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VARIETIES OF SUPERALGEBRAS OF POLYNOMIAL GROWTH*

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Abstract. Let $\mathcal{V}^{\rm gr}$ be a variety of associative superalgebras over a field F of characteristic zero. It is well-known that $\mathcal{V}^{\rm gr}$ can have polynomial or exponential growth. Here we present some classification results on varieties of polynomial growth. In particular we classify the varieties of at most linear growth and all subvarieties of the varieties of almost polynomial growth.

1. Introduction. The superalgebras and their graded identities play a relevant role in the structure theory of varieties developed by Kemer (see [10]). An effective way to distinguish varieties is that of defining invariants measuring the growth of the corresponding identities ([9]). In particular one considers the sequence of graded codimensions $c_n^{\rm gr}(A)$, $n = 1, 2, \ldots$, of a superalgebra A, where the n-th term measures the dimension of the space of multilinear polynomials in n fixed elements of the relatively free superalgebra of countable rank of the

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variety of superalgebras generated by A. It turns out that if an associative superalgebra satisfies an ordinary identity, then its sequence of graded codimensions is exponentially bounded. Moreover, the hyperoctahedral group $\mathbb{Z}_2 \wr S_n$ and its representation theory are a natural tool for studying the graded identities of a superalgebra in characteristic zero ([9], [4]).

For superalgebras satisfying an ordinary polynomial identity, it was shown in [1, 5] that the limit

$$\lim_{n\to\infty} \sqrt[n]{c_n^{\mathrm{gr}}(A)} = \exp^{\mathrm{gr}}(A)$$

exists and is a non-negative integer, called the superexponent of the algebra A.

Moreover $\exp^{\mathrm{gr}}(A)$ can be explicitly computed and it turns out to be equal to the dimension of a suitable finite dimensional semisimple superalgebra over an algebraically closed field.

Given a variety of superalgebras $\mathcal{V}^{\mathrm{gr}}$, the growth of $\mathcal{V}^{\mathrm{gr}}$ is the growth of the sequence of graded codimensions of any algebra A generating $\mathcal{V}^{\mathrm{gr}}$, i.e., $\mathcal{V}^{\mathrm{gr}} = \mathrm{var}^{\mathrm{gr}}(A)$.

The purpose of this paper is to present a survey on varieties of associative superalgebras of polynomial growth.

In such a case, if A is an algebra with 1, it was proved in [13, 15] that

$$c_n^{\mathrm{gr}}(A) = qn^k + O(n^{k-1})$$

is a polynomial with rational coefficients whose leading term satisfies the inequalities

$$\frac{1}{k!} \leq q \leq \sum_{i=0}^{k} 2^{k-i}\frac{(-1)^i}{i!}.$$

Moreover superalgebras realizing the smallest and the largest value of q were constructed.

Concerning the ordinary case, already in [3] it was proved that if A is a unitary algebra and $c_n(A)$ is polynomially bounded, then

$$c_n(A) = qn^k + O(n^{k-1}) \approx qn^k,$$

where q is a rational number satisfying the inequalities

$$\frac{1}{k!} \;\leq\; q \;\leq\; \sum_{j=2}^{k}\frac{(-1)^j}{j!}.$$

Later in [7] the authors constructed PI-algebras realizing the smallest and the largest value of q. They also proved that $q = \dfrac{1}{k!}$ is reached only in case k is even. For $k > 1$ odd the smallest value of q is given by $\dfrac{k-1}{k!}$.

A complete classification of the graded identities whose sequence of graded codimensions is linearly bounded was given in [6]. Moreover for each such ideal I, a finite dimensional superalgebra was exhibited having I as ideal of graded identities.

The problem of characterizing the graded identities of a superalgebra whose sequence of graded codimensions is polynomially bounded was studied in [8]. It was proved that a superalgebra A has such property if and only if its graded identities are not a consequence of the graded identities of five explicit superalgebras. Four of these algebras are the algebras G and UT_2 endowed with suitable $\mathbb{Z}_2$-gradings. In particular these results show that for the superalgebras, as for the ordinary case, no intermediate growth is allowed.

As a consequence, a classification was obtained of the varieties of superalgebras of almost polynomial growth. We recall that a variety has almost polynomial growth if it has exponential growth but any proper subvariety grows polynomially.

In [11, 12, 14] the author classified all subvarieties of the varieties and supervarieties of almost polynomial growth. Such a classification was given in terms of generators of the corresponding ideals of identities. Moreover, a complete list of finite dimensional algebras generating such subvarieties was exhibited. Concerning the ordinary variety generated by the Grassmann algebra, a complete description of its subvarieties was presented in another language in [16].

It is worth pointing out that the results obtained are based on the classification of minimal subvarieties of polynomial growth. These are precisely those varieties $\mathcal{V}^{\mathrm{gr}}$ such that $c_n^{\mathrm{gr}}(\mathcal{V}^{\mathrm{gr}}) \approx qn^k$ for some $k \geq 1, q > 0$, and for any proper subvariety $\mathcal{U}^{\mathrm{gr}} \subsetneqq \mathcal{V}^{\mathrm{gr}}$, $c_n^{\mathrm{gr}}(\mathcal{U}^{\mathrm{gr}}) \approx q'n^t$ with $t < k$.

2. Preliminaries.

Throughout the paper F will denote a field of characteristic zero and A an associative F-algebra satisfying a non-trivial polynomial identity (PI-algebra). Let $F\langle X\rangle$ be the free associative algebra on a countable set $X = \{x_1, x_2, \ldots\}$ and $\mathrm{Id}(A) = \{f \in F\langle X\rangle \mid f \equiv 0 \,\mathrm{in}\, A\}$ the T-ideal of (ordinary) polynomial identities of A. It is well known that in characteristic zero $\mathrm{Id}(A)$ is completely determined by its multilinear polynomials and we denote by

$$P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

the vector space of multilinear polynomials in the variables $x_1, \ldots, x_n$. The non-negative integer

$$c_n(A) = \dim_F \frac{P_n}{P_n \cap \mathrm{Id}(A)}, \ n \geq 1,$$

is called the n-th codimension of A.

Now assume that the algebra $A = A^{(0)} \oplus A^{(1)}$ is an associative $\mathbb{Z}_2$-graded algebra (or a superalgebra) over F.

Recall that the elements of $A^{(0)}$ and of $A^{(1)}$ are homogeneous of degree zero (or even elements) and of degree one (or odd elements), respectively. A subalgebra $B \subseteq A$ is a graded subalgebra if $B = (B \cap A^{(0)}) \oplus (B \cap A^{(1)})$.

The free associative algebra $F\langle X\rangle$ has a natural structure of superalgebra as follows: write $X = Y \cup Z$, the disjoint union of two countable sets. If we denote by $\mathcal{F}^{(0)}$ the subspace of $F\langle Y \cup Z\rangle$ spanned by all monomials in the variables of X having even degree in the variables of Z and by $\mathcal{F}^{(1)}$ the subspace spanned by all monomials of odd degree in Z, then $F\langle Y \cup Z\rangle = \mathcal{F}^{(0)} \oplus \mathcal{F}^{(1)}$ is a $\mathbb{Z}_2$-graded algebra called the free superalgebra on Y and Z over F.

