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ON SOME RECENT RESULTS ABOUT THE GRADED GELFAND-KIRILLOV DIMENSION OF GRADED PI-ALGEBRAS

Lucio Centrone

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This paper is dedicated to Professor Y. A. Bahturin for his 65-th birthday.

Abstract. We survey some recent results on graded Gelfand-Kirillov dimension of PI-algebras over a field F of characteristic 0. In particular, we focus on verbally prime algebras with the grading inherited by that of Vasilovsky and upper triangular matrices, i.e., $UT_n(F)$, $UT_n(E)$ and $UT_{a,b}(E)$, where E is the infinite dimensional Grassmann algebra.

1. Introduction. The interest of algebraists in this invariant started in 1966 from two papers published by Gelfand and Kirillov (see [25], [26]). The first study of the properties of the Gelfand-Kirillov (GK) dimension started in 1976 in a paper of Borho and Kraft ([16]). Now this dimension is a standard tool in the study of non-commutative algebras. Before 1976 a key role has been played by Krull dimension in the investigation over non-commutative rings. Moreover,

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the Gelfand-Kirillov dimension sometimes provides information about the Krull dimension but it is often easier to compute. For a more detailed account of the algebraic properties and related topics of the GK dimension of algebras, we refer to the book of Krause and Lenagan ([33]).

The GK dimension is an important tool also in the theory of algebras with polynomial identities (PI-algebras). It is well known that PI-algebras are a generalization of commutative algebras; in particular they have nice structure theory, similar to that of commutative algebras (see [39]). Due to the fact that the Gelfand-Kirillov dimension is well behaved for commutative algebras, it might be hoped that it could be very useful in the study of PI-algebras. Nevertheless, there are non-commutative PI-algebras whose GK dimensions are real (non integer) numbers, exhibiting one of the possible undesirable behaviors with respect to the commutative case.

Although the previous facts may not realize a full parallelism between commutative algebras and non-commutative PI-algebras, the GK dimension provides a nice tool in order to measure "how big" is the algebra of the polynomials under the polynomial conditions of the identities of a certain algebra. More precisely, if A is a PI-algebra, we use to define the GK dimension of A as the GK dimension of its relatively free algebra in a certain number of variables. In [12] Berele proved that the GK dimension of a finitely generated PI-algebra is finite. Indeed, if A and B are PI-algebras that are PI-equivalent, i.e., with the same ideal of polynomial identities, they have the same GK dimensions. In the light of this fact, Alves ([3], [4]) and Azevedo, Fidelis and Koshlukov ([6]) were able to determine some important PI-non equivalences in positive characteristic proving the fact that the Tensor Product Theorem is no more valid in positive characteristic. We suggest the survey of Drensky ([23]) for a general overlook of problems concerning the GK dimension of PI-algebras.

By the theorem of Kemer ([30]), we have that the T-ideals of polynomial identities of associative algebras over fields of characteristic 0 are finitely basis. Nevertheless the finite set of generators of the T-ideals is well known only for few classes of algebras, then we are allowed to study something "weaker" than the polynomial identities, i.e., the graded polynomial identities. In [2] Aljadeff and Kanel-Belov proved the analog of the theorem of Kemer in the graded case. As well as for the ordinary case, the GK dimension of the relatively free graded algebra can give an idea of "how big" is the set of the graded polynomials under the polynomial conditions of the graded identities of a graded algebra. In this survey we want to recall some standard facts about the GK dimension of PI-algebras and we want to summarize the first results of the author (see [17] and [18]) concerning the graded GK dimension of graded PI-algebras already presented in the occasion of the international workshop "Polynomial identities in algebras II". We also provide sketches of the proofs of the main results and we present some "work in progress" problems linked to this topic.

2. Preliminary results. We fix a field F and we consider associative F-algebras with 1. We start off with the following definition:

Definition 2.1. *Let A be an F-algebra generated by a finite set $\{r_1, \ldots, r_m\}$. Let*

$$V^n = \mathrm{span}\langle r_{i_1} \cdots r_{i_n} \mid i_j = 1, \ldots, m\rangle, n = 0, 1, 2, \ldots$$

we assume $V^0 = F$. The function of the non-negative argument n

$$g_V(n) = \dim_F(V^0 + V^1 + \cdots + V^n), n = 0, 1, 2, \ldots$$

is called the growth function of A (with respect to $V = V^1$). The Gelfand-Kirillov dimension of A is defined by

$$\mathrm{GKdim}(A) := \limsup_{n\to\infty}(\log_n g_V(n)) = \limsup_{n\to\infty} \frac{\log g_V(n)}{\log n}.$$

We shall recall now some of the properties of the GK dimension. The proofs can be found in [33].

Proposition 2.2. *Let A be a finitely generated F-algebra. Then:*

(1) *The Gelfand-Kirillov dimension of A does not depend on the generating space V.*
(2) *If B is either a subalgebra or a homomorphic image of A, then*

$$\mathrm{GKdim}(B) \leq \mathrm{GKdim}(A).$$

(3) $\mathrm{GKdim}(A) = 0$ *if and only if A is finite dimensional. Otherwise*

$$\mathrm{GKdim}(A) = 1 \text{ or } \mathrm{GKdim}(A) \geq 2.$$

(4) *If A is commutative, then its GK dimension equals its transcendence degree.*
(5) *If B is a finitely generated subalgebra of $Z(A)$, i.e., the center of A, such that A is finitely generated as a B-module, then*

$$\mathrm{GKdim}(A) = \mathrm{GKdim}(B).$$

We consider now algebras with polynomial identities. We recall that if $X = \{x_1, x_2, \ldots\}$ is a countable set of variables, a polynomial $f(x_1, \ldots, x_n) \in F\langle X\rangle$ (free associative algebra, freely generated by X) is said to be a polynomial identity for the algebra A if $f(a_1, \ldots, a_n) = 0$ for all $a_1, \ldots, a_n \in A$. The algebra A is said to be an algebra with polynomial identities (or PI-algebra), if there exists a non-trivial polynomial identity for A. The set $T(A)$ of all polynomial identities of A is a two-sided ideal of $F\langle X\rangle$, called T-ideal of A. The algebra

$$F_k(A) = F\langle x_1, \ldots, x_k\rangle/(F\langle x_1, \ldots, x_k\rangle \cap T(A))$$

is called the relatively free algebra of rank k of A. Using the language of variaties, the class $\mathcal{V} = \mathcal{V}(A)$ of all algebras S such that $T(A) \subseteq T(S)$ is called the variety of algebras generated by A.

In the case A is a PI-algebra, it is often used the definition of GK dimension in k variables:

Definition 2.3. *Let A be a PI-algebra, then the GK dimension of A in k variables is*

$$\mathrm{GKdim}_k(A) := \mathrm{GKdim}(F_k(A)).$$

In the next proposition we shall recall some important results concerning the GK dimension of PI-algebras.

Proposition 2.4. *Let A be a PI-algebra, then:*

(1) *If A is finitely generated, then* $\mathrm{GKdim}(A) < \infty$*.*
(2) *If A is prime, then* $\mathrm{GKdim}(A) = \mathrm{tr.deg}_F(A)$ *.*
(3) *If A is finitely generated prime, then* $\mathrm{GKdim}(A)$ *is a non-negative integer.*
(4) $\mathrm{GKdim}_k(A)$ *is a non-negative integer.*
(5) *If $T(A) = T(A_1)T(A_2)$ then* $\mathrm{GKdim}_k(A) = \mathrm{GKdim}_k(A_1) + \mathrm{GKdim}_k(A_2)$*.*

In order to give the definition of graded GK dimension for graded PI-algebras, we shall recall some basic facts about graded polynomial identities. From now on all fields we refer to are of characteristic 0.

