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Disorder in the Sachdev–Ye–Kitaev model

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ABSTRACT

We give qualitative arguments in support of the mesoscopic nature of the Sachdev–Ye–Kitaev (SYK) model in the regime with $q^2/N \ll 1$ with N Majorana particles coupled by antisymmetric and random interactions of range q. Using a stochastic deformation of the SYK model, we show that its characteristic determinant obeys a viscid Burgers equation with a small spectral viscosity in the regime with $q/N = 1/2$. In particular, the stochastic evolution of the SYK model can be mapped exactly onto that of random matrix theory, with universal Airy oscillations at the edges.

1. Introduction

The understanding of how entropy is produced and developed in heavy-ion collisions at ultra-relativistic energies is the subject of intensive interest at collider energies [1]. Key to this is the concept of thermalization and the time it takes to reach it [2–6]. More generally, there is a wide theoretical interest in the understanding of non-perturbative entropy formation in quantum processes ranging from atomic systems at the unitarity limit [7,8] to string theory using black-holes [9].

The SYK model consists of N quantum mechanical fermions with Gaussian distributed random couplings of rank-q and strength J. The model is solvable at large N and fixed J by a saddle point approximation where a special class of Feynman graphs is selected. Originally, this model was proposed by Sachdev and Ye [10] to describe quantum spin fluids. More recently, Kitaev [11] has suggested the model to shed light on holography, by arguing that the large N limit of the model is dual to a black hole in an emergent AdS_2 space–time. A number of investigations have since followed [12–15].

The SYK model offers a simple framework for discussing the formation of black holes in quantum mechanics [11]. In the regime $q^2/N \ll 1$, numerical analyses [14,15] support the existence of a chaotic regime described by random matrix theory at late times. The chaotic regime suggests that the near ground state properties of the SYK model are dual to a black hole.

The purpose of this paper is two-fold: 1/ to suggest qualitatively that the numerical results for the spectral form factor for the SYK model recently reported in [14] for $q^2/N \ll 1$, reflect on a hierarchy of mesoscopic scales much like those encountered in disordered metals; 2/ to show rigorously using spectral determinants that for $q^2/N \gg 1$ a viscid fluid description emerges with a small spectral viscosity that maps exactly on random matrix theory. Our interest in both regimes stems from their relation to random matrix theory for large times in general, with in particular a fermionic ground state that is dual to a black hole in both cases, and very likely for all permissible $q/N \leq 1/2$.

This paper consists of a review of some of the results obtained in the context of the SYK model as presented in section 2, and more qualitative but speculative arguments for the mesoscopic nature of the spectral form factor of the SYK model in section 3. The main and quantitative new results of the paper will be presented in section 4, where we show how to map the SYK model on random matrix theory for the special case of $q/N = 1/2$, and note that the microscopic accumulation of the states at the edge of the spectrum follows from Airy universality. A more speculative observation regarding the ensuing entropy of the SYK model versus that of a string near a black-hole is presented at the end.

Specifically, the organization of the paper is as follows: in section 2 we briefly outline the SYK model and discuss its bulk spectral distribution in the holographic limit with $q^2/N \ll 1$. In section 3, we provide a qualitative description of the mesoscopic nature of the model and give a simple estimate for the ergodic time. In section 4, we discuss the opposite limit with $q^2/N \gg 1$ using the SYK characteristic determinant. We show that it obeys a viscid Burgers equation which is analogous to the one derived

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using random matrix theory for the GUE ensemble. The inviscid equation gives rise to a semi-circular distribution, while the viscid equation gives rise to Airy universality at the edge of the spectrum. We use it to estimate the contribution of the edge states to the partition function at low temperature. Our conclusions are in section 5.

