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FINDING THE PARETO OPTIMAL EQUITABLE ALLOCATION OF HOMOGENEOUS DIVISIBLE GOODS AMONG THREE PLAYERS

We consider the allocation of a finite number of homogeneous divisible items among three players. Under the assumption that each player assigns a positive value to every item, we develop a simple algorithm that returns a Pareto optimal and equitable allocation. This is based on the tight relationship between two geometric objects of fair division: The Individual Pieces Set (IPS) and the Radon–Nykodim Set (RNS). The algorithm can be considered as an extension of the Adjusted Winner procedure by Brams and Taylor to the three-player case, without the guarantee of envy-freeness.

Keywords: *fair division, Pareto optimality, graph theory, adjusted winner procedure*

1. Introduction

In this article we study a fair division problem: how to optimally allocate a (finite) number of items to three persons, usually denoted *players* or *agents* in the economic literature, with heterogeneous preferences.

We will consider items which are attractive to the players, and are therefore called goods. They are completely divisible: any good can be partitioned in any proportion among the three players, and each of them will enjoy a utility proportional to the fraction received. This is typically true, for instance, with edible goods.

When indivisible goods are present, all of the possible arrangements may prove unsatisfactory for one or more agents (consider, for instance, the case of two goods being assigned to three players). Notice, however, that players could overcome this problem by assigning an item according to a lottery with the probability of winning the good

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being proportional to the fraction proposed by a procedure that treats the good as divisible. We refer to Demko and Hill [12] for further details. As an alternative, players may use monetary side payments to simulate a split.

The assumptions and the model that we are considering are the same as under which Brams and Taylor devised the Adjusted Winner (AW) procedure, which works for two players. This procedure, described in [7] and [8], returns an allocation with many appealing properties: i) it is (strongly) Pareto-optimal: neither player can improve his welfare without worsening that of the other player; ii) it is equitable: in their own evaluations, the two players enjoy the same amount of utility, and iii) it is envy-free: neither player prefers the allotment of the other according to the players' preferences.

Moving from a two-player setting to a three-player one is not an easy extension. Dall'Aglio and Hill [11] presented a series of examples, with three or more players, where the three properties ensured by the AW procedure could not coexist[2]. This difficulty was later noted more explicitly by Brams et al. [6].

Since Pareto optimality is an essential requirement, it is necessary to choose between equitability and freedom from envy. Olvera-López and Sánchez-Sánchez [15], adopted a linear programming approach to find the maxmin-optimal allocation, which, under the hypothesis of Mutual Absolutely Continuous (MAC) utilities is also Pareto--optimal and equitable for a finite number of players. This is done by transforming the fair division problem into an optimization problem over a bipartite graph, with the nodes on one side representing players, and the nodes on the other side denoting goods.

Here we propose an alternative graph-based approach, with the graph originating from the tight relationship between two geometric objects, the Individual Pieces Set (IPS) and the Radon–Nikodym Set (RNS) specifically introduced to deal with problems in fair division. This graph is obtained by placing the objects in the RNS and by considering the objects and the intersections between lines joining these objects to the vertices of the RNS as nodes, and these lines as edges.

While we do not achieve the generality of the previous approach, we believe that this approach is more intuitive. In fact, once the objects are plotted on the RNS, every Pareto-optimal allocation can be visualized immediately. The optimal allocation is sought from among all these allocations by moving from one node of the graph to an adjacent one, until a local optimum is found. This is the global optimal allocation for the problem.

The paper is organized as follows: In Section 2, we introduce the model. In Section 3, we introduce the reader to the geometry of efficient fair division (named after [2]) and

[2]One (counter-)example is used in Theorem 3.10 to falsify a claim by Weller, who stated that under mutually absolutely continuous utilities, every maxmin-optimal allocation is envy-free. Under the same hypotheses, the maxmin-optimal allocation is necessarily Pareto-optimal and equitable. This counterexample, therefore rules out the necessary coexistence of the three optimality properties of AW for three players or more.

describe the graph on which we construct the optimization problem. In Section 4, we describe the algorithm and prove its convergence. Section 5 concludes.

2. The problem

Let $M = \{1, 2, ..., m\}$, with $m \in \mathbb{N}$ be the set of divisible and homogeneous objects to be allocated among three players. The set of players will be usually denoted as $N = \{1, 2, 3\}$ but the Roman numerals I, II and III will be employed in figures.