Given a superalgebra A recall that $f(y_1, \ldots, y_n, z_1, \ldots, z_m) \in F\langle Y \cup Z\rangle$ is a graded identity of A if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$ for all $a_1, \ldots, a_n \in A^{(0)}$, $b_1, \ldots, b_m \in A^{(1)}$. Let $\mathrm{Id}^{\mathrm{gr}}(A)$ denote the set of graded identities of A. Notice that $\mathrm{Id}^{\mathrm{gr}}(A)$ is a T_2-ideal of $F\langle Y \cup Z\rangle$, i.e., an ideal invariant under all endomorphisms η of $F\langle Y \cup Z\rangle$ such that $\eta(\mathcal{F}^{(0)}) \subseteq \mathcal{F}^{(0)}$ and $\eta(\mathcal{F}^{(1)}) \subseteq \mathcal{F}^{(1)}$.

It is well known that in characteristic zero, every graded identity is equivalent to a system of multilinear graded identities. Hence if we denote by

$$P_n^{\mathrm{gr}} = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n, \quad w_i = y_i \ \text{ or } \ w_i = z_i, \quad i = 1, \ldots, n\}$$

the space of multilinear polynomials of degree n in $y_1, z_1, \ldots, y_n, z_n$, (i.e., y_i or z_i appears in each monomial at degree 1) the study of $\mathrm{Id}^{\mathrm{gr}}(A)$ is equivalent to the study of $P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)$, for all $n \geq 1$. The non-negative integer

$$c_n^{\mathrm{gr}}(A) = \dim_F \frac{P_n^{\mathrm{gr}}}{P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)}, \ n \geq 1,$$

is called the n-th graded codimension of A.

We define the corresponding complexity function

$$\tilde{c}^{\mathrm{gr}}(A, t) = \sum_{n \geq 0} c_n^{\mathrm{gr}}(A) \frac{t^n}{n!},$$

that is the exponential generating function of the sequence of graded codimensions.

Notice that any F-algebra A can be regarded as a superalgebra with trivial grading, i.e., $A = A^{(0)} \oplus A^{(1)}$ where $A = A^{(0)}$ and $A^{(1)} = 0$. Hence the theory of graded identities generalizes the ordinary theory of polynomial identities. The relation between ordinary codimensions and graded codimensions is given in [9]: given a superalgebra A, $c_n(A) \leq c_n^{\mathrm{gr}}(A)$ for all $n \geq 1$ and, in case A satisfies an ordinary polynomial identity then $c_n^{\mathrm{gr}}(A) \leq 2^n c_n(A)$.

If A is an algebra with 1, by [2] $\mathrm{Id}^{\mathrm{gr}}(A)$ is completely determined by its multilinear proper polynomials. Recall that $f(y_1, z_1, \ldots, y_n, z_n) \in P_n^{\mathrm{gr}}$ is a proper polynomial if it is a linear combination of elements of the type

$$z_{i_1} \cdots z_{i_k} w_1 \cdots w_m,$$

where $w_1, \ldots, w_m$ are left normed (long) Lie commutators in the y_is and z_is.

Let Γ_n^{gr} denote the subspace of P_n^{gr} of proper polynomials in y_1, $z_1, \ldots, y_n, z_n$ and $\Gamma_0^{\mathrm{gr}} = \mathrm{span}\{1\}$.

The sequence of proper graded codimensions is defined as

$$\gamma_n^{\mathrm{gr}}(A) = \dim \frac{\Gamma_n^{\mathrm{gr}}}{\Gamma_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)}, \quad n = 0, 1, 2, \ldots,$$

and

$$\tilde{\gamma}^{\mathrm{gr}}(A, t) = \sum_{n \geq 0} \gamma_n^{\mathrm{gr}}(A) \frac{t^n}{n!}$$

is the corresponding exponential generating function.

For a unitary algebra A, the relation between ordinary graded codimensions and proper graded codimensions (see for instance [2]), is given by

$$c_n^{\mathrm{gr}} A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A), \quad n = 0, 1, 2, \ldots. \tag{1}$$

This easily implies the following result relating the two exponential generating functions.

Corollary 2.1.

$$\tilde{c}^{\mathrm{gr}}(A, t) = \exp(t) \tilde{\gamma}^{\mathrm{gr}}(A, t).$$

As a consequence (see [15]) we derive that, for every $n = 0, 1, \ldots$,

$$\dim \Gamma_n^{\mathrm{gr}} = n! \sum_{i=0}^{n} 2^{n-i} \frac{(-1)^i}{i!}.$$

Given a set $S \subseteq F\langle Y \cup Z\rangle$ of polynomials, let us denote by $\langle S\rangle_{T_2}$ the T_2-ideal of $F\langle Y \cup Z\rangle$ generated by the set S. We say that a set of polynomials S' follows from S if $S' \subseteq \langle S\rangle_{T_2}$.

Lemma 2.1 [15, Lemma 2.2]**.** *Let $k \geq 2$ and $i \geq 1$. If k is odd then $\Gamma_{k+i}^{\mathrm{gr}}$ follows from Γ_k^{gr} plus the polynomial $[y_1, y_2] \cdots [y_k, y_{k+1}]$. Otherwise, $\Gamma_{k+i}^{\mathrm{gr}}$ follows simply from Γ_k^{gr}.*

In particular, as a consequence we have:

Corollary 2.2. *Let A be a superalgebra with 1. If for some $k \geq 1$, $\gamma_{2k}^{\mathrm{gr}}(A) = 0$, then $\gamma_m^{\mathrm{gr}}(A) = 0$ for all $m \geq 2k$.*

Corollary 2.3. *Let A be a superalgebra with 1. If the sequence $c_n^{\mathrm{gr}}(A)$, $n = 0, 1, 2, \ldots$, is polynomially bounded, then*

$$c_n^{\mathrm{gr}}(A) = qn^k + q_1 n^{k-1} + \cdots$$

is a polynomial with rational coefficients. Moreover its leading term satisfies the inequalities

$$\frac{1}{k!} \leq q \leq \sum_{i=0}^{k} 2^{k-i} \frac{(-1)^i}{i!}.$$

Proof. If $\gamma_{2k}^{\mathrm{gr}}(A) \neq 0$ for all $k \geq 0$, then by (1)

$$c_n^{\mathrm{gr}}(A) = \sum_{i=0}^{n} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A) \geq \sum_{k=0}^{[n/2]} \binom{n}{2k} = 2^{n-1},$$

where $[n/2]$ denotes the integer part of $n/2$. Hence, since $c_n^{\mathrm{gr}}(A)$ is polynomially bounded, we must have $\gamma_{2k}^{\mathrm{gr}}(A) = 0$, for some $k \geq 1$.