2. SYK model

The SYK model consists of N-Majorana fermions with q-interactions in $0+1$-dimensions, with random and antisymmetric couplings. The corresponding quantum-mechanical Hamiltonian say for $q=4$ is

$$H = \sum_{a<b<c<d} J_{abcd}\, \psi^a \psi^b \psi^c \psi^d \tag{1}$$

with N-Majorana fermions ψ_a of color $a = 1, ..., N$. The couplings J are antisymmetric in all entries and randomly sampled from a Gaussian distribution with zero mean but fixed variance $N_q = (q-1)!J^2/N^{q-1}$. All units are set by J which will be set to 1. As operators in a Hilbert space, the colored Majorana ψ^a map onto the representations of the Clifford algebra $Cl[\frac{N}{2}]$ as realized by γ^a matrices

$$H \to \sum_{a<b<c<d} J_{abcd}\, \gamma^a \gamma^b \gamma^c \gamma^d \equiv \sum_A J_A \Gamma_A \tag{2}$$

which is $L \times L$ valued with $L = 2^{\frac{N}{2}}$. (2) refers to a sum of sparse matrices with N^4 random weights. (1)–(2) exhibit particle-hole symmetry which is enforced by an anti-unitary operation. As a result the spectra exhibit some degeneracy for some values of N modulo 8 (Bott periodicity) [14,15].

The Hamiltonian in (2) is bounded and symmetric. The edge states map onto the low-lying N-body excitations close to the ground state, and the central states map onto the high lying N-body excitations. The latter follow a Gaussian distribution [14, 15]. Specifically, consider the average partition function

$$\langle \mathbb{Z}(\beta)\rangle_J \equiv \left\langle \mathrm{Tr}\left(e^{-\beta H}\right)\right\rangle_J = \sum_k \frac{\beta^{2k}}{(2k)!}\left\langle H^{2k}\right\rangle_J \tag{3}$$

where uniform convergence is assumed (this is likely upset at the edges [16]). Formally, the moments in (3) are

$$\left\langle H^{2k}\right\rangle_J = \frac{1}{N_q^k}\sum_{i_1...i_k}\sum_{j_1...j_{2k}} \mathrm{Tr}\left(\Gamma_{j_1}\Gamma_{j_2}...\Gamma_{j_{2k}}\right) \tag{4}$$

with typically $(j_1 j_2 ... j_{2k-1} j_{2k}) = (i_1 i_1 ... i_k i_k)$. Since the Γ's need at least one common factor in order to anti-commute, and these pairs form only a small fraction of all pairs, we can throw away the anti-commutators in (4) at large N [15,16]. For a fixed sequence $(i_1 i_2 ... i_k)$, the trace in each term of the j contribution is L and there are $(2k-1)!!$ such contributions. The final sum over the sequences $(i_1 i_2 ... i_k)$ gives $(C_N^q)^k$

$$\left\langle H^{2k}\right\rangle_J \approx L\,(2k-1)!!\left(\frac{C_N^q}{N_q}\right)^k \tag{5}$$

The partition function at high temperature (small β) follows by inserting (5) into (3) and performing the sum. The ensuing inverse Laplace transform gives a Gaussian distribution of the central eigenvalues

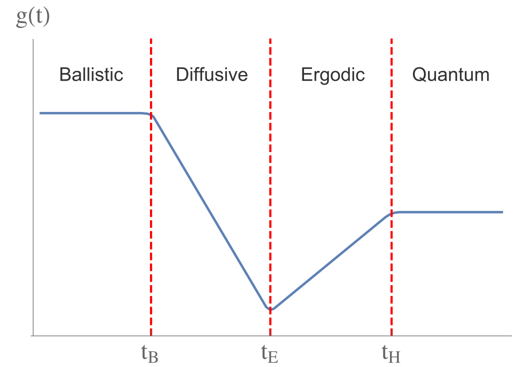

Fig. 1. Schematic rendering of the spectral form factor for the SYK model as evaluated in [14]. Four mesoscopic regimes are identified. See text.

$$\begin{aligned}\rho(E) &= \int_C d\beta\, e^{\beta E}\,\langle \mathbb{Z}(\beta)\rangle_J \\ &\approx \int_C d\beta L\, e^{\beta E + N\beta^2/q} \approx L\, e^{-\frac{qE^2}{2N}}\end{aligned} \tag{6}$$

Overall, (6) is in agreement with the arguments and numerics presented in [13–15]. Eq. (6) fails at the edges of the spectrum [13–15]. For $q^2/N \ll 1$ the symmetric edges expand as $\pm N\lambda_0$ with an exponential growth of states away from the edges given by $\sinh(2cN|E - N\lambda_0|)^{\frac{1}{2}}$ in the triple scaling limit [14].

