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# OUTER AUTOMORPHISMS OF LIE ALGEBRAS RELATED WITH GENERIC $2 \times 2$ MATRICES

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*Communicated by V. Drensky*

*We dedicate this paper to the 65th birthday of Yuri Bahturin.*

Abstract. Let $F_m = F_m(\text{var}(sl_2(K)))$ be the relatively free algebra of rank $m$ in the variety of Lie algebras generated by the algebra $sl_2(K)$ over a field $K$ of characteristic 0. Our results are more precise for $m = 2$ when $F_2$ is isomorphic to the Lie algebra $L$ generated by two generic traceless $2 \times 2$ matrices. We give a complete description of the group of outer automorphisms of the completion $\widehat{L}$ of $L$ with respect to the formal power series topology and of the related associative algebra $\widehat{W}$. As a consequence we obtain similar results for the automorphisms of the relatively free algebra $F_2/F_2^{c+1} = F_2(\text{var}(sl_2(K)) \cap \mathfrak{N}_c)$ in the subvariety of $\text{var}(sl_2(K))$ consisting of all nilpotent algebras of class at most $c$ in $\text{var}(sl_2(K))$ and for $W/W^{c+1}$. We show that such automorphisms are $\mathbb{Z}_2$-graded, i.e., they map the linear combinations of elements of odd, respectively even degree to linear combinations of the same kind.

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**Introduction.** Let $L_m$ be the free Lie algebra of rank $m \geq 2$ over a field $K$ of characteristic 0 and let $G$ be an arbitrary Lie algebra. Let $I(G) = I_m(G)$ be the ideal of $L_m$ consisting of all Lie polynomial identities in $m$ variables for the algebra $G$. The factor algebra $F_m(G) = F_m(\text{var}(G)) = L_m/I(G)$ is the relatively free Lie algebra of rank $m$ in the variety of Lie algebras generated by $G$. Typical examples of relatively free algebras are free solvable of class $k$ Lie algebras when $I(G) = L_m^{(k)}$ (e.g., free metabelian Lie algebras with $I(G) = L_m''$), free nilpotent of class $c$ Lie algebras when $I(G) = L_m^{c+1}$, relatively free algebras in a variety generated by a finite dimensional simple Lie algebra $G$, etc. See the books by Bahturin [1] and Mikhalev, Shpilrain and Yu [11] for a background on relatively free Lie algebras and their automorphisms, respectively.

Cohn [3] showed that every automorphism of the free Lie algebra $L_m$ is tame. In particular, the group of automorphisms $\text{Aut}(L_2)$ is isomorphic to the general linear group $GL_2(K)$. Quite often relatively free algebras $F_m(G)$ possess wild automorphisms and for better understanding of the group $\text{Aut}(L_m/I(G))$ one studies its important subgroups.

When we consider a finite dimensional simple Lie algebra $G$ over $\mathbb{C}$, the general theory gives that the series

$$\exp(\text{ad}\, g) = \sum_{n\geq 0} \frac{(\text{ad}\, g)^n}{n!}$$

which defines inner automorphisms converges for all $g \in G$. Studying the inner automorphisms of a relatively free Lie algebra, the first problem arising is that the formal power series defining inner automorphisms has to be well defined. This means that the operator $\text{ad}\, z$, $z \in F_m(G)$, has to be locally nilpotent. In many important cases $\text{ad}\, z$ is not locally nilpotent for all $z \in F_m(G)$. Hence we have two possibilities to study the inner automorphisms:

(1) to restrict the consideration to the locally nilpotent derivations $\text{ad}\, z$,

or

(2) to consider nilpotent relatively free algebras $F_m(G)/F_m^{c+1}(G) = L_m/(I(G) + L_m^{c+1})$ when $\exp(\text{ad}\, z)$ is well defined for all $z \in F_m(G)/F_m^{c+1}(G)$. Hence the group of inner automorphisms $\text{Inn}(F_m(G)/F_m^{c+1}(G))$ of the algebra $F_m(G)/F_m^{c+1}(G)$ is also defined.

In the latter case it is more convenient to consider the formal power series topology on $F_m = F_m(G)$ and to work in the completion $\widehat{F_m}$ of $F_m$. Then we restrict our considerations to the group $\text{Aut}(\widehat{F_m})$ of the continuous automorphisms of $\widehat{F_m}$. Clearly, it is sufficient to define the automorphisms in $\text{Aut}(\widehat{F_m})$ on the generators of $F_m \subset \widehat{F_m}$.

Baker [2] evaluated the Baker-Campbell-Hausdorff series on several finite dimensional Lie algebras given in their adjoint representations, including the three-dimensional simple Lie algebra $G_3$. In [5] Drensky and the author translated the results of Baker in the language of relatively free algebras and gave a complete description of the group of inner automorphisms of the completion $\widehat{F_2}$ of $F_2 = F_2(sl_2(K)) = F_2(G_3)$ with respect to the formal power series topology. The results on $\mathrm{Inn}(F_2/F_2^{c+1})$ were obtained immediately from the corresponding results on $\mathrm{Inn}(\widehat{F_2})$. In particular, [5] contains a multiplication rule for the inner automorphisms of $\widehat{F_2}$.

Although the structure of $F_m(sl_2(K))$ is known for all $m \geq 2$, we consider the case $m = 2$ only because the case $m > 2$ is more complicated than for $m = 2$. We work in the completion $\widehat{W}$ of the associative algebra $W$ generated by two generic traceless $2 \times 2$ matrices $x = (x_{ij})$ and $y = (y_{ij})$, where $x_{ij}$, $y_{ij}$, $(i,j) = (1,1),(1,2),(2,1)$, are algebraically independent commuting variables, $x_{22} = -x_{11}$, $y_{22} = -y_{11}$. Let $L$ be the Lie subalgebra of $W$ generated by $x$ and $y$. Then $L \cong F_2(sl_2(K))$.

For any Lie algebra $G$ the group $\mathrm{Aut}(F_m(G))$ is a semidirect product of the normal subgroup $\mathrm{IA}(F_m(G))$ of the automorphisms which induce the identity map modulo the commutator ideal of $F_m(G)$ and the general linear group $\mathrm{GL}_m(K)$. The group of inner automorphisms $\mathrm{Inn}(F_m(G))$ is contained in $\mathrm{IA}(F_m(G))$. Hence for the description of the factor group $\mathrm{Out}(\widehat{L}) = \mathrm{Aut}(\widehat{L})/\mathrm{Inn}(\widehat{L})$ it is sufficient to know only $\mathrm{IA}(\widehat{L})/\mathrm{Inn}(\widehat{L})$. We give the explicit form of the coset representatives of the continuous outer automorphisms in $\mathrm{IOut}(\widehat{L})$ and also for $\mathrm{IOut}(\widehat{W})$ and then we transfer the obtained results to the algebra $L/L^{c+1}$ and $W/W^{c+1}$ in order to obtain the description of $\mathrm{IOut}(L/L^{c+1})$ and $\mathrm{IOut}(W/W^{c+1})$.

**1. Preliminaries.** We fix a field $K$ of characteristic 0 and the associative algebra $W$ generated by two generic traceless $2 \times 2$ matrices

$$x = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & -x_{11} \end{pmatrix}, \quad y = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & -y_{11} \end{pmatrix},$$

where $x_{ij}$, $y_{ij}$, $(i,j) = (1,1),(1,2),(2,1)$, are algebraically independent commuting variables. We assume that $W$ is a subalgebra of the $2 \times 2$ matrix algebra $M_2(K[x_{ij}, y_{ij}])$ and identify the polynomial $f \in K[x_{ij}, y_{ij}]$ with the scalar matrix with entries $f$ on the diagonal. In particular, for any matrix $z \in W$ we assume that the trace $\mathrm{tr}(z)$ belongs to the centre of $M_2(K[x_{ij}, y_{ij}])$. Let $L$ be the Lie subalgebra of $W$ generated by $x$ and $y$. This is the smallest subspace of the vector space $W$ containing $x$ and $y$ and closed with respect to the Lie multiplication

$$[z_1, z_2] = z_1 \operatorname{ad} z_2 = z_1 z_2 - z_2 z_1, \quad z_1, z_2 \in L.$$

Similarly we define the associative algebra $W_m$ generated by $m \geq 2$ generic traceless $2 \times 2$ matrices. We assume that all commutators are left normed, i.e.,

$$[z_1, \ldots, z_{n-1}, z_n] = [[z_1, \ldots, z_{n-1}], z_n], \quad n = 3, 4, \ldots.$$

The following results give the description of the algebras $W_m$, $W = W_2$ and $L$ and some equalities in $W$.