We write the matrix of evaluations as $[a_{ij}]_{i\in N,\, j\in M}$, where entry a_{ij} describes the value that player $i \in N$ assigns to item $j \in M$. We assume that utilities are

Non-negative. All of the objects are goods:

$$a_{ij} \geq 0 \;\; \forall i \in N,\;\; j \in M$$

Linear. If player i gets a share of item j and a share $t_k \in [0, 1]$ of item k she gets a total utility of $t_j a_{ij} + t_k a_{ik}$.

Normalized. The agents attach the same value to the whole bundle of goods:

$$\sum_{j\in M} a_{ij} = 1,\;\; \forall i \in N$$

Let $\mathbf{X} = [x_{ij}]_{i\in N,\, j\in M}$, $x_{ij} \geq 0,\; \forall i \in N,\; j \in M$ be an allocation matrix, with $\sum_{i\in N} x_{ij} = 1$, $\forall j \in M$ and $\mathbf{X} \in \mathcal{X}$, where $\mathcal{X}$ is the set of all possible allocation matrices. Let us label by $\widehat{\mathbf{X}}$ any integer allocations where $\hat{x}_{ij} \in \{0, 1\}, \forall i \in N,\; j \in M$ and by $\widehat{\mathcal{X}}$ the set of such allocations. Define now, for any $\mathbf{X} \in \mathcal{X}$,

$$a(\mathbf{X}) = (a_1(\mathbf{X}), a_2(\mathbf{X}), a_3(\mathbf{X})) \;\text{ with }\; a_i(\mathbf{X}) = \sum_{j\in M} a_{ij} x_{ij},\; i = 1, 2, 3$$

The i-th component of this vector is the total value that player i derives from the given allocation $\mathbf{X}$.

We are going to search for an allocation $\mathbf{X}^*$ which simultaneously satisfies

(Strong) Pareto optimality (PO). There is no other allocation $\mathbf{X}' \in \mathcal{X}$ such that $a_i(\mathbf{X}') \geq a_i(\mathbf{X}^*)$ for all $i \in N$, with strict inequality for at least one player.

Equitability (*EQ*).

$$a_1(\mathbf{X}^*) = a_2(\mathbf{X}^*) = a_3(\mathbf{X}^*)$$

The proposed allocation coincides with the Kalai–Smorodinsky solution [13, 14] for bargaining problems. Throughout the rest of the article, we make the following simplifying assumption:

Mutual absolute continuity (MAC). Each player assigns a positive value to every item, i.e.,

$$a_{ij} > 0\ C_e \text{for any } i \in N \text{ and } j \in M$$

When MAC holds, a PO-EQ allocation always exists.

Theorem 1. (Corollary 5.8 in [9]). If the utility of all the players has a common support, i.e., a common set of goods with strictly positive utility for all the players, then the PO-EQ allocation coincides with the maxmin allocation, defined by

$$\mathbf{X}^* \in \operatorname{argmax}_{\mathbf{X} \in \mathcal{X}} \left\{ \min_{i \in N}\ a_i(\mathbf{X}) \right\}$$

which exists for any instance of the problem.

3. Geometrical framework

We are now going to review two geometric structures that are useful for the analysis of PO and PO-EQ allocations. First of all, we characterize PO allocations. Let Δ_2 be the two-dimensional simplex

$$\Delta_2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \colon\ x_i \geq 0,\ i = 1, 2, 3, \text{ and } x_1 + x_2 + x_3 = 1 \right\}$$

and let ri(A) be the relative interior of any subset A of an Euclidean space. In particular,

$$\operatorname{ri}(\Delta_2) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 \colon\ x_i > 0, i = 1, 2, 3, \text{ and } x_1 + x_2 + x_3 = 1 \right\}$$

Theorem 2. (Theorem 1 in [3], Proposition 4.3 in [9]). Under MAC, an allocation $\mathbf{X}=[x_{ik}]$ is PO iff, for some $\gamma=(\gamma_1,\gamma_2,\gamma_3)\in \mathrm{ri}(\Delta_2)$ the following holds: For any $k\in M$

$$x_{ik}>0 \text{ if } \gamma_i a_{ik}\geq \gamma_j a_{jk} \text{ for any } i,\ j\in N \tag{1}$$

We denote by $\mathbf{X}^{\gamma}=[x_{ik}^{\gamma}]$ any allocation satisfying (1).

