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SOME QUESTIONS ABOUT PRODUCTS OF VERBALLY PRIME *T*-IDEALS*

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Communicated by A. Giambruno

With best regards to Yuri Bahturin on his 65th birthday

Abstract. In [1] we studied identities of finite dimensional incidence algebras and showed how they were gotten by products and intersections of identities of matrices and we left open the question of when two incidence algebras satisfy the same identities, a problem which is still open. In the current paper we re-visit this problem: We describe it, give some partial results and some related problems based on the work of Kemer.

1. Incidence algebras. In this paper all graphs will be finite directed graphs with no multiple edges. We impose the additional restriction of transitivity, namely, that if there is an edge from v_1 to v_2 and an edge from v_2 to v_3,

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there must also be an edge from v_1 to v_3. Given such a graph G with vertex set

$$V = \{v_1, \ldots, v_n\}$$

we define the algebra $A(G)$ to be the span of the matrix units e_{ij} for each (i, j) such that there is an edge from v_i to v_j. We will write $I(G)$ for the ideal of polynomial identities of $A(G)$. An important special case is $G = C_n$, the complete graph on n vertices. In this case $A(C_n)$ equals $M_n(F)$, the $n \times n$ matrices over the field F and we write I_n for the identities $I(C_n)$. For technical reasons it is useful to define C_0 to be the graph with one vertex and no edges.

There are three basic operations on graphs. Given graphs G_1 and G_2 we may form their union $G = G_1 \cup G_2$. It is not hard to see that

$$A(G_1 \cup G_2) \cong A(G_1) \oplus A(G_2) \text{ and } I(G) = I(G_1) \cap I(G_2).$$

A second graph formed from G_1 and G_2 is denoted G_1G_2 and it consists of $G_1 \cup G_2$ with additional edges from each vertex in G_1 to each vertex in G_2. We denote $A(G_1G_2)$ as $A(G_1) \circ A(G_2)$. It is not hard to see that

$$A(G_1) \circ A(G_2) = \begin{pmatrix} A(G_1) & * \\ 0 & A(G_2) \end{pmatrix},$$

and the description of the identities follows from Lewin's theorem, see [4]:

$$I(G_1G_2) = I(G_1)I(G_2).$$

The third operation is cross product, and it is not hard to see that $A(G_1 \times G_2) = A(G_1) \otimes A(G_2)$. This operation was important in [1] and elsewhere, but will not play a role in the current paper.

A subset C of a graph is called a chain if given any $v, w \in C$ there is at least one edge joining them. The following theorem from [1] describes the identities of incidence algebras.

Theorem 1.1. *Let G be any finite graph.*

(1) *Let $C \subseteq G$ be a chain. Then there exists non-negative integers $a_1, \ldots, a_n$ such that $C \cong C_{a_1} \cdots C_{a_n}$ and so $I(C) = I_{a_1} \cdots I_{a_n}$.*

(2) *The ideal of polynomial identities of $A(G)$ is the intersection of all $I(C)$ where $C \subseteq G$ is a maximal chain.*

Note that if $C = C_{a_1} \cdots C_{a_n}$ then the corresponding incidence algebra $A(C)$ consists of the block upper triangular matrices of the form

$$\begin{pmatrix} M_{a_1}(F) & * & \cdots & * \\ 0 & M_{a_2}(F) & \cdots & * \\ \vdots & 0 & \ddots & * \\ 0 & 0 & \cdots & M_{a_n}(F) \end{pmatrix}$$

On the one hand, Theorem 1.1 completely describes identities of incidence algebras: They are intersections of products of identities of matrix algebras. Let $\mathcal{A}$ be the set of finite sequences of non-negative integers. For $\alpha = (a_1, \ldots, a_n)$ in $\mathcal{A}$ we will let $C(\alpha)$ denote $C_{a_1} \cdots C_{a_n}$ and $I(\alpha)$ denote $I(C(\alpha)) = I_{a_1} \cdots I_{a_n}$. Then the natural question is:

Question 1. *Given graphs G_1 and G_2, when is $I(G_1) = I(G_2)$, or more generally, when is $I(G_1) \subseteq I(G_2)$? Equivalently, for which $\alpha_1, \ldots, \alpha_n, \beta_1, \ldots, \beta_m \in \mathcal{A}$ do we have*

$$I(\alpha_1) \cap \cdots \cap I(\alpha_n) \subseteq I(\beta_1) \cap \cdots \cap I(\beta_m)?$$

If $G_1 \subseteq G_2$ then $A(G_1)$ is a subalgebra of $A(G_2)$ and so it satisfies all of the identities of $A(G_2)$ implying that $I(G_2) \subseteq I(G_1)$. For $\alpha, \beta \in \mathcal{A}$ we will write $\alpha \prec \beta$ if $C(\alpha)$ embeds into $C(\beta)$ (in which case $I(\beta) \subseteq I(\alpha)$). If the graph $C(\alpha)$ has n vertices, so that $\sum \max\{\alpha_i, 1\} = n$, then $\alpha \prec (n)$ and, we shall show in the next section that, for algebras with 1, the relation $\prec$ is generated by this relation.