3. Mesoscopy

In [14,17] the deformed spectral form factor was defined as

$$g(\beta, t) = \frac{\left\langle |\mathbb{Z}(\beta + it)|^2\right\rangle_J}{\langle \mathbb{Z}(\beta)\rangle_J^2} \tag{7}$$

and analyzed both analytically and numerically in [14]. In Fig. 1 we give a schematic rendering of their numerical results for the SYK model with four mesoscopic regimes identified (logarithmic scales used). The features shown are generic of mesoscopic systems with multi-fermion induced interactions such as those developed in the disordered QCD vacuum [18].

The canonical example of a mesoscopic system is that of electrons in disordered metals [19], to which we refer the reader for a more thorough understanding of the concepts to be qualitatively discussed below. In brief, the spectrum of a disordered metal varies dramatically from a good to a bad conductor. For instance at strong disorder, the electrons undergo a metal-insulator transition with a complex spectral rearrangement and localization. The spectral properties of disordered metals are captured by the time-dependent spectral form factor (the analogue of (7) for $\beta \gg 1$). Numerical simulations using essentially the Anderson model with increasing bandwidth (disorder) have revealed the existence of four mesoscopic regimes pretty much along the lines identified in Fig. 1: 1/ a quantum regime with a plateau for $t > t_H$; 2/ an ergodic and universal regime for $t_E < t < t_H$; 3/ a diffusive and non-universal regime for $t_D < t < t_E$; 4/ a ballistic regime for $t < t_B$. These regimes reflect on the classical underlying dynamics of the electronic motion. In the diffusive regime, the electron wave-packet experiences free diffusion in space. In the ergodic regime, the electronic wave-packet explores all the available space uniformly. This regime is universal and amenable to random matrix theory. The ballistic regime corresponds to a ballistic motion of the electron wave-packet. The classical wave-packet description fails beyond the Heisenberg time where quantum corrections become important.

A useful formula for discussing mesoscopic systems is the semiclassical form of the spectral form factor for $\beta \gg 1$ [19,20]

$$g(t) \approx \frac{|t|p(t)}{(2\pi)^2} \tag{8}$$

for times much smaller than the quantum (Heisenberg) time, $t < t_H = 1/\delta$ with $\delta = 2\pi/L$ the quantum energy spacing. Here $p(t)$ is identified with the return probability, typically of the form

$$p(t) = \left\langle \left|\langle J|\gamma^a(t)\gamma^a(0)|J\rangle\right|^2 \right\rangle_J \tag{9}$$

It consists of first calculating the return amplitude of color-a for fixed disorder J in the ground state, squaring it and then averaging over all disorder by analogy with the analysis in the disordered QCD vacuum [18]. The microscopic and quantitative many-body analysis of (9) will be presented elsewhere [21].

Here instead, we will provide very qualitative estimates using (8) and the fact that (9) can be thought as the probability of return for a given color undergoing (anomalous) diffusion in a 2-dimensional volume $V_2 = L^2$ span by γ^a_{ij}. Although qualitative, these estimates will lead the key features of Fig. 1. A diffusing color undergoing random walk spreads in a time t an effective squared distance $X^2 \approx t^\Delta$ with Δ the anomalous diffusion exponent ($\Delta = 1$ for the canonical Brownian random walk). A simple estimate of the return probability (9) in this random walk approximation is $p(t) \approx V_2/X$. The ergodic time (inverse of Thouless energy) is reached when the random walks fill out the effective volume $X \approx V_2$ causing $p(t) \approx 1$, that is $t_E \approx L^{\frac{4}{\Delta}}$.

In the diffusive regime with $t_B < t < t_E$, (8) becomes

$$g(t_B < t < t_E) \approx \frac{|t|}{(2\pi)^2}\frac{L^2}{t^{\frac{\Delta}{2}}} \tag{10}$$

