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GRADINGS AND GRADED IDENTITIES FOR THE MATRIX ALGEBRA OF ORDER TWO IN CHARACTERISTIC 2

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Dedicated to Professor Yuri Bahturin on the occasion of his 65th birthday

Abstract. Let K be an infinite field and let $M_2(K)$ be the matrix algebra of order two over K. The polynomial identities of $M_2(K)$ are known whenever the characteristic of K is different from 2. The algebra $M_2(K)$ admits a natural grading by the cyclic group of order 2; the graded identities for this grading are known as well. But $M_2(K)$ admits other gradings that depend on the field and on its characteristic. Here we describe the graded identities for all nontrivial gradings by the cyclic group of order 2 when the characteristic of K equals 2. It turns out that there is only one grading to consider. This grading is not elementary. On the other hand the graded identities are the same as for the elementary grading.

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Introduction. The polynomial identities satisfied by matrix algebras have been attracting the attention of algebraists for more than 60 years. One of the first results in that direction is the well known theorem due to Amitsur and Levitzki that describes the polynomial identities of least degree for the matrix algebra $M_n(K)$ of order n over a field K. Nevertheless little is known about the concrete form of the identities satisfied by $M_n(K)$. In 1973 Razmyslov, see [15] (or [16]) found a finite generating set of the T-ideal (the ideal of identities) for $M_2(K)$ when $\operatorname{char} K = 0$. Later on Drensky described a minimal generating set of identities, see [8]. The first named author of this paper found a finite basis of identities of $M_2(K)$ when K is infinite and $\operatorname{char} K = p \neq 2$, see [11]. It turned out that the basis of identities is exactly the same as in characteristic 0, when $p > 3$; one extra identity is needed when $p = 3$, see [4].

From now on we assume that K is an infinite field. The concrete form of the identities of $M_n(K)$ is unknown when $n > 2$. It is a sort of a mystery whether the T-ideal of $M_2(K)$ is finitely generated or not when $\operatorname{char} K = 2$. The difficulties in describing the polynomial identities of $M_n(K)$, $n > 2$, are of technical and theoretical nature, and the solution of this problem seems to be very distant. This is one of the reasons people study other kinds of identities: weak, with trace, with involution, graded, and so on. But the weak identities and the identities with involution are also very hard to describe: these are known only for $M_2(K)$, see [15, 16] for the weak identities of $(M_2(K), sl_2(K))$ in characteristic 0, and [10] in characteristic $p \neq 2$, and [13], [5] for the identities with involution for $M_2(K)$ in characteristic 0 and $p > 2$, respectively. The trace identities for $M_n(K)$ were described in characteristic 0 by Procesi [14] and by Razmyslov, see for example [16]. Later on the trace identities of $M_n(K)$ were described in positive characteristic, see [7, 18].

During the last decades graded identities became a very important topic of study. The matrix algebra of order n admits a natural grading by the cyclic group $\mathbb{Z}_n$ of order n where the matrix units e_{ij} are of degree $j - i \pmod{n}$. The identities for this grading were described by Vasilovsky [17] (in characteristic 0), and by Azevedo [2] in any characteristic. The matrix algebras admit gradings by various groups. In many occasions the corresponding graded identities are known.

In this paper we describe the graded identities of $M_2(K)$ when K is an infinite field of characteristic 2. When the grading is the natural one and $\operatorname{char} K = 0$ these were obtained by Di Vincenzo in [6], and when $\operatorname{char} K > 0$ in [1]. (We mention here that in [1] the authors imposed the restriction $\operatorname{char} K = p \neq 2$ but the proof given there is in fact characteristic-free.) We obtain that the graded identities are the same as for the natural grading. The methods we use resemble those of [1].

It should be noted that here we consider non-elementary gradings (i.e., gradings where the matrix units are not homogeneous). Our gradings depend on the properties of the field. As the gradings are not elementary one cannot extend the scalars to an algebraically closed field nor can one use the results of [12]. In the latter paper the following theorem was proved. Let G be an abelian group and K an algebraically closed field such that the orders of all finite subgroups of G are invertible in K. Then two finite dimensional G-graded simple algebras are isomorphic if and only if they satisfy the same graded identities. But this result cannot be applied to our case since the group is of order 2.

