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ASYMPTOTIC BEHAVIOUR OF FUNCTIONAL IDENTITIES*

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Abstract. We calculate the asymptotics of functional codimensions $\mathrm{fc}_n(A)$ and generalized functional codimensions $\mathrm{gfc}_n(A)$ of an arbitrary not necessarily associative algebra A over a field F of any characteristic. Namely, $\mathrm{fc}_n(A) \sim \mathrm{gfc}_n(A) \sim \dim(A^2) \cdot (\dim A)^n$ as $n \to \infty$ for any finite-dimensional algebra A. In particular, codimensions of functional and generalized functional identities satisfy the analogs of Amitsur's and Regev's conjectures. Also we precisely evaluate $\mathrm{fc}_n(\mathrm{UT}_2(F)) = \mathrm{gfc}_n(\mathrm{UT}_2(F)) = 3^{n+1} - 2^{n+1}$.

This article is dedicated to Professor Yuri Bahturin on the occasion of his 65th birthday.

In 1970–80s, a new direction appeared in the theory of polynomial identities. This direction is concerned with the study of numeric characteristics of polynomial identities.

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Let F be a field, $F\langle X\rangle$ be the free associative algebra on the countable set

$$X = \{x_1, x_2, x_3, \ldots\},$$

i.e. $F\langle X\rangle$ is the algebra of all polynomials in the non-commuting variables from X without a constant term. Let A be an associative F-algebra. We say that $f(x_1, \ldots, x_n) \in F\langle X\rangle$ is a *polynomial identity* of A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. The set $\mathrm{Id}(A)$ of polynomial identities of A is a two-sided ideal of $F\langle X\rangle$. We say that A is a *p.i. algebra* if $\mathrm{Id}(A) \neq 0$. Denote by $P_n \subset F\langle X\rangle$ the subspace of multilinear polynomials in $x_1, x_2, \ldots, x_n$, $n \in \mathbb{N}$. The number $c_n(A) := \dim \frac{P_n}{P_n \cap \mathrm{Id}(A)}$ is called the nth *codimension of ordinary polynomial identities* of A.

Conjecture (S. A. Amitsur)**.** *Let A be a p.i. algebra over a field of characteristic* 0*, then there exists* $\mathrm{PIexp}(A) := \lim\limits_{n\to\infty} \sqrt[n]{c_n(A)} \in \mathbb{Z}_+$.

Conjecture (A. Regev)**.** *Let A be a p.i. algebra over a field of characteristic* 0*, then there exists $C > 0$, $r \in \mathbb{Z}$, $d \in \mathbb{Z}_+$ such that $c_n(A) \sim C n^{\frac{r}{2}} d^n$ as $n \to \infty$. (We write $f \sim g$ if $\lim \frac{f}{g} = 1$.)*

Amitsur's conjecture for ordinary codimensions of associative algebras was proved by A. Giambruno and M. V. Zaicev [9, Theorem 6.5.2] in 1999. In 2002 M. V. Zaicev proved its analog for finite dimensional Lie algebras [12] and in 2011 A. Giambruno, I. P. Shestakov, and M. V. Zaicev proved its analog for finite dimensional Jordan and alternative algebras [8]. Regev's conjecture was proved by A. Berele and A. Regev in 2008 for all unitary associative algebras [5, 4]. The analogues of Amitsur's and Regev's conjectures for generalized polynomial identities of associative algebras was proved by the author [11] in 2009.

In 1995 M. Brešar [6] introduced functional identities. Functional and generalized functional identities were used by Yu. A. Bahturin, M. Brešar, K. I. Beidar, M. A. Chebotar, M. V. Kotchetov, W. S. Martindale, A. V. Mikhalev, and others [3, 7, 1, 2] to solve a number of open problems in the ring theory. Therefore, a natural question arises as to whether the analogs of Amitsur's and Regev's conjectures hold for codimensions of functional identities $\mathrm{fc}_n(A)$ and codimensions of generalized functional identities $\mathrm{gfc}_n(A)$.

In [10] the author proved the analog of Amitsur's conjecture. In the case when a finite dimensional algebra A satisfies the property $A^2 = AaA + Aa + aA$ for some $a \in A$ the analog of Regev's conjecture was proved too [10]. Here $A^2 := \langle ab \mid a, b \in A\rangle_F$.

In this paper we provide a criterion for an algebra to satisfy the analogs of Amitsur's and Regev's conjectures (Theorem 1). It turns out that for every finite dimensional algebra A we have

$$\lim_{n\to\infty} \sqrt[n]{\mathrm{fc}_n(A)} = \lim_{n\to\infty} \sqrt[n]{\mathrm{gfc}_n(A)} = \dim A.$$

Note that in the case of ordinary polynomial identities we have much more complicated formulas for $\mathrm{PIexp}(A)$ (see [9, Section 6.2] and [12, Definition 2]).