For all super-diffusive random walks with $\Delta > 2$, (10) is in qualitative agreement with the slope in Fig. 1 (logarithmic scale). A comparison of (10) with the numerical results in [14] suggests that $\Delta = 8$. The ballistic time is identified as $t_B \approx L^0$ below which the left plateau is seen in Fig. 1. In the ergodic regime with $t_E < t < t_H$ we have $p(t) \approx 1$, and

$$g(t_E < t < t_H) \approx \frac{|t|}{(2\pi)^2} \tag{11}$$

grows linearly with time in overall agreement with the rise in Fig. 1 (logarithmic scale). The slope is universally fixed by (11) where the Dyson index is here set to $\beta_D = 2$. In the Heisenberg regime with $t > t_H$, the spectral form factor is dominated by the self-correlation for a single energy level which is normalized to a delta-function in energy space $g(E \to 0) = \delta(E)$, and translates to a constant in time

$$g(t > t_H) = \frac{1}{2\pi} \tag{12}$$

which is the right plateau in Fig. 1. The occurrence of this plateau is also universal. As in mesoscopic systems, we note the hierarchy of times ($\Delta = 8$)

$$t_B \approx L^0 < t_E \approx L^{\frac{1}{2}} < t_H \approx L \tag{13}$$

The ergodic regime is universal and follows from random matrix theory and symmetries as observed in [14,15]. The ergodic scaling of $L^{\frac{1}{2}}$ is precisely the one noted in [14]. In the presence of time-reversal symmetry the counting of paths in (8) is increased by a factor of 2. The particularly short time $t_S \approx \ln L$ reported in [22] is of the order of the Ehrenfest time and may be a signal for the loss of quantum coherence at the edge of the ballistic regime.

Finally, we note that the small parameter in the SYK model is $1/N$, while in the random matrix theory it is $1/L$. This means that the perturbative effects in the random matrix theory of the form $1/L = e^{-N\frac{1}{2}\ln 2}$ map onto non-perturbative effects in the $1/N$ expansion, and are extremely small.

4. Random matrix limit

In the regime with $q^2/N \ll 1$, the spectrum near the edges of the SYK model appears semi-circular [13–15] and amenable to random matrix theory. This suggests that the low lying excitations of the SYK model are chaotic, with a fermionic ground state dual to a black hole [13,14]. In this section we will show that in the limiting regime with $q/N = 1/2$ the entire SYK spectrum is chaotic and described by random matrix theory. In particular, the fermionic ground state of the SYK model in this regime is likely dual to a black hole.

4.1. Ergodic evolution

To streamline the counting for the case with $q/N = 1/2$, it is more convenient to re-define (2) using the new normalization for the q-range couplings

$$H = \frac{1}{(C_N^{p_N})^{\frac{1}{2}}}\sum_{J_{pn}} \alpha_{J_{pn}}\Gamma_{J_{pn}} \tag{14}$$

with

$$\Gamma_{J_{p_N}} \equiv \gamma_{i_1}\gamma_{i_2}\ldots\gamma_{i_{p_n}}$$
$$J_{p_n} = 1 \le i_1 < i_2 < \ldots < i_{p_n} \le N \tag{15}$$

a typical basis element of rank-p_n in the minimal representation of the Clifford algebra $Cl([\frac{N}{2}])$. There are $C_N^{p_n}$ such basis elements, and they all satisfy $\Gamma_J^2 = 1$. A similar normalization for Pauli matrices were used in [16].

We now define the characteristic determinant for the SYK model as

$$\mathbf{\Psi}[\tau, z] = \langle \det(z - H)\rangle_J = \int d[J]\,\mathbf{P}(\tau, J)\det(z - H) \tag{16}$$

with the measure $\mathbf{P}(\tau, \alpha_J) \approx e^{-\frac{1}{2\tau}\alpha_{J_i}\alpha_{J_i}}$, and $\mathbf{P}(0, \alpha_J) \approx \delta(\alpha_J)$. Note that the measure reduces to a delta-function as $\tau \to 0$, and asymptotes a Gaussian as $\tau > 1$ which is the SYK model. Eq. (16) provides a stochastic deformation of the SYK model with vanishing couplings as $\tau \to 0$, much like in the random matrix deformation in [25].

In general, characteristic determinants, their products or ratios can be used to analyze multipoint correlation functions of spectral densities, including the statistics of eigenvalues in quantum chaotic systems [26]. So they are as useful as (7) for understanding the spectral correlations in the SYK model.

