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Abstract

Let us consider the locus in the moduli space of curves of genus $2k$ defined by curves with a pencil of degree k. Since the Brill–Noether number is equal to -2, such a locus has codimension two. Using the method of test surfaces, we compute the class of its closure in the moduli space of stable curves.

Contents

Introduction

The classical Brill–Noether theory is of crucial importance for the geometry of moduli of curves. While a general curve admits only linear series with non-negative Brill–Noether number, the locus $\mathcal{M}^r_{g,d}$ of curves of genus g admitting a $\mathfrak{g}^r_d$ with negative Brill–Noether number $\rho(g,r,d) := g-(r+1)(g-d+r) < 0$ is a proper subvariety of $\mathcal{M}_g$. Harris, Mumford and Eisenbud have extensively studied the case $\rho(g,r,d) = -1$ when $\mathcal{M}^r_{g,d}$ is a divisor in $\mathcal{M}_g$. They computed the class of its closure in $\overline{\mathcal{M}}_g$ and found that it has slope $6 + 12/(g+1)$. Since for $g \geqslant 24$ this is less than $13/2$ the slope of the canonical bundle, it follows that $\overline{\mathcal{M}}_g$ is of general type for g composite and greater than or equal to 24.

While in recent years classes of divisors in $\overline{\mathcal{M}}_g$ have been extensively investigated, codimension-two subvarieties are basically unexplored. A natural candidate is offered from Brill–Noether theory. Since $\rho(2k,1,k) = -2$, the locus $\mathcal{M}^1_{2k,k} \subset \mathcal{M}_{2k}$ of curves of genus $2k$ admitting a pencil of degree k has codimension two (see [Ste98]). As an example, consider the hyperelliptic locus $\mathcal{M}^1_{4,2}$ in $\mathcal{M}_4$.

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Our main result is the explicit computation of classes of closures of such loci. When $g \geqslant 12$, a basis for the codimension-two rational homology of the moduli space of stable curves $\overline{\mathcal{M}}_g$ has been found by Edidin [Edi92]. It consists of the tautological classes κ_1^2 and κ_2 together with boundary classes. Such classes are still homologically independent for $g \geqslant 6$. Using the stability theorem for the rational cohomology of $\mathcal{M}_g$, Edidin's result can be extended to the case $g \geqslant 7$. While there might be non-tautological generators coming from the interior of $\overline{\mathcal{M}}_g$ for $g = 6$, one knows that Brill–Noether loci lie in the tautological ring of $\mathcal{M}_g$. Indeed in a similar situation, Harris and Mumford computed classes of Brill–Noether divisors in $\overline{\mathcal{M}}_g$ before knowing that $\mathrm{Pic}_{\mathbb{Q}}(\mathcal{M}_g)$ is generated solely by the class λ, by showing that such classes lie in the tautological ring of $\mathcal{M}_g$ (see [HM82, Theorem 3]). Their argument works in arbitrary codimension.

Since in our case $r = 1$, in order to extend the result to the Chow group, we will use a theorem of Faber and Pandharipande, which says that classes of closures of loci of type $\mathcal{M}^1_{g,d}$ are tautological in $\overline{\mathcal{M}}_g$ [FP05].

Then, having a basis for the classes of Brill–Noether codimension-two loci, in order to determine the coefficients we use the method of test surfaces. That is, we produce several surfaces in $\overline{\mathcal{M}}_g$ and, after evaluating the intersections on one hand with the classes in the basis and on the other hand with the Brill–Noether loci, we obtain enough independent relations to compute the coefficients of the sought-for classes.

The surfaces used are bases of families of curves with several nodes, hence a good theory of degeneration of linear series is required. For this, the compactification of the Hurwitz scheme by the space of admissible covers introduced by Harris and Mumford comes into play. The intersection problems thus boil down first to counting pencils on the general curve, and then to evaluating the respective multiplicities via a local study of the compactified Hurwitz scheme.

For instance when $k = 3$, we obtain the class of the closure of the trigonal locus in $\overline{\mathcal{M}}_6$.

Theorem 1. *The class of the closure of the trigonal locus in $\overline{\mathcal{M}}_6$ is*

