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Shift–tail equivalence and an unbounded representative of the Cuntz–Pimsner extension

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Abstract. We show how the fine structure in shift–tail equivalence, appearing in the non-commutative geometry of Cuntz–Krieger algebras developed by the first two listed authors, has an analogue in a wide range of other Cuntz–Pimsner algebras. To illustrate this structure, and where it appears, we produce an unbounded representative of the defining extension of the Cuntz–Pimsner algebra constructed from a finitely generated projective bi-Hilbertian module, extending work by the third listed author with Robertson and Sims. As an application, our construction yields new spectral triples for Cuntz and Cuntz–Krieger algebras and for Cuntz–Pimsner algebras associated to vector bundles twisted by an equicontinuous $*$-automorphism.

Contents

1. *Introduction*

In this paper we study the non-commutative geometry of Cuntz–Pimsner algebras. The end product is an unbounded Kasparov module representing the defining extension which reflects the dynamics encoded in the Cuntz–Pimsner algebra. In addition to the important examples of Cuntz–Krieger algebras which arise from subshifts of finite type, Cuntz–Pimsner algebras also include crossed products by $\mathbb{Z}$, topological graph algebras and Exel crossed products.

Pimsner's construction [**19**] associates to a given bimodule E (or C^*-correspondence) over a C^*-algebra A a new C^*-algebra $\mathcal{O}_E$, which is to be viewed as the crossed product of A by E. This viewpoint is in line with the idea that an A-bimodule E is a generalization of the notion of $*$-endomorphism, and a $*$-endomorphism of a commutative C^*-algebra corresponds to a continuous map of the underlying space. As such, bimodules can be viewed as discrete-time dynamical systems over A. See [**9**] for a detailed discussion supporting this point of view.

By construction, the Cuntz–Pimsner algebra $\mathcal{O}_E$ associated with a finitely generated projective Hilbert-bimodule E over a C^*-algebra A is the quotient in its Toeplitz extension, a short exact sequence of C^*-algebras

$$0 \to \mathcal{K}_A(\mathcal{F}_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0. \tag{1.1}$$

We call the extension (1.1) the *defining extension* of $\mathcal{O}_E$. Here $\mathcal{T}_E$ is the algebra of Toeplitz operators on the Fock module $\mathcal{F}_E$. The C^*-algebra $\mathcal{K}_A(\mathcal{F}_E)$ of A-compact operators is Morita equivalent to A. For A nuclear, the extension (1.1) is semisplit and defines a distinguished class $[\mathrm{ext}] \in KK^1(\mathcal{O}_E, A)$; see [**15**]. Pimsner showed that the Toeplitz algebra $\mathcal{T}_E$ is KK-equivalent to A, and six-term exact sequences relate the K-theory and K-homology of $\mathcal{O}_E$ with that of A through the Pimsner sequence (see [**19**, Theorem 4.8]).

The class of the defining extension [ext] of finitely generated projective bi-Hilbertian modules satisfying an additional technical requirement was represented by a Kasparov module in [**22**]. The work in [**22**] gives a starting point for studying the non-commutative geometry of the corresponding Cuntz–Pimsner algebras, i.e. their spectral triples or more generally unbounded Kasparov modules [**2**].

A detailed study of the non-commutative geometry of Cuntz–Krieger algebras $O_{\boldsymbol{A}}$ associated to a $\{0, 1\}$ matrix $\boldsymbol{A}$ was presented in [**12**]. Using the groupoid model for these algebras, a new ingredient in the form of a function on the groupoid was introduced and used to construct unbounded Kasparov modules. In the present paper we utilize the fact that Cuntz–Krieger algebras admit a Cuntz–Pimsner model over the commutative algebra $C(\Omega_{\boldsymbol{A}})$, where $\Omega_{\boldsymbol{A}}$ is the underlying subshift of finite type (see [**9**, **23**]) to emulate the ideas in [**12**] for a wider class of Cuntz–Pimsner algebras. In particular, we show that the non-commutative geometries that were described in [**12**] in fact arise from the extension (1.1) associated to this particular model. Thus, a key idea in this paper is to place the construction for Cuntz–Krieger algebras in [**12**] into the framework for Cuntz–Pimsner algebras of [**22**].

Algebras that model discrete-time dynamical systems quite generally carry a dual circle action. The case of bimodule dynamics is no different, and every Cuntz–Pimsner algebra carries a canonical circle action inducing a $\mathbb{Z}$-grading. While the Pimsner sequence in KK-theory relates KK-groups of $\mathcal{O}_E$ with the KK-groups of A, the Pimsner–Voiculescu sequence in KK-theory relates KK-groups of $\mathcal{O}_E$ with those of the core $\mathcal{C}_E$ (the fixed point algebra for the circle action). The literature has mostly focussed on Kasparov cycles associated with the core [**4**, **6**, **11**]. In fact, if we consider the $\mathcal{C}_E$ correspondence $E \otimes_A \mathcal{C}_E$

we find that $\mathcal{O}_{E\otimes_A \mathcal{C}_E} \cong \mathcal{O}_E$, so that

$$0 \to \mathcal{K}_{\mathcal{C}_E}(\mathcal{F}_{E\otimes_A \mathcal{C}_E}) \to \mathcal{T}_{E\otimes_A \mathcal{C}_E} \to \mathcal{O}_E \to 0 \tag{1.2}$$

is exact [**19**]. The unbounded model for the Kasparov class of the exact sequence (1.2) is well understood under mild technical assumptions (the spectral subspace condition [**4**]) and arises from the number operator constructed from the grading. See the discussion in Remark 2.2 below. The case of Cuntz–Krieger algebras, and in particular the results of [**12**, §3.4], show that it is impossible to describe non-trivial classes in $K^1(\mathcal{O}_{\mathbf{A}})$ as Kasparov products of the extension (1.2) with classes in $K^0(\mathcal{C}_{\mathbf{A}})$ by general methods. In contrast, in Theorem 3.25 of the present paper we show that the surjective map $K^0(C(\Omega_{\mathbf{A}})) \to K^1(O_{\mathbf{A}})$ constructed in [**12**, Theorem 5.2.3 and Remark 5.2.6] is in fact the boundary map in the Pimsner sequence arising from the model of $O_{\mathbf{A}}$ as a Cuntz–Pimsner algebra over $C(\Omega_{\mathbf{A}})$. This shows that the exact sequences associated to the two extensions (1.1) and (1.2) behave very differently.

Our method employs a refined aspect of the dynamics, arising as part of the data of shift–tail equivalence in the Cuntz–Krieger case, and gives further grading information needed to assemble an unbounded Kasparov module splitting the extension (1.1) and representing the class $[\mathrm{ext}] \in KK^1(\mathcal{O}_E, A)$. This grading is defined on an important module constructed from the algebra $\mathcal{O}_E$ in [**22**]. The technical novelty of this paper is that of a *depth-kore operator*†. The depth-kore operator κ detects 'depth' relative to the inductive limit structure of the core and provides the missing piece when assembling a Dirac operator from the number operator associated with the circle action.

The technical setup for this paper is a finitely generated projective bi-Hilbertian bimodule E over a unital C^*-algebra A. That is, E is equipped with left and right A-valued inner products ${}_A(\cdot|\cdot)$, $(\cdot|\cdot)_A$ both making E full, and for which the respective actions are injective and adjointable, see [**13**]. We place additional technical assumptions on E involving the asymptotic properties of the Jones–Watatani indices of the tensor powers $E^{\otimes_A k}$ (see Assumption 1 in §2.2 and Assumption 2 in §3.2). These assumptions are satisfied in a large class of examples, with no known counter-examples at the present time. The examples for which the assumptions have been verified include Cuntz–Krieger algebras in the model over the one-sided shift space, crossed products by $\mathbb{Z}$ and graph C^*-algebras of primitive graphs.

We let $\mathcal{O}_E$ denote the Cuntz–Pimsner algebra constructed from E, $\Phi_\infty : \mathcal{O}_E \to A$ the conditional expectation constructed in [**22**] assuming Assumption 1 and Ξ_A the completion of $\mathcal{O}_E$ as an A-Hilbert module in the conditional expectation Φ_∞. Under the Assumptions 1 and 2, we prove the following theorem. It appears as Theorem 3.19 below.

THEOREM 1. *The $(\mathcal{O}_E, A)$-bimodule Ξ_A decomposes as a direct sum of finitely generated projective A-modules:*

$$\Xi_A = \bigoplus_{r\geq n} \Xi_A^{n,r}.$$

† Kore is not only phonologically the same as core, as in gauge fixed point subalgebra, but also another name for Persephone—queen of the Underworld.

Here $\Xi_A^{n,r} = P_{n,r}\Xi_A$ *for* $n \in \mathbb{Z}$, $r \in \mathbb{N}$ *and a projection* $P_{n,r} = P_{n,r}^* = P_{n,r}^2 \in \mathcal{K}_A(\Xi_A)$. *The number operator* $c := \sum_{n,r} nP_{n,r}$ *and the depth-kore operator* $\kappa := \sum_{n,r}(r-n)P_{n,r}$ *define self-adjoint regular operators on* Ξ_A *that commute on the common core given by the algebraic direct sum. The operator*

$$\mathcal{D} := \psi(c,\kappa) = \sum_{n,r} \psi(n, r-n)P_{n,r}, \quad \psi(n,k) = \begin{cases} n, & k=0, \\ -(k+|n|) & \text{otherwise} \end{cases}$$

makes $(\mathcal{O}_E, \Xi_A, \mathcal{D})$ *into an unbounded Kasparov module representing the class of the Toeplitz extension* $[\mathrm{ext}] \in KK^1(\mathcal{O}_E, A)$.

The number operator c and the depth-kore operator κ are both canonically constructed from E. There is however some freedom when choosing ψ. This is discussed further in Remarks 3.16 and 3.20. The cycle in Theorem 1 recovers the well-known number operator construction when E is a self-Morita equivalence bimodule or SMEB (see Proposition 3.24 in §3.5.1), as well as the construction for Cuntz–Krieger algebras in [**12**], viewed as Cuntz–Pimsner algebras over the maximal abelian subalgebra coming from a subshift of finite type (§3.5.2). Theorem 1 sheds new light on some of the results obtained in [**12**].

An application of Theorem 1 is the following construction of a spectral triple for the Cuntz–Pimsner algebra of a vector bundle. Such C^*-algebras were previously considered in [**8**, **24**]. Let $V \to M$ be a complex vector bundle on a Riemannian manifold M and $\alpha : C(M) \to C(M)$ a $*$-automorphism induced from an isometric C^1-diffeomorphism. We define ${}_\alpha E := \Gamma(M, V)$ with the ordinary right $C(M)$-action and the left $C(M)$-action defined from α and denote the associated Cuntz–Pimsner algebra by $\mathcal{O}_{\alpha E}$.

Consider a Dirac-type operator $\not{D}$ on a Clifford bundle $S \to M$ and the Hilbert space $\mathcal{H} := {}_\alpha\Xi_{C(M)} \otimes_{C(M)} L^2(M, S)$. The operator $\mathcal{D}$ appearing in Theorem 1 and the Dirac-type operator $\not{D}$ can be assembled to form a self-adjoint operator $\mathcal{D}_E$ on $\mathcal{H}$. For more details regarding the construction, see §4, in particular Lemma 4.2. The following result appears as Theorem 4.3 below.

THEOREM 2. *The triple* $(\mathcal{O}_{\alpha E}, \mathcal{H}, \mathcal{D}_E)$ *is a spectral triple for the Cuntz–Pimsner algebra* $\mathcal{O}_{\alpha E}$ *representing the Kasparov product of the class of*

$$0 \to \mathcal{K}_{C(M)}(\mathcal{F}_{\alpha E}) \to \mathcal{T}_{\alpha E} \to \mathcal{O}_{\alpha E} \to 0$$

in $KK^1(\mathcal{O}_{\alpha E}, C(M))$ *with* $[\not{D}] \in KK^*(C(M), \mathbb{C})$.

In fact, the theorem remains true for ‘almost isometries’, namely C^1-diffeomorphisms inducing automorphisms α such that $\sup_k \|[\not{D}, \alpha^k(f)]\| < \infty$ for each $f \in C^1(M)$: see Proposition 4.8. Spectral triples on the crossed product $C(M) \rtimes \mathbb{Z}$ of an equicontinuous action, as studied in [**3**], arise as a special case.

The contents of the paper are as follows. In §2, we recall the setup of [**22**]. In particular, we recall the construction of the operator-valued weight $\Phi_\infty : \mathcal{O}_E \to A$ used to define the module Ξ_A of Theorem 1. After recalling the motivating example of Cuntz–Krieger algebras (from [**12**]) in §3.1, we proceed in §3 to construct the orthogonal decomposition of Ξ_A (§3.2), the depth-kore operator κ and the unbounded cycle (§3.4) appearing in Theorem 1.

In §3.5, we provide examples in the form of the above-mentioned SMEBs and Cuntz–Krieger algebras. For the latter we use the construction of the Cuntz–Pimsner algebra of a local homeomorphism from [**9**]. This clarifies some K-theoretic statements proved in [**12**]. In §3.5.3, we compare the dynamical approach for the Cuntz algebra O_N with the model using the coefficient algebra $\mathbb{C}$ (the graph C^*-algebra approach). Finally, in §4, we prove Theorem 2.

2. *The Kasparov module representing the extension class*

In this section, we will recall the basic setup of [**22**]. We have a unital separable C^*-algebra A, and a bi-Hilbertian bimodule E over A which is finitely generated and projective for both the right and left module structures. This means that E is a bimodule over A, carries both left and right A-valued inner products ${}_A(\cdot|\cdot)$, $(\cdot|\cdot)_A$ for each of which E is full, and for which the respective actions are injective and adjointable. The two inner products automatically yield equivalent norms (see, for instance, [**22**, Lemma 2.2]). We write ${}_AE$ for E when we wish to emphasize its left module structure and E_A for E when emphasizing the right module structure.

2.1. *Cuntz–Pimsner algebras.*

Regarding E as a right module with a left A-action (a correspondence), we can construct the Cuntz–Pimsner algebra $\mathcal{O}_E$. This we do concretely in the Fock representation. The algebraic Fock space is the algebraic direct sum

$$\mathcal{F}_E^{\mathrm{alg}} = \bigoplus_{k\geq 0}^{\mathrm{alg}} E^{\otimes_A k} = \bigoplus_{k\geq 0}^{\mathrm{alg}} E^{\otimes k} = A \oplus E \oplus E^{\otimes 2} \oplus \cdots ,$$

where the copy of A is the trivial A-correspondence. The Fock space $\mathcal{F}_E$ is the completion of $\mathcal{F}_E^{\mathrm{alg}}$ as an A-Hilbert module. For $\nu \in \mathcal{F}_E^{\mathrm{alg}}$, we define the creation operator T_ν by the formula

$$T_\nu(e_1 \otimes \cdots \otimes e_k) = \nu \otimes e_1 \otimes \cdots \otimes e_k.$$

The expression T_ν extends to an adjointable operator on $\mathcal{F}_E$. The C^*-algebra generated by the set of creation operators $\{T_\nu : \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$ is the Toeplitz–Pimsner algebra $\mathcal{T}_E$. It is straightforward to show that $\mathcal{T}_E$ contains the compact endomorphisms $\mathcal{K}_A(\mathcal{F}_E)$ as an ideal. The defining extension for the Cuntz–Pimsner algebra $\mathcal{O}_E$ is the following short exact sequence:

$$0 \to \mathcal{K}_A(\mathcal{F}_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0. \tag{2.1}$$

For $\nu \in \mathcal{F}_E^{\mathrm{alg}}$, we let S_ν denote the class of T_ν in $\mathcal{O}_E$. If $\nu \in E^{\otimes k}$, we write $|\nu| := k$. We note that Pimsner's general construction [**19**] uses an ideal that in general is larger than $\mathcal{K}_A(\mathcal{F}_E)$. In our case, A acts from the left on E_A by compact endomorphisms, ensuring that Pimsner's ideal coincides with $\mathcal{K}_A(\mathcal{F}_E)$.

Remark 2.1. The Fock module $\mathcal{F}_E$ is not to be confused with the Fock space defined in the context of Cuntz–Krieger algebras and used by Kaminker–Putnam [**14**]. The constructions in [**14**] are related to KK-theoretic duality, whereas our aim is to represent a specific extension class by an unbounded cycle.