To analyze (16) we set $N = 2n$ and specialize to the case $p_n = n$ even (the odd case is discussed in Appendix A). This is the case with maximum range for the random couplings. With this in mind, we unwind the determinant using Grassmannians, and carry the Gaussian integration over the random couplings α_J to obtain

$$\Psi[\tau, z] = \int D\chi D\bar{\chi}\, e^{-z\bar{\chi}\chi + \frac{\tau}{2C_{2n}^n}\sum \bar{\chi}\Gamma_n\chi\bar{\chi}\Gamma_n\bar{\chi}} \tag{17}$$

We now use a Fierz re-arrangement of the 4-Grassmannian induced interaction, the details of which are given in Appendix B, to obtain

$$\frac{1}{2C_{2n}^n}\sum \bar\chi\Gamma_n\chi\,\bar\chi\Gamma_n\bar\chi =$$
$$-\frac{1}{2L}\bar\chi\chi\bar\chi\chi - \frac{(-1)^n}{2L}\bar\chi\Gamma_{2n}\chi\,\bar\chi\Gamma_{2n}\chi$$
$$-\frac{1}{2L}\sum_{A\neq 0,2n}\frac{N_A^+ - N_A^-}{C_{2n}^n}\bar\chi\Gamma_A\chi\,\bar\chi\Gamma_A\chi \tag{18}$$

with

$$N_p^+ = (-1)^{p+n}\sum_{k=0}^{[n/2]} C_p^{2k}C_{2n-2k}^{n-2k}$$
$$N_p^- = (-1)^{p+n}\sum_{k=0}^{[n-1/2]} C_p^{2k+1}C_{2n-2k-1}^{n-2k-1} \tag{19}$$

For any large n, we have

$$N_{2p+1}^+ - N_{2p+1}^- = 0$$
$$\frac{|N_{2p}^+ - N_{2p}^-|}{C_{2n}^n} \approx \frac{(2p-1)!!}{2^p n^p} \tag{20}$$

As a result the third line contributions to the Fierz re-arrangement in (18) are all of order $1/n$ in comparison to the first two lines, and subleading. Therefore (18) simplifies to

$$-\frac{1}{2L}\bar\chi\chi\bar\chi\chi - \frac{(-1)^n}{2L}\bar\chi\Gamma_{2n}\chi\,\bar\chi\Gamma_{2n}\chi \tag{21}$$

Since

$$\Gamma_{2n}^2 = (-1)^{\frac{(2n-1)(2n)}{2}} = (-1)^{n(2n-1)} = (-1)^n \tag{22}$$

$(-1)^{n/2}\Gamma_{2n}$ squares to one and we can write (21) as

$$-\frac{1}{L}\bar\chi_+\chi_+\bar\chi_+\chi_+ - \frac{1}{L}\bar\chi_-\chi_-\bar\chi_-\chi_- \tag{23}$$

The labels $\pm$ refer to the positive-negative eigenvalues of the chirality matrix $(-1)^{n/2}\Gamma_{2n}$.

This result is physically expected as the Fierzing in (18) rescinds the 4-Fermi induced interaction into all spin channels in the large Hilbert space. In leading $1/n$, all the spin bearing channels wash out, except for the scalar and pseudo-scalar channels with each carrying an effective coupling of $1/2L$. The chiral copies in (23) reflect on the particle-hole symmetry noted in [14,15,23,24].

Therefore at large n, the characteristic determinant (17) splits into two chiral copies with

$$\Psi[\tau,z] \approx \Psi_+[\tau,z]\Psi_-[\tau,z]$$
$$\Psi_\pm[\tau,z] = \int D\chi_\pm D\bar\chi_\pm e^{-z\bar\chi_\pm\chi_\pm - \nu_L\tau\bar\chi_\pm\chi_\pm\bar\chi_\pm\chi_\pm} \tag{24}$$

with $\nu_L = 1/L$. It follows that each of the chiral copies in (24) close under ergodic evolution (reverse diffusion)

$$\partial_\tau\Psi_\pm = -\nu_L\partial_{zz}\Psi_\pm \qquad \textit{with} \qquad \Psi_\pm(0,z) = z^{L/2} \tag{25}$$

Using the complex Cole–Hopf transformation for the characteristic determinant $f_L = \partial_z \ln\Psi_\pm/\tilde L$ with $\tilde L = L/2$, (25) for the SYK model maps onto the viscid Burgers equation

$$\partial_\tau f_L + f_L\partial_z f_L = -\nu_L\partial_{zz}f_L \tag{26}$$

with ν_L playing the role of a (negative) spectral viscosity [27]. In terms of the (cold) entropy $S/N \approx \ln 2/2$ [13], the spectral viscosity is $\nu_L = 1/e^S$.