**Theorem 1.** *Let $W_m$, $W$ and $L$ be as above. Then:*

(i) (Razmyslov [12]) *The algebra of generic traceless matrices $W_m$ is isomorphic to the factor-algebra $K\langle x_1, \ldots, x_m\rangle / I(M_2(K), sl_2(K))$ of the free associative algebra $K\langle x_1, \ldots, x_m\rangle$, where the ideal $I(M_2(K), sl_2(K))$ of the weak polynomial identities in $m$ variables for the pair $(M_2(K), sl_2(K))$ consists of all polynomials from $K\langle x_1, \ldots, x_m\rangle$ which vanish on $sl_2(K)$ considered as a subset of $M_2(K)$. As a weak $T$-ideal $I(M_2(K), sl_2(K))$ is generated by the weak polynomial identity $[x_1^2, x_2] = 0$. The Lie subalgebra of $W_m$ generated by the $m$ generic traceless matrices is isomorphic to the relatively free algebra $F_m(sl_2(K))$ in the variety of Lie algebras generated by $sl_2(K)$.*

(ii) (Drensky and Koshlukov [7], see also the comments in [4] and Koshlukov [8, 9] for the case of positive characteristic) *The algebra $W_m$ has the presentation*

$$W_m \cong K\langle x_1, \ldots, x_m \mid [x_i^2, x_j] = [x_i x_j + x_j x_i, x_k] = s_4(x_{i_1}, x_{i_2}, x_{i_3}, x_{i_4}) = 0\rangle,$$

*where $i, j, k, i_l = 1, \ldots, m$, $i \neq j$, $i_1 < i_2 < i_3 < i_4$, and*

$$s_4(x_1, x_2, x_3, x_4) = \sum_{\sigma \in S_4} \operatorname{sign}(\sigma) x_{\sigma(1)} x_{\sigma(2)} x_{\sigma(3)} x_{\sigma(4)}$$

*is the standard polynomial of degree 4. In particular,*

$$W \cong K\langle x_1, x_2 \mid [x_1^2, x_2] = [x_2^2, x_1] = 0\rangle.$$

(iii) (see e.g., Le Bruyn [10]) *The centre of $W$ is generated by*

$$t = \operatorname{tr}(x^2), \quad u = \operatorname{tr}(y^2), \quad v = \operatorname{tr}(xy).$$

*The elements $t, u, v$ are algebraically independent and $W$ is a free $K[t, u, v]$-module with free generators $1, x, y, [x, y]$.*

(iv) (see e.g., Drensky and Gupta [6]) *For $k \geq 1$ the following equalities hold in $W$:*

$$x^2 = \frac{t}{2}; \quad y^2 = \frac{u}{2}; \quad xy + yx = v; \quad [x, y]^2 = v^2 - tu;$$

$$y \operatorname{ad}^{2k} x = 2^k t^{k-1}(-vx + ty); \quad y \operatorname{ad}^{2k+1} x = 2^k t^k [y, x];$$

$$x \operatorname{ad}^{2k} y = 2^k u^{k-1}(ux - vy); \quad x \operatorname{ad}^{2k+1} y = 2^k u^{k-1}[x, y].$$

Theorem 1 (iii) and (iv) gives immediately that $L$ is embedded into the free $K[t, u, v]$-module with free generators $x, y, [x, y]$. The next lemma gives the precise description of the Lie elements in $W$. It also provides an algorithm how to express in Lie form the elements of $L$ given as elements of the free $K[t, u, v]$-module with basis $x, y, [x, y]$.

**Lemma 2** ([5])**.** (i) *The commutator ideal $L'$ of $L \cong F_2(sl_2)$ is a free $K[t, u, v]$-module of rank 3, with free generators*

$$xv - yt, \quad xu - yv, \quad [x, y].$$

(ii) *The elements of*

$$L' = (xv - yt)K[t, u, v] \oplus (xu - yv)K[t, u, v] \oplus [x, y]K[t, u, v]$$

*can be expressed in Lie form using the identities*

$$2^{a+b+c+1}(xv - yt)t^a u^b v^c = [x, y, y](\operatorname{ad} y)^{2b-1}(\operatorname{ad} x)^{2a+1}(\operatorname{ad} y \operatorname{ad} x)^c, \quad b > 0,$$

$$2^{a+c+1}(xv - yt)t^a v^c = [x, y, x](\operatorname{ad} x)^{2a}(\operatorname{ad} y \operatorname{ad} x)^c,$$

$$2^{a+b+c+1}(xu - yv)t^a u^b v^c = [x, y, x](\operatorname{ad} x)^{2a-1}(\operatorname{ad} y)^{2b+1}(\operatorname{ad} x \operatorname{ad} y)^c, \quad a > 0,$$

$$2^{b+c+1}(xu - yv)u^b v^c = [x, y, y](\operatorname{ad} y)^{2b}(\operatorname{ad} x \operatorname{ad} y)^c,$$

$$2^{a+b+c}[x, y]t^a u^b v^c = [x, y](\operatorname{ad} x)^{2a}(\operatorname{ad} y)^{2b}(\operatorname{ad} x \operatorname{ad} y)^c.$$

Let $R$ be a (not necessarily associative) graded $K$-algebra,

$$R = \bigoplus_{n \geq 0} R_{(n)} = R_{(0)} \oplus R_{(1)} \oplus R_{(2)} \oplus \cdots ,$$

where $R_{(n)}$ is the homogeneous component of degree $n$ in $R$, and $R_{(0)} = 0$ or $R_{(0)} = K$. We consider the *formal power series topology* on $R$ induced by the filtration

$$\omega^0(R) \supseteq \omega^1(R) \supseteq \omega^2(R) \supseteq \cdots, \quad \omega^n(R) = \bigoplus_{k\geq n} R_{(k)}, \quad n = 0,1,2,\ldots,$$

where $\omega(R) = R$ if $R_0 = 0$, and $\omega(R)$ is the augmentation ideal of $R$ when $R_0 = K$. This is the topology in which the sets

$$r + \omega^n(R), \quad r \in R, \quad n \geq 0,$$

form a basis for the open sets. We shall denote by $\widehat{R}$ the completion of $R$ with respect to the formal power series topology and shall identify it with the Cartesian sum $\widehat{\bigoplus}_{n\geq 0} R_{(n)}$. The elements $f \in \widehat{R}$ are formal power series

$$f = f_0 + f_1 + f_2 + \cdots, \quad f_n \in R_{(n)}, \quad n = 0,1,2,\ldots,$$

A sequence

$$f^{(k)} = f_{k0} + f_{k1} + f_{k2} + \cdots, \quad k = 1,2,\ldots,$$

where $f_{kn} \in R_{(n)}$, converges to $f = f_0 + f_1 + f_2 + \cdots$, where $f_n \in R_{(n)}$, if for every $n_0$ there exists a $k_0$ such that $f_{kn} = f_n$ for all $n < n_0$ and all $k \geq k_0$, i.e., for all sufficiently large $k$ the first $n_0$ terms of the formal power series $f^{(k)}$ are the same as the first $n_0$ terms of $f$.

Let $F_m = F_m(G)$ be a relatively free algebra freely generated by $x_1, \ldots, x_m$. Then $F_m$ is graded and the $n$th homogeneous component is spanned by all commutators $[x_{i_1}, \ldots, x_{i_n}]$ of length $n$. Hence the elements of $\widehat{F_m}$ are formal series of commutators. Since $[F_m^n, u] = F_m^n \operatorname{ad} u \subset F_m^{n+1}$ for any $u \in F_m$, we derive that the inner automorphisms $\exp(\operatorname{ad} u)$ of $\widehat{F_m}$ are continuous automorphisms and hence it is sufficient to define them on the generators only.

Let $W_{(n)}$ be the subspace of $W$ spanned by all monomials of total degree $n$ in $x, y$. The elements $f \in \widehat{W}$ are formal power series

$$f = f_0 + f_1 + f_2 + \cdots, \quad f_n \in W_{(n)}, \quad n = 0,1,2,\ldots,$$

and $\widehat{W}$ is a free $K[[t,u,v]]$-module with free generators $1, x, y, [x,y]$, where $K[[t,u,v]]$ is the algebra of formal power series in the variables $t, u, v$. Since $\widehat{L}$ is embedded canonically into $\widehat{W}$, Lemma 2 gives that $(\widehat{L})'$ is a free $K[[t,u,v]]$-module with free generators $xv - yt, xu - yv, [x,y]$ and

$$\widehat{L} = \{\alpha x + \beta y + a(xv - yt) + b(xu - yv) + c[x,y] \mid \alpha, \beta \in K, a, b, c \in K[[t,u,v]]\}.$$

Let us denote by $\omega$ the augmentation ideal of the polynomial algebra $K[t,u,v]$ consisting of the polynomials without constant terms and let $\widehat{\omega} \subset K[[t,u,v]]$ be its completion with respect to the formal power formal series. Now we give the next lemma which will be needed in the further proofs.