In addition, functional and generalized functional codimensions of the algebra $\mathrm{UT}_2(F)$ (Theorem 2) are precisely calculated. Moreover, in Theorem 3 we obtain that $\mathrm{fc}_n(M_k(F)) < \mathrm{gfc}_n(M_k(F)) = k^{2(n+1)}$, i.e. functional and generalized functional codimensions do not always coincide. Here $\mathrm{UT}_2(F)$ and $M_k(F)$ are the associative algebras of, respectively, upper triangular matrices 2×2 and all matrices $k \times k$.

1. (Generalized) functional polynomial identities and their codimensions. Now let A be a not necessarily finite dimensional algebra over a field F of arbitrary characteristic.

We call an expression

$$\sum_{i=1}^{n} (G_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i + x_iH_i(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n))$$

a *multilinear functional polynomial* of degree n with coefficients in A. Here $G_i, H_i \colon A^{\otimes(n-1)} \to A$ are arbitrary F-linear maps, $n \geqslant 2$. Expressions $cx + xd$, where $c, d \in A$, we call linear functional polynomials of degree 1.

Denote the vector space of multilinear functional polynomials of degree n by $\mathrm{FP}_n(A)$.

Let $f \in \mathrm{FP}_n(A)$. If $f(p_1,\ldots,p_n) = 0$ for $p_1,\ldots,p_n \in A$, then f is a *functional identity* of A. Clearly, the set $\mathrm{FId}_n(A)$ of multilinear functional identities of degree $n \in \mathbb{N}$ is a linear subspace of $\mathrm{FP}_n(A)$. The codimensions $\mathrm{fc}_n(A) := \dim \dfrac{\mathrm{FP}_n(A)}{\mathrm{FId}_n(A)}$ of functional identities are called *functional codimensions* of A.

Example 1. Let A be an algebra with a center Z. Let $H \colon A \to Z$ be a linear map. Then $H(x)y - yH(x) \in \mathrm{FId}_2(A)$.

Analogously

$$\sum_{i=1}^{n}\sum_{k=1}^{\ell}(G_{ik}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)x_i a_{ik}$$
$$+b_{ik}x_i H_{ik}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n))$$

is a *multilinear generalized functional polynomial* of degree n with coefficients in A. Here $G_{ik}, H_{ik}\colon A^{\otimes(n-1)} \to A$ are arbitrary F-linear maps, $a_{ik}, b_{ik} \in A\cup\{1\}$, $n \geqslant 2$, $\ell \in \mathbb{N}$. An arbitrary arrangement of brackets is fixed on each monomial. Linear generalized functional polynomials of degree 1 are expressions $\sum\limits_{k=1}^{\ell} a_k x b_k + cx + xd$ where $a_k, b_k, c, d \in A$, $\ell \in \mathbb{N}$. Denote the space of multilinear generalized functional polynomials of degree n by $\mathrm{GFP}_n(A)$. Let $f \in \mathrm{GFP}_n(A)$. Then f is *generalized functional identity* of A if $f(p_1,\ldots,p_n) = 0$ for all $p_1,\ldots,p_n \in A$. The set $\mathrm{GFId}_n(A)$ of multilinear generalized functional identities of degree $n \in \mathbb{N}$ is a linear subspace of $\mathrm{GFP}_n(A)$. Codimensions $\mathrm{gfc}_n(A) := \dim \dfrac{\mathrm{GFP}_n(A)}{\mathrm{GFId}_n(A)}$ of generalized functional identities are called *generalized functional codimensions* of A.

Example 2. Let A be the Grassmann (or exterior) algebra with generators e_i, $i \in \mathbb{N}$, $H\colon A \to (Fe_1)$ be a linear map. Then $H(x)ye_1 \in \mathrm{GFId}_2(A)$.

Now we can formulate the analogs of Amitsur's and Regev's conjectures for functional and generalized functional identities. It turns out that in our case the conjectures can be refined and strengthened.

Conjecture (analog of Amitsur's conjecture)**.** *There exist*

$$\lim_{n\to\infty}\sqrt[n]{\mathrm{fc}_n(A)} = \lim_{n\to\infty}\sqrt[n]{\mathrm{gfc}_n(A)} = \dim A.$$

Conjecture (analog of Regev's conjecture)**.**

$$\mathrm{fc}_n(A) \sim \mathrm{gfc}_n(A) \sim \dim\left(A^2\right)\cdot(\dim A)^n \ \textit{as } n\to\infty.$$

2. Main theorem. The following theorem is the main result of the paper.

Theorem 1. *Let A be a not necessary associative algebra over a field F of an arbitrary characteristic.*

(1) *If* $A^2 = 0$, *then* $\mathrm{fc}_n(A) = \mathrm{gfc}_n(A) = 0$ *for all* $n \in \mathbb{N}$.

(2) *If* $A^2 \neq 0$ *and* $\dim A = +\infty$, *then* $\mathrm{fc}_n(A) = \mathrm{gfc}_n(A) = +\infty$ *for all* $n \geqslant 2$.

(3) *If* $A^2 \neq 0$ *and* $\dim A < +\infty$, *then* $\mathrm{fc}_n(A) \sim \mathrm{gfc}_n(A) \sim \dim\left(A^2\right) \cdot (\dim A)^n$ *as* $n \to \infty$.

