Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **38** (2012), 297–312

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

CENTRAL A-POLYNOMIALS FOR THE GRASSMANN ALGEBRA

Antônio Pereira Brandão Jr.*, Dimas José Gonçalves**

Communicated by A. Giambruno

We dedicate this paper to the 65th birthday of Yuri Bahturin.

Abstract. Let F be an algebraically closed field of characteristic 0, and let G be the infinite dimensional Grassmann (or exterior) algebra over F. In 2003 A. Henke and A. Regev started the study of the A-identities. They described the A-codimensions of G and conjectured a finite generating set of the A-identities for G. In 2008 D. Gonçalves and P. Koshlukov answered in the affirmative their conjecture. In this paper we describe the central A-polynomials for G.

1. Introduction. Let F be a field and let $F\langle X\rangle$ be the free unitary associative algebra, freely generated over F by the infinite set $X = \{x_1, x_2, \ldots\}$. The elements of $F\langle X\rangle$ are called polynomials. All algebras considered in this paper will be associative, unitary and over the field F.

2010 *Mathematics Subject Classification:* 16R10, 16R40, 16R50.

Key words: A-identity, central A-polynomial, Grassmann algebra.

*Partially supported by CNPq/Brazil 620150/2008-4, and by INCT.

**Supported by DPP/UnB and by CNPq-FAPDF PRONEX grant 2009/00091-0 (193.000.580/2009).

A polynomial $f(x_1, \ldots, x_n)$ is a *polynomial identity* for an algebra R if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in R$. It is well known that the set $T(R)$ of all polynomial identities for R is a *T-ideal*, that is, an ideal invariant under all endomorphisms of $F\langle X\rangle$. A polynomial $f(x_1, \ldots, x_n)$ is a *central polynomial* for R if $f(a_1, \ldots, a_n) \in Z(R)$, the centre of R, for all $a_1, \ldots, a_n \in R$. The set $C(R)$ of all central polynomials for R is a *T-space* of $F\langle X\rangle$, that is, $C(R)$ is a vector subspace invariant under all endomorphisms of $F\langle X\rangle$.

Recall that if $\mathrm{char}(F) \neq 2$ and V is a vector space over F with an infinite basis $e_1, e_2, \ldots$ then the *Grassmann algebra of* V is the unitary associative algebra G with a basis consisting of 1 and the elements

$$e_{i_1} e_{i_2} \cdots e_{i_n}, \tag{1}$$

where $i_1 < i_2 < \cdots < i_n$. The multiplication in G is induced by $e_i e_j = -e_j e_i$ for all i and j. The centre of G is the subspace G_0 spanned by 1 and the elements (1) with n even. It is well known that the polynomial $[x_1, x_2, x_3] = [[x_1, x_2], x_3]$ is a polynomial identity for G, where $[x, y] = xy - yx$ is the commutator of x and y. A direct consequence of this fact is that the polynomial $[x_1, x_2]$ is a central polynomial for G.

The polynomial identities for G were described in [9] by Krakowski and Regev when char(F)=0, and by various authors in the general case (see [4] and [10]). The central polynomials for the Grassmann algebra were described independently by several authors, see for example [1], [2] and [6].

Let P_n be the set of all multilinear polynomials of degree n in the variables $x_1, \ldots, x_n$. The set formed by all monomials $x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$, where $\sigma \in S_n$, the symmetric group of degree n, is a basis for the vector space P_n. It is known that the multilinear identities for an algebra R generate its T-ideal $T(R)$ when $\mathrm{char}(F) = 0$. In other words, all polynomials identities of R are linear combinations of elements

$$g_0 f(g_1, \ldots, g_n) g_{n+1},$$

where $g_i \in F\langle X\rangle$ for all i and $f \in P_n \cap T(R)$. Due the importance of the multilinear identities, the quotient space

$$\frac{P_n}{P_n \cap T(R)}$$

has become an object of extensive study. Its dimension, $c_n(R)$, is called the n-th *codimension* of R. The codimensions of G were computed explicitly in [9].

In 2003 Henke and Regev [7] started the study of the *A-identities* of an algebra. Let P_n^A be the subspace of P_n spanned by the monomials $x_{\sigma(1)} \cdots x_{\sigma(n)}$, where $\sigma \in A_n$, the alternating group of degree n. The elements in P_n^A have the form

$$f(x_1, \ldots, x_n) = \sum_{\sigma \in A_n} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $\alpha_\sigma \in F$, and they are called *A-polynomials.* If $f \in P_n^A$ is a polynomial identity for an algebra R, then f is called an *A-identity* for R. The n-th *A-codimension* of R is

$$c_n^A(R) = \dim \left(\frac{P_n^A}{P_n^A \cap T(R)} \right).$$

An example of an A-identity for the Grassmann algebra G is the polynomial

$$[x_1, x_2 x_3] x_4 - x_4 [x_1, x_3 x_2]. \tag{2}$$

In [7] Henke and Regev proved the following result.