**Lemma 3.** *Let $a,b,c \in \widehat{\omega}$ and let*

$$f = \frac{1}{\sqrt{c}} \log\left(\frac{1+a+b\sqrt{c}}{1+a-b\sqrt{c}}\right).$$

*Then $f \in K[[t,u,v]]$.*

Proof. Recall that

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}x^n}{n}.$$

Now we have that

$$\begin{aligned}
\log\left(\frac{1+a+b\sqrt{c}}{1+a-b\sqrt{c}}\right) &= \log(1+a+b\sqrt{c}) - \log(1+a-b\sqrt{c}) \\
&= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n}((a+b\sqrt{c})^n - (a-b\sqrt{c})^n) \\
&= \sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{j=1}^{n} \binom{n}{j} a^{n-j} b^j (\sqrt{c})^j (1-(-1)^j) \\
&= 2\sum_{n\geq 1} \frac{(-1)^{n+1}}{n} \sum_{j=1, j odd}^{n} \binom{n}{j} a^{n-j} b^j (\sqrt{c})^j.
\end{aligned}$$

Since $a,b,c \in \widehat{\omega}$, the logarithm $\log\left(\frac{a+b\sqrt{c}}{a-b\sqrt{c}}\right)$ is well defined and divisible by $\sqrt{c}$. Hence $f$ contains only even powers of $\sqrt{c}$ which completes the proof. □

If $\delta$ is an endomorphism of the free $K[[t,u,v]]$-submodule of $\widehat{W}$ with basis $\{x,y,[x,y]\}$, then we denote by $\mathrm{M}(\delta)$ the *associated* matrix of $\delta$ with respect to this basis. If

$$\delta(x) = \sigma_{11}x + \sigma_{21}y + \sigma_{31}[x,y],$$

$$\delta(y) = \sigma_{12}x + \sigma_{22}y + \sigma_{32}[x,y],$$

$$\delta([x,y]) = \sigma_{13}x + \sigma_{23}y + \sigma_{33}[x,y],$$

$\sigma_{ij} \in K[[t,u,v]]$, then

$$\mathrm{M}(\delta) = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}.$$

Clearly $\mathrm{M}(\delta)$ behaves as a matrix of a usual linear operator. In particular,

$$\mathrm{M}(\delta_1\delta_2) = \mathrm{M}(\delta_1)\mathrm{M}(\delta_2).$$

Since the derivation $\operatorname{ad} X$, $X \in \widehat{W}$, acts trivially on the centre of $\widehat{W}$, it is an endomorphism of $\widehat{W}$ as a $K[[t,u,v]]$-module. Its restriction on the submodule generated by $x, y, [x,y]$ satisfies the above conditions. Hence the matrix $\mathrm{M}(\operatorname{ad} X)$ is well defined, and similarly for the matrix $\mathrm{M}(\exp(\operatorname{ad} X))$.

Let $\mathrm{Inn}(\widehat{W})$ denote the set of all inner automorphisms of $\widehat{W}$ which are of the form $\exp(\operatorname{ad} X)$, $X \in \widehat{W}$. As we already discussed, since $\widehat{W}$ is a $K[[t,u,v]]$-module with the generators $1, x, y, [x,y]$ and $\operatorname{ad} X$ acts trivially on 1 it is sufficient to know the action of inner automorphisms only on $x, y, [x,y]$.

**Theorem 4** ([5]). *Let $X = ax + by + c[x,y]$, $a,b,c \in K[[t,u,v]]$, be an element in $\widehat{W}$. Then the associated matrix of* $\exp(\operatorname{ad} X)$ *is of the form*

$$\mathrm{M}(\exp(\operatorname{ad} X)) = I_3 + A(X)\mathrm{M}(\operatorname{ad} X) + B(X)\mathrm{M}^2(\operatorname{ad} X),$$

*where*

$$\mathrm{M}(\operatorname{ad} X) = \begin{pmatrix} -2cv & -2cu & 2(av+bu) \\ 2ct & 2cv & -2(at+bv) \\ b & -a & 0 \end{pmatrix},$$

$$\mathrm{M}^2(\operatorname{ad} X) =$$

$$= \begin{pmatrix} 4c^2w + 2b(av+bu) & -2a(av+bu) & -4acw \\ -2b(at+bv) & 4c^2w + 2a(at+bv) & -4bcw \\ -2c(at+bv) & -2c(av+bu) & 2a(at+bv) + 2b(av+bu) \end{pmatrix},$$

$$A(X) = \frac{\sinh(\sqrt{g(X)})}{\sqrt{g(X)}}, \quad B(X) = \frac{\cosh(\sqrt{g(X)}) - 1}{g(X)},$$

$$g(X) = 2(a^2t + 2abv + b^2u + 2c^2(v^2 - tu), \quad w = v^2 - tu.$$

For any Lie algebra $G$ the group $\operatorname{Aut}(F_m(G))$ is a semidirect product of the normal subgroup $\operatorname{IA}(F_m(G))$ of the automorphisms which induce the identity map modulo the commutator ideal of $F_m(G)$ and the general linear group $\mathrm{GL}_m(K)$. The automorphisms are contained in $\operatorname{IA}(F_m(G))$. Similarly, let $\operatorname{Aut}(\widehat{L})$ and $\operatorname{IA}(\widehat{L})$ be, respectively, the group of continuous automorphisms of $\widehat{L}$ and its subgroup of continuous IA-automorphisms of $\widehat{L}$. For the description of the factor group $\operatorname{Out}(\widehat{L}) = \operatorname{Aut}(\widehat{L})/\operatorname{Inn}(\widehat{L})$ of continuous outer automorphisms of $\widehat{L}$, it is sufficient to know only $\operatorname{IA}(\widehat{L})/\operatorname{Inn}(\widehat{L})$.

Now let $\delta$ be an IA-automorphism of $\widehat{L}$. Then $\delta$ is of the form

$$\begin{aligned}\delta : x &\to x + a_1(xv - yt) + b_1(xu - yv) + c_1[x, y]\\ y &\to y + a_2(xv - yt) + b_2(xu - yv) + c_2[x, y]\end{aligned}$$

where $a_1, a_2, b_1, b_2, c_1, c_2 \in K[[t, u, v]]$. We define the matrix of $\delta$ as

$$\widehat{\delta} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ a_1 & a_2 & 1 + a_1v - a_2t & a_1u - a_2v & p_1 \\ b_1 & b_2 & b_1v - b_2t & 1 + b_1u - b_2v & p_2 \\ c_1 & c_2 & c_1v - c_2t & c_1u - c_2v & p_3 \end{pmatrix},$$

where

$$p_1 = 2c_1(a_2v + b_2u) - 2c_2(1 + a_1v + b_1u),$$

$$p_2 = 2c_1(1 - a_2t - b_2u) + 2c_2(a_1t + b_1v),$$

$$p_3 = (a_1v - a_2t) + (b_1u - b_2v) + (a_2b_1 - a_1b_2)w,$$

$$w = (v^2 - tu).$$

In the expression of $\widehat{\delta}$, the first two columns are the coordinates of $\delta(x)$ and $\delta(y)$ and the other three columns are the coordinates of the image of the basis of the completion of $L'$.

Now we define the *related* matrix $\mathrm{N}(\delta)$ of $\delta$ as below:

$$\mathrm{N}(\delta) = \begin{pmatrix} 1 + a_1v - a_2t & a_1u - a_2v & p_1 \\ b_1v - b_2t & 1 + b_1u - b_2v & p_2 \\ c_1v - c_2t & c_1u - c_2v & p_3 \end{pmatrix},$$

which counts the coordinates of the image of the basis of the completion of $L'$ only. Let $\delta_1, \delta_2$ be two IA-automorphisms of $\widehat{L}$. One can easily check that the matrix $\widehat{\delta_1\delta_2}$ of the composition $\delta_1\delta_2$ is determined by $\mathrm{N}(\delta_1)\mathrm{N}(\delta_2)$. Then it is sufficient to work on the related matrices only.