3.1. The individual pieces set

We consider the individual pieces set $\mathrm{IPS}\subset\mathbb{R}^3$ (see [2]), also known as the partition range, defined as follows

$$\mathrm{IPS}=\left\{a(\mathbf{X}):\mathbf{X}\in\mathcal{X}\right\}$$

Dall'Aglio et al. [10] have shown that

$$\mathrm{IPS}=\mathrm{conv}\left(a(\widehat{\mathbf{X}}):\widehat{\mathbf{X}}\in\widehat{\mathcal{X}}\right) \tag{2}$$

The value of a PO-EQ allocation is given by the point of intersection between the egalitarian ray, i.e., the line of points whose coordinates are equal in $\mathbb{R}^3$ and the upper surface of the IPS denoted by the Pareto boundary, PB. Statement (2) shows that the PB is composed of faces. MAC implies that no Pareto face is parallel to any of the coordinate axes[3].

If we consider the partition range from above, finding the PO-EQ allocation amounts to finding the face of the PB that contains the egalitarian ray (more than one face may be involved if the egalitarian ray "hits" an edge, or coincides with an integer allocation), and then finding the allocation on this Pareto face which yields the optimal value (Fig. 1a).

Consider, for any $x=(x_1,x_2,x_3)\in\mathbb{R}_+^3$ the normalizing operator

$$N(x)=\left(\frac{x_1}{s(x)},\frac{x_2}{s(x)},\frac{x_3}{s(x)}\right) \quad \text{with} \quad s(x)=x_1+x_2+x_3$$

---

[3]Take an allocation on PB. Under MAC, if we move to another allocation which has a higher utility for one player, the other players' utilities cannot remain constant. The assumption of MAC is therefore incompatible with Pareto faces parallel to the axes.

and define the normalized Pareto boundary, NPB, as

$$\text{NPB} = \left\{ N(x) : x \in \text{PB} \right\}$$

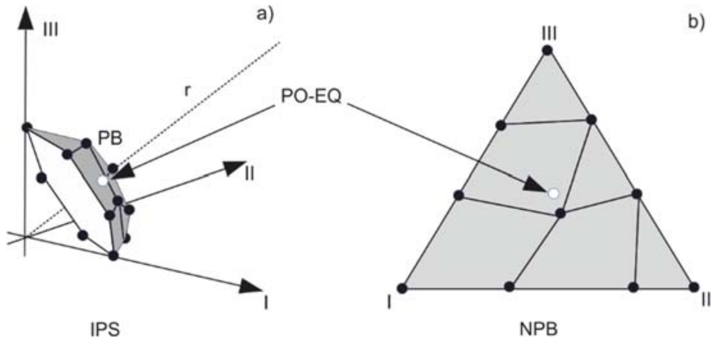

Fig. 1. In a), the PB set is indicated by the grey surface, while the white dot indicates the value of the PO-EQ allocation. In b), the same set seen from above yields the NPB set. The PO-EQ allocation is in the barycenter of the simplex

In general, $\text{NPB} \subseteq \Delta_2$ holds, but it can be easily shown that these two sets coincide when MAC holds. The Pareto faces partition the set NPB, and finding the PO-EQ allocation amounts to finding the allocation corresponding to the centre (1/3, 1/3, 1/3) on the NPB (Fig. 1b).

To find the PO-EQ allocations, we will employ the following result, which is valid for general fair division problems (any number of players, completely divisible and non-homogeneous goods).

Theorem 3. (Proposition 6.1 in [9]). Consider the following function g: $\Delta_2 \to [0, 1]$

$$g(\gamma) = \sum_{i \in N} \gamma_i a_i(\mathbf{X}^\gamma) \quad \gamma = (\gamma_1, \gamma_2, \gamma_3) \in \Delta_2$$

with $\mathbf{X}^\gamma$ being the PO allocation associated to γ according to (1). Then the hyperplane

$$\mathcal{H}(\gamma) = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i \in N} \gamma_i x_i = g(\gamma) \right\}$$

supports IPS at the point $(a_1(\mathbf{X}^\gamma), a_2(\mathbf{X}^\gamma), a_3(\mathbf{X}^\gamma))$, i.e.,

$$(a_1(\mathbf{X}^\gamma), a_2(\mathbf{X}^\gamma), a_3(\mathbf{X}^\gamma)) \in \mathcal{H}(\gamma) \text{ and } \sum_{i \in N} \gamma_i y_i \leq g(\gamma) \;\; \forall (y_1, y_2, y_3) \in \text{IPS}$$

The hyperplane $\mathcal{H}(\gamma)$ intersects the egalitarian ray at the point $g(\gamma)(1,1,1)$.