Theorem 1 ([7])**.** *If F is an algebraically closed field and* $\operatorname{char}(F) = 0$, *then*

$$c_n^A(G) = \dim \left(\frac{P_n^A}{P_n^A \cap T(G)} \right) = 2^{n-1} - 1.$$

Using this theorem D. Gonçalves and P. Koshlukov [5] gave an affirmative answer to the conjecture of Henke and Regev [7] about the description of the A-identities of G. It was shown in [5] that the A-identities of G are determined by the polynomial (2).

Theorem 2 ([5])**.** *Let F be an algebraically closed field of characteristic 0. If $\sigma \in A_n$ and $0 \le r \le n-4$, denote by $u_{r,\sigma}$ the polynomial*

$$u_{r,\sigma} = [x_{\sigma(r+1)}, x_{\sigma(r+2)} x_{\sigma(r+3)}] x_{\sigma(r+4)} - x_{\sigma(r+4)} [x_{\sigma(r+1)}, x_{\sigma(r+3)} x_{\sigma(r+2)}]$$

and denote by $f_{r,\sigma}$ the polynomial

$$f_{r,\sigma} = x_{\sigma(1)} \cdots x_{\sigma(r)} \cdot (u_{r,\sigma}) \cdot x_{\sigma(r+5)} \cdots x_{\sigma(n)}. \tag{3}$$

Then the polynomials $f_{r,\sigma}$ span all A-identities of degree n for G.

In this paper we describe the central A-polynomials for the infinite dimensional Grassmann algebra G. Our main result is the following theorem.

Theorem 3. *Let F be an algebraically closed field of characteristic* 0. *Given $\sigma \in A_n$, consider the polynomials*

$$g_\sigma = [x_{\sigma(1)} \cdots x_{\sigma(n-1)}, x_{\sigma(n)}] \quad \textit{and} \quad h_\sigma = [x_{\sigma(1)} \cdots x_{\sigma(n-2)}, x_{\sigma(n-1)} x_{\sigma(n)}].$$

(a) *If n is odd, then the set*

(4) $$\{f_{r,\sigma} \mid \sigma \in A_n \text{ and } 0 \le r \le n-4\} \cup \{g_\sigma \mid \sigma \in A_n\}$$

spans all central A-polynomials of degree n for G. Futhermore

$$\dim\left(\frac{P_n^A}{P_n^A \cap C(G)}\right) = 2^{n-2} - 1.$$

(b) *If n is even, then the set*

(5) $$\{f_{r,\sigma} \mid \sigma \in A_n \text{ and } 0 \le r \le n-4\} \cup \{h_\sigma \mid \sigma \in A_n\}$$

spans all central A-polynomials of degree n for G. Futhermore

$$\dim\left(\frac{P_n^A}{P_n^A \cap C(G)}\right) = 2^{n-2}.$$

The other main result of this paper is the description of the A_n-cocharacters of $C(G)$.

2. Central A-polynomials for G. In this section F will be a field, $\mathrm{char}(F) \neq 2$. Let U_n be the subspace of P_n^A spanned by

(i) $P_n^A \cap T(G)$ and $\{g_\sigma \mid \sigma \in A_n\}$, if n is odd.

(ii) $P_n^A \cap T(G)$ and $\{h_\sigma \mid \sigma \in A_n\}$, if n is even.

Since $[x_1, x_2] \in C(G)$ we have

(6) $$U_n \subseteq P_n^A \cap C(G).$$

From now on we denote by V_n an arbitrary subspace of P_n^A such that

$$U_n \subseteq V_n \subseteq P_n^A \cap C(G).$$

If $mx_px_qx_rx_sm'$ is a monomial in P_n^A, then

$$\zeta(x_1, \ldots, x_n) = m([x_p, x_qx_r]x_s - x_s[x_p, x_rx_q])m'$$

is an A-identity for G. This polynomial will be called **principal polynomial**. For future reference we write ζ as follows

(7) $$\zeta = +mx_px_qx_rx_sm'$$

(8) $$-mx_sx_px_rx_qm'$$

(9) $$-mx_qx_rx_px_sm'$$

(10) $$+mx_sx_rx_qx_pm'.$$

Note that since ζ is an A-identity for G we have $\zeta \in V_n$.

If $\sigma \in A_n$ we say that $x_{\sigma(i)}$ occupies the i-th position of the monomial $x_{\sigma(1)}x_{\sigma(2)}\cdots x_{\sigma(n)}$. We will study the subspace W_n of P_n^A spanned by monomials whose last position is occupied by x_n.