Now we state a technical lemma which gives the relation between *associated* and *related* matrices of IA-automorphisms of $\widehat{L}$. The proof is straightforward.

**Lemma 5.** *Let $\delta$ be an* IA-*automorphism of $\widehat{L} \subset \widehat{W}$ with associated matrix of the form*

$$\mathrm{M}(\delta) = \begin{pmatrix} 1+\alpha_1 & \alpha_2 & \sigma_1 \\ \beta_1 & 1+\beta_2 & \sigma_2 \\ \gamma_1 & \gamma_2 & \sigma_3 \end{pmatrix},$$

$$\sigma_1 = 2\gamma_1(\alpha_2 v + (1+\beta_2)u) - 2\gamma_2((1+\alpha_1)v + \beta_1 u),$$

$$\sigma_1 = -2\gamma_1(\alpha_2 t + (1+\beta_2)v) + 2\gamma_2((1+\alpha_1)t + \beta_1 v),$$

$$\sigma_3 = (1+\alpha_1)(1+\beta_2) - \alpha_2\beta_1,$$

*for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \widehat{\omega}$, $\gamma_1, \gamma_2 \in K[[t,u,v]]$. Then the related matrix $\mathrm{N}(\delta)$ of $\delta$ is*

$$\mathrm{N}(\delta) = \begin{pmatrix} 1+a_1 & a_2 & \frac{1}{w}(\sigma_1 v + \sigma_2 u) \\ b_1 & 1+b_2 & -\frac{1}{w}(\sigma_1 t + \sigma_2 v) \\ \gamma_1 v - \gamma_2 t & \gamma_1 u - \gamma_2 v & \sigma_3 \end{pmatrix},$$

*where $w = v^2 - tu$,*

$$a_1 = \frac{1}{w}\left((\alpha_1 v - \alpha_2 t)v + (\beta_1 v - \beta_2 t)u\right),$$

$$a_2 = \frac{1}{w}\left((\alpha_1 u - \alpha_2 v)v + (\beta_1 u - \beta_2 v)u\right),$$

$$b_1 = -\frac{1}{w}\left((\alpha_1 v - \alpha_2 t)t + (\beta_1 v - \beta_2 t)v\right),$$

$$b_2 = -\frac{1}{w}\left((\alpha_1 u - \alpha_2 v)t + (\beta_1 u - \beta_2 v)v\right).$$

**2. Outer automorphisms of associative algebras of two generic matrices.** In this section we describe the group $\mathrm{IOut}(\widehat{W}) = \mathrm{IA}(\widehat{W})/\operatorname{Inn}(\widehat{W})$ of outer IA-automorphisms of $\widehat{W}$, where $\mathrm{Aut}(\widehat{W})$ is the group of continuous automorphisms of $\widehat{W}$. For this purpose we find the explicit form of the associated matrix of the outer IA-automorphisms of $\widehat{W}$ and then we transfer the obtained results to the algebra $W/W^{c+1}$ and obtain the description of $\operatorname{Inn}(W/W^{c+1})$.

**Lemma 6.** *Let $\theta$ be a continuous automorphism of $\widehat{W}$. Then $\theta$ is of the form*

$$\theta : x \to a_1 x + b_1 y + c_1[x, y]$$

$$y \to a_2 x + b_2 y + c_2[x, y]$$

*where $a_1, a_2, b_1, b_2, c_1, c_2 \in K[[t, u, v]]$.*

P r o o f. Let $\theta$ be a continuous automorphism of $\widehat{W}$. Since $W$ is a free $K[t, u, v]$-module with free generators $1, x, y, [x, y]$, then $\theta$ is of the form

$$\theta : x \to \alpha + a_1 x + b_1 y + c_1[x, y]$$

$$y \to \beta + a_2 x + b_2 y + c_2[x, y]$$

where $\alpha, \beta \in K$ and $a_1, a_2, b_1, b_2, c_1, c_2 \in K[[t, u, v]]$. Thus the relations

$$(\theta(x))^2 \equiv 0 \pmod{K[[t, u, v]]}, \quad (\theta(y))^2 \equiv 0 \pmod{K[[t, u, v]]})$$

hold true, because $x^2 = t/2$ ad $y^2 = u/2$ are in the center $K[[t, u, v]]$ of $\widehat{W}$. Using the notation $(\operatorname{mod} K[[t, u, v]])$ we mean working in the vector space $\widehat{W}$ modulo the subspace $K[[t, u, v]] = K[[t, u, v]] \cdot 1$. Then we have that

$$\begin{aligned}(\theta(x))^2 &= (\alpha + a_1 x + b_1 y + c_1[x, y])^2 \\ &= \alpha^2 + a_1^2 x^2 + b_1^2 y^2 + c_1^2 [x, y]^2 + a_1 b_1 (xy + yx) + a_1 c_1 (x[x, y] + [x, y]x) \\ &+ b_1 c_1 (y[x, y] + [x, y]y) + 2\alpha(a_1 x + b_1 y + c_1[x, y])\end{aligned}$$

which implies that $\alpha(a_1 x + b_1 y + c_1[x, y]) \in K[[t, u, v]]$ and so $\alpha = 0$. Similarly one can check that $\beta = 0$. $\square$

**Corollary 7.** *Let $\theta$ be an* IA*-automorphism of $\widehat{W}$. Then $\theta$ is of the form*

$$\theta : x \to x + a_1 x + b_1 y + c_1[x, y]$$
$$y \to y + a_2 x + b_2 y + c_2[x, y]$$

*where* $a_1, a_2, b_1, b_2 \in \widehat{\omega}$ *and* $c_1, c_2 \in K[[t,u,v]]$.

Now we shall find the coset representatives of the normal subgroup $\mathrm{Inn}(\widehat{W})$ of the group $\mathrm{IA}(\widehat{W})$ of IA-automorphisms $\widehat{W}$, i.e., we shall find a set of IA-automorphisms $\theta$ of $\widehat{W}$ such that the factor group $\mathrm{IOut}(\widehat{W}) = \mathrm{IA}(\widehat{W})/\,\mathrm{Inn}(\widehat{W})$ of the outer IA-automorphisms of $\widehat{W}$ is presented as the disjoint union of the cosets $\mathrm{Inn}(\widehat{W})\theta$.

**Theorem 8.** *Let* $\Theta$ *be the set of automorphisms* $\theta$ *of* $\widehat{W}$ *with associated matrix of the form*

$$\mathrm{M}(\theta) = \begin{pmatrix} 1+a & b_1 & 0 \\ 0 & 1+b_2 & 0 \\ 0 & 0 & (1+a)(1+b_2) \end{pmatrix},$$

*where* $a, b_1, b_2 \in \widehat{\omega}$ *are formal power series without constant terms. Then* $\Theta$ *consists of coset representatives of the subgroup* $\mathrm{Inn}(\widehat{W})$ *of the group* $\mathrm{IA}(\widehat{W})$ *and* $\mathrm{IOut}(\widehat{W})$ *is a disjoint union of the cosets* $\mathrm{Inn}(\widehat{W})\theta$, $\theta \in \Theta$.

P r o o f. Let

$$A = \begin{pmatrix} 1+a & b_1 & 0 \\ 0 & 1+b_2 & 0 \\ 0 & 0 & (1+a)(1+b_2) \end{pmatrix},$$

where $a, b_1, b_2 \in \widehat{\omega}$ be an $3 \times 3$ matrix satisfying the conditions of the theorem. Applying Corollary 7, it is clear that $A$ is the associative matrix of a certain IA-automorphism of $\widehat{W}$.

Now we shall show that for any $\psi \in \mathrm{IA}(\widehat{W})$ there exists an inner automorphism $\phi = \exp(\mathrm{ad}\, u) \in \mathrm{Inn}(\widehat{W})$ and an automorphism $\theta$ in $\Theta$ such that $\psi = \exp(\mathrm{ad}\, u) \cdot \theta$. Let $\psi$ be an arbitrary element of $\mathrm{IA}(\widehat{W})$ and let

$$\mathrm{M}(\psi) = \begin{pmatrix} 1+a_1 & b_1 & 2v(c_1a_2 - c_2(1+a_1)) + 2u(c_1(1+b_2) - c_2b_1) \\ b_1 & 1+b_2 & -2t(c_1a_2 - c_2(1+a_1)) - 2v(c_1(1+b_2) - c_2b_1) \\ c_1 & c_2 & (1+a_1)(1+b_2) - a_2b_1 \end{pmatrix},$$

where $a_1, a_2, b_1, b_2 \in \widehat{\omega}$ and $c_1, c_2 \in K[[t,u,v]]$.