Let $(G, \cdot) = \{g_1, \ldots, g_s\}$ be any group of finite order s and A be an algebra. Then A is said to be a G-graded algebra if there exist subspaces A^g for each $g \in G$ such that

$$A = \bigoplus_{g\in G} A^g \text{ and } A^g A^h \subseteq A^{gh}.$$

If $0 \neq a \in A^g$ we say that a is homogeneous of G-degree g or G-graded homogeneous of G-degree g, and we write $\|a\| = g$.

Let $\{X^g \mid g \in G\}$ be a family of disjoint countable sets. Let $X = \bigcup_{g\in G} X^g$ and denote by $F\langle X\rangle$ the free associative algebra freely generated by the set X. An indeterminate $x \in X$ is said to be of homogeneous G-degree g, written $\|x\| = g$, if $x \in X^g$. We shall always write x^g if $x \in X^g$. The homogeneous G-degree of a monomial $m = x_{i_1}x_{i_2}\cdots x_{i_k}$ is defined to be $\|m\| = \|x_{i_1}\| \cdot \|x_{i_2}\| \cdot \cdots \cdot \|x_{i_k}\|$. For every $g \in G$, denote by $F\langle X\rangle^g$ the subspace of $F\langle X\rangle$ spanned by all the monomials having homogeneous G-degree g. Notice that $F\langle X\rangle$ is also a G-graded algebra. The elements of the G-graded algebra $F\langle X\rangle$ are referred to as G-graded polynomials or graded polynomials. If A is a G-graded algebra, a G-graded polynomial $f(x_1,\dots,x_n)$ is said to be a graded polynomial identity of A if $f(a_1,a_2,\dots,a_n) = 0$ for all $a_1,a_2,\dots,a_n \in \bigcup_{g\in G} A^g$ such that $a_k \in A^{\|x_k\|}$, $k = 1,\cdots,n$. Moreover, we say that the G-graded algebra A is a G-graded PI-algebra if it satisfies a graded polynomial identity. Notice that if G is a finite group and A is a G-graded PI-algebra, then A is a PI-algebra, too (see [9] for more details). We denote by $T_G(A)$ the ideal of all graded polynomial identities of A.

We shall denote with the symbol $F_k^G(A)$ the relatively free algebra

$$F\langle x_1^{g_1},\dots,x_k^{g_1},\dots,x_1^{g_s},\dots,x_k^{g_s}\rangle/(F\langle x_1^{g_1},\dots,x_k^{g_1},\dots,x_1^{g_s},\dots,x_k^{g_s}\rangle \cap T_G(A)).$$

It is called the relatively free G-graded algebra of A in k variables.

The theory of G-graded PI-algebras is strictly related to the representation theory of the symmetric group. For this purpose, the following set plays a key role:

Definition 2.5. *Let*

$$P_n^G = \mathrm{span}\langle x^{g_{i_1}}_{\sigma(1)}x^{g_{i_2}}_{\sigma(2)}\cdots x^{g_{i_n}}_{\sigma(n)} | g_i \in G, \sigma \in S_n\rangle.$$

The elements of P_n^G are called the multilinear polynomials of degree n of $F\langle X\rangle$.

We observe that P_n^G is a left S_n-module under the natural action of the symmetric group S_n and a vector space over F; we denote the S_n-character of the factor module $P_n^G/(P_n^G \cap T_G(A))$ by $\chi_n^G(A)$, and by $c_n^G(A)$ its dimension over F. We say that

$$\left(\chi_n^G(A)\right)_{n\in\mathbb{N}} \text{ is the } G\text{-}\textit{graded cocharacter sequence} \text{ of } A;$$

$$\left(c_n^G(A)\right)_{n\in\mathbb{N}} \text{ is the } G\text{-}\textit{graded codimension sequence} \text{ of } A.$$

Now, for $l_{g_1},\dots,l_{g_s} \in \mathbb{N}$ let us consider the blended components of the multilinear polynomials in the indeterminates labeled as follows: $x_1^{g_1},\dots,x_{l_{g_1}}^{g_1}$, then $x^{g_2}_{l_{g_1}+1}, \ldots, x^{g_2}_{l_{g_1}+l_{g_2}}$ and so on. We denote this linear space by $P^G_{l_{g_1},\ldots,l_{g_s}}$. Of course, this is a left $S_{l_{g_1}} \times \cdots \times S_{l_{g_s}}$-module. We shall denote by $\chi^G_{l_{g_1},\ldots,l_{g_s}}(A)$ the character of the module $P^G_{l_{g_1},\ldots,l_{g_s}}/(P^G_{l_{g_1},\ldots,l_{g_s}} \cap T_G(A))$.

We remark that, given the cocharacter $\chi^G_{l_{g_1},\ldots,l_{g_s}}(A)$, the graded cocharacter $\chi^G_n(A)$ is known as well as in the next proposition of Di Vincenzo (see [20], Theorem 2):

Proposition 2.6. *Let A be a G-graded algebra with graded cocharacter sequences $\chi^G_{l_{g_1},\ldots,l_{g_s}}(A)$. Then*

$$\chi^G_n(A) = \sum_{\substack{(l_{g_1},\ldots,l_{g_s}) \\ l_{g_1}+\cdots+l_{g_s}=n}} \chi^G_{l_{g_1},\ldots,l_{g_s}}(A)^{\uparrow S_n}.$$

We consider now the following definition:

Definition 2.7 (Graded Gelfand-Kirillov dimension in k variables)**.** *Let $G = \{g_1, \ldots, g_s\}$ be a finite group and A a G-graded PI-algebra. The G-graded Gelfand-Kirillov dimension of A in k variables is*

$$\mathrm{GKdim}^G_k(A) := \mathrm{GKdim}(F^G_k(A))$$

where the new sk non-commutative variables are $x^{g_1}_1, \ldots, x^{g_s}_1, \ldots, x^{g_1}_k, \ldots, x^{g_s}_k$.

Notice that we have defined the G-graded GK dimension of A in k variables as the GK dimension of $F^G_k(A)$ that is generated by sk variables. That is because in the original definition suggested by Di Vincenzo the idea was to "spread" each of the k ordinary variables into $|G| = s$ graded variables.

Definition 2.8. *Let $G = \{g_1, \ldots, g_s\}$ be a finite abelian group of order s. Let A be a G-graded PI-algebra and $F_{k_1,\ldots,k_s}$ be its relatively free G-graded algebra in the variables $T_1 := \{t_{11}, \ldots, t_{1k_1}\}$, $\ldots$, $T_s := \{t_{s1}, \ldots, t_{sk_s}\}$, where the variables in T_1 are of G-degree 1_G and the variables in T_s are of G-degree g_s. It is well known that $F_{k_1,\ldots,k_s}$ is a $\mathbb{Z}^{k_1+\cdots+k_s}$-graded algebra. The formal power series*

$$H^G(A; T_1, \ldots, T_s) =$$

$$= \sum_n \dim F_{k_1,\ldots,k_s}^{\left(n_1,\ldots,n_{k_1},\ldots,n_{\sum_{i=1}^{s-1} k_i+1},\ldots,n_{\sum_{i=1}^{s} k_i}\right)} t_{11}^{n_1} \cdots t_{1k_1}^{n_{k_1}} \cdots t_{s1}^{n_{\sum_{i=1}^{s-1} k_i+1}} \cdots t_{sk_s}^{n_{\sum_{i=1}^{s} k_i}}$$

is called the Hilbert Series of A in the sets of variables $T_1, \ldots, T_s$.