$$\begin{aligned}
[\overline{\mathcal{M}}^1_{6,3}]_Q = {} & \tfrac{41}{144}\kappa_1^2 - 4\kappa_2 + \tfrac{329}{144}\omega^{(2)} - \tfrac{2551}{144}\omega^{(3)} - \tfrac{1975}{144}\omega^{(4)} + \tfrac{77}{6}\lambda^{(3)} \\
& - \tfrac{13}{6}\lambda\delta_0 - \tfrac{115}{6}\lambda\delta_1 - \tfrac{103}{6}\lambda\delta_2 - \tfrac{41}{144}\delta_0^2 - \tfrac{617}{144}\delta_1^2 + 18\delta_{1,1} \\
& + \tfrac{823}{72}\delta_{1,2} + \tfrac{391}{72}\delta_{1,3} + \tfrac{3251}{360}\delta_{1,4} + \tfrac{1255}{72}\delta_{2,2} + \tfrac{1255}{72}\delta_{2,3} \\
& + \delta_{0,0} + \tfrac{175}{72}\delta_{0,1} + \tfrac{175}{72}\delta_{0,2} - \tfrac{41}{72}\delta_{0,3} + \tfrac{803}{360}\delta_{0,4} + \tfrac{67}{72}\delta_{0,5} \\
& + 2\theta_1 - 2\theta_2.
\end{aligned}$$

For all $k \geqslant 3$ we produce a closed formula expressing the class of $\overline{\mathcal{M}}^1_{2k,k}$.

Theorem 2. *For $k \geqslant 3$ the class of the locus $\overline{\mathcal{M}}^1_{2k,k}$ in $\overline{\mathcal{M}}_{2k}$ is*

$$\begin{aligned}
[\overline{\mathcal{M}}^1_{2k,k}]_Q = \frac{2^{k-6}(2k-7)!!}{3(k!)} \Bigg[& (3k^2+3k+5)\kappa_1^2 - 24k(k+5)\kappa_2 \\
& + \sum_{i=2}^{2k-2} (-180i^4 + 120i^3(6k+1) - 36i^2(20k^2+24k-5) \\
& + 24i(52k^2-16k-5) + 27k^2 + 123k + 5)\omega^{(i)} + \cdots \Bigg].
\end{aligned}$$

The complete formula is shown in § 7. We also test our result in several ways, for example by pulling-back to $\overline{\mathcal{M}}_{2,1}$. The computations include the case $g = 4$, which was previously known: the hyperelliptic locus in $\overline{\mathcal{M}}_4$ has been computed in [FP05, Proposition 5].

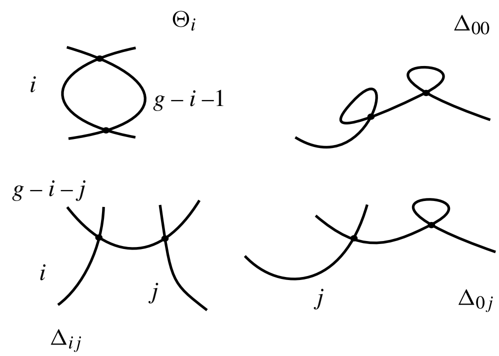


Figure 1. Loci in $\overline{\mathcal{M}}_g$.

1. A basis for $R^2(\overline{\mathcal{M}}_g)$

Let $A^*(\overline{\mathcal{M}}_g)$ be the Chow ring with $\mathbb{Q}$-coefficients of the moduli space of stable curves $\overline{\mathcal{M}}_g$, and let $R^*(\overline{\mathcal{M}}_g) \subset A^*(\overline{\mathcal{M}}_g)$ be the tautological ring of $\overline{\mathcal{M}}_g$ (see [FP05]). In [Edi92], Edidin gives a basis for the space of codimension-two tautological classes $R^2(\overline{\mathcal{M}}_g)$ and he also shows that such a basis holds for the codimension-two rational homology of $\overline{\mathcal{M}}_g$ for $g \geqslant 12$.

Let us quickly recall the notation. There are the tautological classes κ_1^2 and κ_2 coming from the interior $\mathcal{M}_g$; the following products of classes from $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$: $\lambda\delta_0, \lambda\delta_1, \lambda\delta_2, \delta_0^2$ and δ_1^2; the following push-forwards $\lambda^{(i)}, \lambda^{(g-i)}, \omega^{(i)}$ and $\omega^{(g-i)}$ of the classes λ and $\omega = \psi$ respectively from $\mathcal{M}_{i,1}$ and $\mathcal{M}_{g-i,1}$ to $\Delta_i \subset \overline{\mathcal{M}}_g$: $\lambda^{(3)}, \ldots, \lambda^{(g-3)}$ and $\omega^{(2)}, \ldots, \omega^{(g-2)}$; for $1 \leqslant i \leqslant \lfloor (g-1)/2 \rfloor$ the Q-class θ_i of the closure of the locus Θ_i whose general element is a union of a curve of genus i and a curve of genus $g-i-1$ attached at two points; finally the classes δ_{ij} defined as follows. The class δ_{00} is the Q-class of the closure of the locus Δ_{00} whose general element is an irreducible curve with two nodes. For $1 \leqslant j \leqslant g-1$ the class δ_{0j} is the Q-class of the closure of the locus Δ_{0j} whose general element is an irreducible nodal curve of geometric genus $g-j-1$ together with a tail of genus j. Finally, for $1 \leqslant i \leqslant j \leqslant g-2$ and $i+j \leqslant g-1$, the class δ_{ij} is defined as $\delta_{ij} := [\overline{\Delta}_{ij}]_Q$, where Δ_{ij} has as general element a chain of three irreducible curves with the external ones having genus i and j (see Figure 1).

