MINIMAL MODELS OF SELF-ORGANIZED CRITICALITY

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Abstract. The paper deals with an evaluation of the behavior of non equilibrium systems displaying self-organized criticality, according to the concept of Bak-Tang-Wiesenfeld ([3]). One of the fundamental characteristics of a system in a self-organized state is to exhibit a stationary state with a long-range power law of decay of both spatial and temporal correlations.

Keywords: Self organizing criticalities, Markov Chains, Bak Sneppen processes, econophysics, socio-economic evolution staircase.

Introduction

A rather general frame can be the following:

The spatial structure is discretized in N nodes, that are connected according to some distance rules. Bak-Sneppen original structure (see [1] and [2]) foresaw a

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circular disposition, where each node was connected only with the two adjoining nodes. Many models supplied more sophisticated systems of contact between the nodes, but a relevant structural change seemed to happen only in the case of a bipartite system, as pointed out in Piccinini-Lepellere-Chang ([23]) and Piccinini-Lepellere-Chang-Iseppi ([21]).

Also time is discretized. At each step all nodes have one or more properties. These can be expressed by discrete or continuos variables, numerical or ordinal, or even simply by labels, as in the case of political choices. In numerical cases the function will be called fitness, according to the historical definition of Bak-Sneppen.

The transition from one time step to another involves random changes of the quality of one or more nodes, according to structural distance properties. Hence the process may be represented by a Markov Chain. A typical, but not compulsory, feature is that the changing node is selected among the nodes that have the lowest fitness. Hence it is selected according to a global information. Its fitness will change according to some probability law, so that it may even not increase, but this is inessential. What is essential is that one or more neighbors will also randomly change their fitness even if it is already good. Here lies the clue of the process, in the fact that the behavior of neighbors may even damage their fitness. It is anyhow well proved that in spite of the randomness of evolution, the average level of fitness will be superior to the average of the probability distribution, and that depends on the fact that there exists a moment of rational choice, that is the order of changing given to one of the worst nodes.

Many questions may be asked, but one of the most appealing is the average distribution of the fitness levels. Analytic formulas are very rare because of the extreme complication of the model, but statistical models give good evidences, in particular that in the classical Bak Sneppen model, for large number of nodes (100 or more), for high numers of iterations (over 1,000,000), with fitness uniformly distributed in the interval $[0, 1]$, the density of the average distribution follows a sigmoid curve, with values near 0 up to 0.5, and an almost constant asymptotic value in the interval starting from 0.667... ([14]). The sigmoid behavior seems to be very stable and is a characteristic of this type of processes, even if for small systems the left side is less evident.

A recent explicit solving formula for the case of 4 nodes (the smallest non obvious dimension) is to be found in ([24]) and for 5 nodes in ([25]). Small models have been investigated in order to get analytic sharp informations. One of the most celebrated is Barbay-Kanyon [5], with two discrete fitness levels, achieved with a varying probability, so that many models should be approximated. The authors have shown that phaenomena may arise that cannot be explained with Barbay-Kenyon model. Anyhow also this case is by no means trivial, as it was shown by Meester and Znamenski in ([16]), see also ([4]).

The aim of this paper is to give a minimal non-trivial model where the standard behavior is still recognizable, and at the same time a Barbay-Kenyon type method allows exact results.

1. The elementary model

In general there exist L^N configurations, where N is the number of nodes and L is the number of fitness levels. The transition matrix of the associated Markov chain is thus a very large matrix, and depends on the structure of the choices that are allowed and the neighborhood table.

The simplest non-trivial case requires three nodes and one neighbor, so that at each step one node is selected for change, another is changed in view of neighborhood and the third one is left unchanged. Anyhow the complexity of computation in general depends on the number of fitness levels. But in the case of three nodes it is possible to reduce every computation to the case of two levels as it is proved in section 3.