Remark 2.2. The formula $z \cdot S_\nu S_\mu^* := z^{|\nu|-|\mu|} S_\nu S_\mu^*$ extends to a $U(1)$-action on $\mathcal{O}_E$ [**19**]. We denote the fixed point algebra for this action by $\mathcal{C}_E$. The formula $\rho(x) := \int_{U(1)} z \cdot x \, \mathrm{d}z$ (where $\mathrm{d}z$ denotes the normalized Haar measure on $U(1)$) defines a conditional expectation $\rho : \mathcal{O}_E \to \mathcal{C}_E$. The generator of the circle action defines a closed operator N on the completion $X_{\mathcal{C}_E}$ of $\mathcal{O}_E$ as a $\mathcal{C}_E$-Hilbert module in the inner product defined from ρ. Under the spectral subspace assumption (see [**4**, Definition 2.2]), N is a self-adjoint, regular operator with compact resolvent whose commutators with $\{S_\nu : \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$ are bounded. In particular, $(\mathcal{O}_E, X_{\mathcal{C}_E}, N)$ defines an unbounded $(\mathcal{O}_E, \mathcal{C}_E)$-Kasparov module.

There is an equality $\mathcal{C}_E = A$ if and only if E can be given a left inner product making it an SMEB [**16**, Proposition 5.18]. SMEBs are considered further in Example 2.12. This case has been studied in [**22**] as well as in [**11**]. In general, $\mathcal{C}_E$ is substantially larger than A and the generator of the circle action is insufficient for constructing an unbounded $(\mathcal{O}_E, A)$-Kasparov module.

Example 2.3. (Local homeomorphisms) Let $g : V \to V$ be a local homeomorphism of a compact space V. Associated with g, there is a transfer operator

$$\mathfrak{L} : C(V) \to C(V), \quad \mathfrak{L}(f)(x) := \sum_{g(y)=x} f(y).$$

We can define a bimodule structure $E = {}_{\mathrm{id}}C(V)_{g^*}$ on $C(V)$ by

$$(afb)(x) = a(x) f(x) b(g(x)), \quad a, b \in C(V),\ f \in E.$$

The two inner products on E are given by

$$(f_1|f_2)_{C(V)} := \mathfrak{L}(\overline{f_1} f_2) \quad \text{and} \quad {}_{C(V)}(f_1|f_2) = f_1 \overline{f_2}.$$

For more details, see [**9**]. As a source of examples, we will mainly be concerned with a special case: the shift mapping on a subshift of finite type. The reason for this is that the associated Cuntz–Pimsner algebra is a Cuntz–Krieger algebra, and as such it also admits a model as a Cuntz–Pimsner algebra over a finite-dimensional C^*-algebra. This will allow us to compare and contrast our techniques relative to the choice of Cuntz–Pimsner model rather than the isomorphism class of the Cuntz–Pimsner algebra.

Example 2.4. (Graph C^*-algebras) Let $G = (G^0, G^1, r, d)$ be a finite directed graph. We consider the finite-dimensional algebra $A = C(G^0)$ and the A-bimodule $E = C(G^1)$ with the bimodule structure

$$(afb)(g) = a(r(g)) f(g) b(d(g)), \quad a, b \in A,\ f \in E.$$

For $e, f \in E$, the inner products are defined by

$$(e|f)_A(v) := \sum_{s(g)=v} \overline{e(g)} f(g) \quad \text{and} \quad {}_A(e|f)(v) := \sum_{r(g)=v} e(g) \overline{f(g)}.$$

The associated Cuntz–Pimsner algebra coincides with the graph C^*-algebra $C^*(G)$; see [**19**, Example 2, p. 193].

Example 2.5. (Cuntz–Krieger algebras) Assume that $\boldsymbol{A} := (a_{ij})_{i,j=1}^N$ is an $N \times N$ matrix of 0's and 1's. We let $O_{\boldsymbol{A}}$ denote the associated Cuntz–Krieger algebra [**7**]. If $\boldsymbol{A}$ is the edge adjacency matrix of a finite directed graph G, then $O_{\boldsymbol{A}} \cong C^*(G)$. On the other hand, letting $(\Omega_{\boldsymbol{A}}, \sigma)$ denote the associated one-sided subshift of finite type, $O_{\boldsymbol{A}}$ coincides with the Cuntz–Pimsner algebra associated with the local homeomorphism σ as in Example 2.3. Yet another description is in terms of groupoids; $O_{\boldsymbol{A}}$ is isomorphic to the groupoid C^*-algebra of the groupoid

$$\mathcal{R}_{\boldsymbol{A}} := \{(x, n, y) \in \Omega_{\boldsymbol{A}} \times \mathbb{Z} \times \Omega_{\boldsymbol{A}} : \exists k \geq \max\{0, -n\} \text{ with } \sigma^{n+k}(x) = \sigma^k(y)\} \rightrightarrows \Omega_{\boldsymbol{A}}. \tag{2.2}$$

That is, $\mathcal{R}_{\boldsymbol{A}}$ consists of shift–tail equivalent pairs of points with a prescribed lag. The set $\mathcal{R}_{\boldsymbol{A}}$ becomes a groupoid for the operation $(x, n, y)(y, m, z) = (x, n + m, z)$ and can be equipped with an étale topology (see [**20**, **21**]).

A Cuntz–Krieger algebra $O_{\boldsymbol{A}}$ also has a graph C^*-algebra model. However, we use the convention that whenever referring to a Cuntz–Krieger algebra as a Cuntz–Pimsner algebra we mean its model over $C(\Omega_{\boldsymbol{A}})$. We distinguish it from its Cuntz–Pimsner model as a graph C^*-algebra.

2.2. *The conditional expectation.* A Kasparov module representing the class of the extension (2.1) was constructed in [**22**]. Here we recall the salient points. For x and y in a right Hilbert module, we denote the associated rank-one operator by $\Theta_{x,y} := x\langle y, \cdot\rangle$. We choose a frame $(e_\rho)_{\rho=1}^N$ for E_A. By a frame, we mean

$$\sum_{\rho=1}^{N} \Theta_{e_\rho, e_\rho} = \mathrm{Id}_E.$$

The frame $(e_\rho)_{\rho=1}^N$ induces a frame for $E_A^{\otimes k}$, namely $(e_\rho)_{|\rho|=k}$, where ρ is a multi-index and $e_\rho = e_{\rho_1} \otimes \cdots \otimes e_{\rho_k}$.

We use ideas from [**13**] to define an A-bilinear functional $\Phi_\infty : \mathcal{O}_E \to A$. The details of this construction can be found in [**22**, §3.2]. This functional will furnish us with an A-valued inner product on $\mathcal{O}_E$. We define

$$\Phi_k : \mathrm{End}_A^*(E^{\otimes k}) \to A, \quad \Phi_k(T) = \sum_{|\rho|=k} {}_A(Te_\rho|e_\rho).$$

Here we use the notation $\mathrm{End}_A^*(E^{\otimes k})$ for the C^*-algebra of A-linear adjointable operators on $E^{\otimes k}$. It follows from [**13**, Lemma 2.16] that Φ_k does not depend on the choice of frame. We write $\mathrm{e}^{\beta_k} := \Phi_k(\mathrm{Id}_{E^{\otimes k}})$. Since Φ_k is independent of the choice of frame, so is e^{β_k}. Note that e^{β_k} is a positive, central, invertible element of A. Therefore, β_k is a well-defined self-adjoint central element in A. We extend the functional Φ_k to a mapping $\mathrm{End}_A^*(\mathcal{F}_E) \to A$ by compressing along the orthogonally complemented submodule $E^{\otimes k} \subseteq \mathcal{F}_E$.

Naively, we would like to define

$$\Phi_\infty(T) \text{ ``:=''} \lim_{k\to\infty} \Phi_k(T)\mathrm{e}^{-\beta_k} \quad \text{for suitable } T \in \mathrm{End}_A^*(\mathcal{F}_E). \tag{2.3}$$

Indeed, $\Phi_k(T)\mathrm{e}^{-\beta_k}$ is easily shown to be bounded and some 'generalized limit' might exist. In general, we cannot employ the theory of generalized limits as Φ_∞ is not scalar valued. Following [**22**], we work under the following assumption guaranteeing that the limit exists for T in a dense subspace of $\mathcal{T}_E$.

Assumption 1. We assume that for every $k \in \mathbb{N}$, there is a $\delta > 0$ such that whenever $\nu \in E^{\otimes k}$ there exists a $\tilde{\nu} \in E^{\otimes k}$ satisfying

$$\|\mathrm{e}^{-\beta_n} \nu \mathrm{e}^{\beta_{n-k}} - \tilde{\nu}\| = O(n^{-\delta}) \quad \text{as } n \to \infty.$$

Example 2.6. (Assumption 1 and graph C^*-algebras) Assumption 1 is non-trivial for graph C^*-algebras. It was verified in [**22**, Example 3.8] that a graph C^*-algebra with primitive vertex adjacency matrix satisfies Assumption 1. The Jones–Watatani indices of a graph C^*-algebra were computed in [**22**, Example 3.8] by means of its vertex adjacency matrix $\boldsymbol{A}_v$ as

$$\mathrm{e}^{\beta_k} = \sum_{v,w \in G^0} \boldsymbol{A}_v^k(v, w)\delta_v \in A = C(G^0).$$

If G is the graph with N edges on one vertex, $C^*(G) = O_N$ and $\mathrm{e}^{\beta_k} = N^k$. It is an open problem to determine if all graph C^*-algebras satisfy Assumption 1.

Example 2.7. (Assumption 1 for Cuntz–Krieger algebras) Let us verify Assumption 1 for Cuntz–Krieger algebras in the Cuntz–Pimsner model over $C(\Omega_{\boldsymbol{A}})$ for an $N \times N$ matrix $\boldsymbol{A}$. We will choose a frame for $E = {}_{\mathrm{id}}C(\Omega_{\boldsymbol{A}}))_{\sigma^*}$ as follows. A left frame is given by the constant function $1 \in E$. To construct a right frame, choose a covering $(U_j)_{j=1}^M$ such that $\sigma| : U_j \to \sigma(U_j)$ is a homeomorphism. We also pick a subordinate partition of unity $(\chi_j^2)_{j=1}^M$ (i.e. $\mathrm{supp}(\chi_j) \subseteq U_j$ and $\sum_j \chi_j^2 = 1$). For instance, with $M = N$ the cylinder sets

$$U_j := \{x = x_0 x_1 x_2 \cdots \in \Omega_{\boldsymbol{A}} : x_0 = j\}$$

will form a clopen cover and $\chi_j^2 := \chi_{U_j}$ is a subordinate partition of unity. We claim that $e_j := \chi_j$ defines a right frame. Indeed, for any $f \in E$, the following identity holds:

$$\begin{aligned}\Big[\sum_j \chi_j (\chi_j | f)_{C(\Omega_{\boldsymbol{A}})}\Big](x) &= \sum_j \chi_j(x)\mathfrak{L}(\chi_j f)(\sigma(x)) = \sum_j \sum_{\sigma(y)=\sigma(x)} \chi_j(x)\chi_j(y) f(y)\\ &= \sum_j \chi_j^2(x) f(x) = f(x),\end{aligned}$$

where in the second last step we used the fact that σ is injective on U_j. For a multi-index ρ of length r, we use the notation

$$\chi_\rho(x) := \prod_{j=1}^r \chi_{\rho_j}(\sigma^{j-1}(x)).$$

A simple computation gives

$$\mathrm{e}^{\beta_k} = \sum_{|\rho|=k} \chi_\rho^2 = \prod_{j=1}^k \Big(\sum_{k=1}^M \chi_k^2(\sigma^{j-1}(x))\Big) = 1.$$

Therefore, $\beta_k = 0$ for Cuntz–Krieger algebras and Assumption 1 is satisfied. We remark at this point that the Jones–Watatani index is associated to the module and not the Cuntz–Pimsner algebra constructed from the module. For a Cuntz–Krieger algebra, we see enormous differences between the model over $C(\Omega_{\boldsymbol{A}})$ and the model as a graph C^*-algebra.

We assume that Assumption 1 holds for the remainder of the paper.

In [**22**], the reader can find further examples of Cuntz–Pimsner algebras for which Assumption 1 holds. There are no known examples for which Assumption 1 does not hold. When Assumption 1 holds, [**22**, Proposition 3.5] guarantees that the expression in equation (2.3) is well defined on the $*$-algebra generated by the set of creation operators $\{T_\nu : \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$. Indeed, we can under Assumption 1 compute Φ_∞ on the $*$-algebra generated by $\{T_\nu : \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$.

LEMMA 2.8. *For homogeneous elements $\mu,\ \nu \in \mathcal{F}_E^{\mathrm{alg}}$, we have*

$$\Phi_\infty(T_\mu T_\nu^*) = \lim_{k\to\infty} {}_A(\mu|\mathrm{e}^{-\beta_k}\nu\mathrm{e}^{\beta_{k-|\nu|}}). \tag{2.4}$$

In particular, if T is homogeneous of degree n, $|n| > 0$, then $\Phi_\infty(T) = 0$.

Proof. It is proved in [**22**, Lemma 3.2] that for homogeneous $\mu,\ \nu \in \mathcal{F}_E^{\mathrm{alg}}$, we have

$$\Phi_k(T_\mu T_\nu^*) = {}_A(\mu|\nu\mathrm{e}^{\beta_{k-|\nu|}})$$

whenever $k \geq |\mu| = |\nu|$. Therefore, assuming Assumption 1, we have

$$\Phi_\infty(T_\mu T_\nu^*) = \lim_{k\to\infty} \Phi_k(T_\mu T_\nu^*)\mathrm{e}^{-\beta_k} = \lim_{k\to\infty} {}_A(\mu|\nu\mathrm{e}^{\beta_{k-|\nu|}})\mathrm{e}^{-\beta_k} = \lim_{k\to\infty} {}_A(\mu|\mathrm{e}^{-\beta_k}\nu\mathrm{e}^{\beta_{k-|\nu|}}).$$

Now if T is of degree n, $|n| > 0$, then T is a linear combination of elements of the form $T_\mu T_\nu^*$ with $|\mu| \neq |\nu|$ and therefore ${}_A(\mu|\mathrm{e}^{-\beta_k}\nu\mathrm{e}^{\beta_{k-|\nu|}}) = 0$ for all k, giving the desired statement. □

By a positivity argument, the mapping Φ_∞ is continuous in the C^*-norm on the $*$-algebra generated by $\{T_\nu : \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$. We extend by continuity to obtain a unital positive A-bilinear functional $\Phi_\infty : \mathcal{T}_E \to A$. The functional Φ_∞ annihilates the compact endomorphisms, and descends to a well-defined functional on the Cuntz–Pimsner algebra $\mathcal{O}_E$. By an abuse of notation, we also denote this functional by $\Phi_\infty : \mathcal{O}_E \to A$. The reader is referred to [**22**], and in particular [**22**, Proposition 3.5], for further details on Φ_∞. Since Φ_k and e^{β_k} do not depend on the choice of frame, neither does Φ_∞.

In examples, the conditional expectation is computable. For instance, it was proven in [**22**, Example 3.6] that for the graph G with N edges on one vertex, so $A = \mathbb{C}$ and $E = \mathbb{C}^N$, with $C^*(G) = O_E$ being the Cuntz algebra O_N, Φ_∞ coincides with the unique KMS state for the gauge action on O_N, so

$$\Phi_\infty(S_\mu S_\nu^*) = \delta_{\mu,\nu} N^{-|\mu|}. \tag{2.5}$$

For a Cuntz–Krieger algebra, we can also compute Φ_∞.

Convention. Given a simple tensor $\nu \in \mathcal{F}_E^{\mathrm{alg}}$, with $\nu = \nu_1 \otimes \nu_2 \otimes \cdots \otimes \nu_k$, we will write $\nu = \underline{\nu}\overline{\nu}$ with $\underline{\nu} = \nu_1 \otimes \cdots \otimes \nu_m$ and $\overline{\nu} = \nu_{m+1} \otimes \cdots \otimes \nu_k$, where $m \leq k$ will either be clear from context or specified.