Conjecture 1. *Given graphs G_1 and G_2 such that each of $A(G_1)$ and $A(G_2)$ contains 1, $I(G_1) \subseteq I(G_2)$ if and only if every chain C contained in G_2 has an embedding into G_1.*

Note that $A(G)$ contains 1 if and only if there is a loop at each vertex. The hypothesis that the algebras contain 1 is necessary since C_0C_0 does not embed into C_1, yet every identity of C_1 is of degree at least two, and so lies in C_0C_0. Although the more general case is certainly of interest, we will henceforth restrict attention to graphs with loops at each vertex. Say that a graph G is irredundant if $A(G)$ is not p. i. equivalent to any $A(H)$ for any proper subgraph H. Then, in light of Theorem 1.1, Conjecture 1 is equivalent to the statement that if G_1 and G_2 are irredundant, then $I(G_1) \subseteq I(G_2)$ if and only if G_2 embedds into G_1, and in particular $I(G_1) = I(G_2)$ if and only if $G_1 \cong G_2$.

Before starting in on our results we introduce a bit more notation. We let $\mathcal{A}^+$ be the set of finite sequences of positive integers. Given $\alpha = (a_1, \ldots, a_n) \in \mathcal{A}^+$ we let $|\alpha|$ denote the sum $\sum a_i$ and we note that the incidence algebra $A(\alpha)$ is a subalgebra of $|\alpha| \times |\alpha|$ matrices.

2. Positive results. In [2] Giambruno and Zaicev proved the following, see Theorem 8.4.4 of [3]:

Theorem 2.1. *Let $P, Q, J_1, \ldots, J_m$ be T-ideals such that the J_i are verbally prime and PQ is contained in the product $J = J_1 \cdots J_m$. Then either $P \subseteq J$, $Q \subseteq J$ or there exists a k such that $P \subseteq J_1 \ldots J_k$ and $Q \subseteq J_{k+1} \cdots J_m$.*

Given $\alpha = (a_1, \ldots, a_n) \in \mathcal{A}^+$ we define $s(2\alpha)$ to be the polynomial

$$s_{2a_1}(x_1^{(1)}, \ldots, x_{2a_1}^{(1)})y_1 s_{2a_2}(x_1^{(2)}, \ldots, x_{2a_2}a^{(2)})y_2 \cdots y_{n-1}s_{2a_n}(x_1^{(n)}, \ldots, x_{a_n}^{(n)})$$

where s_a denotes the standard polynomial.

Lemma 2.2. *Given $\alpha, \beta \in \mathcal{A}^+$, the polynomial $s(2\alpha)$ is in $I(\beta)$ if and only if $\beta \prec \alpha$.*

Proof. Let S_i denote the T-ideal generated by the standard identity s_i. If $\alpha = (a_1, \ldots, a_n)$, then the T-ideal generated by $s(2\alpha)$ is the product $S_{2a_1} \ldots S_{2a_n}$. Since $S(2\alpha) \subseteq I(\alpha)$ the polynomial $s(2\alpha)$ is an identity for $C(\alpha)$ and for any subalgebra. Hence, if $\beta \prec \alpha$ then $s(2\alpha) \in I(\beta)$.

For the reverse implication we precede by induction on n. If $n = 1$ then $\alpha = a_1$. If $\beta \not\prec \alpha$ then $|\beta| = b > a_1$. But the algebra $C(\beta)$ contains the upper triangular $b \times b$ matrices and so by the standard staircase argument $C(\beta)$ does not satisfy s_{2a_1}.