Let L be the number of levels. For any given level $L(i)$, $p(i)$ denotes the probability of extracting a fitness equal to $L(i)$, so that $\sum_{i=1}^{N} p(i) = 1$. The three nodes are disposed on a circle and the (only) neighbor is the left node, therefore the right node remains unchanged.

Since in this problem we are not investigating the return time between configuration we can ignore rotations, so that the L^3 possible configurations can be reduced to $(L^3 - L)/3 + L$, namely L corresponding to a constant sequence and the remaining corresponding each to 3 configurations.

Three simple examples are shown below: the choice of the examples depends on the use made of them in the third section.

1.1 Example 1. Ternary elementary model

The first matrix is referred to three fitness levels, hence it is a 11×11 matrix. The simplest way is to construct an auxiliary column for each of the unchanged items: so there are three auxiliary columns, denoted by $aux1$, $aux2$, $aux3$, and p, q, r denote the probabilities of the three levels.

					aux3	aux2	aux1	
THREE	a	3	3	3	r^2	0	0	$a = aux3$
	b	3	3	2	$2qr$	r^2	0	$b = aux3$
TWO	c	3	3	1	$2pr$	0	r^2	$c = aux3$
	d	3	2	2	q^2	$2qr$	0	$d = (aux3 + aux2)/2$
	e	3	2	1	pq	pr	qr	$e = aux3$
	f	3	1	2	pq	pr	qr	$f = aux2$
ONE	g	3	1	1	p^2	0	$2pr$	$g = (aux3 + aux1)/2$
	h	2	2	2	0	q^2	0	$h = aux2$
	i	2	2	1	0	$2pq$	q^2	$i = aux2$
	j	2	1	1	0	p^2	$2pq$	$j = (aux2 + aux1)/2$
ZERO	k	1	1	1	0	0	p^2	$k = aux1$

Table 1: *Transition matrix of elementary ternary model*

The choice of the unchanged node depends on the location of the minimum. In case of ties the same probability is assigned to each of the allowed locations, so that the transition column may not be a pure auxiliary column but a weighted combination of different columns. The final transition matrix, see Table 1, is formed gluing together the auxiliary column according to the rule indicated on the right.

1.2 Example 2. Fundamental elementary transition matrix

An important case is obtained from example 1, stating that one of the probabilities is 0, so that it actually reduces to only two fitness levels. It is obtained from the matrix generated by example 1. Here it is written explicitely, developing the construction up to the end, so that the reader can explicitely check the rule.

The number of states reduces to 4, as they correspond to the cardinality of nodes having the highest level of fitness (probability q), see Table 2.

	# 3	# 2	# 1	# 0
# 3=h	q^2	q^2	$q^2/2$	0
# 2=i	$2pq$	$2pq$	$pq + q^2/2$	q^2
# 1=j	p^2	p^2	$p^2/2 + pq$	$2pq$
# 0=k	0	0	$p^2/2$	p^2

Table 2: *Fundamental elementary transition matrix*

We represent also the minimal problem for the traditional Bak Sneppen model with two neighbors. The minimal non trivial number of nodes is 4, so that also in this case one node is left unchanged. The example is referred to three fitness levels.

1.3 Example 3. Fundamental 3 level Bak Sneppen transition

Traditional Bak Sneppen model with four nodes and three fitness levels has the following basic transition matrix (apart rotations). Since at each step only one node is left unchanged, three auxiliary columns are enough, according to the value of the remaining node. The structure is thus achieved according to the description that follows each line. Ignoring rotations 24 configurations are required, see Table 3.

Aggregating the two lower fitness values one gets the standard 2-level table with 6 configurations.