Let us define

$$b = \frac{1}{2\sqrt{2u}} \log\left(\frac{-1 - a_1 + c_1\sqrt{2u}}{-1 - a_1 - c_1\sqrt{2u}}\right)$$

and let

$$p = (1+a_1)bA(by) + (1 + 2b^2uB(by))c_1,$$

where

$$A(by) = \frac{\sinh(\sqrt{2b^2u})}{\sqrt{2b^2u}}, \quad B(by) = \frac{\cosh(\sqrt{2b^2u}) - 1}{2b^2u}.$$

Note that both expressions $-1-a_1+c_1\sqrt{2u}$ and $-1-a_1-c_1\sqrt{2u}$ can not be zero at the same time. We choose that $-1-a_1-c_1\sqrt{2u} \neq 0$ without loss of generality. After easy calculations we have that

$$\exp(2\sqrt{2b^2u}) = \frac{2p\sqrt{2u}\exp(\sqrt{2b^2u}) + 1 + a_1 - c_1\sqrt{2u}}{1 + a_1 + c_1\sqrt{2u}}$$

and

$$2p\sqrt{2u}\exp(\sqrt{2b^2u}) = 0.$$

Since the ring $K[[t,u,v]]$ is an integral domain, then $p = 0$. Now let us define

$$\phi_b = \exp(\operatorname{ad} by).$$

We know that $b \in K[[t,u,v]]$ from Lemma 3. Thus $\phi_b \in \operatorname{Inn}(\widehat{W})$. As a result, $\mathrm{M}(\phi_b\psi)$ is of the form

$$\mathrm{M}(\phi_b\psi) = \begin{pmatrix} 1+a_1' & b_1' & * \\ b_1' & 1+b_2' & * \\ 0 & c_2' & * \end{pmatrix},$$

where $a_1', a_2', b_1', b_2' \in \widehat{\omega}$, $c_2' \in K[[t,u,v]]$. Here we have denoted by $*$ the corresponding entries of the third column of $\mathrm{M}(\phi_b\psi)$.

Again let us define

$$c = \frac{1}{4\sqrt{w}} \log\left(\frac{(1+a_1')t + b_1'v + b_1'\sqrt{w}}{(1+a_1')t + b_1'v - b_1'\sqrt{w}}\right)$$

and let

$$q = 2ct(1+a_1')A(c[x,y]) + (1 + 2cvA(c[x,y]) + 4c^2wB(c[x,y]))b_1',$$

where

$$A(c[x,y]) = \frac{\sinh(\sqrt{4c^2w})}{\sqrt{4c^2w}}, \quad B(c[x,y]) = \frac{\cosh(\sqrt{4c^2w}) - 1}{4c^2w}.$$

Similarly $q = 0$, $c \in K[[t,u,v]]$ and $\phi_c = \exp(\operatorname{ad} c[x,y]) \in \operatorname{Inn}(\widehat{W})$. Calculating the matrix $\mathrm{M}(\phi_c\phi_b\psi)$ we have that

$$\mathrm{M}(\phi_c\phi_b\psi) = \begin{pmatrix} 1+a_1'' & b_1'' & * \\ 0 & 1+b_2'' & * \\ 0 & c_2'' & * \end{pmatrix},$$

for some $a_1'', b_1'', b_2'' \in \widehat{\omega}$, $c_2'' \in K[[t,u,v]]$. Note that $\mathrm{M}(\phi_c)$ preserves $(3,1)-th$ entry of the matrix $\mathrm{M}(\phi_b\psi)$ after calculation.

Finally let us define

$$a = \frac{1}{2\sqrt{2t}} \log\left(\frac{(1+b_2'')t + c_2''\sqrt{2t}}{(1+b_2'')t - c_2''\sqrt{2t}}\right)$$

and let

$$r = -a(1+b_2'')A(ax) + (1+2a^2tB(ax))c_2'',$$

where

$$A(ax) = \frac{\sinh(\sqrt{2a^2t})}{\sqrt{2a^2t}}, \quad B(ax) = \frac{\cosh(\sqrt{2a^2t})-1}{2a^2t}.$$

Similarly $r=0$, $a \in K[[t,u,v]]$ and $\phi_a = \exp(\mathrm{ad}\, ax) \in \mathrm{Inn}(\widehat{W})$. Calculating the matrix $\mathrm{M}(\phi_a\phi_c\phi_b\psi)$ we have that

$$\mathrm{M}(\phi_a\phi_c\phi_b\psi) = \begin{pmatrix} 1+a_1''' & b_1''' & 0 \\ 0 & 1+b_2''' & 0 \\ 0 & 0 & (1+a_1''')(1+b_2''') \end{pmatrix},$$

for some $a_1''', b_1''', b_2''' \in \widehat{\omega}$. Note that $\mathrm{M}(\phi_a)$ preserves both $(2,1)-th$ and $(3,1)-th$ entries of the matrix $\mathrm{M}(\phi_c\phi_b\psi)$ after calculation.

Hence, starting from an arbitrary coset of IA-automorphisms $\mathrm{Inn}(\widehat{W})\psi$, we found that it contains an automorphism $\theta \in \Theta$ with associated matrix prescribed in the theorem. Now, let $\theta_1$ and $\theta_2$ be two different automorphisms in $\Theta$ with $\mathrm{Inn}(\widehat{W})\theta_1 = \mathrm{Inn}(\widehat{W})\theta_2$. Hence, there exists a nonzero element $X = ax + by + c[x,y] \in \widehat{W}$ such that $\theta_1 = \exp(\mathrm{ad}\, X)\theta_2$. Let $\mathrm{M}(\theta_2)$ be of the form

$$\mathrm{M}(\theta_2) = \begin{pmatrix} 1+a' & b_1' & 0 \\ 0 & 1+b_2' & 0 \\ 0 & 0 & (1+a')(1+b_2') \end{pmatrix},$$

for some $a', b_1', b_2' \in \widehat{\omega}$. Then calculating the matrix $\mathrm{M}(\exp(\mathrm{ad}\, X)\theta_2)$ we have the following equations:

$$(2ctA(X) - 2b(at + bv)B(X))(1 + a_1') = 0$$

$$(bA(X) - 2c(at + bv)B(X))(1 + a_1') = 0$$

$$(2(av + bu)A(X) - 4acwB(X))(1 + a_1')(1 + b_2') = 0$$

$$(-2(at + bv)A(X) - 4bcwB(X))(1 + a_1')(1 + b_2') = 0$$

$$(bA(X) - 2c(at + bv)B(X))b_1' + (-aA(X) - 2c(av + bu)B(X))(1 + b_2') = 0$$

Using the fact that

$$1 + a_1' \neq 0, \quad 1 + b_1' \neq 0 \quad (1 + a_1')(1 + b_2') \neq 0, \quad A(X) \neq 0, \quad B(X) \neq 0,$$

direct calculations give

$$2c^2t - b^2 = 0$$

and so $b = c = 0$. Thus the equality

$$(2(av + bu)A(X) - 4acwB(X))(1 + a_1')(1 + b_2') = 0$$

turns to

$$2avA(X) = 0.$$

Hence $a = 0$ and consequently $X = 0$ which is in contradiction with $X \neq 0$. □

Recall that $\omega$ is the augmentation ideal of the polynomial algebra $K[t, u, v]$ and $\widehat{\omega} \subset K[[t, u, v]]$ is its completion with respect to the formal power formal series. Since the elements $t = 2x^2$, $u = 2y^2$, $v = xy + yx$ are of even degree in $W$, the associated matrices of the automorphisms of $\widehat{W}$ modulo $\widehat{\omega(W)}^{c+1}$, $c \geq 3$, contain the entries in the factor algebra $K[t, u, v]/\omega^{[(c+1)/2]}$.

As a consequence of our Theorem 8 for $\mathrm{IOut}(\widehat{W})$ we immediately obtain the description of the group of outer IA-automorphisms of $W/\omega(W)^{c+1}$. We shall give the results for the associated matrices only.