1. Preliminaries. Let K be an infinite field. Unless otherwise stated we assume K is of characteristic 2. All algebras and vector spaces we consider will be over K. Let G be a group. An algebra A is G-graded if $A = \oplus_{g \in G} A_g$ is a direct sum of the vector subspaces A_g, and $A_g A_h \subseteq A_{gh}$ for every g, $h \in G$. All G-gradings on $A = M_2(K)$ were described in [3] and in [9].

We recall the corresponding result of [3, 9]. If $\operatorname{char} K \neq 2$ then any *nontrivial* G-grading on A is isomorphic to one of the following four gradings.

I. $A_1 = \operatorname{sp}(e_{11}, e_{22})$, $A_g = \operatorname{sp}(e_{12}, e_{21})$ for some $g \in G$, $|g| = 2$, and $A_h = 0$ for all $h \in G$, $h \neq 1$, $h \neq g$. Call this the natural grading of A.

II. $A_1 = \operatorname{sp}(I, e_{12} + be_{21})$, $A_g = \operatorname{sp}(e_{11} - e_{22}, e_{12} - be_{21})$ for some $g \in G$, $|g| = 2$, and $A_h = 0$ for all remaining $h \in G$. Here I is the identity matrix in $M_2(K)$ and $b \in K$ is not a square in K.

III. $A_1 = \operatorname{sp}(e_{11}, e_{22})$, $A_g = \operatorname{sp}(e_{12})$, $A_{g^{-1}} = \operatorname{sp}(e_{21})$ where $g \in G$, $|g| > 2$, and $A_h = 0$ for all $h \notin \{1, g, g^{-1}\}$.

IV. $A_1 = \operatorname{sp}(I)$, $A_g = \operatorname{sp}(P)$, $A_h = \operatorname{sp}(Q)$, $A_{gh} = \operatorname{sp}(PQ)$, and $A_t = 0$ for all $t \notin \{1, g, h, gh\}$. Here g, $h \in G$, $|g| = |h| = 2$, and $gh = hg$. Also P^2 and Q^2 are nonzero scalar multiples of I and $PQ = -QP$.

If $\operatorname{char} K = 2$ then any nontrivial G-grading on $A = M_2(K)$ is isomorphic to either (1), or (3), or else to the grading

$$(1) \qquad A_1 = \left\{ \begin{pmatrix} x & x+y \\ b(x+y) & y \end{pmatrix} \right\}, \qquad A_g = \left\{ \begin{pmatrix} bx+y & x \\ y & bx+y \end{pmatrix} \right\},$$

and $A_h = 0$ for all $h \neq 1$, $h \neq g$; $|g| = 2$. Moreover $x, y \in K$, $b \in K$ is not of the form $t^2 + t$ for any $t \in K$. The reader may check that this is indeed a grading. Moreover, if $\operatorname{char} K \neq 2$ then A_1 and A_g as above are vector subspaces of $A = M_2(K)$ but $A = A_1 \oplus A_g$ is *not* a grading on A.

Now we note that in the latter grading one may consider, without loss of generality, $G = \mathbb{Z}_2$, the cyclic group of order 2. Then it is easier to switch to additive notation for G: thus we write A_0 for A_1, and A_1 for A_g.

Let X be an infinite countable set and form the free associative algebra $K\langle X\rangle$ freely generated over K by the set X. It is convenient to think of $K\langle X\rangle$ as the algebra of the polynomials in the noncommuting variables X. Assume now $X = Y \cup Z$ where Y and Z are disjoint infinite sets. The algebra $K\langle X\rangle$ is $\mathbb{Z}_2$-graded in a natural way, assuming the variables from Y to be of degree 0 (i.e., even variables), and those of Z of degree 1 (i.e., odd variables). If A is a $\mathbb{Z}_2$-graded algebra then the polynomial $f(x_1, \ldots, x_n) \in K\langle X\rangle$ is a *graded polynomial identity* (or simply graded identity) for A whenever $f(a_1, \ldots, a_n) = 0$ in A for every choice of $a_i \in A$ such that $a_i \in A_0$ whenever $x_i \in Y$, and $a_i \in A_1$ whenever $x_i \in Z$. In other words f lies in the kernels of all graded homomorphisms from $K\langle X\rangle$ to A. Clearly the set $T_G(A)$ of all graded identities of A is an ideal in $K\langle X\rangle$, and this ideal is invariant under all endomorphisms of $K\langle X\rangle$ that respect the grading. Conversely every ideal in $K\langle X\rangle$ that is invariant under graded endomorphisms, is the ideal of graded identities for some G-graded algebra. We call these ideals T_G-ideals.