4 = a
3 = b, c
2+ (two consecutive maxima) = d, e, f, g
2- (intervalled maxima) = h, i, j
1 = k, l, m, n, o, p, q, r
0 = s, t, u, v, w, x

					aux3	aux2	aux1	
a	3	3	3	3	r^3	0	0	$a = aux3$
b	3	3	3	2	$3qr^2$	r^3	0	$b = aux3$
c	3	3	3	1	$3pr^2$	0	r^3	$c = aux3$
d	3	3	2	2	$2q^2r$	$2qr^2$	0	$d = aux3$
e	3	3	2	1	$2pqr$	pr^2	qr^2	$e = aux3$
f	3	3	1	2	$2pqr$	pr^2	qr^2	$f = aux3$
g	3	3	1	1	$2p^2r$	0	$2pr^2$	$g = aux3$
h	3	2	3	2	q^2r	qr^2	0	$h = aux2$
i	3	2	3	1	$2pqr$	pr^2	qr^2	$i = aux2$
j	3	1	3	1	p^2r	0	pr^2	$j = aux1$
k	3	2	2	2	q^3	$3q^2r$	0	$k = (2aux2 + aux1)/3$
l	3	2	2	1	pq^2	$2pqr$	q^2r	$l = aux2$
m	3	2	1	2	pq^2	$2pqr$	q^2r	$m = aux3$
n	3	2	1	1	p^2q	p^2r	$2pqr$	$n = (aux3 + aux2)/2$
o	3	1	1	2	p^2q	p^2r	$2pqr$	$o = (aux3 + aux2)/2$
p	3	1	2	1	p^2q	p^2r	$2pqr$	$p = aux1$
q	3	1	1	1	p^3	0	$3p^2r$	$q = (aux3 + 2aux2)/3$
r	3	1	2	2	pq^2	$2pqr$	q^2r	$r = aux2$
s	2	2	2	2	0	q^3	0	$s = aux2$
t	2	2	2	1	0	$3pq^2$	q^3	$t = aux2$
u	2	2	1	1	0	$2p^2q$	$2pq^2$	$u = aux2$
v	2	1	2	1	0	p^2q	pq^2	$v = aux1$
w	2	1	1	1	0	p^3	$3p^2q$	$w = (aux2 + 2aux1)/3$
x	1	1	1	1	0	0	p^3	$x = aux1$

Table 3: *Fundamental 3 level Bak Sneppen transition*

2. Questions and solutions

Many questions arise when studying even simple models. A main question is the average probability of an assigned fitness level. The average distribution requires that the ergodic state is achieved, what could require very high times, especially in certain partitioned frames where a long transition period must be foreseen as was proved by the authors in ([21]).

If we have the full transition matrix, with all the fitness levels explicitely listed, it would just be required to find the first normalized eigenvector corresponding to the transition matrix, so that the probabilities of the level could be easily derived according the assigned model. In presence of a rectangular distribution, the same probability would be assigned to each level.

Unfortunately this problem, even avoiding duplications as in examples 1, 2 and 3, would fast become very heavy for computation. The only cases that can be easily dominated are binary frames, where all the fitness levels are identified either in a low level, say 0, or in an high level, say 1. Of course in this case the probability of the two events shall be taylored according to the level of the cut.

The division of fitness levels leeds straightforward to the definition of boxed probability.

Definition 1. *For each box let l and u denote the minimum level and the maximum level belonging to the box. $B(l, u)$, called boxed probability, thus denotes the probability of lying inside the box.*

In case of a set of discrete values usually the lowest minimum is 0, and is never achieved, while the highest level of each box may be an actual value of the fitness.

A binary model would thus be boxed as $B(0,p)$, $B(p,1)$. A ternary model, as in example 1, would lead to three boxes $B(0,p)$, $B(p,p+q)$, $B(p+q,p+q+r)$, where $p+q+r=1$.

Of course the sum of all the boxes must be equal to 1, since they exaust all the possibilities.

Unfortunately the solution of the problem requires full analysis and full knowledge of the process. The simple derivation from some transition matrixes with few fitness levels leads, as it was stated before, to the knowledge of the first normalized eigenvector, hence to the knowledge of the frequencies of a certain number of fitness levels, usually two, as in example 2, sometimes three as in example 1 and 3. We call the frequencies respectively $D(p)$ and $D(q)$, with $p+q=1$ and respectively $T(p)$, $T(q)$, $T(r)$ with $p+q+r=1$. Of course it is possible to consider systems with more levels of fitness.