The physical meaning of (26) follows from the fact that the imaginary part of f_L for $z=\lambda$ real is the density of states as we show next. At large L and for $q/N = 1/2$, the SYK states forms a viscid fluid with a local density $\mathrm{Im} f_L(\tau,\lambda)$. (26) captures the $1+1$ hydrodynamics of this fluid with the deformation τ acting effectively as a time.

Finally, we note that (25)–(26) map onto the ergodic equation for the characteristic determinant of the unitarity ensemble of random matrix theory of finite size $L/2$ and Dyson index $\beta_D = 2$ [27]. This mapping together with the semi-circular distribution (see below) guarantee that the spectral form factor in (7) is also of the general form shown in Fig. 1 in the random matrix regime with $q/N = 1/2$ and in leading order.

4.2. Airy universality

We now focus on one of the two chiral copies and study its spectrum. The formal solution to (25) is

$$\Psi_\pm[\tau,z] = \left(\frac{1}{4\pi\nu_L\tau}\right)^{\frac12}\int_{\mathbb{C}} dz' e^{\frac{1}{4\nu_L\tau}(z-z')^2} z'^{L/2} \tag{27}$$

which is the convolution of the diffusion kernel with the initial condition in L-space. The L-saddle point approximation to (27) yields the Cole–Hopf transform

$$f_L[\tau,z] \approx \frac{1}{\tau}(z - z_+) = \frac{1}{z_+} \tag{28}$$

with $2z_+ = z + \sqrt{z^2 - 4\tau}$. Eq. (28) acts as a Coulomb-like potential for the macroscopic spectral density of eigenvalues with ($\tilde L = L/2$)

$$\rho(\tau,\lambda) = \frac{\tilde L}{\pi}\mathrm{Im} f_L[\tau, z=\lambda] \approx \frac{\tilde L}{2\pi\tau}\left(4\tau - \lambda^2\right)^{\frac12} \tag{29}$$

which is semi-circular.

Eq. (29) can also be shown to follow from the moment analysis of H, if we were to note that all crossing diagrams are suppressed by powers of the ratio in (20), in comparison to the non-crossing diagrams. As a result only the planar contributions are retained for $N = 2n$ and $p_n = n$ at large n, leading to the standard Pastur equation for the resolvent and a semi-circle.

The key feature of the semi-circle are its edges at $\pm\sqrt{4\tau}$ with an accumulation of states of order $L\lambda^{3/2}$ which suggests the microscopic re-scaling (unfolding) at the origin of the Airy universality (soft-edge universality). This follows from either the rescaled expansion around the saddle point in (27) [25], or the shock analysis of the viscid Burgers equation [27], with the result

$$\Psi(\tau, \sqrt{4\tau} + s\sqrt{\tau}/\tilde L^{2/3}) \approx \mathrm{Ai}(-s) \tag{30}$$

The characteristic determinant (30) and the inverse characteristic determinant capture the overall depletion of the eigenvalues at the edges [28–30]. This depletion is universal and for $\beta_D = 2$ it also follows from the method of orthogonal polynomials. Either way, the result is [30,31]

$$\frac{1}{\tilde L^{\frac23}}\rho(E = \sqrt{4\tau} + s\sqrt{\tau}/\tilde L^{\frac23}) \approx$$
$$\frac{\tilde L}{\sqrt{\tau}}\left((\mathrm{Ai}'(-s))^2 + s\,(\mathrm{Ai}(-s))^2\right) \equiv \frac{\tilde L}{\sqrt{\tau}} f(s) \tag{31}$$

The universal contribution of (31) to the partition function at low temperature is ($\lambda_0 = \sqrt{4\tau}$)

$$Z[\beta, L, \tau] \approx \tilde L\, e^{-\beta\lambda_0}\int_0^\infty ds\, e^{-\beta\sqrt{\tau}s/\tilde L^{\frac23}}\, f(s) \tag{32}$$