Lemma 4. *Let W_n be the subspace of P_n^A spanned by*

$$\{x_{\gamma(1)}\cdots x_{\gamma(n)} \mid \gamma \in A_n,\ \gamma(n) = n\}$$

and let $m = x_{\sigma(1)}\cdots x_{\sigma(n)}$, where $\sigma \in A_n$.
(a) *If n is odd, then there exists $f \in W_n$ such that*

$$m \equiv f \mod U_n.$$

Consequently, $P_n^A = W_n + U_n$.
(b) *If n is even and x_n occupies an even position in m (that is if $\sigma(i) = n$ then i is even), then there exists $f \in W_n$ such that*

$$m \equiv f \mod U_n.$$

Proof. (a) Suppose $\sigma(i) = n$, with $i \neq n$. One has

$$x_{\sigma(1)}\cdots x_{\sigma(n-1)}x_{\sigma(n)} = g_\sigma + x_{\sigma(n)}x_{\sigma(1)}\cdots x_{\sigma(n-1)}.$$

Since the cycle $\mu = (n\ \ n-1\ \ \ldots\ \ 2\ \ 1)$ is an even permutation we have $\sigma\mu \in A_n$ and hence $x_{\sigma(n)}x_{\sigma(1)}\ldots x_{\sigma(n-1)} \in P_n^A$. Applying this argument several times, we have the result.
(b) The argument is the same from item (a). Note that

$$x_{\sigma(1)}\cdots x_{\sigma(n-2)}x_{\sigma(n-1)}x_{\sigma(n)} = h_\sigma + x_{\sigma(n-1)}x_{\sigma(n)}x_{\sigma(1)}\cdots x_{\sigma(n-2)}$$

and $\sigma\mu^2 \in A_n$. $\square$

The following result is proved in [2]. Here we give another proof.

Proposition 5. *If $f(x_1,\ldots,x_{n-1})x_n \in P_n$ is a central polynomial for G, then $f(x_1,\ldots,x_{n-1})$ is a polynomial identity for G.*

Proof. Since $g = fx_n \in C(G)$ we have $f \in C(G)$. Thus $[g, x_{n+1}]$ and $[f, x_{n+1}]$ are identities for G. But

$$[g, x_{n+1}] = [fx_n, x_{n+1}] = [f, x_{n+1}]x_n + f[x_n, x_{n+1}]$$

and $[f, x_{n+1}]$ vanishes on G. Hence $f[x_n, x_{n+1}]$ is an identity for G. Let $a_1, a_2, \ldots, a_{n-1}$ be arbitrary elements from the basis (1) of G. Suppose e_i, e_j are different *letters* that do not appear in the composition of the *words* a_l, for all l. Since

$$0 = f(a_1, \ldots, a_{n-1})[e_i, e_j] = 2f(a_1, \ldots, a_{n-1})e_i e_j,$$

we have $f(a_1, \ldots, a_{n-1}) = 0$. Thus f is an identity for G. □

The next result is an immediate consequence of Proposition 5.

Corollary 6. *If $f(x_1, \ldots, x_{n-1})x_n \in P_n^A$ is a central A-polynomial for G, then $f(x_1, \ldots, x_{n-1})$ is an A-identity for G. Consequently,*

$$C(G) \cap W_n \subseteq P_n^A \cap T(G).$$

The next theorem gives the description of the central A-polynomials of odd degree for G.

Theorem 7. *If n is odd, then $P_n^A \cap C(G) = U_n$.*

P r o o f. By (6) it suffices to prove that $P_n^A \cap C(G) \subseteq U_n$. If $f \in P_n^A \cap C(G)$ then by Lemma 4 (a) there exist polynomials $f_1 \in W_n$ and $f_2 \in U_n$ such that $f = f_1 + f_2$. Since $U_n \subseteq P_n^A \cap C(G)$ it follows that $f_1 \in W_n \cap C(G)$. By Corollary 6 we have $f_1 \in P_n^A \cap T(G) \subseteq U_n$ and hence $f \in U_n$. □

Lemma 8. *Let W_n^* be the subspace*

$$W_n^* = \frac{W_n + V_n}{V_n} \tag{11}$$

of the quotient space P_n^A / V_n. If $n \geq 2$ then $\dim W_n^* = c_{n-1}^A(G)$.