The fundamental question is this: when does it hold the equality?

$$B(p,p+q) = D(p+q) - D(p) \tag{1}$$

That is, when the probability of all the levels between p and $p+q$ is just equal to the difference between the probability of a single level $p+q$ minus the probability of a single level p?

Should such a fact happen, all the analysis would be reduced to the (anyhow non obvious) problem of estimating the binary frequence $D(x)$.

The reason why the answer need not to be positive lies in the geometry of Bak-Sneppen type process, where the neighborhood relations condition, though slightly, changes the final averages. On the contrary the given heavy geometric conditions substantially change the return times between two states.

2.1 Example 4. Transition matrix of 4 nodes random elementary model

A simpler case arises when no geometry is at hand, namely when the simplified process consists of finding the worst possible value, change at random its owner, then choose at random one other node, and change it at random, independently on its distance from the worst point.

The binary transition matrix then becomes (in the case of 4 nodes) the Table 4.

Remark that for 3 nodes this matrix corresponds to matrix 2. The random transition matrix is obtained in the same way for all numbers of nodes summing up the shifted copies of the first column multiplied respectively for the ratios $i/(N-1)$, $(N-1-i)/(N-1)$.

This process has no geometry, hence additivity holds, since the only parameter is the number of good (1) nodes with respect to bad (0) nodes. Taking the

	# 4	# 3	# 2	# 1	# 0
# 4	q^2	q^2	$q^2/3$	0	0
# 3	$2pq$	$2pq$	$2pq/3 + 2q^2/3$	$2q^2/3$	0
# 2	p^2	p^2	$p^2/3 + 4pq/3$	$4pq/3 + q^2/3$	q^2
# 1	0	0	$2p^2/3$	$2p^2/3 + 2pq/3$	$2pq$
# 0	0	0	0	$p^2/3$	p^2

Table 4: *Transition for Random model*

differences $D(0.2) - D(0.1)$, $D(0.3) - D(0.2)$, we get the following ten boxes, that have a sigmoid behaviour that resembles the experimental Bak-Sneppen behaviour, see Figure 1.

10 levels	0	0.0519126
	1	0.0569346
	2	0.0642941
	3	0.0747289
	4	0.0886683
	5	0.1055668
	6	0.1232227
	7	0.1380177
	8	0.1468755
	9	0.1497788

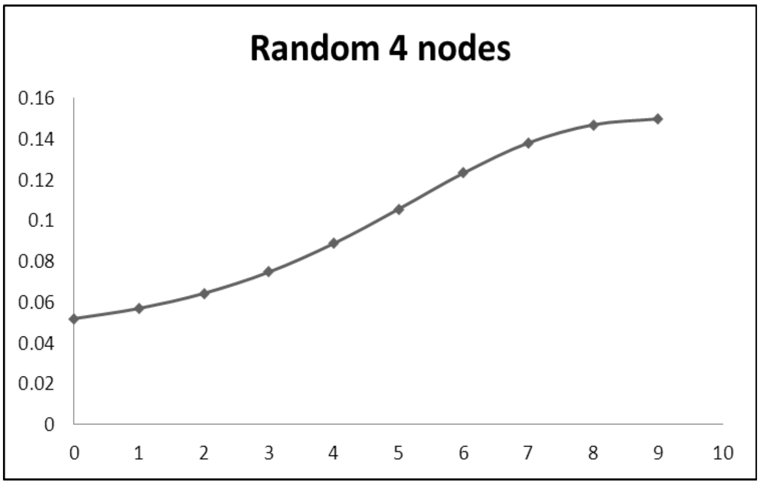

Figure 1: Random model with 4 nodes and 10 levels

3. Some sufficient non trivial conditions for equation 1

The simplest problem is the transition from a ternary model to a dual model in which two values are connected with the ternary values. We call *reducible ternary system* a system where it holds