LEMMA 2.9. *Let $\boldsymbol{A}$ denote an $N \times N$ matrix of 0's and 1's, $\mathcal{R}_{\boldsymbol{A}}$ the associated groupoid as in equation (2.2) and $\Phi_\infty : C^*(\mathcal{R}_{\boldsymbol{A}}) \to C(\Omega_{\boldsymbol{A}})$ the conditional expectation associated with the Cuntz–Pimsner model. For $f \in C_c(\mathcal{R}_{\boldsymbol{A}})$, we have*

$$\Phi_\infty(f)(x) = f(x, 0, x).$$

Proof. It suffices to prove that for $f_{\mu,\nu} \in C_c(\mathcal{R}_{\boldsymbol{A}})$ defined from an element $S_\mu S_\nu^*$, where $\mu, \nu \in E^{\otimes l}$, the identity $\Phi_k(S_\mu S_\nu^*)(x) = f_{\mu,\nu}(x, 0, x)$ holds whenever $k > l$. We note that for general homogeneous $\mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}$,

$$f_{\mu,\nu}(x, n, y) = \begin{cases} \mu(x)\nu^*(y), & \text{if } n = |\mu| - |\nu| \text{ and } \sigma^{|\mu|}(x) = \sigma^{|\nu|}(y), \\ 0 & \text{otherwise.} \end{cases}$$

Here we are using the fact that $E^{\otimes k} \cong C(\Omega_{\boldsymbol{A}})$ as linear spaces for any k to identify μ and ν with functions. We denote the conjugate function by ν^* to avoid notational ambiguity later. Let $(e_j)_{j=1}^M$ denote the frame from Example 2.7, associated with a partition of unity subordinate to the cover $(U_j)_{j=1}^M$. Note that for $\rho = (\rho_1, \rho_2, \ldots, \rho_r)$, we identify e_ρ with the function

$$e_\rho(x) := \chi_\rho(x).$$

We write

$$\Phi_k(S_\mu S_\nu^*) = \sum_{|\rho|=k} {}_{C(\Omega_{\boldsymbol{A}})}(\mu \otimes (\nu|e_{\underline{\rho}})_{C(\Omega_{\boldsymbol{A}})} e_{\overline{\rho}}|e_\rho),$$

where $|\underline{\rho}| = l = |\nu|$. After some short computations, we see that

$$\begin{aligned}
&\left[\sum_{|\rho|=k} {}_{C(\Omega_{\boldsymbol{A}})}(\mu \otimes (\nu|e_{\underline{\rho}})_{C(\Omega_{\boldsymbol{A}})} e_{\overline{\rho}}|e_\rho)\right](x) \\
&\quad = \sum_{|\rho|=k} \mu(x)\mathfrak{L}^l[\nu^*\chi_{\underline{\rho}}](\sigma^l(x))\chi_{\overline{\rho}}(\sigma^l(x))\chi_\rho(x) \\
&\quad = \sum_{|\rho|=k} \mu(x)\mathfrak{L}^l[\nu^*\chi_{\underline{\rho}}](\sigma^l(x))\chi_{\overline{\rho}}^2(\sigma^l(x))\chi_{\underline{\rho}}(x) \\
&\quad = \sum_{|\rho|=k} \sum_{\sigma^l(x)=\sigma^l(y)} \mu(x)\nu^*(y)\chi_{\underline{\rho}}(y)\chi_{\overline{\rho}}^2(\sigma^l(x))\chi_{\underline{\rho}}(x) \\
&\quad = \sum_{|\rho|=k} \mu(x)\nu^*(x)\chi_\rho^2(x) = \mu(x)\nu^*(x) = f_{\mu,\nu}(x, 0, x).
\end{aligned}$$

We used the injectivity of σ on U_j in the third equality. □

2.3. *A bounded Kasparov module for* [ext]. We equip $\mathcal{O}_E$ with the A-valued inner product

$$(S_1|S_2)_A := \Phi_\infty(S_1^* S_2), \qquad S_1, S_2 \in \mathcal{O}_E.$$

Completing $\mathcal{O}_E$ modulo the vectors of zero length (with respect to Φ_∞) yields a right A-Hilbert C^*-module that we denote by Ξ_A. The module Ξ_A carries a left action of $\mathcal{O}_E$ given by extending the multiplication action of $\mathcal{O}_E$ on itself.

Example 2.10. For a Cuntz–Krieger algebra defined from the $N \times N$ matrix $\boldsymbol{A}$, $\Xi_{C(\Omega_{\boldsymbol{A}})}$ coincides with the left regular representation $L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$ of the groupoid model $O_{\boldsymbol{A}} \cong C^*(\mathcal{R}_{\boldsymbol{A}})$ by Lemma 2.9.

By considering the linear span of the image of the generators S_ν, $\nu \in \mathcal{F}_E^{\mathrm{alg}}$, inside the module Ξ_A, we obtain an isometrically embedded copy of the Fock space $\mathcal{F}_E$. This fact follows from the identity $S_\mu^* S_\nu = (\mu|\nu)_A$ in O_E. We let Q be the projection on this copy of the Fock space.

THEOREM 2.11. [**22**, Proposition 3.14] *The tuple* $(\mathcal{O}_E, \Xi_A, 2Q-1)$ *is an odd Kasparov module representing the class of the extension* [ext] *defined in (2.1).*

Example 2.12. A self-Morita equivalence bimodule (SMEB) E is a bimodule over A as above whose left and right inner products satisfy the compatibility condition

$$\mu(\xi|\eta)_A = {}_A(\mu|\xi)\eta \quad \text{for all } \mu, \xi, \eta \in E.$$

Equivalently, E is equipped with a right inner product and there is an isomorphism $A \cong \mathcal{K}_A(E)$ defining the left inner product. In particular, E defines a Morita equivalence $A \sim_M A$. When E is a SMEB, $\Phi_\infty : \mathcal{O}_E \to A$ coincides with the expectation $\rho : \mathcal{O}_E \to \mathcal{C}_E$ discussed in Remark 2.2. Therefore,

$$\Xi_A = \bigoplus_{n\in\mathbb{Z}} E^{\otimes n},$$

where $E^{\otimes(-|n|)} = \overline{E}^{\otimes|n|}$. In general the module Ξ_A is more complicated, and this fact will be captured by the depth-kore operator κ (see below in §3.4).

For $\mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}$, we denote the image of the generator $S_\mu S_\nu^* \in \mathcal{O}_E$ in the module Ξ_A by $[S_\mu S_\nu^*] = W_{\mu,\nu}$. Denote by Ξ_A^0 the completion of the fixed point algebra $\mathcal{C}_E$ in the inner product defined by the restriction of Φ_∞. For $n \in \mathbb{Z}$, we let Ξ_A^n denote the closed linear span of $\{W_{\mu,\nu} : |\mu| - |\nu| = n\}$ inside Ξ_A.

LEMMA 2.13. *Recall the unbounded Kasparov module* $(\mathcal{O}_E, X_{\mathcal{C}_E}, N)$ *from Remark 2.2. The right A-module Ξ_A decomposes as a tensor product*

$$\Xi_A \cong X_{\mathcal{C}_E} \otimes_{\mathcal{C}_E} \Xi_A^0.$$

Consequently, for $z \in U(1)$, the prescription $U_z W_{\mu,\nu} := z^{|\mu|-|\nu|} W_{\mu,\nu}$ defines a $U(1)$-action on Ξ_A. The associated projections $\Psi_n : \Xi_A \to \Xi_A^n$ onto the spectral subspaces are adjointable and there is a direct sum decomposition

$$\Xi_A \cong \bigoplus_{n\in\mathbb{Z}} \Xi_A^n.$$

Proof. Because the multiplication map $\mathcal{O}_E \otimes^{\mathrm{alg}} \mathcal{C}_E \to \mathcal{O}_E$ has dense range in $\mathcal{O}_E$, we only have to verify that the inner products coincide under this map. This follows by computing, for $|\mu_i| = |\nu_i|$, $i = 1, 2$,

$$\begin{aligned}
&\langle S_{\alpha_1} S_{\beta_1}^* \otimes S_{\mu_1} S_{\nu_1}^*, S_{\alpha_2} S_{\beta_2}^* \otimes S_{\mu_2} S_{\nu_2}^* \rangle_{\mathcal{O}_E^\rho \otimes_{\mathcal{C}_E} \Xi_E^0} \\
&\quad = \Phi_\infty(S_{\nu_1} S_{\mu_1}^* \rho(S_{\beta_1} S_{\alpha_1}^* S_{\alpha_2} S_{\beta_2}^*) S_{\mu_2} S_{\nu_2}^*) \\
&\quad = \delta_{|\alpha_1|-|\beta_1|,|\alpha_2|-|\beta_2|} \Phi_\infty(S_{\nu_1} S_{\mu_1}^* S_{\beta_1} S_{\alpha_1}^* S_{\alpha_2} S_{\beta_2}^* S_{\mu_2} S_{\nu_2}^*) \\
&\quad = (S_{\alpha_1} S_{\beta_1}^* S_{\mu_1} S_{\nu_1}^* | S_{\alpha_2} S_{\beta_2}^* S_{\mu_2} S_{\nu_2}^*)_A,
\end{aligned}$$

by Lemma 2.8. The statements on the $U(1)$-action and adjointability of the projections Ψ_n now follow immediately. $\square$

3. *An unbounded representative of the extension class*

In this section, we will use ideas from [**12**] to define an unbounded operator on the module Ξ_A. The issues of self-adjointness and regularity will be rendered trivial by defining our operator in diagonal form. This relies on having an orthogonal decomposition of our module into finitely generated projective submodules.

3.1. *Brief review of the construction for Cuntz–Krieger algebras.* Before going into the general construction, let us briefly recall how the orthogonal decomposition into finitely generated projective submodules is constructed for Cuntz–Krieger algebras. This example is explained in detail in [**12**]. The precise relation to the general construction appearing below in §3.2 can be found in §3.3. The Cuntz–Krieger algebras are Cuntz–Pimsner algebras, but the structure in which said decomposition becomes more transparent is in the picture using the shift–tail equivalence groupoid $\mathcal{R}_{\boldsymbol{A}}$, defined in equation (2.2). To decompose the left regular representation $L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$ into finitely generated projective submodules, both of the parameters n and k have to be taken into account. For $k \geq \max\{0, -n\}$, we define the compact sets

$$\begin{aligned}\mathcal{R}_{\boldsymbol{A}}^{n,k} := \{(x, n, y) \in \mathcal{R}_{\boldsymbol{A}} : \sigma^{n+k}(x) = \sigma^k(y) \text{ and}\\ k = \max\{0, -n\} \text{ or } \sigma^{n+k-1}(x) \neq \sigma^{k-1}(y)\}.\end{aligned} \tag{3.1}$$

The modules $C(\mathcal{R}_{\boldsymbol{A}}^{n,k})$ are finitely generated projective $C(\Omega_{\boldsymbol{A}})$-modules, and $\bigoplus_{n,k} C(\mathcal{R}_{\boldsymbol{A}}^{n,k}) \subseteq C_c(\mathcal{R}_{\boldsymbol{A}})$ gives a dense $C(\Omega_{\boldsymbol{A}})$-submodule of $O_{\boldsymbol{A}}$. These modules are orthogonal for the canonical $C(\Omega_{\boldsymbol{A}})$-valued inner product on $O_{\boldsymbol{A}}$ (for support reasons). The depth-kore operator κ we seek should mimic the multiplication operator by the function $\kappa_{\boldsymbol{A}} \in C(\mathcal{R}_{\boldsymbol{A}})$ defined by

$$\kappa_{\boldsymbol{A}}(x, n, y) := \min\{k \geq \max\{0, -n\} : \sigma^{n+k}(x) = \sigma^k(y)\}. \tag{3.2}$$

The function $\kappa_{\boldsymbol{A}}$ supplements the cocycle c defined by $c(x, n, y) = n$ to provide the orthogonal decomposition of $L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$ via $\mathcal{R}_{\boldsymbol{A}}^{n,k} = c^{-1}(n) \cap \kappa_{\boldsymbol{A}}^{-1}(k)$. We now turn to the case of more general Cuntz–Pimsner algebras and return to this example in §3.3.

3.2. *An orthogonal decomposition.* To construct a self-adjoint regular operator, we first analyse the structure of the module Ξ_A. More precisely, we will construct a densely defined operator κ that tames the wild structure of $\mathcal{F}_E^{\perp} \subset \Xi_A$.

Remark 3.1. Further motivation for the construction below can be found when comparing to the SMEB case (cf. [**22**, Theorem 3.1] and Example 2.12). The negative spectral subspace of Ξ_A for the number operator in the SMEB case is given by the direct sum of all powers of $\overline{E}$. The right module structure on $\overline{E}$ comes from the left module structure on E. For a SMEB, the change of module structure from $E^{\otimes k}$ to $\overline{E}^{\otimes k}$ is harmless, as the left and right module structures are closely related. In the general case, the two module structures are in principle (and in practice) quite different. Therefore, when mapping powers of $\overline{E}$ into Ξ_A by $\bar{e} \to S_e^*$, orthogonality is not preserved and no isometric property holds.

To construct κ, we will add an additional assumption regarding the fine structure of the operation $\nu \mapsto \tilde{\nu}$ in Assumption 1. Under Assumption 1, we can define the operator

$\mathfrak{q}_k : E^{\otimes k} \to E^{\otimes k}$ by

$$\mathfrak{q}_k \nu := \tilde{\nu} = \lim_{n\to\infty} \mathrm{e}^{-\beta_n} \nu \mathrm{e}^{\beta_{n-k}}.$$

The map $Z(A) \otimes Z(A^{\mathrm{op}}) \to \mathrm{End}^*_A(E_A^{\otimes k}) \cap \mathrm{End}^*_A({}_A E^{\otimes k})$ defined by $(a_1 \otimes a_2^{\mathrm{op}})e := a_1 e a_2$ is an injective $*$-homomorphism into the algebra of left and right adjointable operators on $E^{\otimes k}$. In view of this, the definition of $\mathfrak{q}_k$ immediately yields the following.

LEMMA 3.2. *The operator $\mathfrak{q}_k : E^{\otimes k} \to E^{\otimes k}$ does not depend on the choice of frame. Moreover, $\mathfrak{q}_k$ is adjointable and positive with respect to both left and right inner products, and in particular $\mathfrak{q}_k \in Z(A) \otimes Z(A^{\mathrm{op}}) \subset \mathrm{End}^*_A(E_A^{\otimes k}) \cap \mathrm{End}^*_A({}_A E^{\otimes k})$.*

Proof. The operator Φ_k and the element $\mathrm{e}^{\beta_k} = \Phi_k(1)$ are independent of choice of frame, so therefore $\mathfrak{q}_k$ is independent of choice of frame. The fact that $\mathfrak{q}_k$ is adjointable follows from the adjointability of the left and right actions of A, and the centrality of e^{β_n}. For instance, when $\mu, \nu \in E^{\otimes k}$,

$$\begin{aligned} {}_A(\nu|\mathfrak{q}_k\mu) &= \lim_{n\to\infty} {}_A(\nu|\mathrm{e}^{-\beta_n}\mu \mathrm{e}^{\beta_{n-k}}) = \lim_{n\to\infty} {}_A(\nu \mathrm{e}^{\beta_{n-k}}|\mu)\mathrm{e}^{-\beta_n} \\ &= \lim_{n\to\infty} \mathrm{e}^{-\beta_n} {}_A(\nu \mathrm{e}^{\beta_{n-k}}|\mu) = \lim_{n\to\infty} {}_A(\mathrm{e}^{-\beta_n}\nu \mathrm{e}^{\beta_{n-k}}|\mu) \\ &= {}_A(\mathfrak{q}_k\nu|\mu), \end{aligned}$$

and the proof for the right inner product is similar. That $\mathfrak{q}_k \in Z(A) \otimes Z(A^{\mathrm{op}})$ and is positive is because it is the limit of positive operators in $Z(A) \otimes Z(A^{\mathrm{op}})$. □

In addition to the fact that $\mathfrak{q}_k$ is a bimodule morphism, the $\mathfrak{q}_k$ are multiplicative in the following sense. For $\mu \in E^{\otimes m}$, $\nu \in E^{\otimes k}$,

$$\begin{aligned} \mathfrak{q}_{m+k}(\nu \otimes \mu) &= \lim_{n\to\infty} \mathrm{e}^{-\beta_n} \nu \otimes \mu \mathrm{e}^{\beta_{n-m-k}} \\ &= \lim_{n\to\infty} \mathrm{e}^{-\beta_n} \nu \mathrm{e}^{\beta_{n-k}} \otimes \mathrm{e}^{-\beta_{n-k}} \mu \mathrm{e}^{\beta_{n-k-m}} = \mathfrak{q}_k(\nu) \otimes \mathfrak{q}_m(\mu). \end{aligned} \tag{3.3}$$

The conditional expectation Φ_k applied to $\mathfrak{q}_k$ in the tensor power $E^{\otimes k}$ can be computed to be 1_A:

$$\begin{aligned} \Phi_k(\mathfrak{q}_k|_{E^{\otimes k}}) &= \sum_{|\rho|=k} {}_A(\mathfrak{q}_k e_\rho|e_\rho) = \lim_n \sum_{|\rho|=k} {}_A(\mathrm{e}^{-\beta_n} e_\rho \mathrm{e}^{\beta_{n-k}}|e_\rho) \\ &= \lim_n \sum_{|\rho|=k, |\sigma|=n-k} \mathrm{e}^{-\beta_n} {}_A(e_\rho \otimes e_\sigma | e_\rho \otimes e_\sigma) = 1_A. \end{aligned} \tag{3.4}$$

Regardless of all these properties, we need to impose a further technical requirement on the operators $\mathfrak{q}_k$. To describe them better, we first prove a structural result of $\mathfrak{q}_k$ assuming that $\mathfrak{q}_1$ has closed range. Given an A-bimodule E and $c \in Z(A)$, we say that c is *central for the bimodule structure* if for all $e \in E$ the equality $ce = ec$ holds†.