Corollary. *The analogs of Amitsur's and Regev's conjectures hold for codimensions of functional and generalized functional identities.*

Corollary. *If A contains* 1, *then* $\mathrm{fc}_n(A) \sim \mathrm{gfc}_n(A) \sim (\dim A)^{n+1}$ *as* $n \to \infty$.

Remark. We can construct an infinite dimensional algebra B, $B^2 \neq 0$, with $\mathrm{fc}_1(B) = \mathrm{gfc}_1(B) < +\infty$. Consider the algebra B with the basis $b, a_1, a_2, \ldots, a_n, \ldots$ and the relations $a_i a_j = b$, $a_i b = b a_i = b^2 = 0$ for all $i, j \in \mathbb{N}$. Then the factor spaces $\dfrac{\mathrm{FP}_1(B)}{\mathrm{FId}_1(B)}$ and $\dfrac{\mathrm{GFP}_1(B)}{\mathrm{GFId}_1(B)}$ equal the linear span of the image of xa_1 since $xa_i - xa_j \in \mathrm{FId}_1(B)$, $xa_i - a_i x \in \mathrm{FId}_1(B)$, $bx, xb \in \mathrm{FId}_1(B)$, $a_i x a_j \in \mathrm{GFId}_1(B)$. Therefore, $\mathrm{fc}_1(B) = \mathrm{gfc}_1(B) = 1$. At the same time for the countably generated free algebra $F\langle Y\rangle$, $Y = \{y_1, y_2, y_3, \ldots\}$, we have $\mathrm{fc}_1\big(F\langle Y\rangle\big) = \mathrm{gfc}_1\big(F\langle Y\rangle\big) = +\infty$, since the functional polynomials xy_i are linearly independent modulo functional identities. (In order to check this it is sufficient to substitute $x = y_1$.)

We first introduce the notation and prove several auxiliary lemmas used in the proof of Theorem 1.

Let $\mathrm{Hom}_F(A^{\otimes n}; A^2)$ be the space of all n-linear maps from A to A^2. Then we can treat functional and generalized functional polynomials as n-linear maps from A to A^2. The kernel of the corresponding map $\mathrm{FP}_n(A) \to \mathrm{Hom}_F(A^{\otimes n}; A^2)$ equals $\mathrm{FId}_n(A)$ and the kernel of the corresponding map $\mathrm{GFP}_n(A) \to \mathrm{Hom}_F(A^{\otimes n}; A^2)$ equals $\mathrm{GFId}_n(A)$. Moreover, all the n-linear maps from A to A^2 that can be defined by functional polynomials, can be defined by generalized functional polynomials too. Thus we have natural embeddings

$$\frac{\mathrm{FP}_n(A)}{\mathrm{FId}_n(A)} \hookrightarrow \frac{\mathrm{GFP}_n(A)}{\mathrm{GFId}_n(A)} \hookrightarrow \mathrm{Hom}_F(A^{\otimes n}; A^2),$$

which after the identifications become

$$\frac{\mathrm{FP}_n(A)}{\mathrm{FId}_n(A)} \subseteq \frac{\mathrm{GFP}_n(A)}{\mathrm{GFId}_n(A)} \subseteq \mathrm{Hom}_F(A^{\otimes n}; A^2).$$

Hence

$$\mathrm{fc}_n(A) \leqslant \mathrm{gfc}_n(A) \leqslant \dim \mathrm{Hom}_F(A^{\otimes n}; A^2) = \dim \left(A^2\right) \cdot (\dim A)^n. \tag{1}$$

Lemma 1 (see below) and formula (1) imply that for every finite dimensional algebra A there exist $\lim\limits_{n\to\infty} \sqrt[n]{\mathrm{fc}_n(A)} = \lim\limits_{n\to\infty} \sqrt[n]{\mathrm{gfc}_n(A)} = \dim A$. Therefore the analog of Amitsur's conjecture holds for functional and generalized functional codimensions.

Lemma 1. *Let* $A^2 \neq 0$. *Then* $\dim A < +\infty$ *implies*

$$(\dim A)^{n-1} \leqslant \mathrm{fc}_n(A) \leqslant \mathrm{gfc}_n(A).$$

If $\dim A = +\infty$, *then* $\mathrm{fc}_n(A) = \mathrm{gfc}_n(A) = +\infty$ *for all* $n \geqslant 2$.