The function $g(\cdot)$ is convex, and, for any of its minimizing points γ^* the hyperplane $\mathcal{H}(\gamma^*)$ supports IPS at a set of points containing the PO-EQ allocation.

An algorithm that returns the leximin allocation is described in [9]. This can be adapted to return the PO-EQ allocation in the present situation:

1. Find γ^* an absolute minimum of g.
2. Find the Pareto face corresponding to $\mathcal{H}(\gamma^*)$.
3. Find the equitable allocation within the Pareto face.

To fully adapt this algorithm to the present situation, we need to better characterize the Pareto faces.

3.2. The Radon–Nykodim set

Figure 1b shows that the NPB can be represented as a 2-dimensional simplex. We will now consider another 2-dimensional simplex, introduced by Weller [16] and extensively investigated by Barbanel [2] that enables us to represent the items, the efficient partitions and the faces of the PB into a single geometric figure. Following [2], we apply the Radon–Nikodyn set, RNS to define this new simplex.

Each vertex of the RNS represents a player. We next plot the individual items onto the RNS by considering the normalized vectors of the evaluations of these items

$$a_j^n = N(a_j) \quad j \in M$$

The normalized coordinates of all these objects are plotted on a 2-dimensional simplex where each vertex represents a player. Under MAC $a_j^n \in \mathrm{ri}(\Delta_2)$ for each $j \in M$.

Definition 1. For each point $\beta = (\beta_1, \beta_2, \beta_3) \in \mathrm{RNS}$, consider the lines joining β to each vertex. We denote the half open segments on these lines from β to the opposite side of each vertex, with β excluded, disputing segments.

Definition 2. For each $\beta \in \mathrm{ri(RNS)}$ we derive the following Pareto allocation rule based on β, $\mathrm{PAR}(\beta)$ which delivers one or more PO allocations under MAC (see theorem 10.9 in [2]). The disputing segments of β divide the simplex into three parts, each being a neighbourhood of a vertex. The objects in each neighbourhood are assigned to the player associated to the corresponding vertex. Denote any such allocation as $\mathbf{X}^\beta$. In the case where the allocation is integer, we will use $\widehat{\mathbf{X}}^\beta$.

It is important to notice that this allocation rule may not be unique. In the case where an object lies on one of the disputing segments of β, it can be considered to be on both sides of the segment, and therefore can be assigned to any of the corresponding players, or it can be split between these players.

Every Pareto allocation lies on the upper border of the convex set IPS, and a hyperplane supports IPS at this point. A more precise account of the relationship between supporting hyperplanes and the Pareto allocation rule is given by the following result.

Theorem 4. (Theorem 2 in [1]). Assume MAC. If, for any $x = (x_1, x_2, x_3) \in \mathrm{ri}(\Delta_2)$, we denote

$$\mathrm{RD}(x) = N\left(\frac{1}{x_1}, \frac{1}{x_2}, \frac{1}{x_3}\right)$$

then, for any $\gamma \in \mathrm{ri}(\Delta_2)$ the allocation $\mathbf{X}^{\gamma}$ satisfies $\mathrm{PAR}(\beta)$ with $\beta = \mathrm{RD}(\gamma)$. Conversely, for any $\beta \in \mathrm{RNS}$, the allocation rule $\widehat{\mathbf{X}}^{\beta}$ supports IPS through the hyperplane H^{γ} with $\gamma = \mathrm{RD}(\beta)$.

When, given $\beta \in \mathrm{RNS}$, one or more objects lie on disputing segments, the associated hyperplane supports all the corresponding integer allocations and their convex hulls. This fact plays a crucial role in how the Pareto faces are generated.