**Corollary 9.** *Let $\Theta$ be the set of automorphisms $\theta$ of $W/\omega(W)^{c+1} \cong \widehat{W}/\omega(\widehat{W})^{c+1}$ with associated matrix of the form*

$$\mathrm{M}(\theta) = \begin{pmatrix} 1 + a & b_1 & 0 \\ 0 & 1 + b_2 & 0 \\ 0 & 0 & (1 + a)(1 + b_2) \end{pmatrix} \pmod{M_3(\omega^{[(c+1)/2]})},$$

*where* $a, b_1, b_2 \in \widehat{\omega}$ *are formal power series without constant terms and* $M_3(\omega^{[(c+1)/2]})$ *is the* $3 \times 3$ *matrix algebra with entries from the* $[(c+1)/2]$*-th power of the augmentation ideal of* $K[t,u,v]$*. Then* $\Theta$ *consists of coset representatives of the subgroup* $\mathrm{Inn}(W/\omega(W)^{c+1})$ *of the group* $\mathrm{IA}(W/\omega(W)^{c+1})$ *and* $\mathrm{IOut}(W/\omega(W)^{c+1})$ *is a disjoint union of the cosets* $\mathrm{Inn}(W/\omega(W)^{c+1})\theta$, $\theta \in \Theta$.

**3. Outer automorphisms of Lie algebras of two generic matrices.** In this section describe the group $\mathrm{IOut}(\widehat{L}) = \mathrm{IA}(\widehat{L})/\mathrm{Inn}(\widehat{L})$ of outer IA-automorphisms of $\widehat{L}$. For this purpose we find the explicit form of the *related* matrices of the outer IA-automorphisms of $\widehat{L}$ and then we transfer the results to the algebra $L/L^{c+1}$ and obtain the description of $\mathrm{IOut}(L/L^{c+1})$. Throughout this section, we consider the field $K$ to be algebraically closed.

Now we give the description of related matrices of inner automorphisms of $\widehat{L}$ combining Theorem 4 with Lemma 5.

**Lemma 10.** *Let* $X = \alpha x + \beta y + a(xv - yt) + b(xu - yv) + c[x,y]$, $\alpha, \beta \in K$, $a, b, c \in K[[t,u,v]]$*, be an element in* $\widehat{L}$*. Then the related matrix of* $\exp(\mathrm{ad}\, X)$ *is of the form*

$$\mathrm{N}(\exp(\mathrm{ad}\, X)) = I_3 + A(X)\mathrm{D}(X) + B(X)\mathrm{F}(X),$$

*where*

$$\mathrm{D}(X) = \begin{pmatrix} -2cv & -2cu & 2(\alpha + av + bu) \\ 2ct & 2cv & 2(\beta - at - bv) \\ \alpha t + \beta v - bw & \alpha v + \beta u + aw & 0 \end{pmatrix},$$

$$\mathrm{F}(X) = \begin{pmatrix} 4c^2w + 2(\alpha + av + bu)(\alpha t + \beta v - bw) & \sigma_1 & \mu_1 \\ 2(\beta - at - bv)(\alpha t + \beta v - bw) & \sigma_2 & \mu_2 \\ -2c(\beta - at - bv)w & \sigma_3 & \mu_3 \end{pmatrix},$$

*where*

$$\sigma_1 = 2(\alpha + av + bu)(\alpha v + \beta u + aw),$$

$$\sigma_2 = 4c^2w + 2(\beta - at - bv)(\alpha v + \beta u + aw),$$

$$\sigma_3 = 2c(\alpha + av + bu)w,$$

$$\mu_1 = -4c(\alpha v + \beta u + aw),$$

$$\mu_2 = 4c(\alpha t + \beta v - bw),$$

$$\mu_3 = 2(\alpha + av + bu)(\alpha t + \beta v - bw) + 2(\beta - at - bv)(\alpha v + \beta u + aw),$$

$$A(X) = \frac{\sinh(\sqrt{g(X)})}{\sqrt{g(X)}}, \quad B(X) = \frac{\cosh(\sqrt{g(X)}) - 1}{g(X)}, \quad w = v^2 - tu.$$

$$g(X) = 2(\alpha+av+bu)^2 t + 4(\alpha+av+bu)(\beta-at-bv)v + 2(\beta-at-bv)^2 u + 4c^2 w.$$

Proof. Let $X = \alpha x + \beta y + a(xv - yt) + b(xu - yv) + c[x,y]$, $\alpha, \beta \in K$, $a, b, c \in K[[t,u,v]]$, be an element in $\widehat{L}$. Then applying Theorem 4, the associated matrix of $\exp(\operatorname{ad} X)$ is of the form

$$\mathrm{M}(\exp(\operatorname{ad} X)) = I_3 + A(X)\mathrm{M}(\operatorname{ad} X) + B(X)\mathrm{M}^2(\operatorname{ad} X),$$

where

$$\mathrm{M}(\operatorname{ad} X) = \begin{pmatrix} -2cv & -2cu & 2(\alpha v + \beta u + aw) \\ 2ct & 2cv & -2(\alpha t + \beta v - bw) \\ \beta - at - bv & -\alpha - av - bu & 0 \end{pmatrix},$$

$$\mathrm{M}^2(\operatorname{ad} X) = \begin{pmatrix} 4c^2 w + 2(\beta - at - bv)(\alpha v + \beta u + aw) & q_1 & r_1 \\ -2(\beta - at - bv)(\alpha t + \beta v - bw) & q_2 & r_2 \\ -2c(\alpha t + \beta v - bw) & q_3 & r_3 \end{pmatrix},$$

where

$$q_1 = -2(\alpha + av + bu)(\alpha v + \beta u + aw),$$
$$q_2 = 4c^2 w + 2(\alpha + av + bu)(\alpha t + \beta v - bw),$$
$$q_3 = -2c(\alpha v + \beta u + aw),$$
$$r_1 = -4cw(\alpha + av + bu),$$
$$r_2 = -4cw(\beta - at - bv),$$
$$r_3 = 2(\alpha + av + bu)(\alpha t + \beta v - bw) + 2(\beta - at - bv)(\alpha v + \beta u + aw),$$

$$A(X) = \frac{\sinh(\sqrt{g(X)})}{\sqrt{g(X)}}, \quad B(X) = \frac{\cosh(\sqrt{g(X)}) - 1}{g(X)}, \quad w = v^2 - tu.$$

$$g(X) = 2(\alpha+av+bu)^2 t + 4(\alpha+av+bu)(\beta-at-bv)v + 2(\beta-at-bv)^2 u + 4c^2 w.$$

Now applying Lemma 5 for M and $\mathrm{M}^2$ direct calculations immediately give the desired form of $\mathrm{D}(X)$ and $\mathrm{F}(X)$. □

Our next objective is to find the coset representatives of the normal subgroup $\mathrm{Inn}(\widehat{L})$ of the group $\mathrm{IA}(\widehat{L})$ of IA-automorphisms $\widehat{L}$, i.e., we shall find a set of IA-automorphisms $\theta$ of $\widehat{L}$ such that the factor group $\mathrm{IOut}(\widehat{L}) = \mathrm{IA}(\widehat{L})/\mathrm{Inn}(\widehat{L})$ of the outer IA-automorphisms of $\widehat{L}$ is presented as the disjoint union of the cosets $\mathrm{Inn}(\widehat{L})\theta$. We shall give the results for the related matrices only.

**Theorem 11.** *Let $\Theta$ be the set of automorphisms $\theta$ of $\widehat{L}$ with related matrix of the form*

$$\mathrm{N}(\delta) = \begin{pmatrix} 1 + a_1 v - a_2 t & a_1 u - a_2 v & 0 \\ 0 & 1 + bw & 0 \\ 0 & 0 & c \end{pmatrix},$$

*where*

$$c = (1 + a_1 v - a_2 t)(1 + bw), \quad w = v^2 - tu,$$

*$b \in K[[t, u, v]]$ and $a_1, a_2 \in \widehat{\omega}$ formal power series without constant terms. Then $\Theta$ consists of coset representatives of the subgroup $\mathrm{Inn}(\widehat{L})$ of the group $\mathrm{IA}(\widehat{L})$ and $\mathrm{IOut}(\widehat{L})$ is a disjoint union of the cosets $\mathrm{Inn}(\widehat{L})\theta$, $\theta \in \Theta$.*

P r o o f. Let

$$A = \begin{pmatrix} 1 + a_1 v - a_2 t & a_1 u - a_2 v & 0 \\ 0 & 1 + bw & 0 \\ 0 & 0 & c \end{pmatrix},$$

be an $3 \times 3$ matrix satisfying the conditions of the theorem. It is clear that $A$ is the related matrix of a certain IA-automorphism of $\widehat{L}$.