$$D(p+q) = T(p) + T(q), D(r) = T(r) \tag{2}$$

$$D(q+r) = T(q) + T(r), D(p) = T(p) \tag{3}$$

The main property is given by

Theorem 1. *If the dual and ternary representations of a system lead to a reducible ternary sistem, then the original system allows the boxed representation (1), namely*

$$B(p, p+q) = D(p+q) - D(p) \tag{4}$$

Proof. The reducibility of a ternary system leads to additivity on each subinterval, so that (4) holds for any box in the case of a finite number of fitness levels, provided values in the extreme are chosen consistently. The continuous case is obtained passing to the limit. □

The probem of passing from ternary to dual systems mantaining properties (2) and (3) lies in the fact that in the ternary model the sequences 321 and 312 are different, inasmuch the vanishing minimum once lies in the middle position, once in the right. Passing to the corresponding dual model both sequences are reduced to $3AA$, creating a tie. The tie in general cannot be solved at random, since one should a priori know the two separate probabilities of each sequence 321 and 312. So the simplest solution lies in a severe condition as shown in the next theorem.

Theorem 2. *A ternary system is reducible when, for each set of tied configurations, each column of the transition matrix has the same values in the cells corresponding to the elements of the tied set.*

Proof. In this case there is no need to know *a priori* the probability of the tied configurations, since it is enough to know that they have all the same probability. Remark that the condition can be checked directly on the auxiliary columns, without computing explicitely the ternary transition matrix. □

Theorem 3. *The fundamental transition matrix of example 1 leads to a reducible system.*

Proof. In matrix 1 the only tie arises between rows e and f. Looking in the auxiliary columns one finds respectively the equal values of pq, pr, and qr. □

Remark that another way of proof could be showing that symmetry condition is enough for identifying configurations e and f, so that the systems could be reduced to a two level system, as in example 2 (that is a particular case of example 4 where no geometry exist). Numerical results are shown in Figure 2 and 3.

Starting from matrix 2, the 1-eigenvectors are

$$a = \frac{A}{Tot}, \quad b = \frac{B}{Tot}, \quad c = \frac{C}{Tot}, \quad d = \frac{D}{Tot}$$

where

$$\begin{aligned} A &= \frac{p^2}{2(1-p^2)}B \\ C &= \frac{(1-p)(p+4)}{2p(1+p)}B \\ D &= \frac{(1-p)^2}{p^2(1+p)}B \end{aligned}$$

4 levels	0	0.18
	1	0.22
	2	0.276471
	3	0.323529

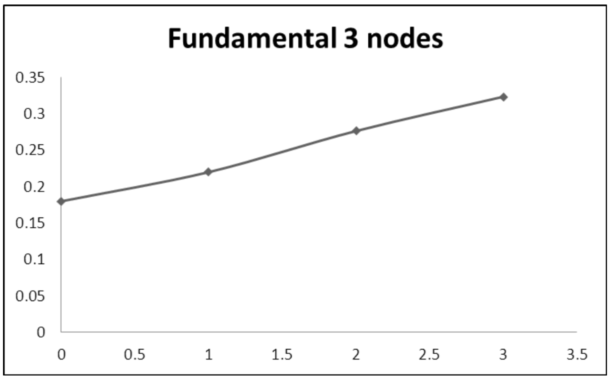

Figure 2: Fundamental model with 3 nodes and 4 levels

10 levels	0	0.068508
	1	0.072955
	2	0.078671
	3	0.085748
	4	0.094118
	5	0.103448
	6	0.113065
	7	0.121948
	8	0.128865
	9	0.132673