LEMMA 3.3. *Assume that the range of $\mathfrak{q}_1$ is closed. Then $\mathfrak{q}_k$ has closed range for any k. Consequently, there is an A-bilinear projection P_k on $E^{\otimes k}$ such that $\mathfrak{q}_k$ is invertible on the range of P_k and $\mathfrak{q}_k = \mathfrak{q}_k P_k$. Furthermore:*

† Note that a central element in A need not be central for the bimodule structure, e.g. in Example 2.3, $a \in C(V)$ is central for the bimodule if and only if $a \circ g = a$.

(a) *if there exists $c_k \in A$ such that $c_k P_k = \mathfrak{q}_k P_k$, then $c_k = \Phi_k(P_k)^{-1}$ is invertible and central in A;*

(b) *if c_1 is given by left multiplication by an element in A which is central for the bimodule structure, then c_k is given by left multiplication by the central invertible element $c_1^k \in A$ for all k.*

Proof. The assumption that $\mathfrak{q}_1$ has closed range guarantees that the range is complemented, and $E = \ker(\mathfrak{q}_1) \oplus \operatorname{im}(\mathfrak{q}_1)$ with $M := \mathfrak{q}_1 E$ a sub-bimodule. We let P_1 denote the projection onto M, so P_1 commutes with A. An easy induction using equation (3.3) shows that $\mathfrak{q}_k E^{\otimes k} = M^{\otimes k}$. Since P_1 is also a bimodule map, the projection onto $M^{\otimes k}$ is easily seen to be $P_k = P_1 \otimes P_1 \otimes \cdots \otimes P_1$, and P_k commutes with A as well.

By definition, $P_k \mathfrak{q}_k = \mathfrak{q}_k$ and thus $\mathfrak{q}_k = \mathfrak{q}_k P_k$ by taking adjoints. Moreover, from the decomposition $\ker(\mathfrak{q}_k) \oplus \operatorname{im}(\mathfrak{q}_k)$, we see that $\mathfrak{q}_k$ is injective on im P_k. It is surjective for if $x = P_k y$ then there exists z such that $x = P_k y = \mathfrak{q}_k z = \mathfrak{q}_k P_k z$.

If there exists $c_k \in A$ whose compression with P_k coincides with $\mathfrak{q}_k$, that is, $\mathfrak{q}_k = P_k c_k P_k = c_k P_k$, then $1_A = c_k \Phi_k(P_k)$, so $\Phi_k(P_k)$ is invertible in A and $c_k = \Phi_k(P_k)^{-1}$. Moreover, $\mathfrak{q}_k$ commutes with both actions of A, so c_k is central in A. This proves (a).

If c_1 is given by left multiplication by a, necessarily central, element in A, then $c_1 = \Phi_1(P_1)^{-1}$. The assumption that c_1 is central for the bimodule structure, along with the fact that $\mathfrak{q}$ and P_1 are bimodule maps, gives

$$\mathfrak{q}_k = c_k P_k = \mathfrak{q}_1 \otimes \cdots \otimes \mathfrak{q}_1 = c_1 P_1 \otimes \cdots \otimes c_1 P_1 = c_1^k P_1 \otimes \cdots \otimes P_1 = c_1^k P_k.$$

This proves (b). □

Remark 3.4. If we can write $\mathfrak{q}_k = c_k P_k = P_k c_k$ for an A-bilinear projection P_k on $E^{\otimes k}$ and a central invertible element $c_k \in A$, then it trivially holds that $\mathfrak{q}_k$ has closed range.

Assumption 2. For any k, we can write $\mathfrak{q}_k = c_k P_k = P_k c_k$, where $P_k \in \operatorname{End}_A^*(E^{\otimes k})$ is a projection and c_k is given by left multiplication by an element in A.

Remark 3.5. It follows from Lemma 3.3 that if a decomposition $\mathfrak{q}_k = c_k P_k$ of the kind in Assumption 2 exists, it is unique and of a very specific form. Indeed, each c_k is central in A, invertible and $c_k = \Phi_k(P_k)^{-1}$. Lemma 3.3 allows one to check Assumption 2 in practice. A sufficient condition for Assumption 2 to hold is that $\mathfrak{q}_1$ is closed with decomposition $\mathfrak{q}_1 = c_1 P_1$ for an element $c_1 \in A$ which is central for the bimodule structure on E. For instance, if β_1 is central for the bimodule structure on E, $c_k = \mathrm{e}^{-\beta_k} = \mathrm{e}^{-k\beta_1}$ is central for the bimodule structure on E and $P_k = \mathrm{Id}_{E^{\otimes k}}$. We remark that it is unclear if the property $c_1 \in A$ suffices to guarantee that Assumption 2 holds.

Example 3.6. Graph C^*-algebras defined from a primitive graph satisfy Assumption 2 by [**22**, equation (3.7)]. Cuntz–Krieger algebras trivially satisfy Assumption 2 because $\beta_k = 0$ for all k in this case and $\mathfrak{q}_k = \mathrm{Id}_{E^{\otimes k}}$ (see Example 2.7).

We assume that Assumption 2 holds for the remainder of the paper.

To simplify notation, we write $P = \sum_k P_k$ interpreted as a strict sum. We also write $\mathfrak{q} = \bigoplus_k \mathfrak{q}_k$, which we interpret as a densely defined operator with domain $\mathcal{F}_E^{\mathrm{alg}}$. We can now turn to generating the direct sum decomposition of Ξ_A. We recall the notation $W_{\mu,\nu}$ for the class of $S_\mu S_\nu^*$ in Ξ_A.

LEMMA 3.7. *For all homogeneous $\mu,\ \nu \in \mathcal{F}_E^{\mathrm{alg}}$, $W_{\mu,\nu} = W_{\mu,P\nu}$ in Ξ_A.*

Proof. We compute the module norm of the difference of $[S_\mu S_\nu^*] = W_{\mu,\nu}$ and $[S_\mu S_{P\nu}^*] = W_{\mu,P\nu}$ and show that it is zero. Write $k := |\nu|$. Using equation (2.4) and the definition of $\mathfrak{q}_k$, we have

$$\begin{aligned}
&(W_{\mu,\nu} - W_{\mu,P\nu} | W_{\mu,\nu} - W_{\mu,P\nu})_A \\
&\quad = \Phi_\infty(S_\nu(\mu|\mu)_A S_\nu^* - S_{P\nu}(\mu|\mu)_A S_\nu^* - S_\nu(\mu|\mu)_A S_{P\nu}^* + S_{P\nu}(\mu|\mu)_A S_{P\nu}^*) \\
&\quad = {}_A(\nu(\mu|\mu)_A | \mathfrak{q}_k \nu) - {}_A(P\nu(\mu|\mu)_A | \mathfrak{q}_k \nu) \\
&\qquad - {}_A(\nu(\mu|\mu)_A | \mathfrak{q}_k P\nu) + {}_A(P\nu(\mu|\mu)_A | \mathfrak{q}_k P\nu) = 0. \qquad \square
\end{aligned}$$

The next result shows that if $\{e_1, e_2, \ldots, e_N\}$ is a frame for E as a right module, then $\{P_k e_\rho\}_{|\rho|=k}$ is a frame for $M^{\otimes k} = P_k E^{\otimes k}$, and similarly for left frames $\{f_1, f_2, \ldots, f_M\}$. We just state the result for the left frame, as this is all we will require below.

LEMMA 3.8. *Let $f_1, \ldots, f_M$ be a frame for the left module ${}_A E$. Then with $Pf_\rho = P_1 f_{\rho_1} \otimes \cdots \otimes P_1 f_{\rho_k}$ we have for all $\mu \in E^{\otimes k}$,*

$$\sum_{|\rho|=k} {}_A(\mu|Pf_\rho) f_\rho = P\mu = \sum_{|\rho|=k} {}_A(P\mu|f_\rho) f_\rho = \sum_{|\rho|=k} {}_A(\mu|f_\rho) Pf_\rho = \sum_{|\rho|=k} {}_A(\mu|Pf_\rho) Pf_\rho. \tag{3.5}$$

Proof. We use Lemma 3.2 to compute

$$\sum_{|\rho|=k} {}_A(\mu|Pf_\rho) f_\rho = \sum_{|\rho|=k} {}_A(P\mu|f_\rho) f_\rho = P\mu,$$

since $\{f_\rho\}$ is a frame for $E^{\otimes k}$. Finally,

$$P\mu = P\bigg(\sum_{|\rho|=k} {}_A(\mu|f_\rho) f_\rho\bigg) = \sum_{|\rho|=k} {}_A(\mu|f_\rho) Pf_\rho,$$

since P is a bimodule map. Applying this identity to $P(P\mu) = P\mu$ gives the last expression. $\square$

We want to build a frame for the module Ξ_A. First, we identify the rank-one operators we need. Recall the notational convention for simple tensors on page 9. We start with the main computational step.

LEMMA 3.9. *For $\mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}$, write $n := |\mu| - |\nu|$. For multi-indices ρ, σ with $|\rho| = r$, $|\sigma| = r - n$, we have the following identities:*

$$W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} S_{e_\rho}^* S_\mu S_\nu^*) = \begin{cases} W_{e_\rho, {}_A(\underline{P\nu}\, {}_A(\mathfrak{q}\overline{\nu}|\overline{\mu})(\underline{\mu}|e_\rho)_A | Pf_\sigma) Pf_\sigma}, & r \le |\mu|, \\ W_{e_\rho, {}_A(P(\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}}) | Pf_\sigma) Pf_\sigma}, & r \ge |\mu|. \end{cases} \tag{3.6}$$

Proof. We first treat the case $r \le |\mu|$ and compute

$$\begin{aligned}
W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} S_{e_\rho}^* S_\mu S_\nu^*) &= W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} (e_\rho|\underline{\mu})_A S_{\overline{\mu}} S_\nu^*) \\
&= W_{e_\rho, Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1} Pf_\sigma (e_\rho|\underline{\mu})_A \overline{\mu}} S_\nu^*)
\end{aligned}$$

$$= \lim_{k\to\infty} W_{e_\rho, Pf_\sigma}\, {}_A(c_{|\sigma|}^{-1} Pf_\sigma (e_\rho|\underline{\mu})_A \overline{\mu}|\mathrm{e}^{-\beta_k}\nu\mathrm{e}^{\beta_{k-|\nu|}})$$

by equation (2.4)

$$= W_{e_\rho, Pf_\sigma}\, {}_A(Pf_\sigma (e_\rho|\underline{\mu})_A \overline{\mu}|c_{|\sigma|}^{-1}\mathfrak{q}\nu)$$

$$= W_{e_\rho, {}_A(\underline{P\nu}\, {}_A(\mathfrak{q}\overline{\nu}|\overline{\mu})(\underline{\mu}|e_\rho)_A|Pf_\sigma)Pf_\sigma}.$$

For $r \geq |\mu|$, we can do a similar calculation:

$$W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} S_{e_\rho}^* S_\mu S_\nu^*) = W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} S_{e_{\overline{\rho}}}^* (e_{\underline{\rho}}|\mu)_A S_\nu^*)$$

$$= W_{e_\rho, Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1} Pf_\sigma} S^*_{\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}}})$$

$$= \lim_{k\to\infty} {}_A(c_{|\sigma|}^{-1} Pf_\sigma|\mathrm{e}^{-\beta_k}\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}} \mathrm{e}^{-\beta_{k-|\nu|-|\overline{\rho}|}})$$

by equation (2.4)

$$= W_{e_\rho, Pf_\sigma}\, {}_A(Pf_\sigma (e_\rho|\underline{\mu})_A (Pf_\sigma|c_{|\sigma|}^{-1}\mathfrak{q}(\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}}))$$

$$= W_{e_\rho, {}_A(P(\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}})|Pf_\sigma)Pf_\sigma},$$

establishing the desired formula. □

This puts us in a position to define the elements of the decomposing frame for the module Ξ_A.

LEMMA 3.10. *For $n \in \mathbb{Z}$ and $r \geq \max\{0, n\}$, we define*

$$Q_{n,r} := \sum_{|\rho|-|\sigma|=n,\ |\rho|=r} \Theta_{W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma}, W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma}}.$$

We have the identity

$$Q_{n,r} W_{\mu,\nu} = \delta_{|\mu|-|\nu|,n} \begin{cases} W_{\underline{\mu}\, {}_A(\overline{\mu}|\mathfrak{q}\overline{\nu}), P\underline{\nu}}, & r \leq |\mu|, \\ W_{\mu, P\nu}, & r \geq |\mu|, \end{cases}$$

where $|\underline{\mu}| = r$. In particular, $Q_{n,r}$ does not depend on the choice of frames.

Proof. In the interests of avoiding at least one subscript, we write P generically for the projection onto $M^{\otimes k} = P_k E^{\otimes k}$, and similarly $\mathfrak{q}$ for $\mathfrak{q}_k$. We have

$$\sum_{|\rho|-|\sigma|=n,\ |\rho|=r} \Theta_{W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma}, W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma}} W_{\mu,\nu}$$

$$= \delta_{|\mu|-|\nu|,n} \sum_{|\rho|-|\sigma|=n,\ |\rho|=r} W_{e_\rho, c_{|\sigma|}^{-1/2} Pf_\sigma} \Phi_\infty(S_{c_{|\sigma|}^{-1/2} Pf_\sigma} S_{e_\rho}^* S_\mu S_\nu^*)$$

$$= \delta_{|\mu|-|\nu|,n} \sum_{|\rho|-|\sigma|=n,\ |\rho|=r} \begin{cases} W_{e_\rho, {}_A(\underline{P\nu}\, {}_A(\mathfrak{q}\overline{\nu}|\overline{\mu})(\underline{\mu}|e_\rho)_A|Pf_\sigma)Pf_\sigma}, & r \leq |\mu| \\ & \text{by equation (3.6)}, \\ W_{e_\rho, {}_A(P(\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}})|Pf_\sigma)Pf_\sigma}, & r \geq |\mu|, \end{cases}$$

$$= \delta_{|\mu|-|\nu|,n} \sum_{|\rho|=r} \begin{cases} W_{e_\rho (e_\rho|\underline{\mu})_A\, {}_A(\overline{\mu}|\mathfrak{q}\overline{\nu}), \underline{P\nu}}, & r \leq |\mu| \text{ by equation (3.5)}, \\ W_{e_\rho, P\nu(\mu|e_{\underline{\rho}})_A Pe_{\overline{\rho}}}, & r \geq |\mu|, \end{cases}$$

$$= \delta_{|\mu|-|\nu|,n} \begin{cases} W_{\underline{\mu}_A(\overline{\mu}|\mathfrak{q}\overline{\nu}),\underline{P\nu}}, & r \leq |\mu| \text{ where } |\underline{\mu}| = r, \\ \sum_{|\rho|=r} W_{e_\rho, P\nu(\mu|e_{\underline{\rho}})_A P e_{\overline{\rho}}}, & r \geq |\mu|, \end{cases}$$

$$= \delta_{|\mu|-|\nu|,n} \begin{cases} W_{\underline{\mu}_A(\overline{\mu}|\mathfrak{q}\overline{\nu}),\underline{P\nu}}, & r \leq |\mu| \text{ where } |\underline{\mu}| = r, \\ W_{\mu,P\nu}, & r \geq |\mu|. \end{cases}$$

In the last step, we are using $W_{\mu,\nu} = W_{\mu,P\nu}$ (see Lemma 3.7) to find that

$$\begin{aligned} \sum_{|\rho|=r} W_{e_\rho, P\nu(\mu|e_{\underline{\rho}})_A P e_{\overline{\rho}}} &= \sum_{|\rho|=r} W_{e_\rho,\nu(\mu|e_{\underline{\rho}})_A e_{\overline{\rho}}} = \sum_{|\rho|=r} S_{e_\rho} S^*_{e_{\overline{\rho}}} W_{0,\nu(\mu|e_{\underline{\rho}})_A} \\ &= \sum_{|\underline{\rho}|=|\mu|} W_{e_{\underline{\rho}},\nu(\mu|e_{\underline{\rho}})_A} \\ &= \sum_{|\underline{\rho}|=|\mu|} W_{e_{\underline{\rho}}(e_{\underline{\rho}}|\mu)_A,\nu} = W_{\mu,\nu} \\ &= W_{\mu,P\nu}. \end{aligned}$$

Our computation of $Q_{n,r} W_{\mu,\nu}$ gives a result that does not depend on the choice of frame; therefore, $Q_{n,r}$ does not depend on the choice of frame. □

With the collection of rank-one operators from Lemma 3.10 in hand, we can construct our orthogonal decomposition.

