H(k,l)$, for all $n \geq 1$, for some $k, l \geq 1$. Then $l_n(L)$ is polynomially bounded.*

As a special case of the above theorem, if L satisfies a Capelli identity, then $c_n(L)$ is exponentially bounded while $l_n(L) \leq Cn^t$. On the other hand, there is an example of a Lie algebra satisfying a Capelli identity and whose PI-exponent is not an integer.

Theorem 30 ([50], [45]). *There exists a Lie algebra solvable of length* 3 *that satisfies a Capelli identity of rank* 5 *such that* $\exp(L)$ *exists and*

$$3 < \exp(L) < 4.$$

Another interesting feature is the following. Let $W_k = \mathrm{Der}F[t_1,\ldots,t_k]$ be the Lie algebra of derivations of the polynomial ring $F[t_1,\ldots,t_k]$. Then we have ([35], [37], [39]):

Theorem 31. *For the algebra W_k we have that the codimensions $c_n(W_k)$ are exponentially bounded but $\chi_n(W_k)$ does not lie in any hook. Moreover*

$$13.1 < \underline{\exp}(W_2), \quad \overline{\exp}(W_2) \leq 13.5.$$

There are only few results concerning codimensions and colengths of Jordan and alternative algebras. The codimensions of a Jordan PI-algebra in general are not exponentially bounded (see [7], [27]). Anyway, as in case of Lie algebras [29], the following result holds.

Theorem 32 ([3]). *For any alternative PI-algebra A and for any $a > 0$ the following inequality holds asymptotically*

$$c_n(A) < \frac{n!}{a^n}.$$

In a series of papers ([23], [25], [18]) the authors studied finite dimensional Jordan and alternative algebras. In case of finite dimensional simple algebras the results resemble the associative case.

Theorem 33. *Let A be a finite dimensional simple Jordan or alternative algebra over an algebraically closed field. Then the PI-exponent of A exists and* $\exp(A) = \dim A$.

A similar conclusion still holds for finite dimensional algebras.

Theorem 34. *Let A be a finite dimensional Jordan or alternative algebra. Then* $\exp(A)$ *exists and is a non-negative integer. Moreover, if A is nilpotent* $\exp(A) = 0$ *and if A is not nilpotent either* $\exp(A) \geq 2$ *or* $\exp(A) = 1$ *and* $c_n(A)$ *is polynomially bounded.*

We should remark that it was also shown that over an algebraically closed field $\exp(A) = \dim A$ if and only if A is simple.

Another area of investigation is that of Lie superalgebras.

If a finite dimensional simple Lie superalgebra has a non-degenerate Killing form, then from Theorem 23 it follows that for such an algebra the PI-exponent exists and equals the dimension of the algebra. Unfortunately most finite dimensional simple Lie superalgebras have degenerate Killing form.

The early computation of the exponent was done for solvable algebras, in particular for metabelian Lie superalgebras [52]. Later these results were generalized as follows [54].

Theorem 35. *Let L be a Lie superalgebra with nilpotent commutator subalgebra. If $[L, L]^{c+1} = 0$, then the PI-exponent of L exists, is an integer and does not exceed $2c$. Moreover, for any integer $1 \leq t \leq 2c$ there exists a Lie superalgebra L such that $[L, L]^{c+1} = 0$ and* $\exp(L) = t$.

For non-solvable infinite dimensional Lie superalgebras in [52] it was proved that if $L = G(sl_2)$ is the Grassmann envelope of the 3-dimensional simple Lie algebra sl_2 with standard $\mathbb{Z}_2$-grading, then $\exp(L) = 3$. This result was generalized in [41] as follows.

Theorem 36. *Let $L = L_0 \oplus L_1$ be a finite dimensional simple Lie algebra over an algebraically closed field with an arbitrary $\mathbb{Z}_2$-grading. Then the PI-exponent of the Lie superalgebra*

$$G(L) = L_0 \otimes G_0 \oplus L_1 \otimes G_1$$

exists and is equal to $\dim L$.

Even in case of Lie superalgebras it can be proved that there is no intermediate growth ([52], [54]).

Theorem 37. *For any Lie superalgebra L, the codimension sequence $c_n(L)$ is either polynomially bounded or $c_n(L) \gtrsim \sqrt{2}^n$, asymptotically. If $\dim L < \infty$, then either $c_n(L) < Cn^t$ or $\overline{\exp}(L) \geq 2$.*

Although Lie superalgebras resemble Lie algebras in some sense, finite dimensional Lie superalgebras have a quite different asymptotic behavior of the codimensions as we shall see below.