Definition 3. For any item $j \in K$, we call the segments joining a_j^n to each of the three vertices supporting segments. Two items, $j, k \in K$ are support independent, or *s*-independent, if neither a_j^n, nor a_k^n lies on any of the supporting segments of the other item. In Figure 2, we illustrate the dividing and supporting segments of one or more items.

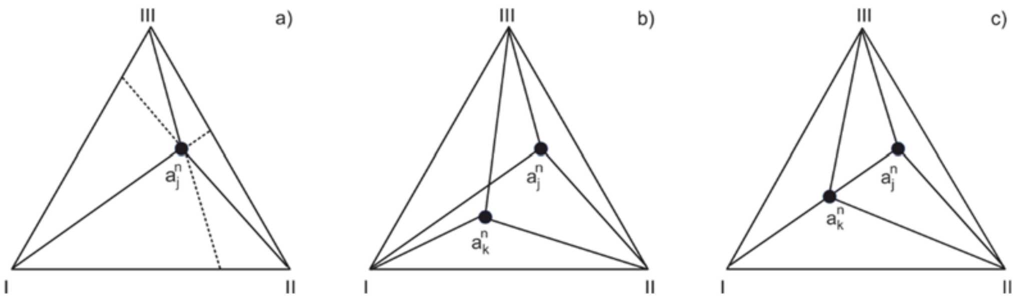

Fig. 2. Supporting segments (solid lines) and disputing segments (dashed lines) of an item (a), two s-independent items (b), two dependent items (c)

In [10], it is shown (Proposition 2) that the supporting segments of two items intersect exactly once if and only if the two items are s-independent.

3.3. The Pareto boundary as a graph

The items and the supporting line segments in the RNS fully define the faces of the NPB. In Dall'Aglio et al. [10] (Theorem 4 and the following considerations) it is shown that, for any $\beta \in \mathrm{RNS}$ that coincides with an item or is positioned at the intersection of two or more s-independent supporting line segments, $\mathrm{PAR}(\beta)$ defines a face on the Pareto border of the IPS, and therefore on the NPB. Conversely, each hyperplane $\mathcal{H}(\gamma)$ with $\gamma \in \mathrm{ri}(\Delta_2)$ supporting a Pareto face on the IPS is such that $RD(\gamma)$ coincides with an item in the RNS, or is positioned at the intersection of two s-independent supporting line segments.

The RNS simplex can therefore be used to build a graph $\mathcal{G} = \{V, E\}$ where each vertex $v \in V$ is a face on the Pareto surface, and two vertices v_i and v_j are connected by an arc[4] $e_{ij} \in E$ if and only if the corresponding faces are adjacent, i.e., they share a common edge. The construction of this graph is based on considering all the goods in RNS with their supporting line segments. The vertices V consist of all the points in the RNS coinciding with a good or with an intersection of supporting line segments. Two vertices $v_i, v_j \in V$ are connected by an arc $e_{ij} \in E$ whenever there is a supporting segment joining the two vertices, with no other vertex of V in between.

In the same reference, [10] (Theorem 6), the authors show that, under MAC, two faces on the Pareto border of IPS are adjacent, i.e., they share a common line segment if and only if the corresponding vertices v_k and v_ℓ are joined by an arc in $\mathcal{G}$.

4. A simple algorithm

We now consider a function $\tilde{g} : RNS \to [0,1]$ defined for each $\beta \in \mathrm{RNS}$ as

$$\tilde{g}(\beta) = g(RD(\beta))$$

This function is defined for each vertex of the graph $\tilde{g}(v)$. The following theorem shows that it suffices to check the value of $\tilde{g}$ on the face/vertices only. Moreover, the

[4]We prefer to use arc in place of the more common edge to avoid confusion with the edges of a face on the Pareto surface.

function $\tilde{g}$ inherits the convexity property of g, according to which it suffices to show that $\tilde{g}$ is a "local" minimum (i.e., a minimum w.r.t. the adjacent vertices) to ensure that it is a global optimum.

Theorem 5. Under MAC, a face/vertex v^* containing the egalitarian ray has the following properties:

- It is the global minimum for $\tilde{g}$:

$$\tilde{g}(v^*) \leq \tilde{g}(v) \text{ for any } v \in V \tag{3}$$

- It suffices to show that it is a local minimum for $\tilde{g}$, i.e.,

$$\tilde{g}(v^*) \leq \tilde{g}(v') \text{ for any } v' \text{ adjacent to } v^* \tag{4}$$

Proof.