Finally, for the induction step, we assume that

$$S_{2a_1} \ldots S_{2a_n} \subseteq I(b_1) \ldots I(b_m),$$

with $n \geq 2$. Letting $P = S_{2a_1}$ and $Q = S_{2a_2} \cdots S_{2a_n}$ and noting that the $I(b_i)$ are verbally prime, we apply the Giambruno-Zaicev theorem together with the induction hypothesis. If P or Q is contained in $I(\beta)$ then $C(\beta)$ embeds into $C(a_1)$ or $C(a_2, \ldots, a_n)$. Either of these two containments implies that $C(\beta)$ embeds in $C(\alpha)$ and so $\beta \prec \alpha$. In the remaining case,

$$P \subseteq I(b_1) \ldots I(b_k) \text{ and } Q \subseteq I(b_{k+1}) \ldots I(b_m)$$

for some k. By induction

$$(b_1, \dots, b_k) \prec a_1 \text{ and } (b_{k+1}, \dots, b_m) \prec (a_2, \dots, a_n)$$

which implies that $\beta \prec \alpha$ as claimed. □

Lemma 2.2 has a number of implications, two of which are immediate. First it proves Conjecture 1 in the case that G_1 and G_2 are chains. This is a special case of Corollary 8.5.5 of [3].

Theorem 2.3. *Let $\alpha, \beta \in \mathcal{A}^+$. Then $\alpha \prec \beta$ if and only if $I(\beta) \subseteq I(\alpha)$.*

Proof. If $I(\beta) \subseteq I(\alpha)$ then $s(2\beta)$ is an identity for $I(\alpha)$. □

Corollary 2.4. *Conjecture 1 is true if and only if whenever $\cap I(\alpha_i) \subset I(\beta)$ then $I(\alpha_i) \subseteq I(\beta)$ for some i.*

Another consequence of Lemma 2.2 is that the relation $\prec$ on $\mathcal{A}^+$ is generated by the three relations:

- $\alpha \prec |\alpha|$
- $n \prec n + 1$
- If $\alpha_1 \prec \beta_1$ and $\alpha_2 \prec \beta_2$ then $\alpha_1\alpha_2 \prec \beta_1\beta_2$.

The proof follows from noting that these are the relations used in the proof of Lemma 2.2.

Here is another positive result about Conjecture 1.

Theorem 2.5. *Let $\beta = (b_1, \dots, b_m) \in \mathcal{A}^+$ where either $b_1 = \cdots = b_m = 1$ or $m \leq 2$, let $\alpha_1, \dots, \alpha_n \in \mathcal{A}^+$, and assume that $\cap_i I(\alpha_i) \subseteq I(\beta)$. Then some $I(\alpha_i)$ is contained in $I(\beta)$ and so $\beta \prec \alpha_i$.*

Proof. We first take the case of $\beta_1 = \cdots = \beta_m = 1$. Then whenever some $I(\alpha_i) \not\subseteq I(\beta)$, we have $(1, 1, \dots, 1) \not\prec \alpha_i$ which happens if and only if $m > |\alpha_i|$. Hence the standard identity S_{2m-2} is an identity for $C(\alpha_i)$ but not for $C(\beta)$. So if no $I(\alpha_i)$ is contained in $I(\beta)$ then the intersection $\cap_i I(\alpha_i)$ contains S_{2m-2}, but $I(\beta)$ does not.

Next we consider $m = 1$, $\beta = b_1$. In general $b_1 \prec \alpha$ for some $\alpha = (a_1, \dots, a_t) \in \mathcal{A}^+$ if and only $b_1 < a_i$ for some i. If this does not happen, then the standard identity $s_{2\beta_1-2}$ is a member of each $I(a_j)$ implying that $s^t_{2\beta_1-2}$ is in

$I(\alpha_i)$ but not $I(\beta)$. Hence, if $\beta \prec \alpha_i$ is false for every i then for large N $s_{2b_1-2}^N$ is in every $I(\alpha_i)$, but is not in $I(\beta)$.

Turning to the $m = 2$ case, if $(b_1, b_2) \not\prec (a_1, \ldots, a_n)$, either $a_i < b_1$ for every i or if k is minimum such that $a_k \geq b_1$ then $a_k < b_1 + b_2$ and $a_i < b_2$ for all $i > k$. Hence if $\alpha \not\prec (b_1, b_2)$ then $\alpha \prec \gamma_n$ where

$$\gamma_n = ((b_1 - 1)^n, b_1 + b_2 - 1, (b_2 - 1)^n),$$

the exponents representing repetitions rather than multiplications. Therefore, if N is large enough, $\alpha_i \prec \gamma_N$ for every i, and so $s(2\gamma_N) \in \cap I(\alpha_i)$. But $\beta \not\prec \gamma_N$ and so $s(\gamma_N) \notin I(\beta)$. □

At this point one might imagine that if $I(G) \not\subseteq I(\alpha)$ then one can find a $s(2\beta)$ which is an identity for $C(\alpha)$ but not $A(G)$. Sadly, this is not the case.