Recall that finite dimensional simple Lie superalgebras were classified by Kac [31] and here we recall the description of the algebras of type $b(t)$ (see [44] for details). Let $M_t(F)$ be the algebra of $t \times t$ matrices over F and let T be the transpose involution. Then L is of type $b(t)$ if L is the Lie superalgebra of $2t \times 2t$ matrices over F of the type

$$\begin{pmatrix} A & B \\ C & -A^T \end{pmatrix},$$

where $A, B, C \in M_t(F)$, $B^T = B, C^T = -C$ and $\operatorname{tr} A = 0$. We write $L = L_0 \oplus L_1$ where

$$L_0 = \left\{ \begin{pmatrix} A & 0 \\ 0 & -A^T \end{pmatrix} \mid A \in M_t(F), tr(A) = 0 \right\},$$

and

$$L_1 = \left\{ \begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix} \mid B^T = B, C^T = -C \in M_t(F) \right\}.$$

The elements of $L_0 \cup L_1$ are called homogeneous. Any $x \in L_0$ (or $x \in L_1$) has homogeneous degree $|x| = 0$ ($|x| = 1$, respectively). The algebra L is a Lie superalgebra with $\mathbb{Z}_2$-grading $L = L_0 \oplus L_1$ if we define a product $[\ ,\ ]$ of two homogeneous elements $x, y \in L$ as

$$[x, y] = xy - (-1)^{|x||y|} yx.$$

The algebra L is simple not only as a superalgebra but also as an algebra for any $t \geq 3$ and $\dim L = 2t^2 - 1$. We have:

Theorem 38 ([26]). *Let L be a finite dimensional Lie superalgebra of type $b(t)$, $t \geq 3$. Then the PI-exponent of L exists and* $\exp(L) < 2t^2 - 1 = \dim L$.

It is unknown if for the algebras L of type $b(t)$ the inequality $2t^2 - 2 < \exp(L)$ holds (thought it seems very reasonable).

When $t = 2$, the Lie superalgebra L of type $b(2)$ is not simple, but the previous theorem can be improved in this case.

Theorem 39 ([26]). *Let L be a Lie superalgebra of type $b(2)$. Then the PI-exponent of L exists and*

$$6 < \exp(L) < 7 = \dim L.$$

Actually it was shown that $6.26 < \exp(L) < 6.47$.

REFERENCES

[1] S. A. Amitsur, A. Regev. P.I. algebras and their cocharacters. *J. Algebra* **78** (1982), 248–254.

[2] Y. Bahturin, V. Drensky. Graded polynomial identities of matrices. *Linear Algebra Appl.* **357** (2002), 15–34.

[3] Y. Bahturin, M. Zaicev. Identities of graded alternative algebras. In: Nonassociative Algebra and Its Applications. Proceedings of the Fourth International Conference, São Paulo, Brazil, 1998 (Eds. Roberto Costa et al.). Lecture Notes in Pure and Appl. Math. vol. **211**, New York, Marcel Dekker, 2000, 9–20.

[4] A. Berele. Properties of hook Schur functions with applications to P.I. algebras. *Adv. in Appl. Math.* **41**, *1* (2008), 52-75.

[5] A. Berele, A. Regev. Applications of hook Young diagrams to P.I. algebras. *J. Algebra* **82** (1983), 559–567.

[6] A. Berele, A. Regev. Asymptotic behaviour of codimensions of P.I. algebras satisfying Capelli identities. *Trans. Amer. Math. Soc.* **360**, *10* (2008), 5155–5172.

[7] V. Drensky. Polynomial identities for the Jordan algebra of a symmetric bilinear form. *J. Algebra* **108** (1987), 66–87.

[8] V. Drensky. Relations for the cocharacter sequences of T-ideals. Proceedings of Int. Conf. Memory A. I. Mal'cev, Novosibirsk/USSR 1989, Contemp. Math. vol. **131**, Part 2, Providence, RI, Amer. Math. Soc., 1992, 285–300.

[9] V. Drensky, A. Regev. Exact asymptotic behaviour of the codimensions of some P.I. algebras. *Israel J. Math.* **96** (1996), 231–242.

[10] A. Giambruno, D. La Mattina. PI-algebras with slow codimension growth. *J. Algebra* **284**, *1* (2005), 371–391.

[11] A. Giambruno, D. La Mattina, V. Petrogradsky. Matrix algebras of polynomial codimension growth. *Israel J. Math.* **158** (2007), 367–378.

[12] A. Giambruno, S. Mishchenko, M. Zaicev. On the colength of a variety of Lie algebras. *Internat. J. Algebra Comput.* **9** (1999), 483–491.

[13] A. Giambruno, S. Mishchenko, M. Zaicev. Algebras with intermediate growth of the codimensions. *Adv. in Applied Math.* **37** (2006), 360–377.

[14] A. Giambruno, S. Mishchenko, M. Zaicev. Codimension growth of two-dimensional non-associative algebras. *Proc. Amer. Math. Soc.* **135**, *11* (2007), 3405–3415.