Proof. There exist $a, b \in A$, that $ab \neq 0$ since $A^2 \neq 0$. Let $e_j \in A$, $1 \leqslant j \leqslant m$, $m \in \mathbb{N}$, be linear independent elements. Let Λ be the set of $(n-1)$-tuples $k = (k_1, \ldots, k_{n-1})$, where $1 \leqslant k_\ell \leqslant m$, $1 \leqslant \ell \leqslant n-1$. For every $k \in \Lambda$ we define a multilinear map G_k such that

$$G_k(e_{i_1}, \ldots, e_{i_{n-1}}) := \begin{cases} a & \text{if } i_\ell = k_\ell \text{ for all } \ell, \\ 0 & \text{otherwise.} \end{cases}$$

Then functional polynomials $f_k := G_k(x_1, x_2, \ldots, x_{n-1})x_n$, $k \in \Lambda$, are linearly independent modulo functional identities. Indeed, suppose $\sum\limits_{k\in\Lambda} \alpha_k f_k \equiv 0$ for some $\alpha_k \in F$. We fix some $t = (t_1, \ldots, t_{n-1}) \in \Lambda$ and substitute $x_1 = e_{t_1}$, $\ldots$, $x_{n-1} = e_{t_{n-1}}$, $x_n = b$. Then we obtain that all f_k, except f_t, vanish. Thus $\alpha_t = 0$. Hence $\mathrm{fc}_n(A) \geqslant |\Lambda| = m^{n-1}$. Therefore, $\dim A < +\infty$ implies $\mathrm{fc}_n(A) \geqslant (\dim A)^{n-1}$. If $\dim A = +\infty$, then $\mathrm{fc}_n(A) = +\infty$. The assertion concerning $\mathrm{gfc}_n(A)$ now follows from (1). $\square$

If $A^2 = 0$, then every (generalized) functional polynomial is an identity since the polynomial involves the multiplication. Hence $\mathrm{fc}_n(A) = \mathrm{gfc}_n(A) = 0$ for all $n \in \mathbb{N}$. This remark and Lemma 1 make the study of the cases $A^2 = 0$ and $\dim A = +\infty$ complete. Without lost of generality we assume further that $\dim A < +\infty$ and $A^2 \neq 0$.

Denote $L_n := \mathrm{Hom}_F(A^{\otimes n}; A^2)$ and $W_n := \dfrac{\mathrm{FP}_n(A)}{\mathrm{FId}_n(A)}$. Fix a basis $a_1, \ldots, a_m$ in A. Then we can choose a basis $w_1, \ldots, w_t$ in A^2 consisting of some elements $a_s a_k$. Let A^* be the space of linear functions on A and let $\varphi^1, \ldots, \varphi^m$ be the dual to $a_1, \dots, a_m$ basis in A^*, i.e. $\varphi^s(a_k) = \delta^s_k$. Then

$$\mathcal{B}_n = \{\varphi^{i_1}(x_1)\dots\varphi^{i_n}(x_n)w_j \mid 1 \leqslant i_s \leqslant m, 1 \leqslant s \leqslant n, 1 \leqslant j \leqslant t\}$$

is a natural basis in L_n. We also use the notation

$$\varphi^{i_1}(x_1)\dots\varphi^{i_n}(x_n)w_j =: \varphi^{i_1} \otimes \dots \otimes \varphi^{i_n} \otimes w_j =: \varphi^{i}(\overline{x}_n)w_j.$$

We introduce the order $\prec$ as the lexicographic order on $(n+1)$-tuples $(i_1, i_2, \dots, i_n, j)$. Let $H \in L_n$. Then $\operatorname{lt} H := v$ where $v \in \mathcal{B}_n$ such that $H = \alpha_v v + \sum\limits_{u \prec v,\ u \in \mathcal{B}_n} \alpha_u u$, $\alpha_u \in F$, $\alpha_v \neq 0$. If $S \subseteq L_n$ is a subset, then $\operatorname{lt} S := \{\operatorname{lt} H \mid H \in S\}$. Also we define the function $\tilde{\jmath}\colon \mathcal{B}_n \to \mathbb{N}$ by $\tilde{\jmath}\Big(\varphi^i(\overline{x}_n)w_j\Big) = j$.

Lemma 2. *For any $1 \leqslant j \leqslant t = \dim\left(A^2\right)$ there exist $n(j) \in \mathbb{N}$ and a map $H_j \in W_{n(j)}$ such that $\operatorname{im} H_j = \langle w_j \rangle_F$. (Here* im *denotes the image of a linear map.)*

Proof. We do not use the ordering on the elements w_i in this lemma. Hence without lost of generality we may assume that $j = t$. The basis (w_i) of the space A^2 was chosen in such a way that there exist $1 \leqslant s, k \leqslant m$ such that $w_t = a_s a_k$.

Consider the map $H_{(0)} \in W_1$ defined by $H_{(0)}(x) := a_s x$. Then

$$H_{(0)}(x) = \varphi^k(x)w_t + \sum_{\substack{1 \leqslant \ell \leqslant m,\\ \ell \neq k}} \varphi^\ell(x)u_\ell, \quad u_\ell \in A^2.$$

Rewriting u_ℓ as linear combinations of w_r and grouping the terms with the same w_r, we obtain $H_{(0)} = \sum\limits_{r=1}^{t} \psi^r_{(0)} \otimes w_r$, where $\psi^r_{(0)} \in A^*$, $\psi^r_{(0)}(a_k) = \delta^r_t$ for all $1 \leqslant r \leqslant m$. In other words, in the presentation of every $\psi^r_{(0)}$ as a linear combination of φ^ℓ, the coefficient near φ^k equals 0 for $r \neq t$ and 1 for $r = t$.