PROPOSITION 3.11. *For $n \in \mathbb{Z}$ and $r \geq \max\{0, n\}$, the operators $Q_{n,r}$ from Lemma 3.10 have the strict limit*

$$\lim_{r\to\infty} Q_{n,r} = \Psi_n.$$

In particular, we have

$$Q_{n,\max\{0,n\}} + \sum_{r=\max\{0,n\}}^{\infty} (Q_{n,r+1} - Q_{n,r}) = \Psi_n,$$

again strictly. Finally, the operators

$$P_{n,r} = \begin{cases} Q_{n,r} - Q_{n,r-1}, & r > \max\{0, n\}, \\ Q_{n,\max\{0,n\}}, & r = \max\{0, n\} \end{cases}$$

define a family of pairwise orthogonal finite rank projections which sum (strictly) to the identity on Ξ_A. The family of projections $P_{n,r}$ does not depend on the choice of frame.

Proof. It follows from Lemma 3.10 that each $Q_{n,r}$ is bounded, self-adjoint and idempotent, and thus $\|Q_{n,r}\| \leq 1$. Since for $r \geq |\mu|$ we have

$$\sum_{|\rho|-|\sigma|=n,\ |\rho|=r} \Theta_{W_{e_\rho, c_{|\sigma|}^{-1/2} P f_\sigma}, W_{e_\rho, c_{|\sigma|}^{-1/2} P f_\sigma}} W_{\mu,\nu} = \Psi_n W_{\mu,\nu},$$

the strict convergence statements follow now because the sequence $Q_{n,r}$ is uniformly bounded and the set $\{W_{\mu,\nu} : \mu, \nu \in \mathcal{F}_E^{\mathrm{alg}}\}$ spans a dense submodule.

The second statement follows from the first by a telescoping argument. We are left with the third statement, and we begin by showing that the $P_{n,r}$ are in fact projections.

Since $P_{n,r}$ is a difference of self-adjoints, $P_{n,r}$ is self-adjoint. As mentioned above, for $P_{n,\max\{0,n\}} = Q_{n,0}$ it follows directly from Lemma 3.10 that $P_{n,\max\{0,n\}}$ is idempotent. We now turn to the generic case $r > \max\{0, n\}$.

To reduce the number of subscripts, we drop the subscript on $\mathfrak{q}$. Using the computations in Lemma 3.10, we have

$$P_{n,r+1}W_{\mu,\nu} = \delta_{|\mu|-|\nu|,n}\begin{cases} W_{\underline{\mu}_{r+1}A(\overline{\mu}^{r+1}|\mathfrak{q}\overline{\nu}^{r+1}),\underline{P\nu}_{r+1}} - W_{\underline{\mu}_{r}A(\overline{\mu}^{r}|\mathfrak{q}\overline{\nu}^{r}),\underline{P\nu}_{r}}, & |\mu| \geq r+1, \\ 0, & |\mu| \leq r, \end{cases} \tag{3.7}$$

where $\underline{\mu}_r$ is the initial segment of length r, and $\overline{\mu}^r$ has length $|\mu| - r$. Now this computation shows that

$$P_{n,r+1}W_{\underline{\mu}_{r}A(\overline{\mu}^{r}|\mathfrak{q}\overline{\nu}^{r}),\underline{P\nu}_{r}} = 0$$

and that with $\mu = \underline{\mu}_r\mu_{r+1}\overline{\mu}^{r+1}$,

$$\begin{aligned} &P_{n,r+1}W_{\underline{\mu}_{r+1}A(\overline{\mu}^{r+1}|\mathfrak{q}\overline{\nu}^{r+1}),\underline{P\nu}_{r+1}} \\ &\quad = W_{\underline{\mu}_{r+1}A(\overline{\mu}^{r+1}|\mathfrak{q}\overline{\nu}^{r+1}),\underline{P\nu}_{r+1}} - W_{\underline{\mu}_{r}A(\mu_{r+1}A(\overline{\mu}^{r+1}|\mathfrak{q}\overline{\nu}^{r+1})|\tilde{\nu}_{r+1}),\underline{P\nu}_{r}} \\ &\quad = W_{\underline{\mu}_{r+1}A(\overline{\mu}^{r+1}|\mathfrak{q}\overline{\nu}^{r+1}),\underline{P\nu}_{r+1}} - W_{\underline{\mu}_{r}A(\overline{\mu}^{r}|\mathfrak{q}\overline{\nu}^{r}),\underline{P\nu}_{r}} \\ &\quad = P_{n,r+1}W_{\mu,\nu}, \end{aligned} \tag{3.8}$$

whence $P^2_{n,r+1} = P_{n,r+1}$. The projection property of $P_{n,r}$ implies, by a standard algebraic computation, that $Q_{n,r}Q_{n,r-1} = Q_{n,r-1}Q_{n,r} = Q_{n,r-1}$, and by induction for $s < r$, $Q_{n,r}Q_{n,s} = Q_{n,s}$. The pairwise orthogonality of the $P_{n,r}$ is now immediate. □

3.3. *Examples of the orthogonal decomposition.* We will in this subsection compute some examples of the orthogonal decomposition defined from the projections in Lemma 3.11. First we consider Cuntz–Krieger algebras.

LEMMA 3.12. *Let $\boldsymbol{A}$ denote an $N \times N$ matrix of 0's and 1's and $\mathcal{R}_{\boldsymbol{A}}$ the associated groupoid as in equation (2.2) decomposed as in equation (3.1). Under the isomorphism $\Xi_{C(\Omega_{\boldsymbol{A}})} \cong L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$,*

$$C(\mathcal{R}^{n,k}_{\boldsymbol{A}}) = P_{n,n+k}\Xi_{C(\Omega_{\boldsymbol{A}})},$$

as Hilbert C^*-modules.

Proof. For support reasons, $L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})} = \bigoplus_{n,k} C(\mathcal{R}^{n,k}_{\boldsymbol{A}})$ is an orthogonal decomposition. Therefore, the theorem follows if we can prove that $P_{n,n+k}f = f$ for $f \in C(\mathcal{R}^{n,k}_{\boldsymbol{A}})$.

To shorten notation, we write $W_{\rho,j} = W_{e_\rho,1^{\otimes j}}$—these are the elements appearing in $Q_{n,r}$ (see Lemma 3.10; cf. Example 2.7). We can identify $W_{\rho,j}$ with functions given by

$$W_{\rho,j}(x,n,y) = \begin{cases} \chi_\rho(x), & \text{if } n = |\rho| - j \text{ and } \sigma^{|\rho|}(x) = \sigma^j(y), \\ 0 & \text{otherwise.} \end{cases}$$

For $f \in C(\mathcal{R}^{n,k}_{\boldsymbol{A}}) \subseteq C_c(\mathcal{R}_{\boldsymbol{A}})$, we compute that

$$\langle W_{\rho,j}, f\rangle_{L^2(\mathcal{R}_{\boldsymbol{A}})}(y) = \sum_{(z,n,y)\in\mathcal{R}_{\boldsymbol{A}}} W_{\rho,j}(z,n,y)f(z,n,y) = \sum_{\sigma^j(y)=\sigma^{|\rho|}(z)} \chi_\rho(z)f(z,n,y).$$

This can be combined into

$$
\begin{aligned}
Q_{n,r}f(x,n,y) &= \sum_{|\rho|-j=n,|\rho|=r} W_{\rho,j}(x,n,y)\langle W_{\rho,j}, f\rangle_{L^2(\mathcal{R}_A)}(y)\\
&= \begin{cases} \displaystyle\sum_{|\rho|-j=n,|\rho|=r}\ \sum_{\sigma^j(y)=\sigma^{|\rho|}(z)} \chi_\rho(x)\chi_\rho(z)f(z,n,y) & \text{for } \sigma^j(y)=\sigma^r(x),\\ 0 & \text{otherwise,}\end{cases}\\
&= \begin{cases} \displaystyle\sum_{|\rho|-j=n,|\rho|=r}\ \sum_{\sigma^k(y)=\sigma^{n+k}(z)} \chi_\rho(x)\chi_\rho(z)f(z,n,y) & \text{for } \sigma^j(y)=\sigma^r(x),\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}
$$

In the last identity we used the fact that on the support of f, $\kappa_A(z,n,y)=k$. If $k=0$, then $r=n\geq 0$ and $j=0$. The injectivity of σ on U_j implies that

$$Q_{n,n}f(x,n,y)=P_{n,n}f(x,n,y)=f(x,n,y).$$

When considering $k>0$, we write

$$
\begin{aligned}
P_{n,r+1}f(x,n,y) &= Q_{n,r+1}f(x,n,y)-Q_{n,r}f(x,n,y)\\
&= \sum_{|\rho|-j=n,\ |\rho|=r+1}\ \sum_{\sigma^k(y)=\sigma^{n+k}(z)} \chi_\rho(x)\chi_\rho(z)f(z,n,y)\delta_{\sigma^{j+1}(y)=\sigma^{r+1}(x)}\\
&\quad - \sum_{|\rho|-j=n,\ |\rho|=r}\ \sum_{\sigma^k(y)=\sigma^{n+k}(z)} \chi_\rho(x)\chi_\rho(z)f(z,n,y)\delta_{\sigma^{j}(y)=\sigma^{r}(x)}.
\end{aligned}
\tag{3.9}
$$

Again using the injectivity of σ on U_j, $P_{n,r+1}f(x,n,y)=f(x,n,y)$ follows from equation (3.9) using a case-by-case analysis. □

We turn to the case of the Cuntz algebra. The case of a general graph C^*-algebra is combinatorially complicated, especially in light of the discussion in Example 2.6.

LEMMA 3.13. *The GNS representation $L^2(O_N)$ of the Cuntz algebra O_N associated with the KMS state coincides with $\Xi_{\mathbb{C}}$ defined from $O_N=O_{\mathbb{C}^N}$. Moreover, $L^2(O_N)$ can be decomposed into orthogonal finite-dimensional subspaces given by*

$$
\begin{aligned}
\mathcal{H}_{n,k} &= P_{n,n+k}\Xi_{\mathbb{C}}\\
&= \begin{cases} \operatorname{span}\{NW_{\mu i,\nu j}-W_{\mu,\nu}\delta_{i,j}:|\nu|=k-1, & \\ \qquad |\mu|-|\nu|=n,\ i,j=1,\ldots,N\}, & n+k,\ k>0,\\ \operatorname{span}\{W_{\mu,\emptyset}:|\mu|=n\}, & k=0,\\ \operatorname{span}\{W_{\emptyset,\nu}:|\nu|=n\}, & n+k=0.\end{cases}
\end{aligned}
$$

Proof. It follows from construction that

$$Q_{n,r}W_{\mu,\nu}=\delta_{|\mu|-|\nu|,n}\begin{cases} W_{\underline{\mu},\underline{\nu}}\delta_{\overline{\mu},\overline{\nu}}N^{r-|\mu|}, & r\leq|\mu|,\\ W_{\mu,\nu}, & r\geq|\mu|.\end{cases}$$

Here we use the convention $|\underline{\mu}| = r$ as in Lemma 3.10. From this, it follows that the space where $(c, \kappa) = (n, 0)$ and $(c, \kappa) = (n, -n)$ is exactly $\mathcal{H}_{n,0}$ and $\mathcal{H}_{n,-n}$, respectively. We also have the identity

$$\begin{aligned}P_{n,r+1}W_{\mu,\nu} &= Q_{n,r+1}W_{\mu,\nu} - Q_{n,r}W_{\mu,\nu}\\ &= \delta_{|\mu|-|\nu|,n}\begin{cases}N^{r-|\mu|}(NW_{\underline{\mu}_{r+1},\underline{\nu}_{r+1}}\delta_{\overline{\mu}^{r+1},\overline{\nu}^{r+1}} - W_{\underline{\mu}_r,\underline{\nu}_r}\delta_{\overline{\mu}^r,\overline{\nu}^r}), & r \le |\mu|,\\ W_{\mu,\nu}, & r \ge |\mu|.\end{cases}\end{aligned}$$

From this computation, we conclude that the space where $(c, \kappa) = (n, k)$ is exactly $\mathcal{H}_{n,k}$ for $n + k$, $k > 0$. $\square$

3.4. *The depth-kore operator and the unbounded Kasparov module.* We fix the decomposition

$$\Xi_A = \bigoplus_{n\in\mathbb{Z}} \bigoplus_{r\ge\max\{0,n\}} P_{n,r}\Xi_A \tag{3.10}$$

established in the last section. We consider the following A-linear operators defined on the algebraic sum $\bigoplus^{\mathrm{alg}}_{r\ge n} P_{n,r}\Xi_A \subseteq \Xi_A$:

$$\kappa_0 := \sum_{n,r}(r-n)P_{n,r} \quad \text{and} \quad c_0 := \sum_n n\Psi_n.$$

Both κ_0 and c_0 are closable. We define κ and c as the closures of κ_0 and c_0, respectively. We call κ the *depth-kore operator*. The following proposition is immediate from the construction.

PROPOSITION 3.14. *The operators c and κ are self-adjoint and regular operators on Ξ_A such that $c + \kappa \ge 0$ on* $\mathrm{Dom}(c) \cap \mathrm{Dom}(\kappa)$*. They commute on the common core* $\mathrm{Dom}(c\kappa) = \mathrm{Dom}(\kappa c)$.

Definition 3.15. (cf. [**12**, equation (5.37)]) For $k \ge \max\{0, -n\}$, we define the function $\psi : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z}$ by

$$\psi(n, k) = \begin{cases} n, & k = 0,\\ -(k + |n|) & \text{otherwise.}\end{cases}$$

We define $\mathcal{D} := \psi(c, \kappa) = \sum_{n,r} \psi(n, r - n)P_{n,r}$ as a densely defined operator on Ξ_A.

Remark 3.16. We remark that $\mathcal{D}$ does not depend on the choice of frame. However, there is freedom in the choice of the function ψ used to assemble $\mathcal{D}$ from c and κ. We will see below that the bounded commutator calculation boils down to some relatively simple estimates. This gives us some freedom in choosing the function ψ. There are reasons for preferring the definitions

$$\psi(n, k) = \begin{cases} n, & k = 0,\\ -\frac{1}{2}(k + n + 2|n|) & \text{otherwise,}\end{cases}$$

$$\psi(n, k) = \begin{cases} n, & k = 0,\\ -\frac{1}{2}(k + |n|) & \text{otherwise.}\end{cases}$$

The main reason is that these definitions restrict to the number operator in the SMEB case, since then $k = \max\{0, -n\}$ always (see more in §3.5.1).

LEMMA 3.17. *The projection onto the isometric copy of the Fock space in Ξ_A is given by*

$$Q = \sum_{n=0}^{\infty} P_{n,n}.$$

Proof. This is just $r = n \Leftrightarrow r = n \geq 0$. □

LEMMA 3.18. *For all homogeneous $\mu, \alpha, \beta \in \mathcal{F}_E^{\mathrm{alg}}$, $n \in \mathbb{Z}$, we have*

$$S_\mu P_{n,r} W_{\alpha,\beta} = \begin{cases} Q_{n+|\mu|,r+|\mu|} W_{\mu\alpha,\beta}, & r = 0, \\ P_{n+|\mu|,r+|\mu|} W_{\mu\alpha,\beta}, & r > 0. \end{cases}$$

Proof. It is straightforward to check the corresponding relations for the $Q_{n,r}$, from which the statement of the lemma follows immediately. □

THEOREM 3.19. *The data $(\mathcal{O}_E, \Xi_A, \mathcal{D})$ defines an odd unbounded Kasparov module which represents the class of the extension*

$$0 \to \mathcal{K}_A(\mathcal{F}_E) \to \mathcal{T}_E \to \mathcal{O}_E \to 0$$

in $KK^1(\mathcal{O}_E, A)$.

Proof. Since $\mathcal{D}$ is given in diagonal form with finitely generated projective eigenspaces, the proof of self-adjointness and regularity is straightforward. The range of each $P_{n,r}$ is finitely generated, and the function ψ is unbounded with value $\pm(r - n + |n|)$ on $P_{n,r}\Xi_A$, and so $(1 + \mathcal{D}^2)^{-1/2}$ is compact. The non-negative spectral projection of $\mathcal{D}$ is precisely the projection onto the isometric copy of the Fock module in Ξ_A by Lemma 3.17, and so if $(\mathcal{O}_E, \Xi_A, \mathcal{D})$ is an unbounded Kasparov module, its class represents the extension.