P r o o f. Consider the linear map $\psi : P_{n-1}^A \to W_n^*$ defined by

$$\psi(f(x_1, \ldots, x_{n-1})) = f(x_1, \ldots, x_{n-1})x_n + V_n.$$

It follows from Corollary 6 that $f \in \ker(\psi)$ if and only if $f \in P_{n-1}^A \cap T(G)$. Since ψ is surjective and $\ker(\psi) = P_{n-1}^A \cap T(G)$, it follows that

$$\dim W_n^* = \dim \left(\frac{P_{n-1}^A}{P_{n-1}^A \cap T(G)} \right) = c_{n-1}^A(G). \qquad \square$$

Proposition 9. *Let n be an even number, $n \geq 4$. Consider $\sigma \in A_n$ such that x_n occupies an odd position in the monomial $y = x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$.*

(a) *If $y = m x_a x_b x_n x_c m'$, where m and m' are monomials, then*

$$y \equiv m x_n x_a x_b x_c m' \mod (W_n + U_n).$$

(b) *If $y = m x_a x_b x_n x_c m'$, where m and m' are monomials, then*

$$y \equiv m x_c x_a x_n x_b m' \mod (W_n + U_n).$$

(c) *If* $y = mx_ax_nx_bx_cm'$, *where* m *and* m' *are monomials, then*

$$y \equiv mx_cx_nx_ax_bm' \bmod (W_n + U_n).$$

P r o o f. (a) If we identify y with the monomial (9) in the principal polynomial, then x_n occupies even positions in the monomials (8) and (10). By Lemma 4 (b) the monomials (8) and (10) belong to $W_n + U_n$. Now observe that the right-hand side in our congruence is the monomial (7).

(b) If we identify y with the monomial (7) in the principal polynomial, then x_n occupies even positions in the monomials (9) and (10). Observe that the right-hand side in the congruence is the monomial (8).

(c) If we identify y with the monomial (9) in the principal polynomial, then x_n occupies even positions in the monomials (7) and (8). Observe that the right-hand side in the congruence is the monomial (10). □

Theorem 10. *If* n *is even, then* $P_n^A \cap C(G) = U_n$.

P r o o f. Suppose n even. Let Q_n be the subspace of P_n^A spanned by the monomial $x_1x_2 \cdots x_{n-4}x_{n-2}x_{n-3}x_nx_{n-1}$ and let $Q_n^* = (Q_n + V_n)/V_n$. We shall prove that

$$\frac{P_n^A}{V_n} = W_n^* \oplus Q_n^*. \tag{12}$$

First we show that $P_n^A = W_n + V_n + Q_n$. Let $\sigma \in A_n$ and let $y = x_{\sigma(1)} \cdots x_{\sigma(n)}$. If x_n occupies an even position in y then $y \in W_n + V_n$ by Lemma 4 (b). Thus suppose x_n occupies an odd position in y. Using Proposition 9 (a) we can "join" x_n and x_1, that is, we can show that

$$y \equiv ux_nx_1v \mod (W_n + U_n) \qquad \text{or} \qquad y \equiv ux_1x_nv \mod (W_n + U_n),$$

where u and v are monomials. We will show that

$$y \equiv x_1m' \mod (W_n + U_n)$$

for some monomial m'. We consider two cases:

Case 1. $y \equiv ux_nx_1v \mod (W_n + U_n)$.
(i) If u is a monomial of lenght 0, then by Proposition 9 (a) we have

$$y \equiv x_nx_1x_b\overline{v} \equiv x_1x_bx_n\overline{v} \mod (W_n + U_n),$$

where $v = x_b\overline{v}$ for some monomial $\overline{v}$.

(ii) If u is a monomial of lenght > 0, then $u = mx_ax_b$, because y_n occupies an odd position in y. By Proposition 9 (b)

$$y \equiv mx_ax_bx_nx_1v \equiv mx_1x_ax_nx_bv \mod (W_n + U_n).$$

Note that x_1 "walked" 3 positions to the left in the new monomial. Now we use Proposition 9 (a) again and we have

$$y \equiv mx_1x_ax_nx_bv \equiv mx_nx_1x_ax_bv \mod (W_n + U_n).$$

Using the same procedure in (i) and (ii), after several steps we will obtain

$$y \equiv x_1m' \mod (W_n + U_n)$$

for some monomial m'.

Case 2. $y \equiv ux_1x_nv \mod (W_n + U_n)$.
Since ux_1x_nv is of even length and x_n occupies an odd position it follows that u and v have lengths ≥ 1. Thus $ux_1x_nv = u'x_ax_1x_nx_cv'$ for some monomials u' and v'. We have

$$y \equiv u'x_ax_1x_nx_cv' \equiv u'x_cx_ax_nx_1v' \mod (W_n + U_n).$$

Since $u'x_cx_ax_nx_1v'$ is a monomial satisfying Case 1 we have

$$y \equiv x_1m' \mod (W_n + U_n)$$

for some monomial m'.