Let $f, g \in K\langle X\rangle$. Then g is a consequence of f (or g follows from f as a graded identity) if g belongs to the ideal of graded identities in $K\langle X\rangle$ generated by f. If in addition f follows from g then f and g are equivalent as graded identities. Let T be a T_G-ideal, then the polynomials $\{f_i \mid i \in I\}$ form a basis of T if the T_G-ideal they generate coincides with T.

It was proved in [6, 1], that for the natural $\mathbb{Z}_2$-grading on $M_2(K)$, the polynomials $y_1y_2 - y_2y_1$, $z_1z_2z_3 - z_3z_2z_1$ form a basis of the corresponding graded identities.

Since our field is infinite every T_G-ideal has a basis consisting of multihomogeneous elements. Therefore we can and shall work with multihomogeneous polynomials only. (Here the adjective *multihomogeneous* refers to the usual multigrading on $K\langle X\rangle$, by the degree in each one of the variables.)

2. Basis of the graded identities.

Here we fix $G = \mathbb{Z}_2$, $\operatorname{char} K = 2$, and $A = M_2(K)$ with the grading given in (1).

Lemma 1. *The graded polynomials*

$$y_1 y_2 + y_2 y_1, \qquad z_1 z_2 z_3 + z_3 z_2 z_1 \tag{2}$$

are graded identities for $M_2(K)$.

Proof. The proof is a straightforward and easy computation and therefore will be omitted. One uses the fact that $\operatorname{char} K = 2$ in order to show these are indeed graded identities. $\square$

Denote by I the T_G-ideal generated by the two polynomials from (2). Then $I \subseteq T_G(M_2(K))$. Running ahead we shall prove that $I = T_G(M_2(K))$.

First we construct a convenient model for the relatively free graded algebra $K\langle X\rangle/T_G(M_2(K))$. Let y_i^j, z_i^j be commuting variables, $i \geq 1$, $j = 1, 2$. Pay attention that j is an upper index, not an exponent. Denote

$$A_i = \begin{pmatrix} y_i^1 & y_i^1 + y_i^2 \\ b(y_i^1 + y_i^2) & y_i^2 \end{pmatrix}, \quad B_i = \begin{pmatrix} bz_i^1 + z_i^2 & z_i^1 \\ z_i^2 & bz_i^1 + z_i^2 \end{pmatrix} \in M_2(K[y_i^j, z_i^j])$$

the corresponding generic graded matrices. Denote further by F the subalgebra of $M_2(K[y_i^j, z_i^j])$ generated by the matrices A_i and B_i, $i \geq 1$. Then F is graded by assuming A_i of degree 0, and B_i of degree 1. The following fact is well known.

Lemma 2. *The graded algebra F is isomorphic to the relatively free graded algebra $K\langle X\rangle/T_G(M_2(K))$.*

The next lemma as well as its proof can be found in [6] and also in [1].

Lemma 3. *Let $g \in K\langle X\rangle_0$, then for every y_i one has $y_i g + g y_i \in I \subseteq T_G(M_2(K))$.*

For the proof of the following proposition, see also [6, 1].

Proposition 4. *The graded algebra $K\langle X\rangle/I$ is spanned by the products of the following three types.*

(a) $y_{a_1} y_{a_2} \cdots y_{a_k}$;

(b) $y_{a_1} y_{a_2} \cdots y_{a_k} z_{c_1} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m} \widehat{z_{c_{m+1}}}$;

(c) $y_{a_1} y_{a_2} \cdots y_{a_k} z_{c_1} y_{b_1} y_{b_2} \cdots y_{b_l} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m} \widehat{z_{c_{m+1}}}$

where $a_1 \leq a_2 \leq \cdots \leq a_k$, $b_1 \leq b_2 \leq \cdots \leq b_l$, $c_1 \leq c_2 \leq \cdots \leq c_m \leq c_{m+1}$, $d_1 \leq d_2 \leq \cdots \leq d_m$. Also $k \geq 0$, $l \geq 0$, $m \geq 0$. In the third type when $k = l = 0$ the total degree is ≥ 2. The hat over a variable means that the corresponding variable may be missing.