The only remaining thing to prove is that we have bounded commutators. Let $\mu, \alpha, \beta \in \mathcal{F}_E^{\mathrm{alg}}$ be homogeneous and consider the generator $S_\mu \in \mathcal{O}_E$. By Lemma 3.18, we have the computation

$$\begin{aligned}
&\mathcal{D}S_\mu W_{\alpha,\beta} - S_\mu \mathcal{D} W_{\alpha,\beta} \\
&\quad = \sum_{r>0,n} \psi(n, r-n) P_{n,r} W_{\mu\alpha,\beta} - S_\mu \sum_{s>0,m} \psi(m, s-m) P_{m,s} W_{\alpha,\beta} \\
&\qquad + \sum_{n} \psi(n, -n) P_{n,0} W_{\mu\alpha,\beta} - S_\mu \sum_{m} \psi(m, -m) P_{m,0} W_{\alpha,\beta} \\
&\quad = \sum_{r>0,n} \psi(n, r-n) P_{n,r} W_{\mu\alpha,\beta} - \sum_{s>0,m} \psi(m, s-m) P_{m+|\mu|,s+|\mu|} W_{\mu\alpha,\beta} \\
&\qquad + \sum_{n} \psi(n, -n) Q_{n,0} W_{\mu\alpha,\beta} - \sum_{m} \psi(m, -m) Q_{m+|\mu|,|\mu|} W_{\mu\alpha,\beta} \\
&\quad = \sum_{s>0,m} (\psi(m+|\mu|, s-m) - \psi(m, s-m)) P_{m+|\mu|,s+|\mu|} W_{\mu\alpha,\beta} \qquad (3.11) \\
&\qquad + \sum_{n \leq |\mu|} \bigg(\psi(n, -n)(Q_{n,0} - Q_{n+|\mu|,|\mu|}) + \sum_{r=1}^{|\mu|} \psi(n, r-n) P_{n,r} W_{\mu\alpha,\beta} \bigg). \qquad (3.12)
\end{aligned}$$

A case-by-case check shows the identity

$$\psi(m+|\mu|, s-m) - \psi(m, s-m) = \begin{cases} |\mu|, & m \geq 0, s = m, \\ -|\mu|, & m \geq 0, s > m, \\ -|\mu| + 2|m|, & m < 0, m + |\mu| \geq 0, \\ |\mu|, & m < 0, m + |\mu| < 0. \end{cases}$$

Therefore, the first sum (3.11) of the commutator defines a bounded operator. For the second two sums (3.12), observe that since $1 \leq r \leq |\mu|$ and $r - n \geq 0$, it suffices to address the case $n < 0$, in which case $r - n > 0$ and thus $\psi(n, r-n) = 2n - r$ always. For fixed $n < 0$, we can compute

$$\begin{aligned} &\bigg(\psi(n, -n)(Q_{n,0} - Q_{n+|\mu|,|\mu|}) + \sum_{r=1}^{|\mu|} \psi(n, r-n) P_{n,r}\bigg) W_{\mu\alpha,\beta} \\ &= \bigg(2n(Q_{n,0} - Q_{n+|\mu|,|\mu|}) + \sum_{r=1}^{|\mu|} (2n - r) P_{n,r}\bigg) W_{\mu\alpha,\beta} \\ &= -\sum_{r=1}^{|\mu|} r P_{n,r} W_{\mu\alpha,\beta} + 2n\bigg(Q_{n,0} - Q_{n+|\mu|,|\mu|} + \sum_{r=1}^{|\mu|} P_{n,r}\bigg) W_{\mu\alpha,\beta} \\ &= -\sum_{r=1}^{|\mu|} r P_{n,r} W_{\mu\alpha,\beta}. \end{aligned}$$

Thus, the second two sums (3.12) define a bounded operator. □

Remark 3.20. We see that the crucial properties of ψ for proving that $[\mathcal{D}, S_\mu]$ is bounded are that ψ satisfies: for every $l > 0$, there is a constant $C_l > 0$ such that

$$|\psi(n+l, k) - \psi(n, k)| \leq C_l \quad \text{for all } (n, k) \in \mathbb{Z} \times \mathbb{N}, n + k \geq 0; \quad \text{and}$$

for every $r > 0$, there is a constant C_r such that

$$|\psi(n, -n) - \psi(n, r-n)| \leq C_r \quad \text{for all } n \in \mathbb{Z} \setminus \mathbb{N}.$$

Remark 3.21. An unbounded representative $[\mathcal{D}]$ of the extension class allows one to use the explicit lift $[\widehat{\mathcal{D}}]$ to the mapping cone of the inclusion $A \hookrightarrow \mathcal{O}_E$ described in [**5**]. This lift allows a concrete comparison, on the level of cycles, of the exact sequences determined by the defining extension of a Cuntz–Pimsner algebra and the mapping cone exact sequence of $A \hookrightarrow \mathcal{O}_E$. This comparison is described in [**1**].

Remark 3.22. The construction of the unbounded Kasparov module is independent of the choices of left and right frames. It does however depend heavily on the choices of left and right inner products on the module E. In certain cases (see the discussion of SMEBs and vector bundles below), there is an obvious choice of left inner product, but of course not the only possible choice. In general the left inner product is part of the data that goes into the construction.

3.5. *Examples of unbounded Kasparov modules and spectral triples.* In this subsection, we will compute examples and compare to the existing works in the literature.

3.5.1. *Self-Morita equivalence bimodules.* For a SMEB, the depth-kore operator κ takes a simple form (cf. Example 2.12).

PROPOSITION 3.23. *Suppose that E is a SMEB. The following mapping defines a unitary isomorphism of A-modules:*

$$\Psi_n : \Xi_A^n \to E^{\otimes n}, \qquad W_{\mu,\nu} \mapsto \begin{cases} \underline{\mu}_A(\overline{\mu}|\nu), & |\mu| \geq |\nu|, \\ {}_A(\mu|\overline{\nu})\underline{\nu}, & |\mu| < |\nu|, \end{cases}$$

where $|\underline{\mu}| = n$ in the first line, $|\underline{\nu}| = -n$ in the second line and in the same line $\underline{\overline{\nu}} \in \overline{E}^{\otimes|n|}$ denotes the image of $\underline{\nu}$ under the anti-linear mapping $E^{\otimes|n|} \to \overline{E}^{\otimes|n|}$.

The proof of the previous proposition follows from the proof of [**22**, Theorem 3.1]. From Proposition 3.23, we deduce the structure of the operators $P_{n,r}$ from Proposition 3.11. We pick a right frame $(e_i)_{i=1}^N$ as in §2 and take $f_j = e_j$ as a left frame. Using the isomorphism of Proposition 3.11, we have that

$$P_{n,r} = \begin{cases} \sum_{|\sigma|=n} \Theta_{W_{e_\sigma,1}, W_{e_\sigma,1}}, & r = n \geq 0, \\ \sum_{|\sigma|=n} \Theta_{W_{1,e_\sigma}, W_{1,e_\sigma}}, & r = 0,\ n < 0, \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \Psi_n, & r = n \geq 0, \\ \Psi_n, & r = 0,\ n < 0, \\ 0 & \text{otherwise.} \end{cases}$$

We sum up the consequences for κ in a proposition.

PROPOSITION 3.24. *If E is a SMEB, then*

$$\kappa = \overline{\sum_{n<0} |n|\Psi_n} = \frac{1}{2}(|c| - c).$$

In particular, for ψ and $\mathcal{D}$ as in Remark 3.16, we get $\mathcal{D} = c$, the usual number operator.

3.5.2. *The depth-kore operator κ for Cuntz–Krieger algebras.* In [**12**, Theorem 5.1.7], a family of unbounded bivariant $(O_{\boldsymbol{A}}, C(\Omega_{\boldsymbol{A}}))$-cycles $(O_{\boldsymbol{A}}, L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}, \mathcal{D}_\lambda)$, parameterized by λ in the set of finite $\boldsymbol{A}$-admissible words was constructed. We let $\circ$ denote the empty word. It was shown that the mapping $K^0(C(\Omega_{\boldsymbol{A}})) \to K^1(O_{\boldsymbol{A}})$ defined by taking the Kasparov product with the cycle $(L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}, D_\circ))$ is surjective. We now give a different perspective on this cycle and identify its class. The case of general surjective local homeomorphisms is dealt with in the context of Smale spaces in work by the first two listed authors with Deeley and Whittaker [**10**].

THEOREM 3.25. *Let $\sigma : \Omega_{\boldsymbol{A}} \to \Omega_{\boldsymbol{A}}$ be a subshift of finite type and $E = {}_{\mathrm{Id}}C(\Omega_{\boldsymbol{A}})_{\sigma^*}$ the associated C^*-bimodule. Under the isomorphism $\Xi_{C(\Omega_{\boldsymbol{A}})} \cong L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$, the unbounded cycle $(\mathcal{O}_E, \Xi_{C(\Omega_{\boldsymbol{A}})}, \mathcal{D})$ constructed in Theorem 3.19 coincides with the unbounded cycle $(O_{\boldsymbol{A}}, L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}, \mathcal{D}_\circ)$ constructed in [**12**, Theorem 5.1.7] for Cuntz–Krieger algebras.*

The theorem is immediate from the following proposition describing the depth-kore operator. The proposition in turn follows from Lemma 3.12. Recall the function $\kappa_{\boldsymbol{A}} \in C(\mathcal{R}_{\boldsymbol{A}})$ from equation (3.2).

PROPOSITION 3.26. *Under the isomorphism* $\Xi_A \cong L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}$*, the operator* κ *of Proposition 3.14 satisfies* $C_c(\mathcal{R}_{\boldsymbol{A}}) \subseteq \mathrm{Dom}(\kappa)$ *and, for* $f \in C_c(\mathcal{R}_{\boldsymbol{A}})$*,*

$$[\kappa f](x, n, y) = \kappa_{\boldsymbol{A}}(x, n, y) f(x, n, y).$$

As a consequence of Theorem 3.25, we see that the cycle $(O_{\boldsymbol{A}}, L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}, \mathcal{D}_\circ)$ represents the extension

$$0 \to \mathcal{K}_{C(\Omega_{\boldsymbol{A}})}(\mathcal{F}_{\boldsymbol{A}}) \to \mathcal{T}_{\boldsymbol{A}} \to O_{\boldsymbol{A}} \to 0$$

obtained from $E = {}_{\mathrm{Id}}C(\Omega_{\boldsymbol{A}})_{\sigma^*}$ and the isomorphism $O_{\boldsymbol{A}} \cong \mathcal{O}_E$. Here $\mathcal{F}_{\boldsymbol{A}}$ denotes the Fock module constructed from $E = {}_{\mathrm{Id}}C(\Omega_{\boldsymbol{A}})_{\sigma^*}$. In particular, the unbounded cycle $(O_{\boldsymbol{A}}, L^2(\mathcal{R}_{\boldsymbol{A}})_{C(\Omega_{\boldsymbol{A}})}, \mathcal{D}_\circ)$ constructed in [**12**, Theorem 5.1.7] represents the boundary maps in the associated Pimsner six-term exact sequence in K-homology:

$$\begin{tikzcd}
K^0(O_A) \arrow[r] & K^0(C(\Omega_{\boldsymbol{A}})) \arrow[r, "1-[E]"] & K^0(C(\Omega_{\boldsymbol{A}})) \arrow[d, "{[\mathcal{D}_\circ]}"] \\
K^1(C(\Omega_{\boldsymbol{A}})) \arrow[u, "{[\mathcal{D}_\circ]}"] & K^1(C(\Omega_{\boldsymbol{A}})) \arrow[l, "1-[E]"'] & K^1(O_{\boldsymbol{A}}) \arrow[l]
\end{tikzcd} \tag{3.13}$$

Here $[E] \in KK_0(C(\Omega_{\boldsymbol{A}}), C(\Omega_{\boldsymbol{A}}))$ denotes the class associated with the bimodule E represented by the (unbounded) Kasparov module $(C(\Omega_{\boldsymbol{A}}), E, 0)$. The six-term exact sequence (3.13) is an example of a Pimsner sequence in KK-theory; for further details, see [**19**].

Because $\Omega_{\boldsymbol{A}}$ is a compact totally disconnected space, we can compute $K^1(C(\Omega_{\boldsymbol{A}})) = 0$. Thus, the sequence (3.13) reduces to

$$0 \to K^0(O_{\boldsymbol{A}}) \to K^0(C(\Omega_{\boldsymbol{A}})) \xrightarrow{1-[E]} K^0(C(\Omega_{\boldsymbol{A}})) \xrightarrow{[\mathcal{D}_\circ]} K^1(O_{\boldsymbol{A}}) \to 0. \tag{3.14}$$

We arrive at a conceptual explanation for the surjectivity of the map $K^0(C(\Omega_{\boldsymbol{A}})) \xrightarrow{[\mathcal{D}_\circ]} K^1(O_{\boldsymbol{A}})$ (cf. [**12**, Remark 5.2.6]). In general, the simple structure of (3.14) cannot be obtained from the Pimsner–Voiculescu sequence. The universal coefficient theorem implies that $K^0(C(\Omega_{\boldsymbol{A}})) = \mathrm{Hom}(C(\Omega_{\boldsymbol{A}}, \mathbb{Z}), \mathbb{Z})$ and $[E]$ acts as $\mathfrak{L}^*$. This gives yet another proof of the fact that $K^1(O_{\boldsymbol{A}}) = \mathbb{Z}^N/(1 - \boldsymbol{A})\mathbb{Z}^N$.

3.5.3. *The two models for* O_N. The odd spectral triples on $O_{\boldsymbol{A}}$ constructed in [**12**, Theorem 5.2.3] are supported on the fibres of the groupoid $\mathcal{R}_{\boldsymbol{A}}$. A consequence of Theorem 3.25 is that these Hilbert spaces are localizations of the module $\Xi_{C(\Omega_{\boldsymbol{A}})}$. For the Cuntz algebra O_N viewed as a Cuntz–Pimsner algebra over $\mathbb{C}$, the method of §3.4 will produce a spectral triple. In view of equation (2.5), this spectral triple will be defined on the GNS space $L^2(O_N)$ associated to the KMS state. In [**12**], the construction of such spectral triples was left as an open problem. We will compare the two approaches in this subsection. Recall the decomposition $L^2(O_N) = \bigoplus_{n,k} \mathcal{H}_{n,k}$ from Lemma 3.13.

THEOREM 3.27. *There is a self-adjoint operator $\mathcal{D}$ on $L^2(O_N)$ defined by $\mathcal{D}|_{\mathcal{H}_{n,k}} := \psi(n,k)$ such that $(O_N, L^2(O_N), \mathcal{D})$ is a θ-summable spectral triple on O_N whose class generates $K^1(O_N)$.*

Proof. The exactness of the Pimsner sequence

$$0 \longrightarrow K^0(O_N) \longrightarrow K^0(\mathbb{C}) \xrightarrow{1-N} K^0(\mathbb{C}) \xrightarrow{[\mathcal{D}]\otimes_{\mathbb{C}}\cdot} K^1(O_N) \longrightarrow 0$$

and Theorem 3.19 imply that the class of $(O_N, L^2(O_N), \mathcal{D})$ generates $K^1(O_N)$. The θ-summability of $(O_N, L^2(O_N), \mathcal{D})$ follows from the fact that the dimensions of $\mathcal{H}_{n,k}$ grow exponentially, and the sequence $\psi(n,k)$ grows linearly. □

Remark 3.28. We remark that by the same argument as in the proof of [**12**, Theorem 5.2.3], the bounded Fredholm module $(O_N, L^2(O_N), \mathcal{D}|\mathcal{D}|^{-1})$ is p-summable for any $p > 0$.

Following §3.5.2, let $(O_N, \Xi^{\Omega_N}, \mathcal{D}^{\Omega_N})$ denote the unbounded Kasparov module that defines the Pimsner extension for O_N over $C(\Omega_N)$, i.e. a class in $KK_1(O_N, C(\Omega_N))$. For $x \in \Omega_N$, we let $\epsilon_x : C(\Omega_N) \to \mathbb{C}$ denote the point evaluation at x. We deduce the following result from [**12**, Theorem 5.2.3].