Using Proposition 9 and similar arguments for x_2 it follows that

$$y \equiv x_1x_2m'' \mod (W_n + U_n),$$

for some monomial m''. In this way we prove that

$$y \equiv x_1x_2 \cdots x_{n-5}x_{n-4}w \mod (W_n + U_n)$$

for some monomial w. Note that x_n occupies either the first or the third position in w. By Proposition 9 (a) we have

$$y \equiv x_1x_2 \cdots x_{n-5}x_{n-4}x_ax_bx_nx_c \mod (W_n + U_n),$$

where $\{x_a, x_b, x_c\} = \{x_{n-1}, x_{n-2}, x_{n-3}\}$. Since $x_1x_2 \cdots x_{n-5}x_{n-4}x_ax_bx_nx_c$ is an A-polynomial it follows that $x_ax_bx_nx_c$ is one of the three monomials

$$x_{n-2}x_{n-3}x_nx_{n-1}, \text{ or } x_{n-3}x_{n-1}x_nx_{n-2}, \text{ or } x_{n-1}x_{n-2}x_nx_{n-3}.$$

Therefore applying twice Proposition 9 (b) it follows that

$$y \equiv x_1x_2 \cdots x_{n-4}x_{n-2}x_{n-3}x_nx_{n-1} \mod (W_n + U_n).$$

Since $U_n \subseteq V_n$ we have the equality $P_n^A = W_n + V_n + Q_n$ as desired.

To finish the proof of equality (12) we shall prove that the intersection of the subspaces involved is $\{0\}$. Let

$$f = \left(\sum_{\sigma \in A_{n-1}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(n-1)} x_n \right) + \beta x_1 x_2 \cdots x_{n-4} x_{n-2} x_{n-3} x_n x_{n-1},$$

where α_σ, $\beta \in F$. If $f \in V_n$ then $[f(x_1, \ldots, x_n), x_{n+1}] \in T(G)$. Looking for the basis elements of G in (1) it follows that

$$[f(e_1, \ldots, e_{n-1}, e_n e_{n+1}), e_{n+2}] = 2(\gamma - \beta) e_1 e_2 \cdots e_{n+2} = 0,$$

where

$$\gamma = \sum_{\sigma \in A_{n-1}} \alpha_\sigma.$$

Hence

$$\gamma - \beta = 0. \qquad (13)$$

Since

$$[f(e_1 e_2, \ldots, e_{2n-3} e_{2n-2}, e_{2n-1}), e_{2n}] = 2(\gamma + \beta) e_1 e_2 \cdots e_{2n} = 0$$

we have

$$\gamma + \beta = 0. \qquad (14)$$

From (13) and (14) it follows that $\beta = 0$ and hence the equality (12) is proved. By Lemma 8 we have $\dim(P_n^A / V_n) = c_{n-1}^A(G) + 1$. Since V_n is any subspace that satisfies $U_n \subseteq V_n \subseteq P_n^A \cap C(G)$ it follows that $P_n^A \cap C(G) = U_n$. $\square$

3. Proof of Theorem 3. Let F be an algebraically closed field of characteristic 0. According to Theorem 2, $T(G) \cap P_n^A$ is spanned by the polynomials $f_{r,\sigma}$, with $0 \leq r \leq n-4$ and $\sigma \in A_n$.

The space $C(G) \cap P_n^A$ is spanned by $T(G) \cap P_n^A$ and $\{g_\sigma \mid \sigma \in A_n\}$ in the case when n is odd (Theorem 7), and by $P_n^A \cap T(G)$ and $\{h_\sigma \mid \sigma \in A_n\}$ in the case of n even (Theorem 10). Thus we have the generating sets (4) and (5) for $C(G) \cap P_n^A$ (as a vector space), in the cases when n is odd, and n is even, respectively.

If n is odd it follows from Lemma 4 (a), Lemma 8 and Theorem 1 that

$$\dim \left(\frac{P_n^A}{P_n^A \cap C(G)} \right) = \dim W_n^* = c_{n-1}^A(G) = 2^{n-2} - 1.$$

If n is even it follows from equality (12) and Theorem 1 that

$$\dim\left(\frac{P_n^A}{P_n^A \cap C(G)}\right) = \dim(W_n^*) + 1 = 2^{n-2}.$$

Thus Theorem 3 is proved. □

4. A_n-cocharacter of $C(G)$. Let F be a field of characteristic 0 and let $\eta : FS_n \to P_n$ be the isomorphism of S_n-modules defined by

$$\eta(\sigma) = \sigma(x_1 \cdots x_n) = x_{\sigma(1)} \cdots x_{\sigma(n)},$$

where $\sigma \in S_n$.