Now we shall show that for any $\psi \in \mathrm{IA}(\widehat{L})$ there exists an inner automorphism $\phi = \exp(\mathrm{ad}\, u) \in \mathrm{Inn}(\widehat{L})$ and an automorphism $\theta$ in $\Theta$ such that $\psi = \exp(\mathrm{ad}\, u) \cdot \theta$. Let $\psi$ be an arbitrary element of $\mathrm{IA}(\widehat{L})$ with related matrix be of the form

$$\mathrm{N}(\delta) = \begin{pmatrix} 1 + \mathrm{a}_1 v - \mathrm{a}_2 t & \mathrm{a}_1 u - \mathrm{a}_2 v & p_1 \\ \mathrm{b}_1 v - \mathrm{b}_2 t & 1 + \mathrm{b}_1 u - \mathrm{b}_2 v & p_2 \\ \mathrm{c}_1 v - \mathrm{c}_2 t & \mathrm{c}_1 u - \mathrm{c}_2 v & p_3 \end{pmatrix},$$

where

$$p_1 = 2\mathrm{c}_1(\mathrm{a}_2 v + \mathrm{b}_2 u) - 2\mathrm{c}_2(1 + \mathrm{a}_1 v + \mathrm{b}_1 u),$$

$$p_2 = 2\mathrm{c}_1(1 - \mathrm{a}_2 t - \mathrm{b}_2 u) + 2\mathrm{c}_2(\mathrm{a}_1 t + \mathrm{b}_1 v),$$

$$p_3 = (1 + \mathrm{a}_1 v + b_1 u)(1 - \mathrm{a}_2 t - \mathrm{b}_2 v) + (\mathrm{a}_1 t + \mathrm{b}_1 v)(\mathrm{a}_2 v + \mathrm{b}_2 u),$$

for some $\mathrm{a}_1, \mathrm{a}_2, \mathrm{b}_1, \mathrm{b}_2, \mathrm{c}_1, \mathrm{c}_2 \in K[[t, u, v]]$.

Let us define

$$\alpha = \mathrm{c}_{20} + \sqrt{\mathrm{c}_{20}^2 + \mathrm{a}_{20}}, \quad \beta = -2\mathrm{c}_{10} - \frac{\mathrm{a}_{10}}{\mathrm{c}_{20} + \sqrt{\mathrm{c}_{20}^2 + \mathrm{a}_{20}}},$$

where $\mathrm{a}_{10}$, $\mathrm{a}_{20}$, $\mathrm{c}_{10}$, $\mathrm{c}_{20} \in K$ are the constant components of the elements $\mathrm{a}_1$, $\mathrm{a}_2$, $\mathrm{c}_1$, $\mathrm{c}_2 \in K[[t, u, v]]$ respectively, and let

$$\begin{aligned} q_0 =& (1 + \mathrm{a}_1 v - \mathrm{a}_2 t)(1 + 2\alpha(\alpha t + \beta v)B(\alpha x + \beta y)) + \\ & 2\alpha(\alpha v + \beta u)(\mathrm{b}_1 v - \mathrm{b}_2 t)B(\alpha x + \beta y) + 2\alpha A(\alpha x + \beta y)(\mathrm{c}_1 v - \mathrm{c}_2 t), \end{aligned}$$

where

$$A(\alpha x + \beta y) = \frac{\sinh(\sqrt{g(\alpha x + \beta y)})}{\sqrt{g(\alpha x + \beta y)}}, \quad B(\alpha x + \beta y) = \frac{\cosh(\sqrt{g(\alpha x + \beta y)}) - 1}{g(\alpha x + \beta y)},$$

$$g(\alpha x + \beta y) = 2(\alpha^2 t + 2\alpha\beta v + \beta^2 u).$$

Note that $\mathrm{c}_{20} + \sqrt{\mathrm{c}_{20}^2 + \mathrm{a}_{20}}$ and $\mathrm{c}_{20} - \sqrt{\mathrm{c}_{20}^2 + \mathrm{a}_{20}}$ cannot be zero at the same time. We fix $\alpha = \mathrm{c}_{20} + \sqrt{\mathrm{c}_{20}^2 + \mathrm{a}_{20}} \neq 0$ without loss of generality. After calculations we obtain that $q_0 - 1$ does not have linear part. Now let

$$\phi_{\alpha\beta} = \exp(\mathrm{ad}(\alpha x + \beta y)).$$

Since the field $K$ is algebraically closed, $\alpha, \beta \in K$ and $\phi_{\alpha\beta} \in \mathrm{Inn}(\widehat{L})$. As a result, $\mathrm{N}(\phi_{\alpha\beta}\psi)$ is of the form

$$\mathrm{N}(\phi_{\alpha\beta}\psi) = \begin{pmatrix} 1 + a_1 v - a_2 t & a_1 u - a_2 v & * \\ b_1 v - b_2 t & 1 + b_1 u - b_2 v & * \\ c_1 u - c_2 v & c_1 u - c_2 v & * \end{pmatrix},$$

for some $b_1, b_2, c_1, c_2 \in K[[t, u, v]]$ and $a_1, a_2 \in \widehat{\omega}$.

Now let us define

$$b = \frac{1}{2\sqrt{-2uw}} \log\left(\frac{(1 + a_1 v - a_2 t)w + (c_1 v - c_2 t)\sqrt{-2uw}}{(1 + a_1 v - a_2 t)w - (c_1 v - c_2 t)\sqrt{-2uw}}\right)$$

and let

$$q_1 = -(1 + a_1 v - a_2 t)bwA(b(xu - yv)) + (1 - 2b^2 uwB(b(xu - yv)))(c_1 v - c_2 t),$$

where

$$A(b(xu-yv)) = \frac{\sinh(\sqrt{-2b^2uw})}{\sqrt{-2b^2uw}}, \quad B(b(xu-yv)) = \frac{\cosh(\sqrt{-2b^2uw})-1}{-2b^2uw}.$$

Note that both expressions $(1+a_1v-a_2t)w+(c_1v-c_2t)\sqrt{-2uw}$ and $(1+a_1v-a_2t)w-(c_1v-c_2t)\sqrt{-2uw}$ can not be zero at the same time. We fix $(1+a_1v-a_2t)w-(c_1v-c_2t)\sqrt{-2uw}\neq 0$ without loss of generality. After easy calculations we have that

$$e^{2\sqrt{-2b^2uw}} = \frac{2q_1\sqrt{-2uw}e^{\sqrt{-2b^2uw}}-(1+a_1v-a_2t)w-(c_1v-c_2t)\sqrt{-2uw}}{-(1+a_1v-a_2t)w+(c_1v-c_2t)}$$

and

$$2q_1\sqrt{-2uw}e^{\sqrt{-2b^2uw}} = 0.$$

Since the ring $K[[t,u,v]]$ is an integral domain, then $q_1=0$. Now let us define

$$\phi_b = \exp(\operatorname{ad} b(xu-yv)).$$

We know that $b\in K[[t,u,v]]$ from Lemma 3. Thus $\phi_b\in \operatorname{Inn}(\widehat{L})$. As a result, $\mathrm{N}(\phi_b\phi_{\alpha\beta}\psi)$ is of the form

$$\mathrm{N}(\phi_b\phi_{\alpha\beta}\psi) = \begin{pmatrix} 1+a_1'v-a_2't & a_1'u-a_2'v & * \\ b_1'v-b_2't & 1+b_1'u-b_2'v & * \\ 0 & c_1'u-c_2'v & * \end{pmatrix}$$

$$= \begin{pmatrix} 1+a_1'v-a_2't & a_1'u-a_2'v & * \\ b_1'v-b_2't & 1+b_1'u-b_2'v & * \\ 0 & k_1w & * \end{pmatrix}.$$

Here we have denoted by $*$ the corresponding entries of the third column of $\mathrm{N}(\phi_b\phi_{\alpha\beta}\psi)$. Note that the $(3,1)-th$ entry, $c_1'v-c_2't$, of the matrix is zero. Therefore $c_1'u-c_2'v=k_1w$ for some $k_1\in K[[t,u,v]]$.