Indeed, the growth function of $F_{k_1,\dots,k_s}$ with respect to the vector space

$$V = \mathrm{span}_F\langle T_1 \cup \cdots \cup T_s\rangle$$

is

$$g_V(n) = \sum_{n_1+\cdots+n_{\sum_{i=1}^s k_i}=n} \dim_F F_{k_1,\dots,k_s}^{\left(n_1,\dots,n_{\sum_{i=1}^s k_i}\right)}.$$

We note that if the coefficients of the G-graded Hilbert series of A are bounded by a polynomial of degree q, then the growth function of $F_{k_1,\dots,k_s}$ is bounded by a polynomial of degree $q+1$ and $\mathrm{GKdim}_k^G(A) \leq q+1$.

Moreover, notice that the Hilbert series $H^G(A; T_1, \dots, T_s)$ is uniquely determined by the integers $m_1, \dots, m_s$ counting the number of variables in $T_1 \cup \cdots \cup T_s$ of G-degrees respectively equal to $1_G,\dots,g_s$. We denote such series by

$$H_{m_1,\dots,m_s}(A; T_1, \dots, T_s).$$

The G-graded cocharacters of a G-graded algebra A are strictly related with the Hilbert series of the relatively free algebra of A. We have the following (see [13],[24]):

Proposition 2.9. *Let A be a G-graded algebra and $m_1, \dots, m_s \in \mathbb{N}$. Suppose that $\chi_{m_1,\dots,m_s} = \sum_{\mu_1,\dots,\mu_s} m_{\mu_1,\dots,\mu_s}\chi_{\mu_1,\dots,\mu_s}$. Then*

$$H_{m_1,\dots,m_s}(A; T_1, \dots, T_s) = \sum_{\sum|\mu_i|=\sum m_i} m_{\mu_1,\dots,\mu_s} S_{\mu_1}(T_1) \cdots S_{\mu_s}(T_s)$$

where $S_{\mu_1}(T_1),\dots,S_{\mu_s}(T_s)$ are the Schur functions with shape $\mu_1, \dots, \mu_s$ in the sets of variables $T_1, \dots, T_s$ respectively.

3. General results about graded GK dimension. In this section we present some general results about the algebraic properties of the graded GK dimension of graded PI-algebras. In order to make a comparison with the ordinary case, these results are to be seen in the light of Proposition 2.4. We start with the following definition:

Definition 3.1. *Let φ and χ be characters of the symmetric group S_n. We consider the decompositions of φ and χ into irreducible characters, say*

$$\varphi = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad \chi = \sum_{\lambda \vdash n} m'_\lambda \chi_\lambda$$

where m_λ and m'_λ are non-negative integers. We shall write

$$\varphi \leq \chi \text{ if and only if } m_\lambda \leq m'_\lambda \text{ for all } \lambda \vdash n.$$

We have now the following proposition (see [20] Theorem 2):

Proposition 3.2. *Let A be a G-graded algebra with ordinary cocharacter sequence $\chi_n(A)$. Then*

$$\chi_n(A) \leq \chi_n^G(A).$$

The next results gives us an upper bound for $\mathrm{GKdim}_k^G(A)$:

Proposition 3.3. *Let G be a finite abelian group such that $|G| = s$ and A be a G-graded algebra. Then for any $k \in \mathbb{N}$, $k \geq 1$,*

$$\mathrm{GKdim}_k^G(A) \leq \mathrm{GKdim}_{sk}(A).$$

Proof. Let $G = \{g_1, \dots, g_s\}$ and let us consider the map

$$\varphi : F_{sk}(A) \to F_k^G(A)$$

such that

$$\begin{array}{rcl} x_1 & \mapsto & x_1^{g_1} \\ x_2 & \mapsto & x_2^{g_1} \\ & \vdots & \\ x_k & \mapsto & x_k^{g_1} \\ x_{k+1} & \mapsto & x_1^{g_2} \\ & \vdots & \\ x_{2k} & \mapsto & x_k^{g_2} \\ & \vdots & \\ x_{(s-1)k+1} & \mapsto & x_1^{g_s} \\ & \vdots & \\ x_{sk} & \mapsto & x_k^{g_s}. \end{array}$$

If we extend φ by linearity it turns out that φ is a onto homomorphism. Now the assertion comes in the light of the point (2) of Proposition 2.2. □

From the previous result it turns out that:

Corollary 3.4. *Let A be a G-graded PI-algebra, where G is a finite abelian group. Then for any $k \geq 1$*

$$\mathrm{GKdim}_k^G(A) < \infty.$$

P r o o f. We have just to observe that for any $t \geq 2$, $\mathrm{GKdim}_t(A) < \infty$, then we use the previous proposition. □

Proposition 3.5. *Let G be a finite abelian group such that $|G| = s$ and A be a G-graded algebra. Then for any $k \in \mathbb{N}$, $k \geq 2$,*

$$\mathrm{GKdim}_k(A) \leq \mathrm{GKdim}_k^G(A).$$

P r o o f. By Proposition 3.2 we have that

$$\chi_n(A) \leq \chi_n^G(A) = \sum_{\sum n_i = n} \chi^G_{n_1,\dots,n_s}(A)^{\uparrow S_n}.$$

The right hand side of the previous inequality equals

$$\sum_{\sum |\lambda_i| = n} m_{\lambda_1,\dots,\lambda_s} \chi_{\lambda_1,\dots,\lambda_s}.$$

In the light of Proposition 2.9, we have that

$$H_n(A) \leq \sum_{\sum n_i = n} H_{n_1,\dots,n_s}(A)$$

if you consider one variable only. Now the result follows comparing the growth functions. □

We have the following proposition:

Proposition 3.6. *Let G be a finite abelian group such that $|G| = s$ and A be a finitely generated G-graded PI-algebra. Then for any $k \in \mathbb{N}$ we have that:*

$$\mathrm{GKdim}_k(A) \leq \mathrm{GKdim}_k^G(A) \leq \mathrm{GKdim}_{sk}(A).$$

We consider now some structural aspects. From the structure theory of PI-algebras we have the following theorem of Posner (see [37]):

Theorem 3.7. *Let A be a prime PI-algebra, then its central quotient algebra is a finite dimensional simple algebra.*

We mention that the Theorem of Posner is one of the main tool in the proof of the point (2) of Proposition 2.4. As well as in the ordinary case, the graded version of the Theorem of Posner of Balaba ([11]) has a key role in the proof of the following (see [18]):

Theorem 3.8. *Let G be an abelian or an ordered group. If A is a G-graded prime PI-algebra, then*

$$\mathrm{GKdim}(A) = \mathrm{tr.deg}(A).$$

In order to sketch the proof of the previous theorem, we recall some definitions and results. If G is any group and A is a G-graded algebra, we denote with the symbol $h(A)$ the set of the G-homogeneous elements of A. We denote by $Z_{gr}(A)$ the maximal graded subalgebra in the center of A, i.e., the graded center of A. Now we have the following definition:

Definition 3.9. *A graded ideal P of a graded ring A is called graded prime if whenever $aAb \subseteq P$, where $a, b \in h(A)$, either $a \in P$ or $b \in P$. Moreover, a graded ring A is called graded prime if (0) is a graded prime ideal of A.*

Recall that by Proposition 1 of [11], the localization A_S of A over S, where S is a set of homogeneous elements of the center $Z(A)$ of A, is a G-graded PI-algebra of central quotients of A. An algebra $Q(A) \supseteq A$ is called the left (right) graded algebra of quotients of A if:

(1) each homogeneous regular element from A is invertible in $Q(A)$;
(2) each homogeneous element $x \in Q(A)$ has the form $a^{-1}b$ (ba^{-1}), where $a, b \in h(A)$ and a is a regular element.