In this section we will study the S_n- and A_n-cocharacters of $C(G)$. The dimension of the vector space $P_n/(P_n \cap C(G))$ is called the *n-th codimension of* $C(G)$. The codimensions of $C(G)$ are given by following result.

Theorem 11 ([2]). *If F is a field of characteristic* 0*, then*

$$\dim\left(\frac{P_n}{P_n \cap C(G)}\right) = 2^{n-2}.$$

Let $\chi_n(C(G))$ be the character of the S_n-module $P_n/(P_n \cap C(G))$. We say that $\chi_n(C(G))$ is the *n-th cocharacter of* $C(G)$.

Let λ_t be the partition of n defined by

$$\lambda_t = (n - t + 1, 1^{t-1}) = (n - t + 1, 1, 1, \dots, 1),$$

where $1 \le t \le n$. We will denote by χ_t and T_t the irreducible character and the standard Young tableau

$$T_t = \begin{array}{|c|c|c|c|c|} \hline 1 & t+1 & t+2 & \cdots & n \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1} \vdots \\ \cline{1-1} t \\ \cline{1-1} \end{array}$$

corresponding to λ_t, respectively. The codimensions and cocharacters of G (or $T(G)$) are described by following result.

Theorem 12 ([9, 11]). *Let F be a field of characteristic* 0*. The n-th cocharacter of $T(G)$ is*

$$\chi_n(T(G)) = \sum_{t=1}^{n} \chi_t.$$

Moreover, the n-th codimension of $T(G)$ is 2^{n-1}.

Let g_t be the element of the group algebra FS_n defined by

$$g_t = \sum_{\sigma \in R_t} \sum_{\gamma \in C_t} (-1)^{\gamma} \sigma\gamma,$$

where R_t is the set of the row permutations and C_t is the set of the column permutations of T_t. From now on we will denote by $\eta_t(x_1, \ldots, x_n)$ the polynomial

$$\eta_t = \eta(g_t) = g_t \cdot (x_1 x_2 \cdots x_n).$$

Note that

$$\eta_1(x_1, \ldots, x_n) = \sum_{\sigma \in S_n} x_{\sigma(1)} x_{\sigma(2)} \cdots x_{\sigma(n)}$$

and

$$\begin{aligned} \eta_t(x_1, \ldots, x_n) &= \sum_{\sigma \in R_t} \sigma(s_t(x_1, x_2, \ldots, x_t) x_{t+1} \cdots x_n) \\ &= \sum_{\sigma \in R_t} s_t(x_{\sigma(1)}, x_2, \ldots, x_t) x_{\sigma(t+1)} \cdots x_{\sigma(n)}, \end{aligned}$$

if $t > 1$, where s_t is the standard polynomial of degree t.

Lemma 13. *If t is odd then η_t is not a central polynomial for G.*

P r o o f. Consider e_1, e_2, e_3, ..., the canonic generators of G. Since

$$\eta_1(e_1 e_2, \ldots, e_{2n-3} e_{2n-2}, e_{2n-1}) = n! e_1 e_2 \cdots e_{2n-1} \notin Z(G)$$

we have $\eta_1(x_1, x_2, \ldots, x_n) \notin C(G)$.

Consider $t > 1$ odd. Suppose that

$$\eta_t(x_1, \ldots, x_n) = \sum_{\sigma \in R_t} s_t(x_{\sigma(1)}, x_2, \ldots, x_t) x_{\sigma(t+1)} \cdots x_{\sigma(n)} \in C(G).$$

Thus

$$\eta_t(x_1, \ldots, x_t, 1, \ldots, 1) = (n-t)!\, s_t(x_1, x_2, \ldots, x_t) \in C(G)$$

and hence $s_t(x_1, \ldots, x_t) \in C(G)$. But

$$s_t(e_1, \ldots, e_t) = t!\, e_1 \cdots e_t \notin Z(G).$$

Contradiction. Therefore η_t is not a central polynomial for G. $\square$

Fix $1 \le t \le n$ odd and consider the map

$$\begin{array}{rccl} f: & (FS_n)\eta_t & \longrightarrow & \dfrac{P_n}{P_n \cap C(G)} \\ & \alpha & \longmapsto & \alpha + (P_n \cap C(G)) \end{array}.$$