Let k be such that $\gamma_k^{\mathrm{gr}}(A) \neq 0$ and $\gamma_m^{\mathrm{gr}}(A) = 0$ for all $m > k$. Such an integer exists by Corollary 2.2. Hence by the above relation we have that

$$c_n^{\mathrm{gr}}(A) = \sum_{i=0}^{k} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A) = \binom{n}{k} \gamma_k^{\mathrm{gr}}(A) + \cdots$$

which is a polynomial in n of degree k with rational coefficients and leading term $q = \dfrac{\gamma_k^{\mathrm{gr}}(A)}{k!}$. Since $1 \le \gamma_k^{\mathrm{gr}}(A) \le \dim \Gamma_k^{\mathrm{gr}}$, we have

$$\frac{1}{k!} \le q \le \sum_{i=0}^{k} 2^{k-i} \frac{(-1)^i}{i!}. \qquad \square$$

In the next section we shall show that the upper bound and the lower bound of q are actually reached for every $k \ge 1$.

The problem of characterizing the graded identities of a superalgebra whose sequence of graded codimensions is polynomially bounded was studied in [8]. The authors generalized Kemer's theorem on the characterization of varieties of polynomial growth in the setting of varieties of superalgebras. We shall describe this result below.

For the infinite dimensional Grassmann algebra

$$G = \langle 1, e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \rangle,$$

we write G to mean G with the trivial grading and G^{gr} to mean G with the grading $(G^{(0)}, G^{(1)})$ where $G^{(0)}$ is the span of all monomials in the e_is of even length and $G^{(1)}$ is the span of all monomials in the e_is of odd length.

Also let UT_2 denote the algebra of 2×2 upper triangular matrices over F with trivial grading and let UT_2^{gr} denote the algebra UT_2 with grading $(UT_2^{(0)}, UT_2^{(1)})$ where $UT_2^{(0)} = Fe_{11} + Fe_{22}$ is the subspace of diagonal matrices and $UT_2^{(1)} = Fe_{12}$. Finally, let $F \oplus tF$ be the commutative algebra with grading (F, tF) where $t^2 = 1$.

The following result describes the graded identities and codimensions of the above superalgebras.

Lemma 2.2.

- $\mathrm{Id}^{\mathrm{gr}}(G) = \langle [y_1, y_2, y_3], z \rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(G) = 2^{n-1}$.
- $\mathrm{Id}^{\mathrm{gr}}(G^{\mathrm{gr}}) = \langle [y_1, y_2], [y, z], z_1 z_2 + z_2 z_1 \rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(G^{\mathrm{gr}}) = 2^n$.
- $\mathrm{Id}^{\mathrm{gr}}(UT_2) = \langle [y_1, y_2][y_3, y_4], z \rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(UT_2) = 2^{n-1}$.
- $\mathrm{Id}^{\mathrm{gr}}(UT_2^{\mathrm{gr}}) = \langle [y_1, y_2], z_1 z_2 \rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(UT_2^{\mathrm{gr}}) = 1 + n 2^{n-1}$.
- $\mathrm{Id}^{\mathrm{gr}}(F \oplus tF) = \langle [y_1, y_2], [y, z], [z_1, z_2] \rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(F \oplus tF) = 2^n$.

The main result of [8] says that the above five superalgebras characterize the graded varieties of polynomial growth.

Theorem 2.1. *Let A be a superalgebra. Then the sequence of graded codimensions $c_n^{\mathrm{gr}}(A)$, $n = 1, 2, \ldots$, is polynomially bounded if and only if*

$$G, G^{\mathrm{gr}}, UT_2, UT_2^{\mathrm{gr}}, F \oplus tF \notin \mathrm{var}^{\mathrm{gr}}(A).$$

Hence $\mathrm{var}^{\mathrm{gr}}(G), \mathrm{var}^{\mathrm{gr}}(UT_2), \mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}}), \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}}), \mathrm{var}^{\mathrm{gr}}(F \oplus tF)$ are the only varieties of superalgebras of almost polynomial growth. In particular these results show that also for the superalgebras no intermediate growth is allowed.

Recall that, given superalgebras A and B, we say that A is T_2-equivalent to B and we write $A \sim_{T_2} B$ if $\mathrm{Id}^{\mathrm{gr}}(A) = \mathrm{Id}^{\mathrm{gr}}(B)$. The following result gives the structure of a finite dimensional generating superalgebra of a given variety of polynomial growth.

Lemma 2.3 [6]**.** *Let A be a superalgebra and suppose that $c_n^{\mathrm{gr}}(A)$ is polynomially bounded. Then $A \sim_{T_2} B$ where $B = B_1 \oplus \cdots \oplus B_m$ with $B_1, \ldots, B_m$ finite dimensional superalgebras over F and* $\dim B_i/J(B_i) \leq 1$ *for all $i = 1, \ldots, m$, where $J(B_i)$ denotes the Jacobson radical of B_i.*

3. Superalgebras with 1 of polynomial codimension growth.

In this section we shall construct, for any fixed $k \geq 1$, a finite dimensional associative $\mathbb{Z}_2$-graded algebras with 1, whose graded codimension sequence behaves asymptotically like qn^k where $q = \frac{1}{k!}$ or $q = \sum_{i=0}^{k} 2^{k-i}\frac{(-1)^i}{i!}$. These are the largest and smallest possible values determined in Corollary 2.3.

Let $U_k = U_k(F)$ be the algebra of $k \times k$ upper triangular matrices with equal entries in the main diagonal. Hence if the e_{ij}s are the usual matrix units and $E = E_{k\times k}$ denotes the identity $k \times k$ matrix,

$$U_k = \left\{ \alpha E + \sum_{1\leq i<j\leq k} \alpha_{ij}e_{ij} \mid \alpha, \alpha_{ij} \in F \right\}.$$

Next we consider an elementary $\mathbb{Z}_2$-grading on U_k. Recall that if $\mathbf{g} = (g_1, \ldots, g_k) \in \mathbb{Z}_2^k$ is an arbitrary k-tuple of elements of $\mathbb{Z}_2$, then $\mathbf{g}$ defines an elementary $\mathbb{Z}_2$-grading on U_k by setting

$$U_k^{(0)} = \operatorname{span}\{E, e_{ij} \mid g_i + g_j = 0\} \text{ and } U_k^{(1)} = \operatorname{span}\{e_{ij} \mid g_i + g_j = 1\}$$

(recall that the equalities are taken modulo 2). We denote by $U_k^{\mathbf{g}}$ the algebra U_k with elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g}$.

Notice that the element $\mathbf{g}' = (g_1+g_1, \ldots, g_k+g_1)$ defines the same grading as $\mathbf{g}$. Hence, without loss of generality, we may assume that $\mathbf{g} = (0, g_2, \ldots, g_k)$. If A is a graded subalgebra of U_k, the induced grading on A is also called elementary.

Let

$$A_k = \bigoplus_{\mathbf{g}\in\mathbb{Z}_2^k} U_k^{\mathbf{g}}$$

be the direct sum of the algebras U_k with all possible elementary $\mathbb{Z}_2$-gradings. Notice that $\operatorname{Id}^{\mathrm{gr}}(A_k) = \bigcap_{\mathbf{g}\in\mathbb{Z}_2^k} \operatorname{Id}^{\mathrm{gr}}(U_k^{\mathbf{g}})$.