COROLLARY 3.29. *For points $x \in \Omega_N$, the localizations*

$$(O_N, \Xi_x^{\Omega_N}, \mathcal{D}_x^{\Omega_N}) := (O_N, \Xi^{\Omega_N} \otimes_{\epsilon_x} \mathbb{C}, \mathcal{D}^{\Omega_N} \otimes_{\epsilon_x} 1)$$

define the same class in $K^1(O_N)$. Moreover, we have

$$[(O_N, L^2(O_N), \mathcal{D})] = [(O_N, \Xi^{\Omega_N}, \mathcal{D}^{\Omega_N})] \otimes_{C(\Omega_N)} [\epsilon_x] \quad \textit{in } K^1(O_N). \tag{3.15}$$

It can be shown that it is not possible to perform a factorization as in equation (3.15) at the level of unbounded cycles.

4. *The Cuntz–Pimsner algebra of a vector bundle on a closed manifold*

Our final example is a construction of spectral triples for Cuntz–Pimsner algebras of vector bundles on a closed manifold. Let M denote a closed Riemannian manifold equipped with an N-dimensional Hermitian vector bundle $V \to M$ and $\phi : M \to M$ an isometric C^1-diffeomorphism. We denote the induced map by $\alpha := \phi^* : C(M) \to C(M)$ and consider the space of continuous sections ${}_\alpha E := \Gamma(M, V)$ as a Hilbert bimodule ${}_\alpha E$ via $(a \cdot f \cdot b)(x) = \alpha(a)(x) f(x) b(x)$. The right $C(M)$-valued inner product is induced from the Hermitian structure on V and the left $C(M)$-valued inner product is defined through

$${}_{C(M)}(f|g) := \alpha^{-1}((g|f)_{C(M)}).$$

Because of the close relationship between the left and right inner products, we will express all formulae using only the right inner product, which will be denoted, unlabelled, by $(\cdot|\cdot)$. Labelled inner products will be used only when necessary.

To work with the module ${}_\alpha\Xi_{C(M)}$, we fix a right frame $(e_\lambda)_\lambda$ for ${}_\alpha E$ as follows. Consider a finite open cover $(U_i)_{i=1}^M$ over which V is trivialized by $\tau_i : V|_{U_i} \to U_i \times \mathbb{C}^N$. Choose C^1-functions χ_i such that $(\chi_i^2)_{i=1}^M$ is a partition of unity subordinate to $(U_i)_{i=1}^M$.

We then take $(\chi_i e^i_j)_{i=1\dots M, j=1,\dots,N}$ as our frame, where the collection $(e^i_j)_{j=1,\dots,N}$ is an orthonormal basis of C^1-sections over each U_i.

The frame $(e_\lambda)_\lambda$ is simultaneously a left and a right frame for ${}_\alpha E$, since

$$\sum_\lambda {}_{C(M)}(e|e_\lambda)e_\lambda = \sum_\lambda \alpha^{-1}((e_\lambda|e)_{C(M)}) \cdot e_\lambda = \sum_\lambda e_\lambda(e_\lambda|e)_{C(M)} = e, \quad e \in {}_\alpha E.$$

As in the case of the Cuntz algebra O_N, we have $e^{\beta_k} = N^k$, which is central for the bimodule structure of E. Thus, it follows that Assumptions 1 and 2 are satisfied (see Remark 3.5). Moreover, the projections P_k are all equal to 1 and $c_k = N^{-k}$.

4.1. *The product operator.* We will now construct a spectral triple on $\mathcal{O}_{\alpha E}$. Let $\not{D} = \not{D}_M$ denote an odd Dirac-type operator acting on a graded Clifford bundle $S \to M$. We note that it is no restriction to assume that S is graded, as our construction commutes with a formal suspension. The module ${}_\alpha \Xi_{C(M)}$ decomposes as a direct sum ${}_\alpha \Xi_{C(M)} = \bigoplus_{n,r} {}_\alpha \Xi^{n,r}_{C(M)}$ of finitely generated projective $C(M)$ modules ${}_\alpha \Xi^{n,r}_{C(M)}$ and we denote the associated vector bundles by ${}_\alpha \Xi^{n,r}_V \to M$ and the full field of Hilbert spaces by ${}_\alpha \Xi_V \to M$. We consider the graded Hilbert space

$$\mathcal{H} := {}_\alpha \Xi_{C(M)} \otimes_{C(M)} L^2(M, S) = \bigoplus_{r \geq n} L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S).$$

The C^*-algebra $\mathcal{O}_{\alpha E}$ acts on $\mathcal{H}$ via its adjointable action on ${}_\alpha \Xi_{C(M)}$. The densely defined operator $\mathcal{D}$ on $\Xi_{C(M)}$ and the grading operator γ on S induce a densely defined self-adjoint operator D on $\mathcal{H}$. The domain of D is clearly

$$\mathrm{Dom}(D) := \Bigg\{ f = (f_{n,r})_{n \geq r} \in \bigoplus_{r \geq n} L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S) : \\ \sum_{n \geq r} (|r| + |n|)^2 \| f_{n,r} \|^2_{L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S)} < \infty \Bigg\},$$

and $D(f_{n,r})_{n \geq r} := (\mathcal{D} \otimes \gamma)(f_{n,r})_{n \geq r} = (\gamma \psi(n, n-r) f_{n,r})_{n \geq r}$.

To construct a connection on the module ${}_\alpha \Xi_{C(M)}$, we observe that by Lemma 3.10, a frame for $Q_{n,r\alpha} \Xi_{C(M)}$ is given by $\{N^{(r-n)/2} W_{e_\rho, e_\sigma}\}_{|\rho|=r, |\sigma|=r-n}$. For notational convenience, we will write $W_{\rho,\sigma} := W_{e_\rho, e_\sigma}$ for multi-indices ρ, σ. The single indices ι and λ will be used in the same way.

LEMMA 4.1. *The collection of vectors*

$$x_{\rho,\sigma} := \begin{cases} N^{|\sigma|/2} W_{\emptyset,\sigma}, & |\rho| = 0, \\ W_{\rho,\emptyset}, & |\sigma| = 0, \\ N^{|\sigma|/2} W_{\rho,\sigma} - N^{|\sigma|/2-1} \alpha^{-|\rho|}(e_{\sigma_{|\sigma|}} | e_{\rho_{|\rho|}}) W_{\underline{\rho}, \underline{\sigma}}, & |\rho| > 0 \textit{ and } |\sigma| > 0 \end{cases}$$

is a frame for ${}_\alpha \Xi_{C(M)}$. *Indeed, for fixed r and n, $(x_{\rho,\sigma})_{|\rho|=r, |\rho|-|\sigma|=n}$ forms a frame for* ${}_\alpha \Xi^{n,r}_{C(M)}$.

Proof. The projections $P_{n.r} \leq Q_{n,r}$ are mutually orthogonal and thus the frame $y_{\rho,\sigma} = N^{(r-n)/2} W_{e_\rho, e_\sigma}$ for $Q_{n,r\alpha} \Xi_{C(M)}$ yields a frame for $P_{n,r\alpha} \Xi_{C(M)}$ by computing $x_{\rho,\sigma} := P_{n,r} y_{\rho,\sigma}$ for $|\rho| = r$, $|\sigma| = r - n$. We distinguish the three cases $|\rho| = 0$, $|\sigma| = 0$ and

$\min\{|\rho|, |\sigma|\} > 0$. Since $P_{n,0} = Q_{n,0}$, we find that $x_{\emptyset,\sigma} = y_{\emptyset,\sigma} = N^{|\sigma|/2} W_{\emptyset, e_\sigma}$. For $|\sigma| = 0$, we have that $Q_{n,r-1} W_{\rho,\emptyset} = 0$ and thus in this case $x_{\rho,\sigma} = y_{\rho,\sigma}$ as well. The generic case follows from a straightforward application of Lemma 3.10:

$$\begin{aligned} Q_{n,r-1} N^{|\sigma|/2} W_{\rho,\sigma} = Q_{n,r-1} N^{|\sigma|/2} W_{e_\rho, e_\sigma} &= N^{|\sigma|/2} W_{e_{\underline{\rho}} C(M)(e_{\rho_{|\rho|}} | \mathfrak{q} e_{\sigma_{|\sigma|}}), e_{\underline{\sigma}}} \\ &= N^{|\sigma|/2} W_{e_{\underline{\rho}} \alpha^{-1}(N^{-1} e_{\sigma_{|\sigma|}} | e_{\rho_{|\rho|}}), e_{\underline{\sigma}}} = N^{|\sigma|/2 - 1} \alpha^{-|\rho|} (e_{\sigma_{|\sigma|}} | e_{\rho_{|\rho|}}) W_{\underline{\rho}, \underline{\sigma}}, \end{aligned}$$

where $|\rho| = r > 0$. The formula for the frame now follows readily. □

Denote by ${}_\alpha \Xi^{n,r}_{C^1(M)}$ the $C^1(M)$-submodule of ${}_\alpha \Xi^{n,r}_{C(M)}$ generated by $x_{\rho,\sigma}$ as in Lemma 4.1 with $|\rho| = r$ and $|\sigma| = r - n$. The frame induces a connection $\nabla^{n,r}$ on each finite projective module ${}_\alpha \Xi^{n,r}_{C^1(M)}$. The connections $\nabla^{n,r}$ allow us to define twisted Dirac operators $T_{n,r} := 1 \otimes_{\nabla^{n,r}} \not{D}$ on ${}_\alpha \Xi^{n,r}_V \otimes S$. We let T denote the densely defined operator on $\mathcal{H}$ with domain

$$\operatorname{Dom}(T) := \left\{ f = (f_{n,r})_{n \geq r} \in \bigoplus_{r \geq n} L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S) : \sum_{n \geq r} \|T_{n,r} f_{n,r}\|^2_{L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S)} < \infty \right\},$$

defined by $T(f_{n,r})_{n \geq r} := (T_{n,r} f_{n,r})_{n \geq r}$.

LEMMA 4.2. *The operators D and T are self-adjoint and anti-commute with each other on their common core*

$$X := \left(\bigoplus_{n,r}^{\mathrm{alg}} {}_\alpha \Xi^{n,r}_{C^1(M)} \right) \bigotimes_{C^1(M)}^{\mathrm{alg}} \operatorname{Dom} \not{D}.$$

Moreover, $\mathcal{D}_E := D + T$ is a self-adjoint operator $\mathcal{D}_E$ with compact resolvent.

Proof. It is clear from their definitions that X is a common core for D and T. Both T and D respect the decomposition $\mathcal{H} = \bigoplus_{r \geq n} L^2(M, {}_\alpha \Xi^{n,r}_V \otimes S)$ in the sense that

$$D, T : {}_\alpha \Xi^{n,r}_{C^1(M)} \bigotimes_{C^1(M)}^{\mathrm{alg}} \operatorname{Dom} \not{D} \to {}_\alpha \Xi^{n,r}_{C^1(M)} \bigotimes_{C^1(M)}^{\mathrm{alg}} L^2(M, S).$$

In fact, D maps X into itself whereas T maps X into $\operatorname{Dom} D$. Therefore, the anti-commutator $DT + TD$ is defined on X and is easily seen to vanish there. It then follows that the sum $\mathcal{D}_E := D + T$ is closed and $D + T$ is an essentially self-adjoint operator on the initial domain X [**18**, Theorem 6.1.8]. The resolvent of $\mathcal{D}_E^2$ can be written as

$$(1 + \mathcal{D}_E^2)^{-1} = \bigoplus_{r \geq n} (1 + \psi(n, n-r)^2 + T_{n,r}^2)^{-1}.$$

For each n, r, $(1 + \psi(n, n-r)^2 + T_{n,r}^2)^{-1}$ is compact with

$$\|(1 + \psi(n, n-r)^2 + T_{n,r}^2)^{-1}\| \leq (1 + \psi(n, n-r)^2)^{-1} \to 0.$$

Therefore, $(1 + \mathcal{D}_E^2)^{-1}$ is compact. □

In the sequel, we will show that the commutators $[\mathcal{D}_E, S_\eta \otimes 1]$ for $\eta \in \Gamma^1(M, V)$ are bounded. From Lemma 4.2, and by checking the conditions of [**17**, Theorem 13], we then deduce the following theorem.

THEOREM 4.3. *Let $V \to M$ be a Hermitian vector bundle on a closed manifold, $\phi : M \to M$ an isometric C^1-diffeomorphism, $\mathcal{D}_E$ the operator constructed from a Dirac operator on M as in Lemma 4.2 and $\mathcal{A}$ the dense $*$-subalgebra of $\mathcal{O}_{\alpha E}$ generated by S_η with $\eta \in \Gamma^1(M, V)$. The triple $(\mathcal{A}, \mathcal{H}, \mathcal{D}_E)$ is a spectral triple for the Cuntz–Pimsner algebra $\mathcal{O}_{\alpha E}$ representing the Kasparov product of the class of*

$$0 \to \mathcal{K}_{C(M)}(\mathcal{F}_{\alpha E}) \to \mathcal{T}_{\alpha E} \to \mathcal{O}_{\alpha E} \to 0$$

in $KK^1(\mathcal{O}_{\alpha E}, C(M))$ with $[\not{D}] \in KK^*(C(M), \mathbb{C})$. The statements remain true if ϕ is a C^1-diffeomorphism such that for all $a \in \mathcal{A}$ there exists $C_a > 0$ such that $\sup_k \|[\not{D}, \alpha^k(a)]\| \leq C_a$.

4.2. *Proof of Theorem 4.3.* We turn to the proof of Theorem 4.3 by proving boundedness of the commutators $[\mathcal{D}_E, S_\eta \otimes 1]$ for $\eta \in \Gamma^1(M, V)$. In the special case when the isometric diffeomorphism ϕ is the identity, boundedness of the commutators $[\mathcal{D}_E, S_\eta \otimes 1]$ can be proved by a quick geometric argument. We prove the general case of a general isometric C^1-diffeomorphism $\phi : M \to M$ directly using the frame in Lemma 4.1. To this end, we first establish some algebraic relations describing the interaction of the algebra $C(M)$ and the operators S_{e_ι} with the global frame $x_{\rho,\sigma}$ constructed in Lemma 4.1.

LEMMA 4.4. *For $a \in C(M)$, we have the identity $ax_{\rho,\sigma} = x_{\rho,\sigma}\alpha^{|\rho|-|\sigma|}(a)$.*

Proof. The relation is obtained from the corresponding relations for $S_{\rho,\sigma}$ by writing

$$aW_{\rho,\sigma} = aS_{\rho,\sigma}W_{\emptyset,\emptyset} = S_{\rho,\sigma}\alpha^{|\rho|-|\sigma|}(a)W_{\emptyset,\emptyset} = S_{\rho,\sigma}W_{\emptyset,\emptyset}\alpha^{|\rho|-|\sigma|}(a) = W_{\rho,\sigma}\alpha^{|\rho|-|\sigma|}(a),$$

and then using that $|\rho| - |\sigma| = |\underline{\rho}| - |\underline{\sigma}|$, so that the relation passes to the $x_{\rho,\sigma}$ in all cases. □

LEMMA 4.5. *For $|\iota| = 1$, we have the relations*

$$S_{e_\iota}x_{\rho,\sigma} = \begin{cases} x_{\iota,\sigma} + N^{-1/2}\alpha^{-1}(e_{\sigma_{|\sigma|}}|e_\iota)x_{\emptyset,\underline{\sigma}}, & |\rho| = 0, \\ x_{\iota\rho,\sigma}, & |\rho| > 0, \end{cases}$$

$$S_{e_\iota}^* x_{\rho,\sigma} = \begin{cases} N^{-1/2}x_{\emptyset,\sigma\iota}, & |\rho| = 0, \\ (e_\iota|e_\rho)x_{\emptyset,\sigma} - N^{-1}(e_{\sigma_{|\sigma|}}|e_\rho)x_{\emptyset,\underline{\sigma}\iota}, & |\rho| = 1, \\ (e_\iota|e_{\rho_1})x_{\overline{\rho},\sigma}, & |\rho| > 1, \end{cases}$$

with the convention that $e_{\emptyset_{|\emptyset|}} = 0$.