Now by induction on $1 \leqslant \gamma \leqslant t-1$, we define the maps $H_{(\gamma)} \in L_{\beta(\gamma)}$ where $\beta(\gamma) \in \mathbb{N}$, $\beta(0) = 1$. We claim that $H_{(\gamma)} = \sum\limits_{r=\gamma+1}^{t} \psi^r_{(\gamma)} \otimes w_r$ for some $\psi^r_{(\gamma)} \in (A^*)^{\otimes \beta(\gamma)}$, i.e.

$$\operatorname{im} H_{(\gamma)} \subseteq \langle w_r \mid \gamma < r \leqslant t \rangle.$$

If $\psi^{\gamma}_{(\gamma-1)} \neq 0$, then

$$H_{(\gamma)}(x_1,\dots,x_{2n}) := H_{(\gamma-1)}(x_1,\dots,x_n)\cdot\psi^{\gamma}_{(\gamma-1)}(x_{n+1},\dots,x_{2n}) \\ - \psi^{\gamma}_{(\gamma-1)}(x_1,\dots,x_n)\cdot H_{(\gamma-1)}(x_{n+1},\dots,x_{2n}),$$

$\beta(\gamma) = 2n$ where $n = \beta(\gamma-1)$.

If $\psi^{\gamma}_{(\gamma-1)} = 0$, then $H_{(\gamma)} := H_{(\gamma-1)}$, $\beta(\gamma) = \beta(\gamma-1)$.

Clearly, $H_{(\gamma)} \in W_{\beta(\gamma)}$ since each multiplication by a multilinear function just changes the multilinear maps, and we again obtain multilinear functional polynomials. Moreover,

$$\psi^{r}_{(\gamma)} = \psi^{r}_{(\gamma-1)}\otimes\psi^{\gamma}_{(\gamma-1)} - \psi^{\gamma}_{(\gamma-1)}\otimes\psi^{r}_{(\gamma-1)},$$

i.e. on the γth step the coefficient near w_γ indeed cancels. In order to finish the proof, it is sufficient to show that in each step $\psi^{t}_{(\gamma)} \neq 0$. Note that φ^k occurs in the decomposition of $\psi^{r}_{(\gamma)}$ only for $r = t$. Moreover, φ^k appears in each product no more than once. Let χ be the greatest term of $\psi^{t}_{(\gamma-1)}$ with φ^k, and let v be the greatest term of $\psi^{\gamma}_{(\gamma-1)}$, both in the lexicographic order. Then the greatest term of $\psi^{t}_{(\gamma-1)}\otimes\psi^{\gamma}_{(\gamma-1)} - \psi^{\gamma}_{(\gamma-1)}\otimes\psi^{t}_{(\gamma-1)}$ containing φ^k equals either χv or $v\chi$. But $\chi v \neq v\chi$ since φ^k occurs in the first product closer to the beginning, than in the second one. Hence the greatest term containing φ^k cannot cancel, and we may put $H_t = H_{(t-1)}$. $\square$

Lemma 3. $\dim W_n = |\operatorname{lt} W_n|$.

Proof. We choose a basis $\mathcal{W}_n$ in W_n and rewrite the elements of $\mathcal{W}_n$ as linear combinations of elements of $\mathcal{B}_n$. Then we put the components of this decomposition to the rows of a matrix. Note that for every $H \in \mathcal{W}_n$, the first nonzero element of the corresponding row occurs in the column that corresponds to $\operatorname{lt} H$. Now we apply the Gauss elimination process to the matrix. After these transformations, the rows of the matrix contain the components of the decomposition for the elements of some basis $\mathcal{S}$ of the space W_n. Note that the matrix has the row echelon form. Thus $\operatorname{lt}\mathcal{S} = \operatorname{lt} W_n$ since $\operatorname{lt} G_1 \neq \operatorname{lt} G_2$ for all $G_1, G_2 \in \mathcal{S}$, $G_1 \neq G_2$. Hence $\dim W_n = |\mathcal{S}| = |\operatorname{lt}\mathcal{S}| = |\operatorname{lt} W_n|$. $\square$

Proof of Theorem 1. The case $\dim A = +\infty$ has been considered in Lemma 1. Thus we assume $\dim A < +\infty$.

For every j we choose $H_j \in W_{n(j)}$ form Lemma 2. We define $h_j := (i_1, \dots, i_{n(j)})$ by the formula

$$\operatorname{lt} H_j =: \varphi^{i_1}(x_1) \dots \varphi^{i_{n(j)}}(x_{n(j)}) w_j.$$

Let $N := \max\limits_{1 \leqslant j \leqslant t} n(j)$. Fix $n \geqslant N$ and $1 \leqslant j \leqslant t := \dim\left(A^2\right)$. Then

$$\mathcal{H}_{jn} := \Big\{ \varphi^{s_1}(x_1) \dots \varphi^{s_{\ell n(j)}}(x_{\ell n(j)}) H_j(x_{\ell n(j)+1}, \dots, x_{(\ell+1)n(j)}) \cdot$$

$$\varphi^{s_{(\ell+1)n(j)+1}}(x_{(\ell+1)n(j)+1}) \dots \varphi^{s_n}(x_n) \,\Big|\, 1 \leqslant s_i \leqslant m, 0 \leqslant \ell < \left[\frac{n}{n(j)}\right] \Big\} \subseteq W_n \backslash \{0\}.$$

Here we multiply the multilinear maps in H_j by the products of linear functions. The number of the variables grows, but the expression is still a functional polynomial.