The next theorem shows that the graded codimension sequence of A_k realizes the largest possible value for q. For every $j \geq 1$, set

$$\theta_j = \frac{\dim \Gamma_j^{\mathrm{gr}}}{j!} = \sum_{i=0}^{j} 2^{j-i}\frac{(-1)^i}{i!}.$$

Theorem 3.1 [15, Theorem 3.1]**.** *For every $k \geq 2$ we have:*

1) $\operatorname{Id}^{\mathrm{gr}}(A_k) = \langle \Gamma_k^{\mathrm{gr}} \rangle_{T_2}$, *if k is even and* $\operatorname{Id}^{\mathrm{gr}}(A_k) = \langle \Gamma_k^{\mathrm{gr}}, [y_1, y_2] \cdots [y_k, y_{k+1}] \rangle_{T_2}$ *in case k is odd.*

2) $c_n^{\mathrm{gr}}(A_k) = \sum_{j=0}^{k-1} \frac{n!}{(n-j)!}\,\theta_j \approx \theta_{k-1} n^{k-1}, \quad n \to \infty.$

The relevance of A_k is shown in the following.

Theorem 3.2. *Let A be a unitary $\mathbb{Z}_2$-graded algebra such that $c_n^{\mathrm{gr}}(A) \approx an^k$ for some $a \in \mathbb{Q}$ and $k \geq 1$. Then* $\operatorname{Id}^{\mathrm{gr}}(A) \supseteq \operatorname{Id}^{\mathrm{gr}}(A_{k+1})$.

P r o o f. By (1) we have that $c_n^{\mathrm{gr}}(A) = \sum_{i=0}^{k} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A)$ and $\gamma_{k+i}^{\mathrm{gr}}(A) = 0$, $i \geq 1$. This says that $\Gamma_{k+i}^{\mathrm{gr}} = \Gamma_{k+i}^{\mathrm{gr}} \cap \operatorname{Id}^{\mathrm{gr}}(A)$, i.e., $\Gamma_{k+i}^{\mathrm{gr}} \subseteq \operatorname{Id}^{\mathrm{gr}}(A)$, $i \geq 1$ and, so, by the previous theorem, $\operatorname{Id}^{\mathrm{gr}}(A_{k+1}) \subseteq \operatorname{Id}^{\mathrm{gr}}(A)$. $\square$

We now turn to the problem of constructing finite dimensional $\mathbb{Z}_2$-graded algebras of polynomial graded codimension growth realizing the minimal possible value for q.

For $k \geq 1$, let G_k be the Grassmann algebra with 1 on a k-dimensional vector space over a field F of characteristic not equal to two. Recall that

$$G_k = \langle 1, e_1, \ldots, e_k \mid e_i e_j = -e_j e_i \rangle.$$

In the ordinary case the lower bound $\frac{1}{k!}$ is reached only in case k is even by G_k [7]. If G_k denotes the algebra G_k endowed with the trivial $\mathbb{Z}_2$-grading, then the graded codimensions are equal to the ordinary codimensions for all $n \geq 1$. Hence the lower bound is realized by G_k. More precisely we have the next result which follows from [7].

Theorem 3.3. *For every $k \geq 1$ we have:*

1) $\mathrm{Id}^{\mathrm{gr}}(G_{2k}) = \langle [y_1, y_2, y_3],\ [y_1, y_2] \cdots [y_{2k+1}, y_{2k+2}], z \rangle_{T_2}$

2) $c_n^{\mathrm{gr}}(G_{2k}) = \displaystyle\sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}, \quad n \to \infty.$

Next we show that the lower bound is reached for every $k \geq 1$ by a commutative subalgebra of U_k with a suitable non-trivial $\mathbb{Z}_2$-grading. We define the commutative subalgebra

$$C_k = C_k(F) = \{\alpha E + \sum_{1 \leq i < k} \alpha_i E_1^i \mid \alpha, \alpha_i \in F\} \subseteq U_k,$$

of U_k with elementary grading induced by $\mathbf{g} = (0, 1, 0, 1, \ldots) \in \mathbb{Z}_2^k$, where $E_1 = \displaystyle\sum_{i=1}^{k-1} e_{i,i+1}$.

Theorem 3.4 [15, Theorem 3.4]**.** *Let $k \geq 2$. Then*

1) $\mathrm{Id}^{\mathrm{gr}}(C_k) = \langle [y_1, y_2], [y, z], [z_1, z_2], z_1 \cdots z_k \rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(C_k) = \displaystyle\sum_{j=0}^{k-1} \binom{n}{j} \approx \frac{1}{(k-1)!} n^{k-1}$.

4. Classifying varieties of polynomial growth. The purpose of this section is to present a classification of the subvarieties of $\mathrm{var}^{\mathrm{gr}}(F \oplus tF)$, $\mathrm{var}^{\mathrm{gr}}(G)$ and $\mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$, where G and G^{gr} denote the Grassmann algebra endowed with the trivial and natural $\mathbb{Z}_2$-grading, respectively.

In [11, 12, 14] such a classification was given by exhibiting a complete list of finite dimensional algebras generating their subvarieties. The complete description of the subvarieties of $\mathrm{var}^{\mathrm{gr}}(G)$ was presented in a different language in [16].

The following theorem gives a classification of the subvarieties of the variety of superalgebras generated by G. Notice that, since $\mathrm{var}^{\mathrm{gr}}(G) = \mathrm{var}(G)$, this is equivalent to the classification of the ordinary subvarieties of $\mathrm{var}(G)$.

Theorem 4.1. *Let $A \in \mathrm{var}^{\mathrm{gr}}(G)$. Then either $A \sim_{T_2} G$ or $A \sim_{T_2} G_{2k} \oplus N$ or $A \sim_{T_2} G_1 \oplus N$ or $A \sim_{T_2} N$, where N is a nilpotent superalgebra and $k \geq 1$.*

Notice that the previous theorem allows us to classify all graded codimension sequences of the superalgebras lying in the variety generated by G.

Corollary 4.1. *Let $A \in \mathrm{var}^{\mathrm{gr}}(G)$ be such that $\mathrm{var}^{\mathrm{gr}}(A) \subsetneq \mathrm{var}^{\mathrm{gr}}(G)$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n^{\mathrm{gr}}(A) = 0$ or*

$$c_n^{\mathrm{gr}}(A) = \sum_{j=0}^{k} \binom{n}{2j} \approx \frac{1}{(2k)!} n^{2k}$$

for some $k \geq 0$.

Recall that if $\mathcal{V}^{\mathrm{gr}} = \mathrm{var}^{\mathrm{gr}}(A)$ is the variety of superalgebras generated by A, then $c_n^{\mathrm{gr}}(\mathcal{V}^{\mathrm{gr}}) = c_n^{\mathrm{gr}}(A)$ and the growth of $\mathcal{V}^{\mathrm{gr}}$ is the growth of the graded codimensions of A.