Theorem 2.6. *If $\alpha \in \mathcal{A}^+$ is such that $s(2\alpha)$ is an identity for $A(2,2) \oplus A(3)$, then $s(2\alpha)$ is also an identity for $A(1,2,1)$. Yet, $I(2,2) \cap I(3)$ is not contained in $I(1,2,1)$.*

Perhaps this is not so sad, for if it were true that $I(2,2) \cap I(3) \subseteq I(1,2,1)$ then Conjecture 1 would be false.

Proof. By Lemma 2.2, $s(2\alpha)$ is an identity for $A(2,2) \cap A(3)$ if and only if $(2,2) \prec \alpha$ and $(3) \prec \alpha$. The former implies that either α has two parts greater than or equal to 2, or one part greater than or equal to 4. If some part of α is greater than 4, then $(1,2,1) \prec \alpha$, so we assume instead that α has at least two parts equal to 2 or 3. But we also have the condition $3 \prec \alpha$ so that α must have at least one part equal to 3. Say for convenience that the 3 comes first: $\alpha = \alpha_1 3 \alpha_2 a \alpha_3$, where $a = 2$ or 3. Then since $(1,2) \prec 3$ and $1 \prec a$, $(1,2,1) \prec \alpha$.

To complete the proof we need a polynomial in $I_2 I_2 \cap I_3$ which is not an identity for $C(1,2,1)$. Such a polynomial is given by

$$s_6(x_1, s_4(x_2, \ldots, x_5), x_6, x_7, s_4(x_8, \ldots, x_{11}), x_{12}).$$

Since this polynomial is a consequence of s_6 it lies in I_3, and since it is a sum of terms each involving two evaluations of s_4 it is also in $I_2 I_2$. Finally, the algebra $C(1,2,1)$ is the span of

$$\{e_{ij} | \text{ either } 1 \leq i \leq j \leq 4 \text{ or } (i,j) = (3,2)\}$$

and the above polynomial is non-zero under the substitution

$$s_6(e_{11}, s_4(e_{11}, e_{13}, e_{32}, e_{22}), e_{22}, e_{23}, s_4(e_{32}, e_{22}, e_{24}, e_{44}), e_{44}) =$$

$$s_6(e_{11}, e_{12}, e_{22}, e_{23}, e_{34}, e_{44}) =$$

$$e_{14}.$$

□

3. More open questions. The ideals I_n of identities of matrices are verbally prime ideals. The other verbally prime ideals are the identities of the algebras $M_n(E)$ of matrices over the Grassmann algebra and $M_{k,\ell}$, which we will not define here. See [3]. Verbally prime ideals not equal to I_o are called proper. The analogue of $C(\alpha)$ are block upper triangular algebras of the form

$$\begin{pmatrix} A_1 & * & \cdots & * \\ 0 & A_2 & \cdots & * \\ \vdots & 0 & \cdots & * \\ 0 & 0 & \cdots & A_n \end{pmatrix}$$

where each of the A_i is a verbally prime algebra. We denote this algebra as $A_1 \circ \cdots \circ A_n$. It is known that the T-ideal of identities equals the product $I_1 \cdots I_n$ where each $I_i = Id(A_i)$.

Theorem 2.1 says that in order to determine when one product of verbally prime T-ideals is contained in another, we need only determine when a single verbally prime T-ideal is contained in a product of others:

Question 2. *For which proper verbally prime ideals $P, Q_1, \ldots, Q_n$ do we have $P \subseteq Q_1 \cdots Q_n$? In particular, when is one verbally prime T-ideal contained in another?*

The following inclusions of proper verbally prime T-ideals are known:

- $Id(M_n(F)) \subseteq Id(M_{n-1}(F))$
- $Id(M_n(E)) \subseteq Id(M_n(F))$
- $Id(M_{n,k}) \subseteq Id(M_n(F))$
- $Id(M_{k,\ell}) \subseteq Id(M_{k',\ell'})$ if $k \geq k'$ and $\ell \geq \ell'$
- $Id(M_{k+\ell}(E)) \subseteq Id(M_{k,\ell})$
- $Id(M_n(E)) \subseteq Id(M_{n-1}(E))$
- $Id(M_{n,n}) \subseteq Id(M_n(E))$

Conjecture 2. *The above inclusions describe all inclusions of proper verbally prime ideals.*

Any T-ideal defines a codimension sequence $c_n(I)$ which in turn defines the exponential rate of growth $\exp(I)$ which equals the limit of $c_n(I)^{1/n}$ as $n \to \infty$.