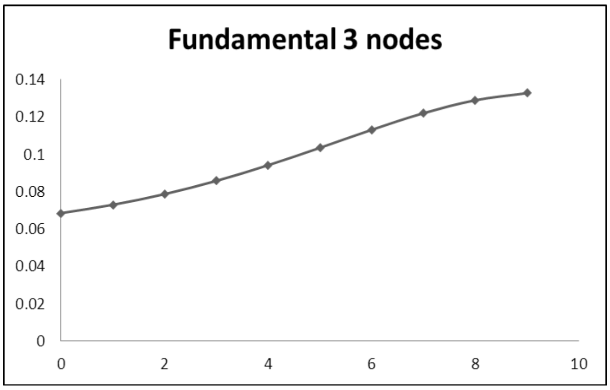

Figure 3: Fundamental model with 3 nodes and 10 levels

and

$$Tot = A + B + C + D$$

so that

$$B(0, p) = 1 - \frac{a \cdot 0 + b \cdot 1 + c \cdot 2 + d \cdot 3}{3}.$$

The numerical result of continuous fundamental model with 3 nodes is shown in Figure 4.

Theorem 4. *The fundamental transition matrix of example 3 leads to a reducible system.*

Proof. A first set of ties is in e, f, i with aux values respectively $2pqr, pr^2, qr^2$. A second set of ties is in l, m, r with aux values respectively $pq^2, 2pqr, q^2r$. The last set of ties is in n, o, p with aux values respectively $p^2q, p^2r, 2pqr$. The use of Theorem 2 leads to the conclusion. The numerical results of discrete fundamental model with 3 nodes and 4 and 10 levels, are shown in Figure 5 and 6 respectively. □

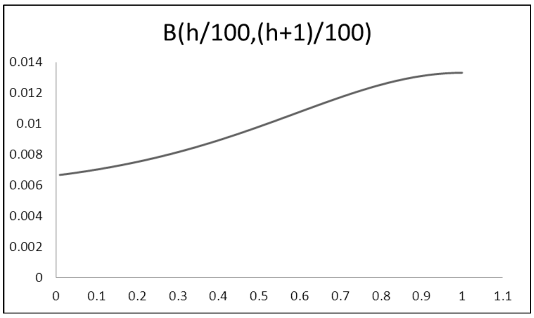

Figure 4: Continuous fundamental model with 3 nodes

4 levels	0	0.2052
	1	0.2433
	2	0.2709
	3	0.2805

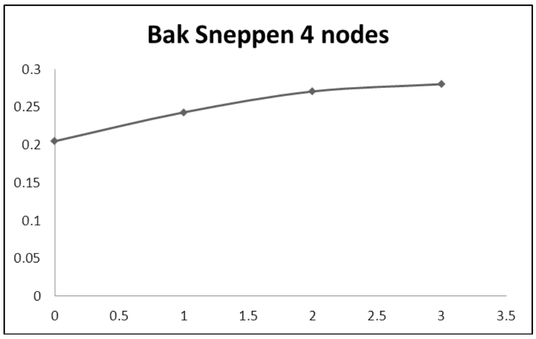

Figure 5: Bak Sneppen model with 4 nodes and 4 levels

10 levels	0	0.07766
	1	0.08349
	2	0.08978
	3	0.096
	4	0.1016
	5	0.10612
	6	0.10935
	7	0.1113
	8	0.1122
	9	0.11248