Definition 4.1. *A variety $\mathcal{V}^{\mathrm{gr}}$ is minimal of polynomial growth if $c_n^{\mathrm{gr}}(\mathcal{V}^{\mathrm{gr}}) \approx qn^k$ for some $k \geq 1$, $q > 0$, and for any proper subvariety $\mathcal{U}^{\mathrm{gr}} \subsetneq \mathcal{V}^{\mathrm{gr}}$ we have that $c_n^{\mathrm{gr}}(\mathcal{U}^{\mathrm{gr}}) \approx q'n^t$ with $t < k$.*

As a consequence of Theorem 4.1 we have

Corollary 4.2. *A superalgebra $A \in \mathrm{var}^{\mathrm{gr}}(G)$ generates a minimal variety if and only if $A \sim_{T_2} G_{2k}$, for some $k \geq 1$.*

For $k \geq 1$, let G_k^{gr} denote the algebra G_k endowed with the grading induced by G^{gr}.

Next we describe explicitly the identities of G_k^{gr} for any $k \geq 1$.

Theorem 4.2. *Let $k \geq 1$. Then*

1) $\mathrm{Id}^{\mathrm{gr}}(G_k^{\mathrm{gr}}) = \langle [y_1, y_2], [y, z], z_1 z_2 + z_2 z_1, z_1 \cdots z_{k+1} \rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(G_k^{\mathrm{gr}}) = \sum_{j=0}^{k} \binom{n}{j} \approx \frac{1}{(k)!} n^k$.

Theorem 4.3 [14, Theorem 7.2]**.** *For any $k \geq 1$, G_k^{gr} generates a minimal variety.*

In the following theorem all the subvarieties of $\mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$ are classified.

Theorem 4.4 [14, Theorem 7.3]**.** *Let $A \in \mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$. Then either $A \sim_{T_2} G^{\mathrm{gr}}$ or $A \sim_{T_2} N$ or $A \sim_{T_2} C \oplus N$ or $A \sim_{T_2} G_k^{\mathrm{gr}} \oplus N$, for some $k \geq 1$, where N is a nilpotent superalgebra and C is a commutative superalgebra with trivial grading.*

Proof. If $A \sim_{T_2} G^{\mathrm{gr}}$ there is nothing to prove. Now let A generate a proper subvariety of $\mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$. Since $\mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$ has almost polynomial growth, $\mathrm{var}^{\mathrm{gr}}(A)$ has polynomial growth and let $c_n^{\mathrm{gr}}(A) \approx qn^r$ for some $r \geq 0$.

By Lemma 2.3 we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional superalgebras such that $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Moreover (see for instance [14])

$$A = A_1 \oplus \cdots \oplus A_n = B \oplus N,$$

where B is a unitary superalgebra, N is a nilpotent superalgebra and, for n large enough,

$$c_n^{\mathrm{gr}}(A) = c_n^{\mathrm{gr}}(B) = \sum_{i=0}^{r} \binom{n}{i} \gamma_i^{\mathrm{gr}}(B).$$

In particular we derive that $\Gamma_{r+1}^{\mathrm{gr}} \subseteq \mathrm{Id}^{\mathrm{gr}}(B)$. This implies that $B \in \mathrm{var}^{\mathrm{gr}}(G_r^{\mathrm{gr}})$. Since G_r^{gr} generates a minimal variety and $c_n^{\mathrm{gr}}(G_r^{\mathrm{gr}}) \approx q'n^r$, we obtain that $B \sim_{T_2} G_r^{\mathrm{gr}}$, and, so, $A \sim_{T_2} G_r^{\mathrm{gr}} \oplus N$. $\square$

As a consequence we have the following corollaries.

Corollary 4.3.Let $A \in \mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$ be such that $\mathrm{var}^{\mathrm{gr}}(A) \subsetneq \mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n^{\mathrm{gr}}(A) = 0$ or

$$c_n^{\mathrm{gr}}(A) = \sum_{j=0}^{k} \binom{n}{j} \approx \frac{1}{(k)!} n^k$$

for some $k \geq 0$.

Corollary 4.4. *A superalgebra $A \in \mathrm{var}^{\mathrm{gr}}(G^{\mathrm{gr}})$ generates a minimal variety if and only if $A \sim_{T_2} G_k^{\mathrm{gr}}$, for some $k \geq 1$.*

P r o o f. The proof follows from Theorem 4.3 and the previous theorem. □

Recall that if $A = F + J$ is a finite dimensional superalgebra over F, where B is a semisimple graded subalgebra and $J = J(A)$ is its Jacobson radical, then J can be decomposed into the direct sum of graded B-bimodules

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11},$$

where for $i \in \{0, 1\}$, J_{ik} is a left faithful module or a 0-left module according as $i = 1$ or $i = 0$, respectively. Similarly, J_{ik} is a right faithful module or a 0-right module according as $k = 1$ or $k = 0$, respectively. Moreover, for $i, k, l, m \in \{0, 1\}$, $J_{ik}J_{lm} \subseteq \delta_{kl}J_{im}$ where δ_{kl} is the Kronecker delta and $J_{11} = BN$ for some nilpotent subalgebra N of A commuting with B.

Next we state a result that will be needed for the proof of the minimality of varieties inside the variety generated by $F \oplus tF$.

Lemma 4.1. *Let $A = F + J$ be a superalgebra with $J = J_{10} + J_{01} + J_{11} + J_{00}$. If A satisfies the graded identity $[y_1, y_2, \ldots, y_r] \equiv 0$ (respectively $[z_1, y_2, \ldots, y_r] \equiv 0$) for some $r \geq 2$, then $J_{10}^{(0)} = J_{01}^{(0)} = 0$ (respectively $J_{10}^{(1)} = J_{01}^{(1)} = 0$). In particular if $[y_1, y_2, \ldots, y_r] \equiv 0$ and $[z_1, y_2, \ldots, y_r] \equiv 0$ are graded identities of A, then $A = (F + J_{11}) \oplus J_{00}$, a direct sum of algebras.*

P r o o f. The proof is obvious because

$$J_{01} = J_{01}^{(0)} + J_{01}^{(1)} = [J_{01}^{(0)}, \underbrace{F, \cdots, F}_{r-1}] + [J_{01}^{(1)}, \underbrace{F, \cdots, F}_{r-1}]$$

and

$$J_{10} = J_{10}^{(0)} + J_{10}^{(1)} = [J_{10}^{(0)}, \underbrace{F, \cdots, F}_{r-1}] + [J_{10}^{(1)}, \underbrace{F, \cdots, F}_{r-1}].$$

Now we are in a position to prove that the superalgebras C_k, introduced in the previous section, generate minimal varieties. $\square$

Theorem 4.5. *For any $k \geq 2$, C_k generates a minimal variety.*

Proof. Suppose that the algebra $A \in \mathrm{var}^{\mathrm{gr}}(C_k)$ generates a subvariety of $\mathrm{var}^{\mathrm{gr}}(C_k)$ and $c_n^{\mathrm{gr}}(A) \approx qn^{k-1}$ for some $q > 0$. We shall prove that in this case $A \sim_{T_2} C_k$ and this will complete the proof.