The following theorems generalize the Theorem of Posner for the graded case:

Theorem 3.10. *Let A be a G-graded graded prime PI-algebra, $Z(A)$ the center of A and S the set of homogeneous regular elements of $Z(A)$. Then:*

(1) $S = h(Z(A))$;
(2) *the algebra of quotients A_S is a G-graded graded prime PI-algebra;*
(3) $Z_{gr}(A_S) = Z_{gr}(A)_S$.

Theorem 3.11. *Let A be a G-graded graded prime PI-algebra and A_0 the algebra of central quotients of A. Then:*

(1) *A_0 is finite dimensional graded-simple over its graded center Z and Z is the graded field of quotients of $Z_{gr}(A)$;*
(2) *A_0 is the graded algebra of quotients of A;*
(3) *A and A_0 satisfy the same identities.*

We consider the G-graded algebra A, then by the graded version of the Theorem of Posner we can consider its graded algebra of quotients, to say, Q such that $\mathrm{GKdim}(Q) = \mathrm{GKdim}(A)$. In the light of Theorems 3.10 and 3.11, A

has a graded algebra of quotients Q satisfying the same identities of A. By the point (2) of Theorem 3.11, we have that Q is finite dimensional over its graded center $Z_{gr}(Q) = Z(Q)$. Hence $\mathrm{GKdim}(Q) = \mathrm{GKdim}(Z(Q))$ but $\mathrm{GKdim}(Z(Q)) = \mathrm{tr.deg}_F(Z(Q))$ then the Theorem 3.8 follows.

4. The verbally prime algebras. In order to motivate the study of the invariant of graded GK dimension for the verbally prime algebras, we give a short account of the structure theory of T-ideals developed by Kemer [29].

Definition 4.1. *The T-ideal S of $F\langle X\rangle$ is called T-semiprime or verbally semiprime if any T-ideal U such that $U^k \subseteq S$ for some k, lies in S, i.e., $U \subseteq S$. The T-ideal P is T-prime or verbally prime if the inclusion $U_1U_2 \subseteq P$ for some T-ideals U_1 and U_2 implies $U_1 \subseteq P$ or $U_2 \subseteq P$.*

The Grassmann algebra E of an infinite dimensional vector space with basis $\{e_1, e_2, \ldots\}$ has a natural $\mathbb{Z}_2$-grading $E = E^{(0)} \oplus E^{(1)}$. Let a, b, where $a \geq b$, be positive integers and let $M_{a\times b}(E^{(1)})$ be the vector space of all $a \times b$ matrices with entries from $E^{(1)}$. The vector subspace of $M_{a+b}(E)$, where $M_n(E)$ is the $n \times n$ matrix algebra with entries from the Grassmann algebra,

$$M_{a,b}(E) := \left\{ \begin{pmatrix} p & q \\ s & t \end{pmatrix} \mid p \in M_a(E^{(0)}), \quad q \in M_{a\times b}(E^{(1)}), \right.$$
$$\left. t \in M_b(E^{(0)}), \quad s \in M_{b\times a}(E^{(1)}) \right\}$$

is an algebra. The building blocks in the theory of Kemer are the polynomial identities of the matrix algebras, the Grassmann algebra and the algebras $M_{a,b}(E)$. In fact, we have the following theorem:

Theorem 4.2.

(1) *For every T-ideal U of $F\langle X\rangle$ there exist a T-semiprime T-ideal S and a positive integer k such that*

$$S^k \subseteq U \subseteq S$$

(2) *Every T-semiprime T-ideal S is an intersection of a finite number of T-prime T-ideals $Q_1, \ldots, Q_m$,*

$$S = Q_1 \cap \cdots \cap Q_m.$$

(3) *A T-ideal P is T-prime if and only if P coincides with one of the following T-ideals:*

$$T(M_n(F)),\ T(M_n(E)),\ T(M_{a,b}(E)),\ (0),\ F\langle X\rangle.$$

By the theory of Kemer one can appreciate the importance of the verbally prime algebras. In [38] Procesi and in [14] Berele gave a natural construction of the relatively free algebras of $M_n(F)$, $M_n(E)$ and $M_{a,b}(E)$ in terms of generic matrix algebras. These descriptions of the relatively free algebras simplified the computation of the GK dimension in k variables of the verbally prime algebras. In particular, we have the following theorem (Procesi ([38]) for (1) and Berele ([14]) for (2) and (3)):

Theorem 4.3. *Let $k \geq 2$, then:*
(1) $\mathrm{GKdim}_k(M_n(F)) = (k-1)n^2 + 1$;
(2) $\mathrm{GKdim}_k(M_n(E)) = (k-1)n^2 + 1$;
(3) $\mathrm{GKdim}_k(M_{a,b}(E)) = (k-1)(a^2 + b^2) + 2$.

Concerning the graded case, in [17] and [18], the author considered the verbally prime algebras endowed with the grading induced by that of Vasilovsky. We recall that if we consider $M_n(F)$, the grading of Vasilovsky is an elementary $\mathbb{Z}_n$-grading over $M_n(F)$ obtained by the n-tuple $(0, 1, \ldots, n-1)$. This elementary grading extends in a natural way to a $\mathbb{Z}_n \times \mathbb{Z}_2$-grading over $M_n(E)$ and a $\mathbb{Z}_{a+b} \times \mathbb{Z}_2$-grading over $M_{a,b}(E)$.

Let X and Y be two countable sets of variables and we consider the free associative algebra $F\langle X \cup Y\rangle$. We observe that the latter algebra is a $\mathbb{Z}_2$-graded algebra, where $\|x\| = 0$ and $\|y\| = 1$ for all $x \in X$ and $y \in Y$. Moreover, we consider the following relation on $F\langle X \cup Y\rangle$: if a,b are $\mathbb{Z}_2$-homogeneous elements, then $ab = (-1)^{\|a\|\|b\|} ba$. We denote this algebra with the symbol $F[X; Y]$.

Let $k, n \in \mathbb{N}$ and a, b such that $a + b = n$ and set

$$X = \{x_{ij}^{(r)} | j = 1, \ldots, n, r = 1, \ldots, k\}$$

and

$$Y = \{y_{ij}^{(r)} | i, j = 1, \ldots, n, r = 1, \ldots, k\}.$$

In analogy with the ordinary case, for every $s = 1, \ldots, k$, let us consider the following:

$$A_s^{(i)} = \sum_{\|e_{pq}\| = (i)} x_{pq}^{(s)} e_{pq},$$

$$B_s^{(i,\alpha)} = \begin{cases} \sum\limits_{\|e_{pq}\|=(i)} x_{pq}^{(s)} e_{pq} & \text{if } \alpha = 0 \in \mathbb{Z}_2 \\ \sum\limits_{\|e_{pq}\|=(i)} y_{pq}^{(s)} e_{pq} & \text{if } \alpha = 1 \in \mathbb{Z}_2 \end{cases},$$

$$C_s^{(i,\alpha)} = \begin{cases} \sum\limits_{\|e_{pq}\|=(i)} x_{pq}^{(s)} e_{pq} \\ \text{if } \alpha = 0 \in \mathbb{Z}_2 \text{ and } 1 \le p,\ q \le a \text{ or } a+1 \le p,\ q \le a+b \\ \sum\limits_{\|e_{pq}\|=(i)} y_{pq}^{(s)} e_{pq} \\ \text{if } \alpha = 1 \in \mathbb{Z}_2 \text{ and } 1 \le q \le a,\ a+1 \le p \le a+b \\ \qquad\qquad\qquad\qquad \text{or } 1 \le p \le a,\ a+1 \le q \le a+b. \end{cases}$$

Theorem 4.4.