For large L, the integration is dominated by the large s-asymptotic of the Airy functions

$$f(s) \approx \sqrt{s} - \frac{1}{4s} \cos\left(\frac{4s^{\frac{3}{2}}}{3}\right) \tag{33}$$

The first term yields the leading contribution to the partition function

$$Z[\beta, L, \tau] \approx \frac{\sqrt{\pi}}{2} \frac{\tilde{L}^2}{(\beta\sqrt{\tau})^{\frac{3}{2}}} e^{-\beta\lambda_0} \tag{34}$$

which results in a leading contribution to the entropy at low temperature

$$S \approx N\ln 2 - \frac{3}{2}\ln(\beta\lambda_0) \tag{35}$$

This is twice the entropy noted in the holographic regime in leading order. In the regime with $q^2/N \gg 1$, the number of random degrees of freedom grows as L^2 and not as L.

Finally, we observe that in the regime $q^2/N \ll 1$, the analogue of (35) in the large N limit is

$$S \approx 2\pi E_N - \frac{3}{2}\ln(\beta E_N) \tag{36}$$

with $E_N/N = e_0 \approx 0.0406$ [13,14]. We note that (36) is of the form suggested in the context of the correspondence between a black hole and a highly excited string, with E_N identified as the Rindler energy [32]. In contrast, (35) is dual to a black hole only if the (negative) ground state energy or spectrum edge $\lambda_0 \to N\ln 2/(2\pi)$ for $q^2/N \gg 1$. This can be checked numerically.

5. Conclusions

In the regime with $q^2/N \ll 1$, we have presented qualitative arguments in support of the mesoscopic nature of the SYK model. In particular, the pre-chaotic phase appears super-diffusive, a point that deserves a more rigorous analysis using (8)–(9). In the opposite regime with $q^2/N \gg 1$ the SYK model maps exactly on random matrix theory in leading order. For that, we have shown that the characteristic determinant obeys a viscid Burgers equation with a small spectral viscosity $\nu_L = 1/L$. In this regime, all the SYK spectrum is chaotic with universal Airy oscillations at the edges. As both regimes, support a fermionic ground state that appears dual to a black hole, this is likely to be the case for all permissible $q/N \leq 1/2$.

While all our analysis was carried out for the complex representations with Dyson index $\beta_D = 2$, we expect that the results carry for $\beta_D = 1, 4$ including the soft edge universality, after careful analysis of the real and quaternion representations of $Cl([\frac{N}{2}])$ respectively. An open problem is how the planar approximation established in the random matrix limit, can be used to organize the quantum diagrammatic analysis. Also, the analogy between the emergent black hole and a mesoscopic "quantum dot" offers the intriguing possibility for its realization in mesoscopic systems.

Finally, the observation that (36) is the entropy of a highly excited string at the Rindler horizon leads to the exciting possibility that the approach to thermal equilibrium in the SYK model may exhibit similarities with a falling string on a thermal black-hole [33,34]. If so, the model may provide a simple quantum understanding for the rate at which the entropy is deposited, and the time it takes to thermalize in current collider experiments.

Note added

After the submission of this paper, new studies have appeared confirming some of our observations. In [35] a similar behavior to one rendered in Fig. 1 was observed for the supersymmetric extension of the SYK model. In [36] the authors have confirmed independently the relationship between mesoscopy and black holes using different arguments. In [37] the authors have extended their analysis in [15] for finite N. Their results for $q/N = 1/2$ agree with ours.

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Appendix A. $n = 2p + 1$

The case with odd $n = 2p + 1$ follows a similar reasoning using instead the more general form

$$\Psi(z_+, z_-, \tau) = \langle \det(z_+ P_+ + z_- P_- - H(\tau)) \rangle_J \tag{37}$$

with the projectors $2P_\pm = 1 \pm \Gamma_{2n}$. A rerun of the Fierzing arguments after the Grassmannian representation, allow to re-write (37) in the form

$$\Psi(z_+, z_-, \tau) = \int d\bar{\chi}\, d\chi\, e^{-z_+ \bar{\chi}_+ \chi_+ - z_- \bar{\chi}_- \chi_- - \frac{2\tau}{L} \bar{\chi}_+ \chi_+ \bar{\chi}_- \chi_-} \tag{38}$$

which is seen to satisfy also a diffusion equation

$$\partial_\tau \Psi = -\frac{2}{L} \partial_{z_+} \partial_{z_-} \Psi \tag{39}$$