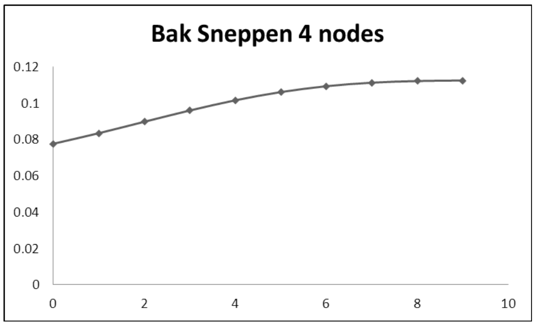

Figure 6: Bak Sneppen model with 4 nodes and 10 levels

4. Conclusions

Bak-Sneppen like models are characterized by some geometry of neighborhood, that overcomes a purely random evolution. Anyhow usually the neighborhood law does not deeply change the distribution of the average of fitness values. What changes in large systems is the speed of propagation of the so called avalanches, and consequently the time elapsed before a section achieves again a stable situation (compare [21]). In that paper, on the contrary, partitioned frames in Bak Sneppen models were introduced and discussed. The situation is as follows: the set of nodes is divided in some distinct sections, and the neighborhood rule allows only evolution in the inside of the section, so that the root change due to minimal fitness can influence only its own section, leaving all the remaining subsystems unchanged, for good and evil. On the contrary, the rule of minimum is global, so that no section should be free from random evolution. But if the local minimum of a section is already equal to the maximum, evolution cannot reach that section unless all other sections reach the maximum, so that maximality is an extremely strong condition of stability.

Of course the condition of maximality of minima can be less strong, inasmuch the minimum in the given section be higher than all minima elsewhere. It can anyhow happen that the minimizing section suddenly jumps to values that are all higher than the previous optimal level of minima, so that overtaking takes place.

The condition for overtaking is the existence of at least two subsections, with no less than two elements, since otherwise the frame becomes trivial. Suppose therefore to consider a simplified system of four nodes, where nodes 1 and 2 belong to the set A, and nodes 3 and 4 belong to the set B. Suppose that fitness levels are 3, otherwise no overtaking can happen, then the transition matrix becomes the Table 5.

Left digit is referred to the set A, right digit to the set B. 0 means that the minimum is 0, 1 that the minimum is 1, 2 that both nodes have the fitness 2. Y means that transition is possible, o that it is impossible.

Start Arrival	00	01	10	11	02	20	12	21	22
00	Y	o	o	o	o	o	o	o	o
01	Y	Y	o	Y	o	o	o	o	o
10	Y	o	Y	Y	o	o	o	o	o
11	o	Y	Y	Y	o	o	o	o	o
02	Y	o	o	o	Y	o	Y	o	Y
20	Y	o	o	o	o	Y	o	Y	Y
12	o	o	Y	Y	Y	o	Y	o	Y
21	o	Y	o	Y	o	Y	o	Y	Y
22	o	o	o	o	Y	Y	Y	Y	Y

Table 5: Transition and overtaking

Remark that the stable set of possible configurations is 02, 20, 12, 21, 22. Overtaking is to be found in the passage from 01 to 21 and from 10 to 12. Of course in the case of more levels a richer choice of cases may arise.

The average value of fitness in the partitioned case is much higher than in non partitioned frames, independently on their detailed geometry.

The confirmed conclusion is that in any Bak-Sneppen model, partition allows higher average levels, but progress may be slower, so that poor subsets only exceptionally make a meaningful jump, an example of weak connection can be found in [15] and [18]. Rich countries have no particular reason to change, somehow following the *happiness paradox* of Easterlin (see [12] and [13]).

In economical history the phenomenon is not frequent but happens and leads to important changes in the world economy. An example of the past is Germany with respect to England, while in the present there is China with respect to USA. Financial evolution, thanks to its volatily and virtuality, is even more subject to sudden changes. Section 5 of [21] gives a useful survey (see also [20]).

A curious phenomenon of overtaking happens with a certain frequence in science, even in hard sciences, where winning ideas arise from *virgin* lands. The case of De Giorgi's theorem is well known and some of its features are explored in detail in [17] in the chapter *Il grande problema* pp. 127-134.

The authors discussed some strange cases of scientific cultural interactions lying between discovery, fashion, teaching (see [22] and [19]).

The problem of economic evolution in restricted frames rather than enlarged ones was analysed in [10] with a long period analysis in Leontief's frame, while in [6] and [7] short period analysis was performed, and efficient strategies were envisaged. Production manifacturing chains are widely subject to Bak-Sneppen phenomena, and are characterized by a continuous attempt to change the structure of neighborhoods, as was well pointed out in [11] and in [8].

Finally it is worth to mention a connection between culture, economy and health management arising in the evolution of food styles, with special reference to the dynamics of Mediterranean Adequacy Index (see [9]).

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