(1) *The algebra generated by* $A_s^{(i)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n}(M_n(F))$*.*

(2) *The algebra generated by* $B_s^{(i,\alpha)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(M_n(E))$*.*

(3) *The algebra generated by* $C_s^{(i,\alpha)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(M_{a,b}(E))$*.*

Using direct computations with generic algebras, in [18], the author found the growth functions of the following relatively free graded algebras:

Proposition 4.5.

(1)

$$g(n)(F_1^{\mathbb{Z}_2}(M_2(F))) = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{4}{3}n + 1$$

and

$$\mathrm{GKdim}_1^{\mathbb{Z}_2}(M_2(F)) = 3$$

(2)

$$g(n)(F_1^{\mathbb{Z}_2\times\mathbb{Z}_2}(M_{1,1}(E))) = n^2 + n + 2$$

and

$$\mathrm{GKdim}_1^{\mathbb{Z}_2\times\mathbb{Z}_2}(M_{1,1}(E)) = 2.$$

The knowledge of the $\mathbb{Z}_2$-cocharacter sequence of $M_2(F)$ (Di Vincenzo [19]) gave the possibility to generalize the previous result. In particular (see [18]):

Proposition 4.6.

$$g(n)(F_k^{\mathbb{Z}_2}(M_2(F))) =$$

$$= \sum_{m=0}^{n}\sum_{s=0}^{m}\sum_{i=0}^{s}\sum_{j=0}^{m-s}(s-i+1)\frac{s-2i+1}{k-1}\binom{s-i+k-1}{k-2}\binom{i+k-2}{k-2}\cdot$$

$$\cdot\frac{m-s-2j+1}{k-1}\binom{m-s-j+k-1}{k-2}\binom{j+k-2}{k-2}$$

that is a polynomial in n of degree $4k-1$, then

$$\mathrm{GKdim}_k^{\mathbb{Z}_2}(M_2(F)) = 4k-1.$$

An easy computation shows that $\mathrm{tr.deg}_F(F_k^{\mathbb{Z}_n}(M_n(F))) = kn^2-n+1$. Combining the latter equation with Theorem 3.8, one has the following theorem (see [17]):

Theorem 4.7.

$$\mathrm{GKdim}_k^{\mathbb{Z}_n}(M_n(F)) = kn^2-n+1.$$

Finally, a joint combination of computation with graded cocharacters and the previous theorem, gives the following theorem (see [18]):

Theorem 4.8.

(1) $\mathrm{GKdim}_k^{\mathbb{Z}_n}(M_n(F)) = kn^2-n+1.$
(2) $\mathrm{GKdim}_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(M_n(E)) = kn^2-n+1.$
(3) $\mathrm{GKdim}_k^{\mathbb{Z}_{a+b}\times\mathbb{Z}_2}(M_{a,b}(E)) = k(a^2+b^2)-a+1.$

We would like to sketch the technique used in [18] in order to prove the point (3) of the previous theorem. For any $r=1,\dots,k$ and for any $i=0,\dots,n-1$, let

$$D_r^{(i)} = C_r^{(i,0)} + C_r^{(i,1)}$$

and consider the algebra D_k generated by

$$\{D_r^{(i)} | r=1,\dots,k \text{ and } i=0,\dots,n-1\}.$$

It is easy to observe that D_k embeds into $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(M_{a,b}(E))$, then

$$\mathrm{GKdim}(D_k) \leq \mathrm{GKdim}_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(M_{a,b}(E)).$$

We also observe that D_k is a homomorphic image of $F_k(A)$ by the homomorphism φ such that

$$A_r^{(i)} \mapsto D_r^{(i)}.$$

In order to compute the GK dimension of D_k we are allowed to study the asymptotic behavior of the 0-component of D_k because in D_k as for $F_k(A)$, any monomial of degree i is determined by its $(1, i+1)$-entry.

First of all, we observe that there exists a one-to-one correspondence between the monomials M of $F_k(A)$ of $\mathbb{Z}_n$-degree 0 that are independent modulo the $T_{\mathbb{Z}_n}$-ideal of $M_n(F)$, and the monomials $m = x^{(i_1)}_{p_1q_1} \cdots x^{(i_t)}_{p_tq_t}$ in the commuting variables $\{x^{(i_h)}_{p_hq_h}\}$, appearing in the $(1,1)$-entry of M, such that $q_h = p_{h+1}$ for any h, and $\|q_t - p_1\| = 0$. Let M^0_k be the algebra generated by the monomials M and m^0_k be the algebra generated by the monomials m, then M^0_k and m^0_k are isomorphic F-algebras through the homomorphism ψ. Analogously, there exists a one-to-one correspondence between the monomials M' of D_k of $\mathbb{Z}_n$-degree 0 that are independent modulo the $T_{\mathbb{Z}_n}$-ideal of D_k, and the monomials $m' = z^{(i_1)}_{p_1q_1} \cdots z^{(i_t)}_{p_tq_t}$ in the commuting variables $\{x^{(i_h)}_{p_hq_h}, y^{(j_v)}_{p_vq_v}\}$, appearing in the $(1,1)$-entry of M', such that $q_h = p_{h+1}$, for any h, and $\|q_t - p_1\| = 0$. Let W^0_k be the algebra generated by the monomials M' and w^0_k be the algebra generated by the monomials m', then W^0_k and w^0_k are isomorphic F-algebras through the homomorphism ψ'. It is easy to see that $W^0_k = \varphi(M^0_k)$, then there exists an unique onto homomorphism $\gamma : m^0_k \to w^0_k$ such that the following diagram is commutative:

$$\begin{array}{ccc} M^0_k & \xrightarrow{\varphi} & W^0_k \\ {\scriptstyle\psi}\big\downarrow & & \big\downarrow{\scriptstyle\psi'} \\ m^0_k & \xrightarrow{\gamma} & w^0_k \end{array}$$

We note that the homomorphism γ is such that $x^{(r)}_{pq} \mapsto x^{(r)}_{pq}$ if $(p,q) \in J$, where J is the set of indices lying in the left upper corner of size a or in the lower right corner of size b, and $x^{(r)}_{pq} \mapsto y^{(r)}_{pq}$ otherwise.

For any $\nu \in \mathbb{N}$ consider the F-subspace $M^0_{k,\nu}$ of M^0_k formed by all monomials in M^0_k of total degree ν. If $R = \{p_1, \ldots, p_d\}$ is a basis of $M^0_{k,\nu}$, due to the fact that γ is an epimorphism, $\gamma(\psi(R))$ is a system of generators of the vector space $w^0_{k,\nu}$, then the elements in $\gamma(\psi(R))$ are monomials $\pm x^{m_{11}}_{11} x^{m_{12}}_{12} \cdots y^{l_{1a}}_{1a} \cdots y^{l_{1n}}_{1n} x^{m_{21}}_{21} \cdots x^{m_{nn}}_{nn}$ such that $\sum_{(i,j)\in J} m_{ij} + \sum_{(i,j)\notin J} l_{ij} = \nu$. Notice that for any $(i,j) \notin J$, l_{ij} is limited by 1. Then the number of variables that give a non-zero contribute to the Gelfand-Kirillov dimension of D_k is the total number of variables minus the number of variables $y^{(r)}_{pq}$ such that $(p,q) \notin J$ and, from the latter, we have not to take into account those $x^{(1)}_{1l}$, $a+1 \le l \le n$. In particular, they are in number of $k(a+b)^2 - n + 1 - 2kab + b$ that is $k(a^2+b^2) - a + 1$. In other words:

$$\mathrm{GKdim}(D_k) = k(a^2 + b^2) - a + 1.$$

Now the theorem follows from Proposition 5.8 of [17].