[15] A. Giambruno, S. Mishchenko, M. Zaicev. Codimensions of Algebras and Growth Functions. *Adv. in Math.* **217** (2008), 1027–1052.

[16] A. Giambruno, S. Mishchenko, M. Zaicev. Polynomial identities of algebras of small dimension. *Comm. Algebra* **37** (2009), 1934–1948.

[17] A. Giambruno, A. Regev, M. Zaicev. Simple and semisimple Lie algebras and codimension growth. *Trans. Amer. Math. Soc.* **352**, *4* (2000), 1935–1946.

[18] A. Giambruno, I. Shestakov, M. Zaicev. Finite dimensional nonassociative algebras and codimension growth. *Adv. in Appl. Math.* **47** (2011), 125–139.

[19] A. Giambruno, M. Zaicev. On codimension growth of finitely generated associative algebras. *Adv. in Math.* **140** (1998), 145–155.

[20] A. Giambruno, M. Zaicev. Exponential codimension growth of P.I. algebras: an exact estimate. *Adv. in Math.* **142** (1999), 221–243.

[21] A. Giambruno, M. Zaicev. Codimension growth and minimal superalgebras. *Trans. Amer. Math. Soc.* **355** (2003), 5091–5117.

[22] A. Giambruno, M. Zaicev. Polynomial Identities and Asymptotic Methods. Mathematical Surveys and Monographs, vol. **122**, Providence, RI, Amer. Math. Soc., 2005.

[23] A. Giambruno, M. Zaicev. Multialternating Jordan polynomials and codimension growth of matrix algebras. *Linear Algebra Appl.* **422** (2007), 372–379.

[24] A. Giambruno, M. Zaicev. Proper identities, Lie identities and exponential codimension growth. *J. Algebra* **320**, *5* (2008), 1933–1962.

[25] A. Giambruno, M. Zaicev. Codimension growth of special simple Jordan algebras. *Trans. Amer. Math. Soc.* **362**, *6* (2010), 3107-3123.

[26] A. Giambruno, M. Zaicev. On codimension growth of finite dimensional Lie superalgebras. *J. London Math. Soc.* **85**, *2* (2012), 534–548.

[27] A. Giambruno, E. Zelmanov. On growth of codimensions of Jordan algebras. In: Groups, Algebras and Applications. Proceedings of XVIII Latin American Algebra Colloquium, São Pedro, Brazil, August 3–8, 2009 (Ed. C. Polcino Milies). Contemp. Math., vol. **537**, Providence, RI, Amer. Math. Soc., 2011, 205–210.

[28] A. S. Gordienko. Regev's conjecture and codimensions of P.I. algebras. *Acta Appl. Math.* **108**, **1** (2009), 33–55.

[29] A. N. Grishkov. Growth of varieties of Lie algebras. *Mat. Zametki* **44**, *1* (1988), 51–54 (in Russian); English translation in *Math. Notes* **44**, *1–2* (1988), 515–517.

[30] G. James, A. Kerber. The Representation Theory of the Symmetric Group. Encyclopedia of Mathematics and Its Applications, vol. **16**, Addison-Wesley, London, 1981.

[31] V. Kac. Lie superalgebras. *Adv. in Math.* **26** (1977), 8–96.

[32] A. R. Kemer. T-ideals with power growth of the codimensions are Specht. *Sibirsk. Mat. Zh.* **19** (1978), 54–69 (in Russian); English translation in *Siberian Math. J.* **19** (1978), 37–48.

[33] D. La Mattina. Varieties of almost polynomial growth: classifying their subvarieties. *Manuscripta Math.* **123**, *2* (2007), 185–203.

[34] M. Lothaire. Algebraic Combinatorics on Words. Encyclopedia of Mathematics and Its Applications, vol. **90**, Cambridge, Cambridge University Press, 2002.

[35] S. P. Mishchenko. On the Engel problem. *Mat. Sb., N.S.* **124(166)**, *1(5)* (1984), 56–67 (in Russian); English translation in *Math. USSR, Sb.* **52** (1985), 53–62.

[36] S. P. Mishchenko. On varieties of Lie algebras of intermediate growth. *Izv. Akad. Nauk BSSR, Ser. Fiz.-Mat. Nauk 2* (1987), 42–45 (in Russian).

[37] S. P. Mishchenko. Growth of varieties of Lie algebras. *Usp. Mat. Nauk* **45**, *6(276)* (1990), 25–45 (in Russian); English translation in *Russian Math. Surveys* **45**, *6* (1990), 27–52.

[38] S. P. Mishchenko, M. V. Zaicev. Asymptotic behaviour of colength of varieties of Lie algebras. *Serdica Math. J.* **26**, *2* (2000), 145–154.