Proof. For the operator S_{e_ι}, the action on $x_{\rho,\sigma}$ for $|\rho| > 0$ is straightforward to check. For $|\rho| = 0$, we compute

$$\begin{aligned} S_{e_\iota}x_{\emptyset,\sigma} &= S_{e_\iota}N^{|\sigma|/2}W_{\emptyset,\sigma} = N^{|\sigma|/2}W_{\iota,\sigma} \\ &= N^{|\sigma|/2}W_{\iota,\sigma} - N^{|\sigma|/2-1}\alpha^{-1}(e_{\sigma_{|\sigma|}}|e_\iota)W_{\emptyset,\underline{\sigma}} + N^{|\sigma|/2-1}(e_{\sigma_{|\sigma|}}|e_\iota)W_{\emptyset,\underline{\sigma}} \\ &= x_{\iota,\sigma} + N^{-1/2}\alpha^{-1}(e_{\sigma_{|\sigma|}}|e_\iota)x_{\emptyset,\underline{\sigma}}. \end{aligned}$$

For $S^*_{e_\iota}$, the relations for $|\rho| = 0$ and $|\rho| > 1$ are straightforward to check. For $|\rho| = 1$, we compute

$$\begin{aligned} S^*_{e_\iota} x_{\rho,\sigma} &= S^*_{e_\iota}(N^{|\sigma|/2} W_{\rho,\sigma} - N^{|\sigma|/2-1}(e_{\sigma_{|\sigma|}}|e_\rho) W_{\emptyset,\underline{\sigma}}) \\ &= N^{|\sigma|/2}(e_\iota|e_\rho) W_{\emptyset,\sigma} - N^{|\sigma|/2-1}(e_{\sigma_{|\sigma|}}|e_\rho) W_{\emptyset,\underline{\sigma}\iota} \\ &= (e_\iota|e_\rho) x_{\emptyset,\sigma} - N^{-1}(e_{\sigma_{|\sigma|}}|e_\rho) x_{\emptyset,\underline{\sigma}\iota}, \end{aligned}$$

as claimed. □

LEMMA 4.6. *The following relations hold:*

(1) $\sum_{|\lambda|=1} x_{\lambda\rho,\sigma} \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota) = x_{\iota\rho,\sigma}$ *for all* ρ;

(2) $\sum_{|\lambda|=1} x_{\emptyset,\sigma\lambda} \alpha^{-|\sigma|-1}(e_\iota|e_\lambda) = x_{\emptyset,\sigma\iota}$;

(3) $\sum_{|\lambda|=1} x_{\lambda,\sigma\lambda} = 0$.

Proof. The identities all rely on the frame relation. For (1) and $|\rho| > 0$,

$$\begin{aligned} \sum_{|\lambda|=1} x_{\lambda\rho,\sigma} \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota) &= \sum_{|\lambda|=1} N^{|\sigma|/2} W_{\lambda\rho,\sigma} \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota) \\ &\quad - N^{|\sigma|/2-1} \alpha^{-|\rho|-1}(e_{\sigma_{|\sigma|}}|e_\lambda) W_{\lambda\underline{\rho},\underline{\sigma}} \alpha^{|\underline{\rho}|-|\underline{\sigma}|}(e_\lambda|e_\iota) \\ &= N^{|\sigma|/2} W_{\iota\rho,\sigma} - N^{|\sigma|/2-1} \alpha^{-|\rho|-1}(e_{\sigma_{|\sigma|}}|e_\lambda) W_{\iota\underline{\rho},\underline{\sigma}} = x_{i\rho,\sigma} \end{aligned}$$

and, for $|\rho| = 0$,

$$\begin{aligned} \sum_{|\lambda|=1} x_{\lambda,\sigma} \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota) &= \sum_{|\lambda|=1} N^{|\sigma|/2} W_{\lambda,\sigma} \alpha^{-|\sigma|}(e_\lambda|e_\iota) \\ &\quad - N^{|\sigma|/2-1} \alpha^{-1}(e_{\sigma_{|\sigma|}}|e_\lambda) W_{\emptyset,\underline{\sigma}} \alpha^{-|\sigma|}(e_\lambda|e_\iota) \\ &= N^{|\sigma|/2} W_{\iota,\sigma} - \sum_{|\lambda|=1} N^{|\sigma|/2-1} \alpha^{-1}((e_{\sigma_{|\sigma|}}|e_\lambda)(e_\lambda|e_\iota)) W_{\emptyset,\underline{\sigma}} \\ &= N^{|\sigma|/2} W_{\iota,\sigma} - N^{|\sigma|/2-1} \alpha^{-1}((e_{\sigma_{|\sigma|}}|e_\iota)) W_{\emptyset,\underline{\sigma}} = x_{\iota,\sigma}. \end{aligned}$$

Identity (2) relies on similar considerations, observing that

$$\sum_{|\lambda|=1} x_{\emptyset,\sigma\lambda} \alpha^{-|\sigma|-1}(e_\iota|e_\lambda) = \sum_{|\lambda|=1} x_{\emptyset,\sigma\lambda(e_\lambda|e_\iota)} = x_{\emptyset,\sigma\iota}.$$

For (3), we also use α-invariance of the Jones–Watatani index:

$$\begin{aligned} \sum_{|\lambda|=1} x_{\lambda,\sigma\lambda} &= \sum_{|\lambda|=1} N^{(|\sigma|+1)/2} W_{\lambda,\sigma\lambda} - N^{(|\sigma|-1)/2} \alpha^{-1}(e_\lambda|e_\lambda) W_{\emptyset,\sigma} \\ &= N^{(|\sigma|+1)/2}\left(\sum_{|\lambda|=1} W_{\lambda,\sigma\lambda}\right) - N^{(|\sigma|-1)/2}\alpha^{-1}\left(\sum_{|\lambda|=1}(e_\lambda|e_\lambda)\right) W_{\emptyset,\sigma} \\ &= N^{(|\sigma|+1)/2} W_{\emptyset,\sigma} - N^{(|\sigma|-1)/2} N W_{\emptyset,\sigma} = 0. \end{aligned}$$

□

For $C^1(M, V) \subset E$, the $C^1(M)$-submodule of C^1-sections of V, the tensor products $C^1(M, V)^{\otimes k}$ are understood to be algebraic tensor products balanced over the action of $C^1(M)$ through α. It is then automatic that for $f, g \in C^1(M, V)^{\otimes k}$, we have $(f|g) \in C^1(M)$.

LEMMA 4.7. *For* $|\iota| = 1$, $\mu \in C^1(M, V)^{\otimes|\mu|}$, $\nu \in C^1(M, V)^{\otimes|\nu|}$ *and* $\xi \in \operatorname{Dom} \not{D}$, *we have the identity*

$$\begin{aligned}[T, S_{e_\iota} \otimes 1]W_{\mu,\nu} \otimes \xi = &\sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &+ \sum_{\sigma} N^{-1/2} x_{\emptyset,\underline{\sigma}} \otimes [\not{D}, \alpha^{-|\sigma|}(e_{\sigma_{|\sigma|}}|e_\iota)](x_{\emptyset,\sigma}|W_{\mu,\nu})\xi \\ &+ \sum_{|\lambda|=1} \sum_{\sigma} x_{\lambda,\sigma} \otimes [\not{D}, \alpha^{-|\sigma|}(e_\lambda|e_\iota)](x_{\emptyset,\sigma}|W_{\mu\nu})\xi \\ &- N^{-1} x_{\lambda,\sigma} \otimes [\not{D}, \alpha^{-|\sigma|}(e_\lambda|e_{\sigma_{|\sigma|}})](x_{\emptyset,\underline{\sigma}\iota}|W_{\mu,\nu})\xi. \end{aligned} \tag{4.1}$$

Proof. We let $[T, S_{e_\iota} \otimes 1]$ act on $W_{\mu,\nu}$ and compute

$$\begin{aligned} &[T, S_{e_\iota} \otimes 1]W_{\mu,\nu} \otimes \xi \\ &= \sum_{\rho,\sigma} x_{\rho,\sigma} \otimes \not{D}(S_{e_\iota}^* x_{\rho,\sigma}|W_{\mu,\nu})\xi - S_{e_\iota} x_{\rho,\sigma} \otimes \not{D}(x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &= \sum_{|\rho|>1,\sigma} x_{\rho,\sigma} \otimes \not{D}(S_{e_\iota}^* x_{\rho,\sigma}|W_{\mu,\nu}) \otimes \xi & (4.2) \\ &\quad + \sum_{|\lambda|=1} \sum_{\sigma} x_{\lambda,\sigma} \otimes \not{D}(S_{e_\iota}^* x_{\lambda,\sigma}|W_{\mu,\nu})\xi & (4.3) \\ &\quad + \sum_{\sigma} x_{\emptyset,\sigma} \otimes \not{D}(S_{e_\iota}^* x_{\emptyset,\sigma}|W_{\mu,\nu})\xi & (4.4) \\ &\quad - \sum_{|\rho|>0,\sigma} S_{e_\iota} x_{\rho,\sigma} \otimes \not{D}(x_{\rho,\sigma}|W_{\mu,\nu})\xi & (4.5) \\ &\quad - \sum_{\sigma} S_{e_\iota} x_{\emptyset,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi. & (4.6) \end{aligned}$$

We proceed with (4.2), using Lemmas 4.5 and 4.6(1) with $n = |\rho| - |\sigma|$:

$$\begin{aligned} (4.2) &= \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes \not{D}((e_\iota|e_\lambda)x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &= \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^n(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &\quad + \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma}\alpha^n(e_\lambda|e_\iota) \otimes \not{D}(x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &= \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^n(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &\quad + \sum_{|\rho|>0,\sigma} x_{\iota\rho,\sigma} \otimes \not{D}(x_{\rho,\sigma}|W_{\mu,\nu})\xi \\ &= \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^n(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi - (4.5), \end{aligned}$$

from which we see that (4.2) and (4.5) add up to

$$\sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi. \tag{4.7}$$

We proceed with (4.6) using Lemmas 4.4, 4.5 and 4.6(2):

$$
\begin{aligned}
&\sum_\sigma -S_{e_\iota} x_{\emptyset,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi \\
&\quad = \sum_\sigma -x_{\iota,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi - N^{-1/2}\alpha^{-1}(e_{\sigma_{|\sigma|}}|e_\iota)x_{\emptyset,\underline{\sigma}} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi \\
&\quad = \sum_\sigma -x_{\iota,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi - N^{-1/2}x_{\emptyset,\underline{\sigma}} \otimes \not{D}(x_{\emptyset,\sigma}\alpha^{-|\sigma|}(e_\iota|e_{\sigma_{|\sigma|}})|W_{\mu,\nu})\xi \\
&\qquad + N^{-1/2}x_{\emptyset,\underline{\sigma}} \otimes [\not{D}, \alpha^{-|\sigma|}(e_{\sigma_{|\sigma|}}|e_\iota)](x_{\emptyset,\sigma}|W_{\mu,\nu})\xi \\
&\quad = \sum_\sigma -x_{\iota,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu,\nu})\xi - N^{-1/2}x_{\emptyset,\sigma} \otimes \not{D}(x_{\emptyset,\sigma\iota}|W_{\mu,\nu})\xi && (4.8) \\
&\qquad + N^{-1/2}x_{\emptyset,\underline{\sigma}} \otimes [\not{D}, \alpha^{-|\sigma|}(e_{\sigma_{|\sigma|}}|e_\iota)](x_{\emptyset,\sigma}|W_{\mu,\nu})\xi. && (4.9)
\end{aligned}
$$

Next, we turn to (4.3), again applying Lemma 4.5:

$$
\begin{aligned}
(4.3) &= \sum_{|\lambda|=1}\sum_\sigma x_{\lambda,\sigma} \otimes \not{D}((e_\iota|e_\lambda)x_{\emptyset,\sigma}|W_{\mu\nu})\xi \\
&\quad - \sum_{|\lambda|=1}\sum_\sigma N^{-1}x_{\lambda,\sigma} \otimes \not{D}((e_{\sigma_{|\sigma|}}|e_\lambda)x_{\emptyset,\underline{\sigma}\iota}|W_{\mu,\nu})\xi \\
&= \sum_{|\lambda|=1}\sum_\sigma x_{\lambda,\sigma}\alpha^{-|\sigma|}(e_\lambda|e_\iota) \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu\nu})\xi \\
&\quad - \sum_{|\lambda|=1}\sum_\sigma N^{-1}x_{\lambda,\sigma}\alpha^{-|\sigma|}(e_\lambda|e_{\sigma_{|\sigma|}}) \otimes \not{D}(x_{\emptyset,\underline{\sigma}\iota}|W_{\mu,\nu})\xi && (4.10) \\
&\quad + \sum_{|\lambda|=1}\sum_\sigma x_{\lambda,\sigma} \otimes [\not{D}, \alpha^{-|\sigma|}(e_\lambda|e_\iota)](x_{\emptyset,\sigma}|W_{\mu\nu})\xi \\
&\quad - \sum_{|\lambda|=1}\sum_\sigma N^{-1}x_{\lambda,\sigma} \otimes [\not{D}, \alpha^{-|\sigma|}(e_\lambda|e_{\sigma_{|\sigma|}})](x_{\emptyset,\underline{\sigma}\iota}|W_{\mu,\nu})\xi. && (4.11)
\end{aligned}
$$

Considering (4.10) and applying Lemma 4.6(1), (2) and (3), we find

$$
\begin{aligned}
(4.10) &= \sum_\sigma x_{\iota,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu\nu})\xi - \sum_\sigma\sum_{|\lambda|=1} N^{-1}x_{\lambda,\sigma\lambda} \otimes \not{D}(x_{\emptyset,\sigma\iota}|W_{\mu,\nu})\xi \\
&= \sum_\sigma x_{\iota,\sigma} \otimes \not{D}(x_{\emptyset,\sigma}|W_{\mu\nu})\xi. && (4.12)
\end{aligned}
$$

Lastly, we compute (4.4):

$$
(4.4) = \sum_\sigma x_{\emptyset,\sigma} \otimes \not{D}(S_{e_\iota}^* x_{\emptyset,\sigma}|W_{\mu,\nu})\xi = \sum N^{-1/2}x_{\emptyset,\sigma} \otimes \not{D}(x_{\emptyset,\sigma\iota}|W_{\mu,\nu})\xi. \quad (4.13)
$$

Now we see that (4.8), (4.12) and (4.13) add up to 0. Thus, we are left with (4.7), (4.9) and (4.11), which together yield the expression (4.1). □

From Lemma 4.7, we deduce the following proposition, providing a proof of Theorem 4.3.

PROPOSITION 4.8. *For $|\iota| = 1$, the operator $S_{e_\iota} \otimes 1$ maps the core X described in Lemma 4.2 into* $\operatorname{Dom}\mathcal{D}_E$. *Moreover, if $\phi : M \to M$ is a C^1-isometric diffeomorphism or more generally for each $a \in C^1(M)$ we have $\sup_{k\in\mathbb{Z}} \|[\not{D}, \alpha^k(a)]\| \le C_a$ (cf. [3]), then the commutator $[\mathcal{D}_E, S_{e_\iota} \otimes 1]$ extends to a bounded operator.*

Proof. It is clear that S_{e_ι} maps X into itself. For the commutator, observe that

$$[\mathcal{D}_E, S_{e_\iota} \otimes 1] = [D, S_{e_\iota} \otimes 1] + [T, S_{e_\iota} \otimes 1],$$

and $[D, S_{e_\iota} \otimes 1]$ is bounded by construction. For $[T, S_{e_\iota} \otimes 1]$, we use Lemma 4.7 and analyse the four terms in equation (4.1). All terms can be shown to be bounded by a similar method. For instance, consider

$$W_{\mu,\nu} \otimes \xi \mapsto \sum_{|\lambda|=1} \sum_{|\rho|>0,\sigma} x_{\lambda\rho,\sigma} \otimes [\not{D}, \alpha^{|\rho|-|\sigma|}(e_\lambda|e_\iota)](x_{\rho,\sigma}|W_{\mu,\nu})\xi. \tag{4.14}$$

Consider the partial isometry

$$V : \Big(\bigoplus_{r,n} {}_\alpha \Xi^{n,r}_{C(M)}\Big) \otimes_{C(M)} L^2(M, S) \to \bigoplus_{|\rho|>0,\sigma} L^2(M, S),$$
$$W_{\mu,\nu} \otimes \xi \to ((x_{\rho,\sigma}|W_{\mu,\nu})\xi)_{\rho,\sigma},$$

and the map M_ι defined through $M_\iota := \mathrm{diag}_{n,r}(M_\iota^{n,r})$, where

$$M_\iota^{n,r} : \bigoplus_{|\rho|=r,|\sigma|=r-n} L^2(M, S) \to \bigoplus_{|\rho|=r+1,|\sigma|=r-n} L^2(M, S),$$
$$(M_\iota^{n,r}\xi)_{\rho,\sigma} := ([\not{D}, \alpha^{|\rho|-|\sigma|-1}(e_{\rho_1}|e_\iota)])\xi_{\overline{\rho},\sigma}.$$

The operator in equation (4.14) then coincides with the composition $V^* M_\iota V$. It thus suffices to show that $\sup_{n,r} \|M_\iota^{n,r}\| < \infty$, which follows from the assumption that $\sup_k \|[\not{D}, \alpha^k(a)]\| \le C_a$ for each a and the fact that the frame e_λ has finitely many elements. Thus, (4.14) defines a bounded operator. The other summands in equation (4.1) can be shown to be bounded by a similar argument. □

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