5. Upper triangular matrices. We shall present the results concerning the ordinary and the graded GK dimension of the upper triangular matrices, i.e., $UT_n(F)$, $UT_n(E)$ and $UT_{a,b}(E)$ that is $UT_n(E) \cap M_{a,b}(E)$. We recall that the algebra of upper triangular matrices $UT_n(F)$ is a central object in the theory of PI-algebras satisfying a non-matrix polynomial identity or, equivalently, having identities that do not hold for $M_2(F)$. For example, in [34] Latyshev proved that every finitely generated PI-algebra satisfying a non-matrix identity satisfies the identities of $UT_n(F)$ for a certain n. It follows that the polynomial identities of $UT_n(F)$ may serve as a measure of the complexity of the polynomial identities of finitely generated algebras with non-matrix identity in the same way as the polynomial identities of $M_n(F)$ measure the complexity of the identities of arbitrary PI-algebras. Moreover, the T-ideals of $UT_n(F)$ and $UT_n(E)$ are also important because they are examples of maximal T-ideals of a given exponent of the codimension sequences (for more details, see the work of Drensky [22]).

We start with the following easy result:

Proposition 5.1. *For any $k \geq 2$ and any $n \geq 1$,*

$$\mathrm{GKdim}_k(UT_n(F)) = kn.$$

As an easy consequence of the Theorem of Lewin ([35]) and of the previous result, one has that:

Proposition 5.2. *For any $k \geq 2$ and any $n \geq 1$,*

(1)

$$\mathrm{GKdim}_k(UT_n(E)) = kn,$$

(2)

$$\mathrm{GKdim}_k(UT_{a,b}(E)) = k(a + b).$$

The next three results of Markov (see [36]) concern the GK dimension of finitely generated PI-algebras satisfying non-matrix polynomial identities.

Theorem 5.3. *Let A be a finitely generated algebra satisfying a non-matrix polynomial identity. Then $\mathrm{GKdim}_k(A) = kd$, where d is the maximal integer such that $UT_d(F) \in \mathcal{V} = \mathcal{V}(A)$.*

Proposition 5.4. *The following conditions on a variety of algebras $\mathcal{V}$ are equivalent:*

(1) $UT_2(F) \notin \mathcal{V}$;
(2) $\mathrm{GKdim}(S) = \mathrm{GKdim}(S/J(S))$ *for any finitely generated algebra* S *from* $\mathcal{V}$, *where* $J(S)$ *is the Jacobson radical of* S.

Proposition 5.5. *The following conditions on a variety of algebras* $\mathcal{V}$ *are equivalent:*
(1) $UT_3(F) \notin \mathcal{V}$;
(2) $\mathrm{GKdim}(S)$ *is an integer for any finitely generated algebra* S *from* $\mathcal{V}$.

We consider now the graded case. In [21], Di Vincenzo, Koshlukov and Valenti, found an explicit basis for the T_G-ideal of the algebra $UT_n(F)$ with any elementary G-grading and where G is a finite group. In particular, $UT_n(F)$ inherits the Vasilovsky $\mathbb{Z}_n$-grading from $M_n(F)$. We have that (see [32] and [21]):

Theorem 5.6. *The* $T_{\mathbb{Z}_n}$*-ideal of* $UT_n(F)$ *with the Vasilovsky grading is generated as a* $\mathbb{Z}_n$*-ideal by*

$$[x_1^{(0)}, x_2^{(0)}]$$

and

$$x^{(r)}x^{(s)}, \text{ where } n \le r + s.$$

As long as for the verbally prime algebras, for every $s = 1, \dots, k$, let us consider the following:

$$UA_s^{(i)} = \sum_{\|e_{pq}\|=(i), p\le q} x_{pq}^{(s)} e_{pq},$$

$$UB_s^{(i,\alpha)} = \begin{cases} \sum_{\|e_{pq}\|=(i), p\le q} x_{pq}^{(s)} e_{pq} & \text{if } \alpha = 0 \in \mathbb{Z}_2 \\ \sum_{\|e_{pq}\|=(i)} y_{pq}^{(s)} e_{pq} & \text{if } \alpha = 1 \in \mathbb{Z}_2 \end{cases},$$

$$UC_s^{(i,\alpha)} = \begin{cases} \sum_{\|e_{pq}\|=(i), p\le q} x_{pq}^{(s)} e_{pq} \\ \text{if } \alpha = 0 \in \mathbb{Z}_2 \text{ and } 1 \le p \le q \le a \text{ or } a+1 \le p \le q \le a+b \\ \\ \sum_{\|e_{pq}\|=(i)} y_{pq}^{(s)} e_{pq} \\ \text{if } \alpha = 1 \in \mathbb{Z}_2 \text{ and } 1 \le p \le a, a+1 \le q \le a+b. \end{cases}$$

Theorem 5.7.
(1) *The algebra generated by* $UA_s^{(i)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n}(UT_n(F))$.

(2) *The algebra generated by* $UB_s^{(i,\alpha)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(UT_n(E))$.

(3) *The algebra generated by* $UC_s^{(i,\alpha)}$*'s is isomorphic to* $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(UT_{a,b}(E))$.

In the light of the previous theorem, the next propositions ([17]) give a complete description of the multi-homogeneous monomials of $F_k^{\mathbb{Z}_n}(UT_n(F))$, $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(UT_n(E))$ and $F_k^{\mathbb{Z}_n\times\mathbb{Z}_2}(UT_{a,b}(E))$.

Proposition 5.8. *Let* $M := M(x_1,\dots,x_m)$ *and* $N := N(x_1,\dots,x_m)$ *be two non-zero monomials of* $\mathcal{R} := F\langle x_1^0,\dots,x_1^{n-1},\dots,x_k^{n-1}\rangle$*. Suppose further that* M, N *belong to the same multi-homogeneous component of* $\mathcal{R}$*. If* $M = x_{i_1}\cdots x_{i_m}$ *and* $N = x_{j_1}\cdots x_{j_m}$*, let* $S =: \{s|\|x_{i_s}\| = 0\}$ *and* $T := \{s|\|x_{j_s}\| = 0\}$*. Then* $M \equiv N(\bmod I)$ *if and only if for any* $k \notin S$*,* $i_k = j_k$ *and if* $\{s, s+1, s+2,\dots,s+k\} \subseteq S$*, for any* $l = 0,\dots,k$ *one has that* $x_{i_{s+l}} = x_{j_{s+m}}$ *for some* $m = 0,\dots,k$.