By re-defining $z_\pm = (z_1 \pm i z_2)/\sqrt{2}$, we have

$$\partial_\tau \Psi = -\nu_L (\partial_{z_1}^2 + \partial_{z_2}^2) \Psi \tag{40}$$

subject to the initial condition

$$\Psi(\tau = 0) = \left(\frac{z_1^2 + z_2^2}{2}\right)^{\frac{L}{2}} \tag{41}$$

which is now a standard 2-dimensional diffusion problem. The solution reads

$$\Psi = \frac{1}{4\pi \nu_L \tau} \int dz_1' dz_2'\, e^{\frac{(z_1 - z_1')^2 + (z_2 - z_2')^2}{4\nu_L \tau}} \left(\frac{z_1'^2 + z_2'^2}{2}\right)^{\frac{L}{2}} \tag{42}$$

To analyze the saddle point we define $\vec{r} = (z_1, z_2)$ and $\vec{r}' = (z_1', z_2')$. The saddle point of (42) is then located at

$$\frac{1}{2\tau}(\vec{r} - \vec{r}') = \frac{\vec{r}'}{|r'|^2} \tag{43}$$

For $z_2 = 0$ and $z_1 = \sqrt{2}\lambda$ we find

$$\begin{aligned} &\frac{1}{\tau}(\lambda - \lambda') = \frac{1}{\lambda'} \\ &f_L(\lambda) = \frac{1}{\tau}(\lambda - \lambda') \\ &\rho = \frac{L}{\pi} \mathrm{Im} f_L = \frac{L}{2\pi\tau}(4\tau - \lambda^2)^{\frac{1}{2}} \end{aligned} \tag{44}$$

which is identical to the even n case discussed in the text. The edge universality can be readily shown to follow from the expansion around the saddle point (43) leading to an Airy function as well. Specifically, the expansion around the right edge with $z_2 = 0$, corresponds to $\delta z'_1 \approx \mathcal{O}(1/L^{\frac{1}{3}})$ and $z'_2 \approx \mathcal{O}(1/L^{\frac{1}{2}})$. The expansion around the saddle point starts at level $z'^2_2 \approx \mathcal{O}(1/L)$, so to leading order the fluctuations of $\delta z'_1$ and $\delta z'_2$ never mix. The integration over $\delta z'_1$ produces the Airy function.

Appendix B. Fierz re-arrangement

We note that the Clifford-algebra with the base

$$\Gamma_A = \alpha_A \Gamma_{i_1} ... \Gamma_{i_p}, \ \ i_1 < i_2 .. < i_p, p \le 2n$$
$$\mathrm{Tr}(\Gamma_A \Gamma_B) = L\delta_{AB} \tag{45}$$

is complete in matrix space such that any matrix in the representation space with dimension $L \times L$ can be uniquely expressed as

$$B = \frac{1}{L}\sum_A \Gamma_A \,\mathrm{Tr}(\Gamma_A B) \tag{46}$$

We can now write for a given Γ_n

$$\Gamma_n \chi \bar{\chi} \Gamma_n = +\frac{1}{L}\sum_A \Gamma_A \,\mathrm{Tr}(\Gamma_A \Gamma_n \chi \bar{\chi} \Gamma_n)$$
$$= -\frac{1}{L}\sum_A \Gamma_A \bar{\chi} \Gamma_n \Gamma_A \Gamma_n \chi \tag{47}$$

The negative sign stems from the Grassmannian nature of χ. Summing over all n, sandwiching between $\bar{\chi}, \chi$ and interchanging the summation order yield

$$\frac{1}{C^n_{2n}} \sum \bar{\chi} \Gamma_n \chi \bar{\chi} \Gamma_n \chi =$$
$$\frac{1}{LC^n_{2n}} \sum_A \bar{\chi} \Gamma_A \chi \sum_{\Gamma_n} \bar{\chi} \Gamma_n \Gamma_A \Gamma_n \chi \tag{48}$$

For a given Γ_A, Γ_n either commute or anti-commute with it. Now just counting the number of commuting or anti-commuting terms we obtain (18), with (21) counting the number of commuting or anti-commuting Γ_n for a given $A = p$.

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