Now we count the number of different leading terms $\operatorname{lt} H$ in such polynomials $H \in \mathcal{H}_{jn}$ for fixed j and n. This allows us to get the lower bound for $q_{jn} := \left|\{v \in \operatorname{lt} W_n \mid \tilde{\jmath}(v) = j\}\right|$. We present the $(n+1)$-tuple (i, j) corresponding to $\varphi^i(\overline{x}_n) w_j \in \mathcal{B}_n$, $i \in \mathbb{N}^n$,as a tuple of smaller tuples: $(i, j) = (u_1, \dots, u_s, u_{s+1}, j)$ where $u_k \in \mathbb{N}^{n(j)}$ for $1 \leqslant k \leqslant s := \left[\frac{n}{n(j)}\right]$, $u_{s+1} \in \mathbb{N}^r$, $r := n - n(j)s$, $0 \leqslant r < n(j)$. The tuples corresponding to the leading terms of the functional polynomials $H \in \mathcal{H}_{jn}$ have $u_\ell = h_j$ for at least one $1 \leqslant \ell \leqslant s$. The number of tuples (i, j) with $u_\ell \neq h_j$ for all $1 \leqslant \ell \leqslant s$, equals $(m^{n(j)} - 1)^s m^r$ where $m = \dim A$. Since h_j is fixed,

$$|\operatorname{lt} \mathcal{H}_{jn}| = m^n - (m^{n(j)} - 1)^s \cdot m^r \geqslant m^n - (m^{n(j)} - 1)^{\frac{n}{n(j)}} m^{n(j)} \sim m^n \text{ as } n \to \infty.$$

Furthermore,

$$q_{jn} \leqslant \left|\{v \in \mathcal{B}_n \mid \tilde{\jmath}(v) = j\}\right| = m^n.$$

Together with $q_{jn} \geqslant |\operatorname{lt} \mathcal{H}_{jn}|$, this implies $q_{jn} \sim m^n = (\dim A)^n$. Applying Lemma 3, we have

$$\text{(2)} \qquad \operatorname{fc}_n(A) = \dim W_n = \sum_{j=1}^{t} q_{jn} \sim \dim\left(A^2\right) \cdot (\dim A)^n \text{ as } n \to \infty.$$

The assertion, concerning the asymptotics of $\operatorname{gfc}_n(A)$, follows from (1) and (2). $\square$

3. Functional codimensions of matrix algebras. In Theorem 1 we have calculated the asymptotics of functional and generalized functional codimensions. In Theorem 2 we evaluate these codimensions for $\mathrm{UT}_2(F)$ precisely. In Theorem 3 we show that functional and generalized functional codimensions do not always coincide.

Theorem 2. *Let F be any field. Then*

$$\mathrm{fc}_n(\mathrm{UT}_2(F)) = \mathrm{gfc}_n(\mathrm{UT}_2(F)) = 3^{n+1} - 2^{n+1} \text{ for all } n \in \mathbb{N}. \tag{3}$$

Theorem 3. *Let F be any field, $k \geqslant 2$. Then*

$$\mathrm{fc}_n(M_k(F)) < \mathrm{gfc}_n(M_k(F)) = k^{2(n+1)}$$

for all $n \in \mathbb{N}$.

Let e_{ij} be the matrix units of $M_k(F)$. Denote the basis of $M_k(F)^*$ dual to (e_{ij}) by $(\varphi_{m\ell})$. In other words, $\varphi^{m\ell}(e_{ij}) = \delta_i^m \delta_j^\ell$. We keep the same notation in the case of $\mathrm{UT}_2(F)$.

P r o o f o f T h e o r e m 2. Consider the natural basis

$$\left(\varphi^{i_1 j_1}(x_1) \ldots \varphi^{i_n j_n}(x_n) e_{m\ell}\right)$$

of the space $\mathrm{Hom}_F(\mathrm{UT}_2(F)^{\otimes n}; \mathrm{UT}_2(F))$. Note that

$$\dim \mathrm{Hom}_F(\mathrm{UT}_2(F)^{\otimes n}; \mathrm{UT}_2(F)) = (\dim \mathrm{UT}_2(F))^{n+1} = 3^{n+1}$$

and the number of functions

$$\psi^1(x_1)\psi^2(x_2) \ldots \psi^n(x_n) e_{11} \text{ where } \psi^i \in \{\varphi^{12}, \varphi^{22}\},$$

and

$$\mu^1(x_1)\mu^2(x_2) \ldots \mu^n(x_n) e_{22} \text{ where } \mu^i \in \{\varphi^{11}, \varphi^{12}\},$$

equals 2^{n+1}. We claim that any non-trivial combination H of these functions does not belong to $\dfrac{\mathrm{GFP}_n(\mathrm{UT}_2(F))}{\mathrm{GFId}_n(\mathrm{UT}_2(F))}$.