In the next several statements we shall prove that the elements defined in Proposition 4 are linearly independent as elements of the algebra F.

Proposition 5. *The monomials of type* (a) *that appear in Proposition* 4 *are linearly independent as elements of the algebra* F.

Proof. By using the graded identity $y_1y_2 + y_2y_1 = 0$ one can order the variables in a monomial. Now the proof follows since K is infinite and we consider multihomogeneous elements only. $\square$

Proposition 6. *The monomials of type* (b) *from Proposition* 4 *are linearly independent modulo the graded identities of* $M_2(K)$.

Proof. The proof of the proposition consists in substituting the variables by generic matrices. The computation is quite far from being "nice" and due to this reason we split it in parts.

Case 1. Suppose no variables y appear in our monomial M, i.e., $k = 0$ and $M = z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}z_{d_m}$, $c_1 \le \cdots \le c_m$ and $d_1 \le \cdots \le d_m$. Substitute z_i by the generic matrix B_i and form the product $B = B_{c_1}B_{d_1}B_{c_2}B_{d_2}\cdots B_{c_m}B_{d_m}$. Then the $(1,1)$-entry of B equals

$$\begin{aligned}
&z^2_{c_1}z^2_{d_1}z^2_{c_2}z^2_{d_2}\cdots z^2_{c_m}z^2_{d_m} + z^1_{c_1}z^2_{d_1}z^2_{c_2}z^2_{d_2}\cdots z^2_{c_m}z^2_{d_m} + z^2_{c_1}z^2_{d_1}z^1_{c_2}z^2_{d_2}\cdots z^2_{c_m}z^2_{d_m}\\
&+\cdots + z^1_{c_1}z^2_{d_1}z^1_{c_2}z^2_{d_2}\cdots z^2_{c_m}z^2_{d_m} + \cdots + z^1_{c_1}z^2_{d_1}z^1_{c_2}z^2_{d_2}\cdots z^1_{c_m}z^2_{d_m}\\
&+bg(z^i_{c_1}, z^i_{d_1}, z^i_{c_2}, z^i_{d_2}, \ldots, z^i_{c_m}, z^i_{d_m}).
\end{aligned}$$

Here g is a polynomial whose monomials are with coefficients 0 or 1, the upper indices i in g may assume values 1 and 2 independently, and moreover none of its monomials equals the remaining monomials in the above expansion.

Using the above form of the $(1,1)$-entry of the product we can recover uniquely our monomial. We note that the upper index 1 appears for the variables $\{z_{c_i}\}$ and only for them. (If one suspects that computing the $(2,2)$-entry would yield a similar result but exchanging c_i and d_i one's guess will be correct.)

In a similar manner we deal with $M = z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}z_{d_m}z_{c_{m+1}}$. As above the $(1,1)$-entry of the matrix B will be a sum of monomials (with coefficients 1) where the upper index 1 comes from the variables z_{d_i}, in all possible ways, plus some polynomial multiplied by b, as above.

Case 2. Suppose $k \ge 1$ and $M = y_{a_1}y_{a_2}\cdots y_{a_k}z_{c_1}z_{d_1}z_{c_2}z_{d_2}\cdots z_{c_m}z_{d_m}$.

Consider first the part $y_{a_1}y_{a_2}\cdots y_{a_k}$. Substituting y_i by A_i we get at position $(1,1)$ of the resulting matrix the element $y^1_{a_1}y^1_{a_2}\cdots y^1_{a_k} + bq_k$ where q_k is a sum of monomials in the y^j_i and q_k does not contain the monomial $y^1_{a_1}\cdots y^1_{a_k}$.