Proof. (Sketches) Suppose $M \equiv N \pmod I$ and, for simplicity, that $\|M\| = \|N\| = 0$. If $A_1,\dots,A_m$ are graded homogeneous elements, we shall denote

$$\overline{M} := M(A_1,\dots,A_m)$$

and

$$\overline{N} := N(A_1,\dots,A_m)$$

then, $\overline{M}$ and $\overline{N}$ have at the same positions the same non zero entries. We shall compare the $(1,1)$ entry of $\overline{M}$ with the $(1,1)$ entry of $\overline{N}$. Then

$$\overline{M}_{1,1} = (x_{1,1}^{(1)})^{l_{1,1}}\cdots(x_{1,1}^{(n-1)})^{l_{1,n-1}}\cdot$$

$$\cdot X_{1,\alpha_1}\cdots X_{\alpha_p,\alpha_{p+1}}(x_{\alpha_{p+1},\alpha_{p+1}}^{(1)})^{l_{p+2,1}}\cdots(x_{\alpha_{p+1},\alpha_{p+1}}^{(n-1)})^{l_{p+2,n-1}},$$

$$\overline{N}_{1,1} = (x_{1,1}^{(1)})^{m_{1,1}}\cdots(x_{1,1}^{(n-1)})^{m_{1,n-1}}\cdot$$

$$\cdot X_{1,\beta_1}\cdots X_{\beta_p,\beta_{p+1}}(x_{\beta_{p+1},\beta_{p+1}}^{(1)})^{m_{p+2,1}}\cdots(x_{\beta_{p+1},\beta_{p+1}}^{(n-1)})^{m_{p+2,n-1}},$$

where we indicate with capitol X the variables appearing in the A_i's with $\mathbb{Z}_n$-degree different from 0. We have that the sets

$$A := \{X_{1,\alpha_1}, X_{\alpha_1,\alpha_2},\dots,X_{\alpha_p,\alpha_{p+1}}\}$$

and

$$B := \{X_{1,\beta_1}, X_{\beta_1,\beta_2},\dots,X_{\beta_p,\beta_{p+1}}\},$$

where $1 < \alpha_1 < \alpha_2 < \cdots < \alpha_{p+1}$ and $1 < \beta_1 < \beta_2 < \cdots < \beta_{p+1}$ must coincide and it is easy to verify that $X_{\alpha_i,\alpha_{i+1}} = X_{\beta_i,\beta_{i+1}}$ for any $i = 0,\dots,p$. Finally, we observe that the homogeneous element of $\mathbb{Z}_n$-degree 0 commute and we are done. $\square$

Proposition 5.9. *Let $M := M(x_1, \ldots, x_m)$ and $N := N(x_1, \ldots, x_m)$ be two non-zero monomials of*

$$\mathcal{S} := F\langle x_1^{(0,0)}, \ldots, x_k^{(0,0)}, \ldots, x_1^{(n-1,0)}, \ldots, x_k^{(n-1,0)}, x_1^{(n-1,1)}, \ldots, x_k^{(n-1,1)}\rangle.$$

Suppose further that M, N belong to the same multi-homogeneous component. If $M = x_{i_1} \cdots x_{i_m}$ and $N = x_{j_1} \cdots x_{j_m}$, let $S_1 =: \{s | \|x_{i_s}\| = (0,0)\}$, $S_2 =: \{s | \|x_{i_s}\| = (0,1)\}$, $T_1 =: \{s | \|x_{j_s}\| = (0,0)\}$ and $T_2 := \{s | \|x_{j_s}\| = (0,1)\}$. If $j \in \{1,2\}$, then $M \equiv N (\mathrm{mod}\ I_j)$ if and only if for any $k \notin S_1 \cup S_2$, $i_k = j_k$, if $\{s, s+1, s+2, \ldots, s+k\} \subseteq S_1$, for any $l = 0, \ldots, k$ one has that $x_{i_{s+l}} = x_{j_{s+m}}$ for some $m = 0, \ldots, k$ and if $\{s, s+1, s+2, \ldots, s+k\} \subseteq S_2$, for any $l = 0, \ldots, k$ one has that $x_{i_{s+l}} = x_{j_{s+m}}$ for some $m = 0, \ldots, k$. Moreover, $|S_2| \leq 2$.

Now we have the following result:

Theorem 5.10. $\mathrm{GKdim}_k^{\mathbb{Z}_n}(UT_n(F)) = nk$*.*

P r o o f. (Sketches) In order to compute the growth function of $F_k^{\mathbb{Z}_n}(UT_n(F))$, we may compute the number $n(m)$ of monomials of degree m of the type

$$w = Y \cdots YXY \cdots YXY \cdots YXY \cdots Y$$

where the variables Y have $\mathbb{Z}_n$-degree 0 and the X have $\mathbb{Z}_n$-degree different from 0, for any $m \in \mathbb{N}$. In the light of Proposition 5.8, we have that $n(m)$ depends only on the variables of $\mathbb{Z}_n$-degree 0. Suppose w has l variables of $\mathbb{Z}_n$-degree different from 0, to say, $X_1, \ldots, X_l$ and let $m_1, \ldots, m_{l+1}$ be respectively the number of variables of w of $\mathbb{Z}_n$-degree 0 lying respectively before X_1, between X_1 and X_2 and so on. If $n(m_1), \ldots, n(m_{l+1})$ are the numbers of commutative monomials in k variables of degree respectively $m_1, \ldots, m_{l+1}$, then the number of different monomials w of degree m with l element of $\mathbb{Z}_n$-degree different from 0 is

$$\alpha(X) \sum_{m_1=0}^{m-l} \sum_{m_2=0}^{m-m_1-l} \cdots \sum_{m_l=0}^{m-\sum_{i=1}^{l-1} m_i - l} \binom{m_1+k-1}{k-1} \cdots \binom{m - \sum_{i=1}^{l-1} m_i + k - 1}{k-1}$$

where $\alpha(X)$ is limited by $(n-1)^l$. Standard combinatorial arguments show that $n(m)_l$ is a polynomial in m of degree $k(l+1)-1$. It turns out that the growth function of $F_k^{\mathbb{Z}_n}(UT_n(F))$ is a polynomial in m of degree kn and the theorem follows. $\square$

Analogously we have that:

Theorem 5.11. *For any $k, n \geq 1$ and $a, b \in \mathbb{N}$ such that $a \geq b$ and $a+b=n$,*

(1)
$$\mathrm{GKdim}_k^{\mathbb{Z}_n \times \mathbb{Z}_2}(UT_n(E)) = nk;$$

(2)
$$\mathrm{GKdim}_k^{\mathbb{Z}_n \times \mathbb{Z}_2}(UT_{a,b}(E)) = nk.$$

We can draw a sort of parallelism between the ordinary case and the graded one. In fact, the graded GK dimensions seem to preserve the "relations" among the ordinary GK dimensions of a certain PI-algebra. For example, in the ordinary case $\mathrm{GKdim}_k(M_n(F)) = \mathrm{GKdim}_k(M_n(E))$ and this equality holds also in the graded case. Moreover,

$$\mathrm{GKdim}_k(UT_n(F)) = \mathrm{GKdim}_k(UT_n(E)) = \mathrm{GKdim}_k(UT_{a,b}(E))$$

but it is also true that

$$\mathrm{GKdim}_k^{\mathbb{Z}_n}(UT_n(F)) = \mathrm{GKdim}_k^{\mathbb{Z}_n \times \mathbb{Z}_2}(UT_n(E)) = \mathrm{GKdim}_k^{\mathbb{Z}_n \times \mathbb{Z}_2}(UT_{a,b}(E)).$$

This means that the definition of graded GK dimension is, in a certain sense, quite "natural".

6. Open problems. In what follows we shall draw some problems arising as advances of the recent studies and results in the theory of algebras with polynomial identities.