Indeed, if we substitute

$$x_1 = x_2 = \ldots = x_n = e_{12},$$

the functions from $\frac{\mathrm{GFP}_n(\mathrm{UT}_2(F))}{\mathrm{GFId}_n(\mathrm{UT}_2(F))}$ take values in the ideal $\langle e_{12}\rangle$. At the same time, $H(e_{12},\dots,e_{12}) \in \langle e_{11}, e_{22}\rangle$. Thus the coefficients in H near $\varphi^{12}(x_1)\varphi^{12}(x_2)\dots\varphi^{12}(x_n)e_{11}$ and $\varphi^{12}(x_1)\varphi^{12}(x_2)\dots\varphi^{12}(x_n)e_{22}$ equal zero.

If we substitute at least one e_{11} and maybe some e_{12}, then the functions from $\frac{\mathrm{GFP}_n(\mathrm{UT}_2(F))}{\mathrm{GFId}_n(\mathrm{UT}_2(F))}$ take values in the ideal $\langle e_{11}, e_{12}\rangle$. The image of the same elements under H belongs to $\langle e_{22}\rangle$. Thus the coefficients in H near $\mu^1(x_1)\mu^2(x_2)\dots\mu^n(x_n)e_{22}$ where $\mu^i \in \{\varphi^{11},\varphi^{12}\}$, equal zero.

If we substitute at least one e_{22} and maybe some e_{12}, then the functions from $\frac{\mathrm{GFP}_n(\mathrm{UT}_2(F))}{\mathrm{GFId}_n(\mathrm{UT}_2(F))}$ take values in the ideal $\langle e_{12}, e_{22}\rangle$. The image of the same elements under H belongs to $\langle e_{11}\rangle$. Thus the coefficients in H near $\psi^1(x_1)\psi^2(x_2)\dots\psi^n(x_n)e_{11}$ where $\psi^i \in \{\varphi^{12},\varphi^{22}\}$, equal zero.

In order to finish the proof, it sufficient to show that all the other basis elements of $\mathrm{Hom}_F(\mathrm{UT}_2(F)^{\otimes n};\mathrm{UT}_2(F))$ belong to $\frac{\mathrm{FP}_n(\mathrm{UT}_2(F))}{\mathrm{FId}_n(\mathrm{UT}_2(F))}$.

First, consider a map $\psi^1(x_1)\dots\psi^n(x_n)e_{12}$ where $\psi^j \in \{\varphi^{11},\varphi^{12},\varphi^{22}\}$. Depending on ψ^1, it can be defined by one of the following three functional polynomials:

$$\varphi^{11}(x_1)\Big(\psi^2(x_2)\dots\psi^n(x_n)\Big)e_{12} = x_1\Big(\psi^2(x_2)\dots\psi^n(x_n)e_{12}\Big),$$

$$\begin{aligned}\varphi^{12}(x_1)&\Big(\psi^2(x_2)\dots\psi^n(x_n)\Big)e_{12}\\ &= \Big(\psi^2(x_2)\dots\psi^n(x_n)e_{11}\Big)x_1 - x_1\Big(\psi^2(x_2)\dots\psi^n(x_n)e_{11}\Big),\end{aligned}$$

$$\varphi^{22}(x_1)\Big(\psi^2(x_2)\dots\psi^n(x_n)\Big)e_{12} = \Big(\psi^2(x_2)\dots\psi^n(x_n)e_{12}\Big)x_1.$$

If $\psi^j \in \{\varphi^{11},\varphi^{12},\varphi^{22}\}$ for all $1 \leqslant j \leqslant n$ but the inclusion $\psi^i \in \{\varphi^{12},\varphi^{22}\}$ is false for at least one i, then we have $\psi^i = \varphi^{11}$, and $\psi^1(x_1)\dots\psi^n(x_n)e_{11}$ can be defined by

$$\begin{aligned}\Big(\psi^1(x_1)\dots\psi^{i-1}(x_{i-1})\Big)&\varphi^{11}(x_i)\Big(\psi^{i+1}(x_{i+1})\dots\psi^n(x_n)\Big)e_{11}\\ &= x_i\Big(\psi^1(x_1)\dots\psi^{i-1}(x_{i-1})\psi^{i+1}(x_{i+1})\dots\psi^n(x_n)e_{11}\Big).\end{aligned}$$

If $\psi^j \in \{\varphi^{11}, \varphi^{12}, \varphi^{22}\}$ for all $1 \leqslant j \leqslant n$ but the inclusion $\psi^i \in \{\varphi^{11}, \varphi^{12}\}$ is false for at least one i, then we have $\psi^i = \varphi^{22}$, and $\psi^1(x_1)\dots\psi^n(x_n)e_{22}$ can be defined by

$$\Big(\psi^1(x_1)\dots\psi^{i-1}(x_{i-1})\Big)\varphi^{22}(x_i)\Big(\psi^{i+1}(x_{i+1})\dots\psi^n(x_n)\Big)e_{22}$$
$$= \Big(\psi^1(x_1)\dots\psi^{i-1}(x_{i-1})\psi^{i+1}(x_{i+1})\dots\psi^n(x_n)e_{22}\Big)x_i.$$