By Lemma 2.3 we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional superalgebras such that $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Notice that this says that either $A_i \cong F + J(A_i)$ or $A_i = J(A_i)$ is a nilpotent algebra. Since

$$c_n^{\mathrm{gr}}(A) \leq c_n^{\mathrm{gr}}(A_1) + \cdots + c_n^{\mathrm{gr}}(A_m),$$

then there exists A_i such that $c_n^{\mathrm{gr}}(A_i) \approx bn^{k-1}$ for some $b > 0$. Hence

$$\mathrm{var}^{\mathrm{gr}}(C_k) \supseteq \mathrm{var}^{\mathrm{gr}}(A) \supseteq \mathrm{var}^{\mathrm{gr}}(F + J(A_i)) \supseteq \mathrm{var}^{\mathrm{gr}}(F + J_{11}(A_i))$$

and $c_n^{\mathrm{gr}}(F + J(A_i)) \approx bn^{k-1}$ for some $b > 0$. By Lemma 4.1, since $F + J(A_i)$ satisfies the identities $[z_1, y_2] \equiv 0$ and $[y_1, y_2] \equiv 0$, we obtain that $F + J(A_i) = (F + J_{11}(A_i)) \oplus J_{00}(A_i)$ and $c_n^{\mathrm{gr}}(F + J(A_i)) = c_n^{\mathrm{gr}}(F + J_{11}(A_i))$ for n large enough. Hence, in order to prove that $A \sim_{T_2} C_k$, it is enough to show that $F + J_{11}(A_i) \sim_{T_2} C_k$. Therefore, without loss of generality, we may assume that A is a unitary algebra. Hence

$$c_n^{\mathrm{gr}}(A) = \sum_{i=0}^{k} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A),$$

and, by Lemma 2.1, we get $\gamma_i^{\mathrm{gr}}(A) \neq 0$ for all $i \geq 2$. Now, since $A \in \mathrm{var}^{\mathrm{gr}}(C_k)$, we have that $\gamma_i^{\mathrm{gr}}(A) \leq \gamma_i^{\mathrm{gr}}(C_k) = 1$. It follows that $c_n^{\mathrm{gr}}(A) = c_n^{\mathrm{gr}}(C_k)$ for all n and so, $A \sim_{T_2} C_k$. $\square$

As in the proof of Theorem 4.3 we can easily prove the following theorem.

Theorem 4.6. *Let $A \in \mathrm{var}^{\mathrm{gr}}(F \oplus tF)$. Then either $A \sim_{T_2} F \oplus tF$ or $A \sim_{T_2} N$, or $A \sim_{T_2} C \oplus N$, or $A \sim_{T_2} C_k \oplus N$ for some $k \geq 2$, where N is a nilpotent superalgebra and C is a commutative superalgebra with trivial grading.*

Notice that the previous theorem allows us to classify all codimension sequences of the superalgebras belonging to the variety generated by $F \oplus tF$.

Corollary 4.5. *Let $A \in \mathrm{var}^{\mathrm{gr}}(F \oplus tF)$ be such that $\mathrm{var}^{\mathrm{gr}}(A) \subsetneq \mathrm{var}^{\mathrm{gr}}(F \oplus tF)$. Then there exists n_0 such that for all $n > n_0$ we must have either $c_n^{\mathrm{gr}}(A) = 0$ or*

$$c_n^{\mathrm{gr}}(A) = \sum_{j=0}^{k-1} \binom{n}{j}$$

for some $k \geq 0$.

We can also classify all superalgebras generating minimal varieties.

Corollary 4.6. *A superalgebra $A \in \mathrm{var}^{\mathrm{gr}}(F \oplus tF)$ generates a minimal variety if and only if $A \sim_{T_2} C_k$ for some $k \geq 2$.*

5. Classifying varieties of slow growth. In this section we present a classification, up to T_2-equivalence, of all the superalgebras generating varieties of at most linear growth.

In [6] the authors gave a complete list of finite dimensional superalgebras generating varieties of at most linear growth.

Moreover they also found a list of 24 superalgebras M_i, $1 \leq i \leq 24$, characterizing the supervarieties of linear growth, i.e., $c_n^{\mathrm{gr}}(A) \leq kn$ if and only if $M_1, \ldots, M_{24} \notin \mathrm{var}^{\mathrm{gr}}(A)$.

Before stating their results, we start by constructing the superalgebras involved in the classification. These are associative superalgebras belonging to the variety generated by UT_2 or UT_2^{gr} and whose graded codimensions grow polynomially ([14]).

For $k \geq 2$, let

$$N_k = \mathrm{span}\{E, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq U_k,$$

where E denotes the $k \times k$ identity matrix and $E_1 = \sum_{i=1}^{k-1} e_{i,i+1}$.

We write N_k to mean N_k with trivial grading.

We next state the following result characterizing the graded polynomial identities and the graded codimensions of N_k (see [7]).

Theorem 5.1. *Let $k \geq 3$. Then*

1) $\mathrm{Id}^{\mathrm{gr}}(N_k) = \langle [y_1, \ldots, y_k],\ [y_1, y_2][y_3, y_4], z\rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(N_k) = 1 + \displaystyle\sum_{j=2}^{k-1}(j-1)\binom{n}{j} \approx \frac{k-2}{(k-1)!}\, n^{k-1}, \quad n \to \infty.$

Notice that $N_k \in \mathrm{var}^{\mathrm{gr}}(UT_2)$ for any $k \geq 2$.

Definition 5.1. *Let $k \geq 2$ and $\mathbf{g} = (0, 1, \ldots, 1) \in \mathbb{Z}_2^k$. Let N_k^{gr} be the algebra N_k with the elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g}$.*

Notice that $N_k^{(0)}$ is a commutative subalgebra of N_k. This says that $[y_1, y_2] \equiv 0$ is a graded identity of N_k^{gr}. Moreover, since $N_k^{(1)} = \mathrm{span}\{e_{12}, e_{13} \ldots, e_{1k}\}$, we have that $z_1 z_2 \equiv 0$ is also a graded identity for N_k^{gr}.

Since by Lemma 2.2 $\mathrm{Id}^{\mathrm{gr}}(UT_2^{\mathrm{gr}}) = \langle [y_1, y_2], z_1 z_2\rangle_{T_2}$, we have that $N_k^{\mathrm{gr}} \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$ for any $k \geq 2$.

The following theorem describes the graded identities and the codimensions of N_k^{gr}.

Theorem 5.2. *If $k \geq 2$, then*

1) $\mathrm{Id}(N_k^{\mathrm{gr}}) = \langle [y_1, y_2], [z, y_1, \ldots, y_{k-1}], z_1 z_2\rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(N_k^{\mathrm{gr}}) = 1 + \displaystyle\sum_{j=1}^{k-1}\binom{n}{j} j \approx \frac{1}{(k-2)!}\, n^{k-1},\ n \to \infty.$

Let $UT_k = UT_k(F)$ be the algebra of $k \times k$ upper triangular matrices over F.

Given $A \subseteq UT_k$, we shall denote by A^* the subalgebra of UT_k obtained by flipping A along its second diagonal. Suppose that A is a subalgebra of UT_k, endowed with some $\mathbb{Z}_2$-grading. Then A^* is also a graded subalgebra of UT_k. This is easily seen by observing that if $A = A^{(0)} \oplus A^{(1)}$, then $A^* = (A^{(0)})^* \oplus (A^{(1)})^*$ is a $\mathbb{Z}_2$-grading on A^*.