We recall that an F-algebra A is said to be representable if it is isomorphic to a subalgebra of $M_n(S)$, where S is a commutative ring. For example, the relatively free algebras are representable (see [31]). We consider the following result of Markov (see [36]):

Proposition 6.1. *Finitely generated representable algebras over infinite fields have integer GK dimension.*

As a corollary, we have the following (see point (4) of Proposition 2.4):

Corollary 6.2. *The GK dimension of any finitely generated relatively free algebra over an infinite field is a non-negative integer.*

We suggest to generalize the notion of representability to the graded case in the following way:

Definition 6.3. *Let A be a finitely generated G-graded algebra, where G is a finite group. We say that A is G-representable if there exists a G-graded isomorphism from A to a G-graded subalgebra of $M_n(S)$, where S is a commutative ring.*

Following the idea of Markov, we state the following conjectures:

Conjecture 6.4. *Finitely generated G-representable algebras over infinite fields have integer GK dimension.*

Conjecture 6.5. *The G-graded relatively free algebras over infinite fields are G-representable. Moreover, the graded GK dimension of a G-graded PI-algebra is a non-negative integer.*

Given a G-graded algebra A, where G is a finite group, the PI-properties of the component A^{1_G} to a large extent predetermine the PI-properties of A. It is well known that if A^{1_G} is PI, then A satisfies a non-trivial polynomial identity (see [15]). More precisely, we have the following ([9], Theorem 5.3):

Theorem 6.6. *Let G be a finite group. Consider a G-graded algebra $A = \bigoplus_{g\in G} A^g$. Suppose that A^{1_G} satisfies a polynomial identity of degree d. Then A satisfies a polynomial identity of degree n where n is any integer satisfying the inequality*

$$\frac{|G|^n\,(|G|\,d-1)^{2n}}{(|G|\,d-1)!} < n!.$$

In particular, if n is the least integer such that $e\,|G|\,(|G|\,d-1)^2 \leq n$ then A satisfies a polynomial identity of degree n where e is the basis of the natural logarithm.

We recall that in [28], [27] Giambruno and Zaicev proved that:

Theorem 6.7. *If A is any PI-algebra, then there exists the limit*

$$\exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$$

and it is a non-negative integer called the PI-exponent of A.

The relations between the identities of A^{1_G} and A have been studied in a number of papers. For example, it is easy to show that if G is a finite group and A^{1_G} is nilpotent then A is also nilpotent. If A^{1_G} satisfies a standard identity of degree m then A satisfies some power of the standard identity of degree dm, where d is the order of G (see [15]).

The relations between identities of A and A^{1_G} can be defined in terms of PI-exponents of A and A^{1_G}. Clearly, $\exp(A^{1_G}) \leq \exp(A)$ since A^{1_G} is a subalgebra of A. On the other hand, there are a lot of examples of graded algebras satisfying the inequality

$$\exp(A) \leq |G|^2 \exp(A^{1_G}). \tag{1}$$

In [10], Bahturin and Zaicev conjectured that the inequality (1) holds for any graded algebra with a non-trivial polynomial identity. Recently, this conjecture has been solved into affirmative in the case G is a finite abelian group in a joint work of Aljadeff, Giambruno and La Mattina (see [1]).

In the spirit of 1 and in the light of the result of [1], Aljadeff and the author conjectured that:

Conjecture 6.8. *Let G be a finite abelian group and A be a G-graded PI-algebra, then for any $k \geq 1$*

$$\mathrm{GKdim}_k^G(A) \leq |G|^2 \mathrm{GKdim}_k(A^{1_G}).$$

We recall that if two algebras A and B satisfy the same polynomial identities we say that A is PI-equivalent to B and denote by $A \sim B$. An important corollary to the structure theory of Kemer is the Tensor Product Theorem (TPT):

Theorem 6.9. *Let $a, b, c, d \in \mathbb{N}$ such that $a \geq b$ and $c \geq d$ and F be a field of characteristic 0, then:*

(1) $M_{a,b}(E) \otimes E \sim M_{a+b}(E)$;
(2) $M_{a,b}(E) \otimes M_{c,d}(E) \sim M_{ac+bd,ad+bc}(E)$;
(3) $M_{1,1}(E) \sim E \otimes E$.

We focus on the fact that the behavior of the corresponding T-ideals in positive characteristic has been studied in [6], [7] and [8]. It has also been proved that the Tensor product theorem is still valid over infinite fields of characteristic $p > 2$ as long as one considers multilinear polynomials only. In the paper [8] the authors constructed an appropriate model for the relatively free algebra in the variety of algebras determined by $E \otimes E$ when the field F has characteristic $p > 2$. This model is the generic algebra of $A = F \oplus M_{1,1}(E')$ where E' stands for the Grassmann algebra without unit. It turned out that $E \otimes E$ and A satisfy the same graded and hence ordinary polynomial identities. In [6] the authors used the properties of A in order to show that that $T(M_{1,1}(E)) \subsetneq T(E \otimes E)$ in positive characteristic. Using the ordinary GK dimension Alves and Koshlukov (see [5]) were able to prove the PI non-equivalence of some of the T-prime algebras in positive characteristic. For example, we have the following:

Proposition 6.10. *Let F be a field of characteristic $p > 2$. Then for any $k \geq 2$*

$$\mathrm{GKdim}_k(E \otimes E) = k$$

and

$$\mathrm{GKdim}_k(M_{1,1}(E) \otimes E) = 2k.$$

Due to the fact that for any $k \geq 2$ one has that $\mathrm{GKdim}_k(M_{1,1}(E)) = 2k$ and $\mathrm{GKdim}_k(M_2(E)) = 4k - 2$, it turns out that:

Corollary 6.11. *Let F be a field of characteristic $p > 2$, then*

$$E \otimes E \nsim M_{1,1}(E)$$

and

$$M_{1,1}(E) \otimes E \nsim M_2(E).$$

In [7] the subalgebras $A_{a,b}$ of $M_{a+b}(E)$ were introduced and these were useful in establishing that $T(M_2(E)) \subsetneq T(M_{1,1}(E))$. We recall that $A_{a,b}$ is the subalgebra of $M_{a+b}(E)$ of all block matrices whose diagonal blocks of size $a \times a$ and $b \times b$ have entries from E while the other two blocks are with entries from E'. It was shown in [7] that $M_{1,1}(E) \otimes E \sim A_{1,1}$. In [7] the authors asked whether $M_{a,b}(E) \otimes E \sim A_{a,b}$. In [4] Mota Alves proved that:

Proposition 6.12. *Let F be a field of characteristic $p > 2$ and $a, b \in \mathbb{N}$ such that $a \geq b$, then $A_{a,b}$ and $M_{a+b}(E)$ are not PI equivalent.*

In the spirit of these results and keeping in mind that to show $E \otimes E \sim A$ one uses the graded identities, Koshlukov suggested to compute $\mathrm{GKdim}_k^{Z_{a+b} \times Z_2}(M_{a,b}(E) \otimes E)$ and $\mathrm{GKdim}_k^{Z_{a+b} \times Z_2}(M_{a+b}(E))$. For this purpose, we notice that the graded identities of the latter algebras (either in characteristic 0 or in characteristic $p > 2$) are well known. For the same reason it should be useful to compute the graded GK dimensions of $M_{a,b}(E) \otimes M_{c,d}(E)$ and $M_{ac+bd,ad+bc}(E)$.

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Dipartimento Di Matematica
Universitá Degli Studi Di Bari
via Orabona 4
70125 Bari, Italy
e-mail: `centrone@ime.unicamp.br`

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