Since $(FS_n)\eta_t$ is an irreducible S_n-module and $f(\eta_t) \neq 0$ (by Lemma 13), it follows that f is an injective S_n-homomorphism. Taking $M_t = \mathrm{Im}\,(f)$ we have $M_t \simeq (FS_n)\eta_t$. Thus M_1, M_3, …, M_{n_1} are minimal S_n-submodules of $P_n/(P_n \cap C(G))$ (where n_1 is the greatest odd number in $\{1, 2, \dots, n\}$) and pairwise nonisomorphic. Hence the sum $M_1 + \cdots + M_{n_1}$ is direct. By the Hook formula we have

$$\dim M_t = \dim[(FS_n)\eta_t] = \dim[(FS_n)g_t] = \binom{n-1}{t-1}.$$

Opening the expression $(1+1)^{n-1} + (1-1)^{n-1}$ as a sum of binomial coefficients we have that

$$\sum_{\substack{t=1\\ t\,\mathrm{odd}}}^{n} \binom{n-1}{t-1} = 2^{n-2}.$$

Thus

$$2^{n-2} = \dim\left(\frac{P_n}{P_n \cap C(G)}\right) \ge \dim\left(\bigoplus_{\substack{t=1\\ t\,\mathrm{odd}}}^{n} M_t\right) = \sum_{\substack{t=1\\ t\,\mathrm{odd}}}^{n} \binom{n-1}{t-1} = 2^{n-2}.$$

Therefore the following result is proved.

Theorem 14. *If F is a field of characteristic 0, then*

$$\frac{P_n}{P_n \cap C(G)} \simeq \bigoplus_{\substack{t=1\\ t\,\mathrm{odd}}}^{n} M_t. \tag{15}$$

Moreover the n-th cocharacter of $C(G)$ is

$$\chi_n(C(G)) = \sum_{\substack{t=1\\ t\,\mathrm{odd}}}^{n} \chi_t.$$

From now on F will be algebraically closed of characteristic 0. We will make a summary extracted from [7], on some basic results of the theory of representations of the alternating group A_n. For more details on the subject see [8].

Given a S_n-character χ, denote by $\overline{\chi}$ its restriction to the group A_n. If $\lambda \vdash n$, denote by χ_λ the irreducible character associated with λ and denote by d_λ the degree of such character. Let λ' be the conjugate partition of λ. Below we describe the irreducible characters of A_n:

(i) If $\lambda \neq \lambda'$, then

$$\overline{\chi_\lambda} = \overline{\chi_{\lambda'}}.$$

Moreover, $\overline{\chi_\lambda}$ is A_n-irreducible.

(ii) If $\lambda = \lambda'$, then

$$\overline{\chi_\lambda} = \overline{\chi_\lambda}^+ + \overline{\chi_\lambda}^-,$$

where $\overline{\chi_\lambda}^\pm$ are A_n-irreducibles. Moreover $^\circ\overline{\chi_\lambda}^\pm = d_\lambda/2$.

(iii) All irreducible A_n-characters were found in items (i) and (ii).

Denote by $\chi_n^A(C(G))$ the A_n-character corresponding to the quotient

$$\frac{P_n^A}{P_n^A \cap C(G)}.$$

We say that it is the A_n-*cocharacter of* $C(G)$.

Theorem 15. *Let F be an algebraically closed field of characteristic 0. The decomposition of the A_n-cocharacter of $C(G)$ as sum of irreducible A_n-characters is given as follows:*

(a) *If $n \geq 2$ is even, then*

$$\chi_n^A(C(G)) = \sum_{\substack{t=1 \\ t\,\text{odd}}}^{n} \overline{\chi_t}.$$

(b) *If $n \geq 5$ is odd, then*

$$\chi_n^A(C(G)) = \overline{\chi_1} + \overline{\chi_{(n+1)/2}}^+ + \overline{\chi_{(n+1)/2}}^- + \sum_{\substack{t=3 \\ t\,\text{odd}}}^{(n-1)/2} 2 \cdot \overline{\chi_t}.$$

(c) *If $n = 3$, then $\chi_n^A(C(G)) = \overline{\chi_1}$.*

P r o o f. Let

$$\iota : P_n^A \to \frac{P_n}{P_n \cap C(G)}$$

be the A_n-homomorphism defined by $\iota(u) = u + (P_n \cap C(G))$. Since $\ker(\iota) = P_n^A \cap C(G)$ we have the isomorphism of A_n-modules:

$$\frac{P_n^A}{P_n^A \cap C(G)} \cong \iota(P_n^A).$$

If n is even then by the Theorem 11 and Theorem 3 we have the isomorphism of A_n-modules

$$\frac{P_n^A}{P_n^A \cap C(G)} \cong \frac{P_n}{P_n \cap C(G)}.$$

Therefore

$$\chi_n^A(C(G)) = \overline{\chi_n(C(G))} = \sum_{\substack{t=1 \\ t \text{ odd}}}^{n} \overline{\chi_t}$$

by Theorem 14.