In addition to Theorem 2.1 Giambruno and Zaicev prove the following theorems about products of proper verbally prime ideals, theorems 8.5.5, and 8.5.6, respectively, in their book [3].

Theorem 3.1. *Let $P_1, \ldots, P_n$ be proper verbally prime ideals and let Q be any T-ideal. If $P_1 \cdots P_n \subsetneq Q$ then $exp(Q) < exp(P_1 \cdots P_n)$.*

Theorem 3.2. *Let $P_1, \ldots, P_n, Q_1, \ldots, Q_m$ be proper verbally verbally prime ideals such that*

$$P_1 \cdots P_n = Q_1 \cdots Q_m.$$

Then $n = m$ and $P_i = Q_i$ for all $i = 1, \ldots, n$.

Using Theorem 3.1 we can now prove that products of proper verbally prime ideals are varietally irreducible. A T-ideal is said to be varietally irreducible if it cannot be written as a finite intersection of strictly larger T-ideals.

Theorem 3.3. *Let $P_1, \ldots, P_n$ be proper verbally verbally prime ideals and let $Q_1, \ldots, Q_m$ be any T-ideals such that*

$$P_1 \cdots P_n = Q_1 \cap \cdots \cap Q_m.$$

Then $P_1 \cdots P_n = Q_i$ for some i.

Proof. The codimension sequence satisfies

$$\max\{c_n(I), c_n(J)\} \leq c_n(I \cap J) \leq c_n(I) + c_n(J)$$

and hence $\exp(I \cap J)$ equals $\max\{\exp(I), \exp(J)\}$. In our case

$$\exp(P_1 \cdots P_n) = \exp(\cap_i Q_i) = \max\{\exp(Q_i)\}$$

and so $\exp(P_1 \cdots P_n) = \exp(Q_i)$ for some i. But since $P_1 \cdots P_n \subseteq Q_i$, Theorem 3.1 implies that $P_1 \cdots P_n = Q_i$. □

We conjecture that a stronger version of this theorem should hold:

Conjectute 3. *Let $P_1, \ldots, P_n$ be proper verbally verbally prime ideals and let $Q_1, \ldots, Q_m$ be any T-ideals such that*

$$Q_1 \cap \cdots \cap Q_m \subseteq P_1 \cdots P_n.$$

Then $Q_i \subseteq P_1 \cdots P_n$ for some i.

By Corollary 2.4, Conjecture 3 would imply Conjecture 1.

We note that there are varietally irreducible ideals which are not products of verbally prime ideals. This follows from the fact that every T-ideal is an intersection of varietally irreducible ideals and the following lemma.

Lemma 3.4. *Let $I = S_{2n}$ be the T-ideal generated by a standard identity s_{2n} for some $n \geq 2$. Then I is not an intersection of products of verbally prime T-ideals.*

Proof. The only products of verbally prime T-ideals containing S_{2n} are the $I(\alpha)$ with $\alpha \prec (n)$ and the intersection of all such is I_n. But since the identities of $n \times n$ matrices are not all consequences of s_{2n}, the ideal S_{2n} is not the intersection of the products of verbally prime T-ideals. □

Question 3. *Find natural examples of varietally irreducible T-ideals which are not products of verbally prime T-ideals.*

In the proof of the Specht conjecture, Kemer defined algebras called PI-basic. Let A be a finite dimensional algebra over the algebraically closed field F. By Wedderburn's theorem we can write A as a vector space direct sum $\overline{A} + J$ where J is the Jacobson radical and $\overline{A}$ is a semisimple subalgebra, and so $\overline{A} = A_1 \oplus \cdots A_n$, a direct sum of simple algebras. The algebra A is said to be full if there is a permutation $\sigma \in S_n$ such that $A_{\sigma(1)} J \cdots J A_{\sigma(n)} \neq 0$; and it is said to be subdirectly irreducible if no intersection of non-zero ideals is zero. Kemer proved that any T-ideal I containing a standard identity can be written as the intersection of $\cap J_i$ where each J_i is the ideal of identities for some full, subdirectly irreducible algebra. Finally, a full, subdirectly irreducible algebra A with radical J is called PI-basic if there does not exist an algebra A' with radical J' which which is p.i. equivalent to A and which is smaller in the sense of either $\dim A'/J' < \dim A/J$ or $\dim A'/J' = \dim A/J$ and J' nilpotent of lower index than J. It follows that every varietally irreducible T-ideal containing a standard identity is the ideal of identities of a PI-basic algebra. In fact, it follows from his work that the $C(\alpha)$ are all PI-basic.

Question 4. *Are there PI-basic algebras whose ideals of identities are not varietally irreducible?*

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