I. The PO-EQ allocation belongs to one or more Pareto faces. According to Theorem 3 (iii), one of the minimizing arguments of g (and therefore of $\tilde{g}$) will correspond to such a Pareto face. The corresponding vertex on the graph will be associated with the same absolute minimum.

II. We need to prove two preliminary claims.

We consider two hyperplanes: $\mathcal{H}(\text{face})$ passing through a Pareto face, and $\mathcal{H}(\text{edge})$ passing through an edge of the face. Denote by $g(\text{face})$ and $g(\text{edge})$ the value of g corresponding to $\mathcal{H}(\text{face})$ and $\mathcal{H}(\text{edge})$, respectively, and denote by ℓ_{ed} the line formed by the intersection of the two (non-parallel) hyperplanes containing this edge. Consider now the following projections onto the NPB, obtained by normalizing the points in geometrical objects: p_{face}, p_{ed}, p_{eq} are the projections of the Pareto face, the line ℓ_{ed} and the bisector, respectively. We prove the following claims:

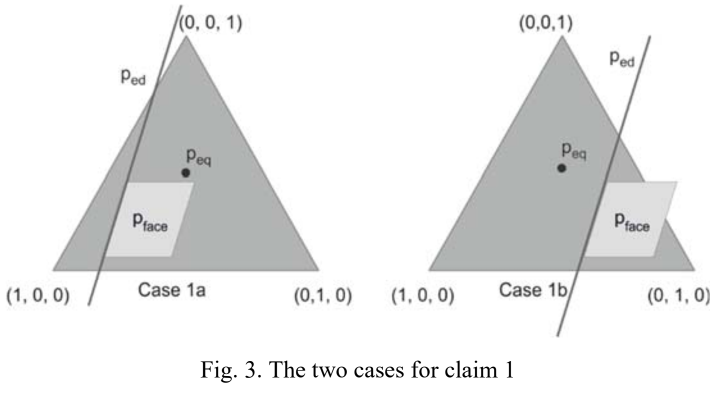

Fig. 3. The two cases for claim 1

Claim 1a. If p_{face} and p_{eq} are on the same side of p_{ed} (see Fig. 3, case 1a) then $g(\text{edge}) \geq g(\text{face})$.

Claim 1b. If p_{face} and p_{eq} are on opposite sides of p_{ed} (see Fig. 3, case 1b) then $g(\text{edge}) \leq g(\text{face})$.

Proof of claims 1a and 1b

1a. Suppose $g(\text{edge}) < g(\text{face})$. Then the hyperplane $\mathcal{H}(\text{edge})$ passes through the edge and $g(\text{edge})(1,1,1)$ separating the IPS from the Pareto face and $g(\text{face})(1, 1, 1)$ (Fig. 4, Case 1a). This is a contradiction.

1b. Suppose $g(\text{edge}) > g(\text{face})$. Then the hyperplane $\mathcal{H}(\text{edge})$ passes through the edge and $g(\text{edge})(1,1,1)$, separating the IPS from the Pareto face and $g(\text{face})(1, 1, 1)$ (Fig. 4, Case 1b). This is again a contradiction. ■

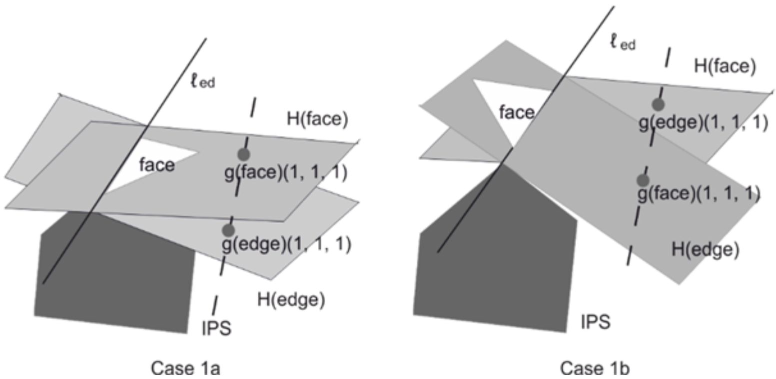

Fig. 4. Proofs of claims 1a and 1b

As a consequence of these claims, suppose that $\tilde{g}(v') < \tilde{g}(v'')$ for two adjacent faces. Hence, projecting the Pareto face v', the line containing the edge and the bisector onto the NPB, the projected Pareto face and the projected bisector must lie on the same side of the projected line.