Again let us define

$$c = \frac{1}{4\sqrt{w}}\log\left(\frac{-(1+a_1'v-a_2't)t-(b_1'v-b_2't)v+(b_1'v-b_2't)\sqrt{w}}{-(1+a_1'v-a_2't)t-(b_1'v-b_2't)v-(b_1'v-b_2't)\sqrt{w}}\right)$$

and let

$$q_2 = 2ct(1+a_1'v-a_2't)A(c[x,y])+(b_1'v-b_2't)(1+2cvA(c[x,y])+4c^2wB(c[x,y])),$$

where

$$A(c[x,y]) = \frac{\sinh(\sqrt{4c^2w})}{\sqrt{4c^2w}}, \quad B(c[x,y]) = \frac{\cosh(\sqrt{4c^2w}) - 1}{4c^2w}.$$

Similarly $q_2 = 0$, $c \in K[[t,u,v]]$ and $\phi_c = \exp(\operatorname{ad} c[x,y]) \in \operatorname{Inn}(\widehat{L})$. Calculating the matrix $\mathrm{N}(\phi_c\phi_b\phi_{\alpha\beta}\psi)$ we have that

$$\mathrm{N}(\phi_c\phi_b\phi_{\alpha\beta}\psi) = \begin{pmatrix} 1 + a_1''v - a_2''t & a_1''u - a_2''v & * \\ 0 & 1 + b_1''u - b_2''v & * \\ 0 & k_1w & * \end{pmatrix}$$

$$= \begin{pmatrix} 1 + a_1''v - a_2''t & a_1''u - a_2''v & * \\ 0 & 1 + k_2w & * \\ 0 & k_1w & * \end{pmatrix},$$

where $b_1'' = -k_2t, b_2'' = -k_2u$ for some $k_1, k_2, a_1'', a_2'' \in K[[t,u,v]]$. Note that $\mathrm{N}(\phi_c)$ preserves $(3,1) - th$ entry $(= 0)$ of the matrix $\mathrm{N}(\phi_b\phi_{\alpha\beta}\psi)$ after calculation.

Finally let us define

$$a = \frac{1}{2\sqrt{-2tw}} \log\left(\frac{(1+k_1w)w + k_2w\sqrt{-2tw}}{(1+k_1w)w - k_2w\sqrt{-2tw}}\right)$$

and let

$$q_3 = -a(1 + b_2'')A(ax) + (1 + 2a^2tB(ax))c_2'',$$

where

$$A(a(xv - yt)) = \frac{\sinh(\sqrt{-2a^2tw})}{\sqrt{-2a^2tw}}, \quad B(a(xv - yt)) = \frac{\cosh(\sqrt{-2a^2tw}) - 1}{-2a^2tw}.$$

Similarly $q_3 = 0$, $a \in K[[t,u,v]]$ and $\phi_a = \exp(\operatorname{ad} a(xv - yt)) \in \operatorname{Inn}(\widehat{L})$. Calculating the matrix $\mathrm{N}(\phi_a\phi_c\phi_b\psi)$ we have that

$$\mathrm{N}(\phi_a\phi_c\phi_b\phi_{\alpha\beta}\psi) = \begin{pmatrix} 1 + a_1'''v - a_2'''t & a_1'''u - a_2'''v & 0 \\ 0 & 1 + bw & 0 \\ 0 & 0 & (1 + a_1'''v - a_2'''t)(1 + bw) \end{pmatrix},$$

for some $a_1''', a_2''', b \in K[[t,u,v]]$. Note that $\mathrm{N}(\phi_a)$ preserves both $(2,1) - th$ and $(3,1) - th$ entries $(= 0)$ of the matrix $\mathrm{N}(\phi_c\phi_b\phi_{\alpha\beta}\psi)$ after calculation.

Hence, starting from an arbitrary coset of IA-automorphisms $\operatorname{Inn}(\widehat{L})\psi$, we found that it contains an automorphism $\theta \in \Theta$ with related matrix prescribed in the theorem. Now, let $\theta_1$ and $\theta_2$ be two different automorphisms in $\Theta$ with $\mathrm{Inn}(\widehat{L})\theta_1 = \mathrm{Inn}(\widehat{L})\theta_2$. Hence, there exists a nonzero element

$$X = \alpha x + \beta y + a(xv - yt) + b(xu - yv) + c[x, y] \in \widehat{L}$$

such that $\theta_1 = \exp(\mathrm{ad}\, X)\theta_2$. Let $\mathrm{N}(\theta_2)$ be of the form

$$\mathrm{N}(\theta_2) = \begin{pmatrix} 1 + a_1'v - a_2't & a_1'u - a_2'v & 0 \\ 0 & 1 + b'w & 0 \\ 0 & 0 & (1 + a_1'v - a_2't)(1 + b'w) \end{pmatrix},$$

for some $a_1', a_2' \in \widehat{\omega}$ and $b \in K[[t, u, v]]$. Then calculating the matrix $\mathrm{N}(\exp(\mathrm{ad}\, X)\theta_2)$ we have the following equations:

$$(2ctA(\alpha x + \beta y) + 2(\beta - at - bv)(\alpha t + \beta v - bw)B(\alpha x + \beta y))(1 + a_1'v - a_2't) = 0$$

$$((\alpha t + \beta v - bw)A(\alpha x + \beta y) - 2c(\beta - at - bv)wB(\alpha x + \beta y))(1 + a_1'v - a_2't) = 0$$

$$((\alpha + av + bu)A(\alpha x + \beta y) \\ - 4c(\alpha v + \beta u + aw)B(\alpha x + \beta y))(1 + a_1'v - a_2't)(1 + b'w) = 0$$

$$((\beta - at - bv)A(\alpha x + \beta y) \\ + 4c(\alpha t + \beta v - bw)B(\alpha x + \beta y))(1 + a_1'v - a_2't)(1 + b'w) = 0$$

$$((\alpha t + \beta v - bw)A(\alpha x + \beta y) - 2c(\beta - at - bv)wB(\alpha x + \beta y))(a_1'u - a_2'v) \\ + ((\alpha v + \beta u + aw)A(\alpha x + \beta y) + 2c(\alpha + av + bu)wB(\alpha x + \beta y))(1 + b'w) = 0$$

and the expression

$$(1-2cvA(\alpha x+\beta y)+4c^2w+2(\alpha+av+bu)(\alpha t+\beta v-bw)B(\alpha x+\beta y))(1+a_1'v-a_2't)-1$$

does not have linear part. Here

$$A(X) = \frac{\sinh(\sqrt{g(X)})}{\sqrt{g(X)}}, \quad B(X) = \frac{\cosh(\sqrt{g(X)}) - 1}{g(X)}, \quad w = v^2 - tu,$$

$$g(X) = 2(\alpha + av + bu)^2 t + 4(\alpha + av + bu)(\beta - at - bv)v + 2(\beta - at - bv)^2 u + 4c^2 w.$$

From the last equation we get that $\alpha = 0$ and using the fact that

$$1 + a_1'v - a_2't \neq 0, \quad 1 + b'w \neq 0 \quad , \quad A(X) \neq 0, \quad B(X) \neq 0,$$

direct calculations give that $X = 0$ which is in contradiction with $X \neq 0$. □

As a consequence of our Theorem 11 for $\mathrm{IOut}(\widehat{L})$ we immediately obtain the description of the group of outer IA-automorphisms of $L/L^{c+1}$. We shall give the results for the related matrices only.

**Corollary 12.** *Let $\Theta$ be the set of automorphisms $\theta$ of $L/L^{c+1} \cong \widehat{L}/\widehat{L}^{c+1}$ with related matrix of the form*

$$\mathrm{N}(\theta) = \begin{pmatrix} 1 + a_1v - a_2t & a_1u - a_2v & 0 \\ 0 & 1 + bw & 0 \\ 0 & 0 & (1 + a_1v - a_2t)(1 + bw) \end{pmatrix}$$

*modulo $M_3(\omega^{[(c+1)/2]})$, where $a_1, a_2 \in \widehat{\omega}$ are formal power series without constant terms, $b \in K[[t, u, v]]$ and $M_3(\omega^{[(c+1)/2]})$ is the $3 \times 3$ matrix algebra with entries from the $[(c+1)/2]$-th power of the augmentation ideal of $K[t, u, v]$. Then $\Theta$ consists of coset representatives of the subgroup $\mathrm{Inn}(L/L^{c+1})$ of the group $\mathrm{IA}(L/L^{c+1})$ and $\mathrm{IOut}(L/L^{c+1})$ is a disjoint union of the cosets $\mathrm{Inn}(L/L^{c+1})\theta$, $\theta \in \Theta$.*

**Remark 13.** Let $G$ be the algebra $L/L^{c+1}$ or $W/\omega(W)^{c+1}$ and let $\theta$ be an outer IA-automorphism of $G$. Then one can observe that $\theta$ is $\mathbb{Z}_2$-graded, i.e., it maps the linear combinations of elements of odd, respectively even degree to linear combinations of the same kind.

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