Remark. Notice that f is a graded identity of A if and only if f^*, which is the polynomial obtained by reversing the order of the variables in each monomial of f, is a graded identity of A^*.

For $k \geq 2$, let

$$A_k = A_k(F) = \mathrm{span}\{e_{11}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{12}, e_{13}, \ldots, e_{1k}\} \subseteq UT_k.$$

We write A_k to mean the algebra A_k with trivial grading. Hence

$$A_k^* = \mathrm{span}\{e_{kk}, E_1, E_1^2, \ldots, E_1^{k-2}; e_{1k}, e_{2k}, \ldots, e_{k-1,k}\}$$

is endowed with trivial grading.

The following result describes explicitly the graded identities of A_k and A_k^* for any $k \geq 2$.

Theorem 5.3. *If $k \geq 2$, then*

1) $\mathrm{Id}^{\mathrm{gr}}(A_k) = \langle [y_1, y_2][y_3, y_4], [y_1, y_2]y_3 \ldots y_{k+1}, z\rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(A_k) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l-1) + 1 \approx qn^{k-1}$, *where $q \in \mathbb{Q}$ is a non-zero constant.*

Hence $\mathrm{Id}^{\mathrm{gr}}(A_k^*) = \langle [y_1, y_2][y_3, y_4], y_3 \ldots y_{k+1}[y_1, y_2], z\rangle_{T_2}$ *and* $c_n^{\mathrm{gr}}(A_k^*) = c_n^{\mathrm{gr}}(A_k)$.

Notice that A_k and A_k^* belong to the variety generated by UT_2.

Next we construct two algebras without unity with elementary $\mathbb{Z}_2$-grading in the variety generated by UT_2^{gr}.

Definition 5.2. *For $k \geq 2$, A_k^{gr} is the algebra A_k with elementary $\mathbb{Z}_2$-grading induced by $\mathbf{g} = (0, 1, \ldots, 1)$.*

Hence $(A_k^{\mathrm{gr}})^*$ is a superalgebra with grading $((A_k^{(0)})^*, (A_k^{(1)})^*)$.

We have the following:

Theorem 5.4. *If $k \geq 2$, then*

1) $\mathrm{Id}^{\mathrm{gr}}(A_k^{\mathrm{gr}}) = \langle [y_1, y_2], z_1 y_2 \cdots y_k, z_1 z_2\rangle_{T_2}$.

2) $c_n^{\mathrm{gr}}(A_k^{\mathrm{gr}}) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l) + 1 \approx qn^{k-1}$, *where $q \in \mathbb{Q}$ is a non-zero constant.*

Hence $\mathrm{Id}^{\mathrm{gr}}((A_k^{\mathrm{gr}})^*) = \langle [y_1, y_2], y_2 \cdots y_k z_1, z_1 z_2\rangle_{T_2}$ *and*

$$c_n^{\mathrm{gr}}((A_k^{\mathrm{gr}})^*) = \sum_{l=0}^{k-2} \binom{n}{l}(n-l) + 1.$$

Theorem 5.5. *The superalgebras N_r, N_k^{gr}, A_k, A_k^*, A_k^{gr}, and $(A_k^{\mathrm{gr}})^*$ generate minimal varieties for any $r > 2$ and $k \geq 2$.*

In what follows we state the results given in [6] in our notation. We start by giving a classification, up to T_2-equivalence, of the superalgebras whose sequence of codimensions is bounded by a constant.

Theorem 5.6. *For a superalgebra A, the following conditions are equivalent:*

1) $A_2, A_2^{\mathrm{gr}}, A_2^*, (A_2^{\mathrm{gr}})^*, N_2^{\mathrm{gr}}, N_3 \notin \mathrm{var}^{\mathrm{gr}}(A)$.
2) *A is T_2-equivalent to either N or $C \oplus N$, where N is a nilpotent superalgebra and C is a commutative algebra with trivial grading.*
3) *$c_n^{\mathrm{gr}}(A) \leq k$ for some constant $k \geq 0$, for all $n \geq 1$.*
4) *$c_n^{\mathrm{gr}}(A) = c_n(A) \leq 1$ for n large enough.*

The classification, up to T_2-equivalence, of the superalgebras whose sequence of codimensions is linearly bounded is given in the following theorem.

Theorem 5.7. *For a superalgebra A the following conditions are equivalent.*

1) *A is T_2-equivalent to either N, a nilpotent superalgebra, or $C \oplus N$, where C is a commutative algebra with trivial grading, or $N_2^{\mathrm{gr}} \oplus N$, or $B \oplus N$, or $B \oplus N_2^{\mathrm{gr}} \oplus N$, or $B_1 \oplus B_2 \oplus N$, or $B_1 \oplus B_2 \oplus N_2^{\mathrm{gr}} \oplus N$, where $B \in \mathcal{C}_1 \cup \mathcal{C}_2$, $B_1 \in \mathcal{C}_1$ and $B_2 \in \mathcal{C}_2$ with $\mathcal{C}_1 = \{A_2^*, (A_2^{\mathrm{gr}})^*, A_2^* \oplus (A_2^{\mathrm{gr}})^*\}$, $\mathcal{C}_2 = \{A_2, A_2^{\mathrm{gr}}, A_2 \oplus A_2^{\mathrm{gr}}\}$.*
2) *$c_n^{\mathrm{gr}}(A) \leq kn$ for all $n \geq 1$ for some constant k.*

As a consequence it is easily seen that the only allowed linearly bounded sequences of graded codimensions are the following, for n sufficiently large:

$$0, 1, n, n+1, 2n-1, 2n, 2n+1, 3n-1, 3n, 3n+1, 4n-1, 4n, 5n-1.$$

6. Classifying the subvarieties of $\mathrm{var}^{\mathrm{gr}}(UT_2)$ and $\mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$.

The following theorem gives a classification of the subvarieties of the variety generated by UT_2 ([11, 12, 14]).

Theorem 6.1. *If $A \in \mathrm{var}^{\mathrm{gr}}(UT_2)$, then A is T_2-equivalent to one of the following superalgebras:*

$$UT_2, N, N_t \oplus N, N_t \oplus A_k \oplus N, N_t \oplus A_r^* \oplus N, N_t \oplus A_k \oplus A_r^* \oplus N,$$

where N is a nilpotent superalgebra and $k, r, t \geq 2$.

The previous theorem allows to classify all graded codimension sequences of the superalgebras belonging to the variety generated by UT_2.

As a consequence we have:

Corollary 6.1. *Let $A \in \mathrm{var}^{\mathrm{gr}}(UT_2)$. Then A generates a minimal variety if and only if either $A \sim_{T_2} N_t$ or $A \sim_{T_2} A_k$ or $A \sim_{T_2} A_k^*$ for some $k \geq 2, t > 2$.*

Next we state some technical lemmas that will be needed for the classification of the proper subvarieties of UT_2^{gr} (see for instance [14]).