Claim 2. Suppose $\tilde{g}(v') = \tilde{g}(v'')$ for two adjacent faces. Then the bisector intersects the line generated by the common edge.

Proof of claim 2. Denote by $\mathcal{H}(v')$, $\mathcal{H}(v'')$ the hyperplanes passing through face v' and face v'', respectively. Since $v' \neq v''$, the two hyperplanes are neither parallel nor coincident and intersect in a line ℓ_{ed} that includes the common edge. Suppose the bisector does not intersect this line. Hence, the two hyperplanes $\mathcal{H}(v')$ and $\mathcal{H}(v'')$ have more than three non-aligned points in common: those in ℓ_{ed} and $g(v')(1,1,1)$. Since they are distinct hyperplanes, this is impossible. The bisector must intersect the common line ℓ_{ed}. ■

To end the proof of this theorem, we distinguish between three cases:

Case A. Equation (4) holds with strict inequality for all the adjacent edges.

Consider the projection of the faces and the bisector onto the NPB. For each adjacent edge, the Pareto face and the bisector lie on the same side of the line generated by the edge. The projected bisector must belong to the projected face and the same is true on the Pareto boundary. By Theorem 3 (iii), v^* is an absolute minimum of g, and therefore of $\tilde{g}$.

Case B. Equation (4) holds with strict inequality for all the adjacent edges but one, i.e., $\tilde{g}(v^*) = \tilde{g}(v'')$ only for one v'' adjacent to v^*.

Considering the projections onto the NPB, the bisector and the Pareto face v^* must lie on the same side of the lines generated by each of the adjacent edges different from v''. Moreover, the bisector must lie on the (projected) line generated by the edge between the faces v^* and v''. Once again, the bisector intersects with the Pareto face v^*, and the theorem holds.

Case C. Equation (4) holds with $\tilde{g}(v^*) = \tilde{g}(v'')$ for two or more adjacent vertices v''.

The bisector must belong to all the lines generated by the edges of the adjacent faces for which equality holds. The bisector belongs to their intersection and thus must be a vertex of the face. Once again, the bisector intersects with the Pareto face v^*, and the theorem holds. ■

4.1. The algorithm

The following algorithm is based on Theorem 5.

Beginning. Start from any $v^0 \in V$ (for instance, the one closest to the centre in the RNS).

Body. For the current $v^k \in V$, compute the value $\tilde{g}(v')$ at each adjacent vertex v':

- If (4) holds $\Rightarrow$ v^k is optimal $\Rightarrow$ End.
- Otherwise move to the adjacent vertex v^{k+1} with the lowest value of $\tilde{g} \Rightarrow$ Repeat step with v^{k+1}.

End. From the optimal vertex/face $\Rightarrow$ find the optimal allocation.

Theorem 6. The algorithm described above converges in a finite number of steps.

Proof. Since each change of node marks a strict improvement in the objective function, the algorithm cannot cycle. Furthermore, since the number of nodes in the graph is finite, the algorithm will stop at the optimum in a finite number of steps. ■

4.2. An example

Consider the following evaluation matrix

$$A = \begin{bmatrix} 0.2 & 0.3 & 0.5 \\ 0.1 & 0.5 & 0.4 \\ 0.4 & 0.15 & 0.45 \end{bmatrix}$$

From the items plotted on the RNS, we obtain the graph in Fig. 5, where the light grey (1, 3, 2) dots indicate the items and the dark grey ones (13, 12, 23) indicate the intersections of the supporting lines.