Again, we have a functional polynomial in the right side of the equality. $\square$

P r o o f o f T h e o r e m 3. Note that for every $1 \leqslant i_r, j_r, m, \ell \leqslant k$ the expression

$$(e_{mi_1}\, x_1\, e_{j_1 i_2}\, x_2\, e_{j_2 i_3}\, x_3 \dots\, e_{j_{n-1} i_n})\, x_n\, e_{j_n \ell}$$

can be considered as a generalized functional monomial since the multiplication is multilinear. Its image in $\mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F))$ equals $\varphi^{i_1 j_1}(x_1)\varphi^{i_2 j_2}(x_2)\dots \varphi^{i_n j_n}(x_n)e_{m\ell}$. In other words, we can obtain an arbitrary element of the natural basis in $\mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F))$. Hence

$$\frac{\mathrm{GFP}_n(M_k(F))}{\mathrm{GFId}_n(M_k(F))} = \mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F))$$

for all $k, n \in \mathbb{N}$, i.e. $\mathrm{gfc}_n(M_k(F)) = \dim \mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F)) = k^{2(n+1)}$. However

$$\varphi^{11}(x_1)\varphi^{11}(x_2)\dots\varphi^{11}(x_n)e_{22} \notin \frac{\mathrm{FP}_n(M_k(F))}{\mathrm{FId}_n(M_k(F))}.$$

Indeed, let $H \in \mathrm{Hom}_F(M_k(F)^{\otimes(n-1)}; M_k(F))$ be an arbitrary map. If we substitute $x_1 = x_2 = \ldots = x_n = e_{11}$, then the values of

$$H(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)x_i$$

and

$$x_i H(x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n)$$

belong to $\langle e_{1j}, e_{j1} \mid 1 \leqslant j \leqslant k\rangle \not\ni e_{22}$, i.e. there is no linear combination of functional polynomials that takes the value e_{22}. Thus

$$\frac{\mathrm{FP}_n(M_k(F))}{\mathrm{FId}_n(M_k(F))} \neq \mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F))$$

and

$$\mathrm{fc}_n(M_k(F)) < \dim \mathrm{Hom}_F(M_k(F)^{\otimes n}; M_k(F)) = \mathrm{gfc}_n(M_k(F)). \qquad \square$$

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REFERENCES

[1] Yu. A. Bahturin, M. Brešar. Lie gradings on associative algebras. *J. Algebra* **321**, *1* (2009), 264–283.

[2] Yu. A. Bahturin, M. Brešar, M. V. Kochetov. Group gradings on finitary simple Lie algebras. `arXiv:1106.2638v1 [math.RA]`

[3] K. I. Beidar, A. V. Mikhalev, M. A. Chebotar. Functional identities in rings and their applications. *Usp. Mat. Nauk* **59**, *3* (2004), 3–30; English translation in: *Russian Math. Surveys* **59**, *3* (2004), 403–428.

[4] A. Berele. Properties of hook Schur functions with applications to p.i. algebras. *Adv. in Appl. Math.* **41**, *1* (2008), 52–75.

[5] A. Berele, A. Regev. Asymptotic behaviour of codimensions of p.i. algebras satisfying Capelli identities. *Trans. Amer. Math. Soc.* **360** (2008), 5155–5172.

[6] M. Brešar. Functional identities of degree two. *J. Algebra* **172** (1995), 690–720.

[7] M. Brešar, M. Chebotar, W. S. Martindale. Functional Identities. Frontiers in Mathematics, Basel, Birkhäuser Verlag, 2007.

[8] A. Giambruno, I. P. Shestakov, M. V. Zaicev. Finite-dimensional nonassociative algebras and codimension growth. *Adv. in Appl. Math.* **47** (2011), 125–139.

[9] A. Giambruno, M. V. Zaicev. Polynomial Identities and Asymptotic Methods. Mathematical Surveys and Monographs, vol. **122**, Providence, RI, Amer. Math. Soc., 2005.

[10] A. S. Gordienko. Codimensions of functional identities. *Usp. Mat. Nauk* **64**, *1* (2009), 141–142; English translation in: *Russian Math. Surveys* **64**, *1* (2009), 148–149.

[11] A. S. Gordienko. A finiteness criterion and asymptotics for codimensions of generalized identities. *Mat. Zametki* **86**, *5* (2009), 681–685; English translation in: *Math. Notes* **86**, *5*, (2009), 645–649.

[12] M. V. Zaitsev. Integrality of exponents of growth of identities of finite-dimensional Lie algebras. *Izv. Ross. Akad. Nauk, Ser. Mat.* **66**, *3* (2002), 23–48; English translation in: *Izv. Math.* **66**, *3* (2002), 463–487.

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