Lemma 6.1. *Let $A \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$ be a superalgebra with 1 such that $\mathrm{var}^{\mathrm{gr}}(A) \subsetneqq \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$. Then either $A \sim_{T_2} C$ or $A \sim_{T_2} N_k^{\mathrm{gr}}$ for some $k \geq 2$, where C is a commutative superalgebra with trivial grading.*

P r o o f. Since A generates a proper subvariety of UT_2^{gr}, then

$$c_n^{\mathrm{gr}}(A) = \sum_{i=0}^{k-1} \binom{n}{i} \gamma_i^{\mathrm{gr}}(A) \approx an^{k-1}$$

for some $k \geq 1$. If $k = 1$ then by Theorem 5.6 $A \sim_{T_2} C$, where C is a commutative superalgebra with trivial grading. Now we assume that $k > 1$. Since $\gamma_k^{\mathrm{gr}}(A) = 0$ then $[z_1, y_2 \ldots, y_k] \in \mathrm{Id}^{\mathrm{gr}}(A)$. Hence

$$\mathrm{Id}^{\mathrm{gr}}(N_k^{\mathrm{gr}}) = \langle [y_1, y_2], [z, y_1, \ldots, y_{k-1}], z_1 z_2 \rangle_{T_2} \subseteq \mathrm{Id}^{\mathrm{gr}}(A).$$

Since by Theorem 5.5, N_k^{gr} generates a minimal variety and $c_n^{\mathrm{gr}}(N_k^{\mathrm{gr}}) \approx qn^{k-1}$ for some constant q, it follows that $A \sim_{T_2} N_k^{\mathrm{gr}}$. $\square$

Lemma 6.2. *Let $A = F + J \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$. Then*

$$A \sim_{T_2} (F + J_{11} + J_{10} + J_{00}) \oplus (F + J_{11} + J_{01} + J_{00}).$$

Lemma 6.3. *Let $A = F + J_{11} + J_{10} + J_{00} \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$ with $J_{10} \neq 0$ (respectively $A = F + J_{11} + J_{01} + J_{00} \in \mathrm{var}(UT_2^{\mathrm{gr}})$ with $J_{01} \neq 0$). Then there exist constants $k, u \geq 2$ such that*

1) *if* $J_{11}^{(1)} = 0$, $A \sim_{T_2} A_k^{\mathrm{gr}} \oplus N$ *(respectively* $A \sim_{T_2} (A_k^{\mathrm{gr}})^* \oplus N$*), where* N *is a nilpotent superalgebra;*

2) *if* $J_{11}^{(1)} \neq 0$, $A \sim_{T_2} N_u \oplus A_k^{\mathrm{gr}} \oplus N$ *(respectively* $A \sim_{T_2} N_u \oplus (A_k^{\mathrm{gr}})^* \oplus N$*).*

From the previous lemmas we derive the following.

Lemma 6.4. *Let* $A = F + J \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$ *with* $J_{10} \neq 0$ *and* $J_{01} \neq 0$*. Then either* $A \sim_{T_2} A_k^{\mathrm{gr}} \oplus (A_r^{\mathrm{gr}})^* \oplus N$ *or* $A \sim_{T_2} N_t^{\mathrm{gr}} \oplus A_k^{\mathrm{gr}} \oplus (A_r^{\mathrm{gr}})^* \oplus N$*, where* N *is a nilpotent superalgebra for some constants* $k, r, t \geq 2$*.*

Now we are in a position to classify all the subvarieties of the variety generated by UT_2^{gr}.

Theorem 6.2 [14, Theorem 6.1]**.** *If* $A \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$ *then* A *is* T_2*-equivalent to one of the following algebras:* UT_2^{gr}, N, $C \oplus N$, $N_t^{\mathrm{gr}} \oplus N$, $A_k^{\mathrm{gr}} \oplus N$, $(A_r^{\mathrm{gr}})^* \oplus N$, $N_t^{\mathrm{gr}} \oplus A_k^{\mathrm{gr}} \oplus N$, $N_t^{\mathrm{gr}} \oplus (A_r^{\mathrm{gr}})^* \oplus N$, $A_k^{\mathrm{gr}} \oplus (A_r^{\mathrm{gr}})^* \oplus N$, $N_t^{\mathrm{gr}} \oplus A_k^{\mathrm{gr}} \oplus (A_r^{\mathrm{gr}})^* \oplus N$*, where* N *is a nilpotent superalgebra,* C *is a commutative superalgebra with trivial grading and* $k, r, t \geq 2$*.*

P r o o f. If $A \sim_{T_2} UT_2^{\mathrm{gr}}$ there is nothing to prove. Hence we may assume that A generates a proper subvariety of UT_2^{gr} and, so, $c_n^{\mathrm{gr}}(A)$ is polynomially bounded. By Lemma 2.3 we may assume that

$$A = A_1 \oplus \cdots \oplus A_m,$$

where $A_1, \ldots, A_m$ are finite dimensional superalgebras such that $\dim A_i/J(A_i) \leq 1$, $1 \leq i \leq m$. Now, if $\dim A_i/J(A_i) = 0$, A_i is nilpotent. Suppose that i is such that $\dim A_i/J(A_i) = 1$. Then $A_i = F + J(A_i)$ and let $J(A_i) = J_{11} + J_{10} + J_{01} + J_{00}$.

If $J_{10} = J_{01} = 0$, then by Lemma 6.1, either $A_i \sim_{T_2} C \oplus N$ or $A_i \sim_{T_2} N_{t_i}^{\mathrm{gr}} \oplus N$ for some $t_i \geq 2$, where N is a nilpotent superalgebra and C is a commutative superalgebra with trivial grading. Otherwise, by Lemmas 6.3 and 6.4, A_i is T_2-equivalent to one of the algebras $A_{k_i}^{\mathrm{gr}} \oplus N$, $N_{t_i}^{\mathrm{gr}} \oplus A_{k_i}^{\mathrm{gr}} \oplus N$, $(A_{r_i}^{\mathrm{gr}})^* \oplus N$, $N_{t_i}^{\mathrm{gr}} \oplus (A_{r_i}^{\mathrm{gr}})^* \oplus N$, $A_{k_i}^{\mathrm{gr}} \oplus (A_{r_i}^{\mathrm{gr}})^* \oplus N$, $N_{t_i}^{\mathrm{gr}} \oplus A_{k_i}^{\mathrm{gr}} \oplus (A_{r_i}^{\mathrm{gr}})^* \oplus N$ for some $k_i, r_i, t_i \geq 2$. Since $A = A_1 \oplus \cdots \oplus A_m$, by collecting together these results we obtain the desired conclusion. □

It is worth noticing that the previous theorem allows us to classify all algebras generating minimal varieties.

Corollary 6.2. *Let* $A \in \mathrm{var}^{\mathrm{gr}}(UT_2^{\mathrm{gr}})$*. Then* A *generates a minimal variety if and only if either* $A \sim_{T_2} N_k^{\mathrm{gr}}$ *or* $A \sim_{T_2} A_k^{\mathrm{gr}}$*, or* $A \sim_{T_2} (A_k^{\mathrm{gr}})^*$ *for some* $k \geq 2$*.*

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