[39] S. S. Mishchenko. New example of a variety of Lie algebras with a fractional exponent. *Vestnik Moskov. Univ. Ser. I Mat. Mekh.* *6* (2011), 44–47 (in Russian); English translation in *Moscow Univ. Math. Bull.* **66**, *6* (2011), 264–266. DOI: 10.3103/S0027132211060088

[40] S. S. Mishchenko. The growth of codimensions of commutative and anticommutative linear algebras. In: Algebra and Mathematical Logic, Proceedings of the International Conference Dedicated to Anniversary of V.V. Morozov, Kazan, Federal University, 2011, 144–145 (in Russian).

[41] D. Repovs, M. Zaicev. On identities of infinite dimensional Lie superalgebras. *Proc. Amer. Math. Soc.* (to appear).

[42] A. Regev. Existence of identities in $A \otimes B$. *Israel J. Math.* **11** (1972), 131–152.

[43] A. Regev. Codimensions and trace codimensions of matrices are asymptotically equal. *Israel J. Math.* **47** (1984), 246–250.

[44] M. Scheunert. The Theory of Lie Superalgebras. An Introduction. Lecture Notes in Math. vol. **716**, Berlin-Heidelberg-New York, Springer-Verlag, 1979.

[45] A. B. Verevkin, M. V. Zaicev, S. P. Mishchenko. A sufficient condition for existence of lower and upper exponents of the variety of linear algebras. *Vestnik Moskov. Univ. Ser. I Mat. Mekh. 2* (2011), 36–39 (in Russian); English translation in *Moscow Univ. Math. Bull.* **66**, *2* (2011), 86–89.

[46] I. B. Volichenko. Varieties of Lie algebras with the identity $[[X_1, X_2, X_3], [X_4, X_5, X_6]] = 0$ over a field of characteristic zero. *Sibirsk. Mat. Zh.* **25**, *3* (1984), 40–54 (in Russian); English translation in *Siberian Math. J.* **25** (1984), 370–382.

[47] M. Zaicev. Varieties and identities of affine Kac-Moody algebras. In: Methods in Ring Theory. Proceedings of the Trento conference, Trento, Italy, 1997 (Eds. V. Drensky et al.). Lect. Notes Pure Appl. Math. vol. **198**, New York, NY: Marcel Dekker, 1998, 303–314.

[48] M. V. Zaicev. Integrality of exponents of growth of identities of finite-dimensional Lie algebras. *Izv. Ross. Akad. Nauk Ser. Mat.* **66** (2002), 23–48 (in Russian); English translation in *Izv. Math.* **66** (2002), 463–487.

[49] M. V. Zaicev. Identities of unitary finite-dimensional algebras. *Algebra Logika* **50**, *5* (2011), 563–594 (in Russian); English translation in *Algebra and Logic* **50**, *5* (2011), 381–404.

[50] M. V. Zaicev, S. P. Mishchenko. An example of a variety of Lie algebras with a fractional exponent. *J. Math. Sci. (New York)* **93**, *6* (1999), 977–982; translated from *Itogi Nauki i Tekhniki, Seriya Sovremennaya Matematika i Ee Prilozheniya. Tematicheskie Obzory. Algebra* **11** (1998).

[51] M. V. Zaicev, S. P. Mishchenko. the colength of varieties of linear algebras. *Mat. Zametki* **79**, *4* (2006), 553–559 (in Russian); English translation in *Math. Notes* **79**, *3–4* (2006), 511–517.

[52] M. V. Zaicev, S. P. Mishchenko. The growth of some varieties of Lie superalgebras. *Izv. Ross. Akad. Nauk Ser. Mat.* **71**, *4* (2007), 3–18 (in Russian); English translation in *Izv. Math.* **71**, *4* (2007), 657–672.

[53] M. V. Zaicev, S. P. Mishchenko. An example of a variety of linear algebras with fractional polynomial growth. *Vestnik Moskov. Univ. Ser. I Mat. Mekh. 1* (2008), 25–31, (in Russian); English translation in *Moscow Univ. Math. Bull.* **63**, *1* (2008), 27–32.

[54] M. V. Zaicev, S. P. Mishchenko. Identities of Lie superalgebras with a nilpotent commutator. *Algebra Logika* **47**, *5* (2008), 617–645 (Russian); English translation in *Algebra Logic* **47**, *5* (2008), 348–364.

Antonio Giambruno
Dipartimento di Matematica e Informatica
Università di Palermo
Via Archirafi 34, 90123 Palermo, Italy
e-mail: `antonio.giambruno@unipa.it`

Sergey Mishchenko
Department of Algebra
and Geometric Computations
Ulyanovsk State University
Ulyanovsk 432970, Russia
e-mail: `mishchenkosp@ulsu.ru`

Mikhail Zaicev
Department of Algebra
Faculty of Mathematics and Mechanics
Moscow State University
Moscow 119992, Russia
e-mail: `zaicevmv@mail.ru`

Received February 14, 2012