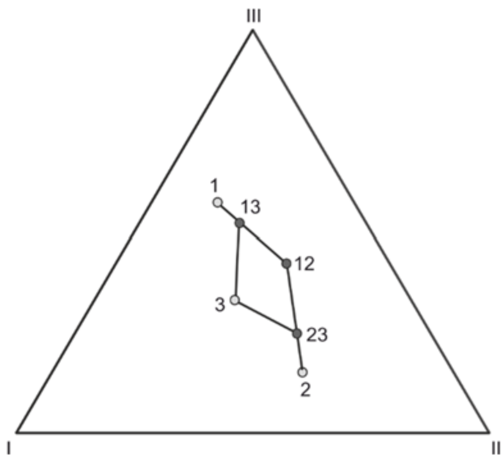

Fig. 5. The graph on the RNS

Figure 6 shows the value of $\tilde{g}$ associated with each node. We start from node 13, then proceed to node 3 and finally stop at node 23.

If we consider the corresponding faces on the NPB, using Fig. 7, we can verify that the face corresponding to node 23 contains the barycentre $(1/3, 1/3, 1/3)$.

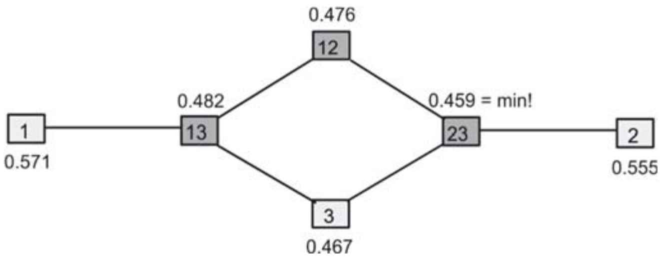

Fig. 6. Evaluating the objective function at the nodes of the graph

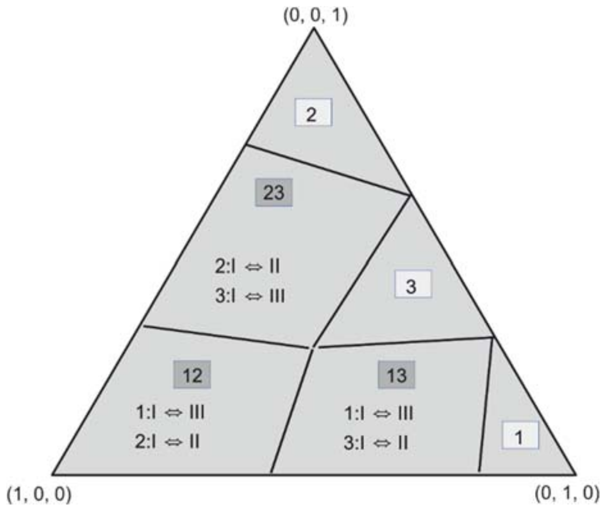

Fig. 7. The faces on the NPB and their corresponding nodes on the graph

The corresponding allocation matrix is

$$X = \begin{bmatrix} 0 & 0.2362 & 0.8947 \\ 0 & 0.7638 & 0 \\ 1 & 0 & 0.1053 \end{bmatrix}$$

with a common utility for the three players 0.4474.

5. Conclusions

In many game-theoretic contexts, the number of players matters, and, in particular, there is often a large increase in complexity going from a two-player to a three-player model. Fair division procedures are no exception in this respect.

The algorithm that we propose is one (but not the only possible) extension of the celebrated AW procedure. Using the AW procedure, the items can be conveniently arranged along the real line, according to their utility ratio. However, here we need a (planar) graph to position the items. Also, using the AW procedure, for each possible Pareto arrangement of the items, we only need to measure the disparity between the overall payoffs of the two players. Here, we need a more structured objective function to guide improvements to the overall allocation of goods.

A further extension to more than three players looks possible, but not trivial. Such an extension should result in a graph of higher dimension, with the dimension determined by the number of players. Such an extension would also require the removal of the MAC assumption. It seems reasonable to make this assumption for two- or three--player games with a limited number of items. However, as the number of players and items grows, it seems natural that at least one player has no interest in one of the disputed goods.

Also, the same setting and graph can be used to find other optimal fair allocations. Very recently, Bogomolnaia and Moulin [4] established the equivalence, for any number of players, between the Nash Bargaining and competitive equilibrium under equivalent income (CEEI) solutions, and have shown that this exhibits several desirable properties that the PO-EQ allocation fails to satisfy. Furthermore, Bogomolnaia et al. [5] explored an interesting setting where the objects to be distributed may be goods for some players and "bads" for others. Finding what becomes of the IPS and the RNS in such a new setting is a challenging enterprise.

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