Now substitute y_i by A_i and z_i by B_i for all variables that appear in M, and compute once again the $(1,1)$-entry. It will be, according to Case 1 and to the above considerations, $y_{a_1}^1 y_{a_2}^1 \cdots y_{a_k}^1$ multiplied by the sum

$$\begin{aligned} &z_{c_1}^2 z_{d_1}^2 z_{c_2}^2 z_{d_2}^2 \cdots z_{c_m}^2 z_{d_m}^2 + z_{c_1}^1 z_{d_1}^2 z_{c_2}^2 z_{d_2}^2 \cdots z_{c_m}^2 z_{d_m}^2 + z_{c_1}^2 z_{d_1}^2 z_{c_2}^1 z_{d_2}^2 \cdots z_{c_m}^2 z_{d_m}^2 \\ &+ \cdots + z_{c_1}^1 z_{d_1}^2 z_{c_2}^1 z_{d_2}^2 \cdots z_{c_m}^2 z_{d_m}^2 + \cdots + z_{c_1}^1 z_{d_1}^2 z_{c_2}^1 z_{d_2}^2 \cdots z_{c_m}^1 z_{d_m}^2 \end{aligned}$$

plus some polynomial of the type $bh(y_{a_j}^i, z_{c_n}^i, z_{d_n}^i)$. The polynomial h does not contain any of the above monomials.

The monomials $M = y_{a_1} y_{a_2} \cdots y_{a_k} z_{c_1} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m} z_{c_{m+1}}$ are dealt with in the same manner. (The upper index 1 will appear for the variables z_{d_i} and not for z_{c_i}.)

Thus we finish the proof of Proposition 6. □

Proposition 7. *The monomials of the type* (c) *from Proposition* 4 *are linearly independent modulo the graded identities of* $M_2(K)$.

Proof. As in the previous proposition we shall consider separately two cases.

Case 1. There are no leading y in the monomial. In other words we consider $M = z_{c_1} y_{b_1} y_{b_2} \cdots y_{b_l} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m}$. As above we substitute y_i by A_i and z_i by B_i and then compute the $(1,1)$-entry of the matrix thus obtained. We split the resulting polynomial into three parts. The first part is obtained as the product of $z_{c_1}^2 y_{b_1}^2 \cdots y_{b_l}^2 z_{d_1}^2 \cdots z_{d_m}^2$ and of the sum of all monomials (with coefficient 1) of the type $z_{c_2}^* \cdots z_{c_m}^*$ where the upper indices assume all possible values of 1 and 2. The second part consists of $z_{c_1}^1 y_{b_1}^2 \cdots y_{b_l}^2 z_{d_1}^2 \cdots z_{d_m}^2$ multiplied by the same sum as in the first part. The third part is of the kind $bh(y_{b_j}^*, z_{c_j}^*, z_{d_j}^*)$ where the coefficients of the monomials in h are also 1 and 0, and no monomial of the above listed appears in h.

Case 2. $M = y_{a_1} y_{a_2} \cdots y_{a_k} z_{c_1} y_{b_1} y_{b_2} \cdots y_{b_l} z_{d_1} z_{c_2} z_{d_2} \cdots z_{c_m} z_{d_m}$. Once again we substitute y_i by A_i and z_i by B_i, and compute the $(1,1)$-entry of the resulting matrix. The only difference with Case 1 will be the multiple $y_{a_1}^1 \cdots y_{a_k}^1$.

Now observe that one can recover uniquely the monomial M knowing the $(1,1)$-entry of the matrix.

In order to finish the proof we have to consider the monomials with $z_{c_{m+1}}$. The computation is quite similar to that of Proposition 6, and is done using the above part of the proof of this proposition. That is why we leave it to the reader. □

Theorem 8. *The graded identities from* (2) *form a basis of the graded identities of $M_2(K)$ when K is an infinite field of characteristic 2, and the grading is the one given in* (1).

Proof. Propositions 5, 6, 7, together with the fact $I \subseteq T_G(M_2(K))$ imply that the only thing we have to prove is that the monomials of type 2 and of type 3 are independent. We draw the reader's attention to the fact that we work with multihomogeneous elements only. Therefore the monomials of type 1 (that is no variables z) are independent from the remaining types.

But this is easily seen since the monomials of type 2 yield only a multiple $y^1_{a_1} \cdots y^1_{a_k}$, i.e., all upper indices are equal to 1 in the part without the multiple b, if the resulting monomial is even in the grading. (This corresponds to the case where there is no c_{m+1}). If on the other hand we have an even monomial of type 3 (that is even number of variables z) then the leading block of variables y will contribute with $y^1_{a_1} \cdots y^1_{a_k}$ and the second block of variables y will yield $y^2_{b_1} \cdots y^2_{b_l}$ (that is upper index 2).

Analogously for the monomials that are odd in the grading one applies the above reasoning exchanging the upper indices 1 and 2 for the variables y.

All this finishes the proof of the theorem. □

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