Suppose $n \geq 5$ odd. If t is odd and $1 \leq t \leq (n-1)/2$ then $\lambda_t \neq \lambda_t' = \lambda_{n-t+1}$. Hence

$$\overline{\chi_t} = \overline{\chi_{n-t+1}}.$$

If $t = (n+1)/2$ then

$$\overline{\chi_t} = \overline{\chi_t}^+ + \overline{\chi_t}^-.$$

By Theorem 14 we have

$$\overline{\chi_n(C(G))} = \overline{\chi_{(n+1)/2}}^+ + \overline{\chi_{(n+1)/2}}^- + \sum_{\substack{t=1 \\ t \text{ odd}}}^{(n-1)/2} 2 \cdot \overline{\chi_t}. \tag{16}$$

Hence

$$\chi_n^A(C(G)) = a^+ \cdot \overline{\chi_{(n+1)/2}}^+ + a^- \cdot \overline{\chi_{(n+1)/2}}^- + \sum_{\substack{t=1 \\ t \text{ odd}}}^{(n-1)/2} a_t \cdot \overline{\chi_t}, \tag{17}$$

where $0 \leq a^+, a^- \leq 1$ and $0 \leq a_t \leq 2$. Let d_t be the degree of character χ_t. By

(16) and Theorem 14 we have the following

$$2^{n-2} = \frac{d_{(n+1)/2}}{2} + \frac{d_{(n+1)/2}}{2} + 2d_1 + \sum_{\substack{t=3 \\ t\,\mathrm{odd}}}^{(n-1)/2} 2d_t.$$

By (17) and Theorem 3 we have

$$2^{n-2} - 1 = a^+ \frac{d_{(n+1)/2}}{2} + a^- \frac{d_{(n+1)/2}}{2} + a_1 d_1 + \sum_{\substack{t=3 \\ t\,\mathrm{odd}}}^{(n-1)/2} a_t d_t.$$

Since

$$d_t = \binom{n-1}{t-1}$$

we have $d_1 = d_n = 1$ and $d_t \geq 4$ otherwise. Hence $a^+ = a^- = a_1 = 1$ and $a_t = 2$ otherwise.

We leave to the reader the proof of the case $n = 3$. $\square$

Acknowledgement. The authors would like to thank Plamen Koshlukov for proposing the study of the A_n-cocharacters of $C(G)$.

REFERENCES

[1] C. Bekh-Ochir, S. A. Rankin. The central polynomials of the infinite dimensional unitary and nonunitary Grassmann algebras. *J. Algebra Appl.* **9** (2010), 687–704.

[2] A. Brandão Jr., P. Koshlukov, A. Krasilnikov, E. A. Silva. The central polynomials for the Grassmann algebra. *Israel J. Math.* **179** (2010), 127–144.

[3] V. Drensky. Free Algebras and PI-Algebras. Graduate Course in Algebra. Singapore, Springer, 2000.

[4] A. Giambruno, P. Koshlukov. On the identities of the Grassmann algebras in characteristic $p > 0$. *Israel J. Math.* **122** (2001), 305–316.

[5] D. J. Gonçalves, P. Koshlukov. A-identities for the Grassmann algebra: the conjecture of Henke and Regev. *Proc. Amer. Math. Soc.* **136** (2008), 2711–2717.

[6] A. V. Grishin. On the structure of the centre of a relatively free Grassmann algebra. *Uspekhi Mat. Nauk* **65**, *4* (2010), 191–192 (in Russian); English translation in *Russ. Math. Surv.* **65** (2010), 781–782.

[7] A. Henke, A. Regev. A-codimensions and A-cocharacters. *Israel J. Math.* **133** (2003), 339–355.

[8] G. D. James, A. Kerber. The Representation Theory of the Symmetric Group. Encyclopedia of Mathematics and its Applications, vol. **16**, Addison-Wesley, 1981.

[9] D. Krakowski, A. Regev. The polynomial identities of the Grassmann algebra. *Trans. Amer. Math. Soc.* **181** (1973), 429–438.

[10] V. N. Latyshev. On the choice of basis in a T-ideal. *Sibirsk. Mat. Zh.* **4** (1963), 1122–1126 (in Russian).

[11] J. B. Olsson, A. Regev. An application of representation theory to PI-algebras. *Proc. Amer. Math. Soc.* **55** (1976), 253–257.

Antônio Pereira Brandão Jr.
UAME/CCT, UFCG, P.O.Box 10044
58109-970 Campina Grande, PB, Brazil
e-mail: brandao@dme.ufcg.edu.br

Dimas José Gonçalves
Departamento de Matemática
Universidade de Brasília
70910-900 Brasília, DF, Brazil
e-mail: dimasjog@gmail.com

Received February 7, 2012