The above classes generate $R^2(\overline{\mathcal{M}}_g)$ and Edidin shows that they are homologically independent for $g \geqslant 6$. It follows that for $g \geqslant 6$ the space of codimension-two tautological classes $R^2(\overline{\mathcal{M}}_g)$ has dimension

$$\lfloor (g^2-1)/4 \rfloor + 3g - 1.$$

When $g \geqslant 12$, to conclude that the above classes also form a basis for $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$, Edidin gives an upper bound on the rank of $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$ using that $H^4(\mathcal{M}_g, \mathbb{Q}) = \mathbb{Q}^2$ for $g \geqslant 12$ as shown by Harer. By the stability theorem for the rational cohomology of $\mathcal{M}_g$, we know that

$$H^k(\mathcal{M}_g, \mathbb{Q}) \cong H^k(\mathcal{M}_{g+1}, \mathbb{Q}) \cong H^k(\mathcal{M}_{g+2}, \mathbb{Q}) \cong \cdots$$

for $3k \leqslant 2(g-1)$ (see for instance [Wah12]). It follows that the above classes form a basis for $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$ when $g \geqslant 7$.

While for $g = 6$ there might be non-tautological generators coming from the interior of $\overline{\mathcal{M}}_g$, using an argument similar to [HM82, Theorem 3] one knows that classes of Brill–Noether loci $\mathcal{M}^r_{g,d}$ lie in the tautological ring of $\mathcal{M}_g$. It follows that classes in $H_{2(3g-3)-4}(\overline{\mathcal{M}}_g, \mathbb{Q})$ of closures of Brill–Noether loci of codimension two can be expressed as linear combinations of the above classes for $g \geqslant 6$.

In the case $r=1$, we know more; classes of closures of Brill–Noether loci $\overline{\mathcal{M}}^1_{g,d}$ lie in the tautological ring of $\overline{\mathcal{M}}_g$ (see [FP05, Proposition 1]). Hence for $g=2k\geqslant 6$ we can write

$$\begin{aligned}[\overline{\mathcal{M}}^1_{2k,k}]_Q &= A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1 + A_{\lambda\delta_2}\lambda\delta_2 \\ &\quad + \sum_{i=2}^{g-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{g-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} + \sum_{i=1}^{\lfloor (g-1)/2\rfloor} A_{\theta_i}\theta_i \end{aligned} \tag{1.1}$$

in $R^2(\overline{\mathcal{M}}_g,\mathbb{Q})$, for some rational coefficients A.

2. On the method of test surfaces

The method of test surfaces has been developed in [Edi92]. See [Edi92, §§ 3.1.2, 3.4 and Lemma 4.3] for computing the restriction of the generating classes to cycles parametrizing curves with nodes. In this section we summarize some results which will be used frequently in § 6.

In order to compute the restriction of κ_1^2 to test surfaces, we will use Mumford's formula for κ_1: if $g>1$ then $\kappa_1 = 12\lambda - \delta$ in $\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_g)$ (see [Mum77]). In the following proposition we note how to compute the restriction of the class κ_2 and the classes $\omega^{(i)}$ and $\lambda^{(i)}$ to a certain kind of surface which will appear in § 6 in (S1)–(S14).

Proposition 3. *Let $\pi_1\colon X_1 \to B_1$ be a one-dimensional family of stable curves of genus i with section $\sigma_1\colon B_1\to X_1$ and similarly let $\pi_2\colon X_2\to B_2$ be a one-dimensional family of stable curves of genus $g-i$ with section $\sigma_2\colon B_2\to X_2$. Next, obtain a two-dimensional family of stable curves $\pi\colon X\to B_1\times B_2$ as the union of $X_1\times B_2$ and $B_1\times X_2$ modulo glueing $\sigma_1(B_1)\times B_2$ with $B_1\times\sigma_2(B_2)$. Then the class κ_2 and the classes $\omega^{(i)}$ and $\lambda^{(i)}$ restrict to $B_1\times B_2$ as follows*

$$\begin{aligned} \kappa_2 &= 0, && \\ \omega^{(i)} = \omega^{(g-i)} &= -\pi_{1*}(\sigma_1^2(B_1))\pi_{2*}(\sigma_2^2(B_2)) && \text{if } 2\leqslant i < g/2,\\ \omega^{(g/2)} &= -2\pi_{1*}(\sigma_1^2(B_1))\pi_{2*}(\sigma_2^2(B_2)) && \text{if } g=2i,\\ \omega^{(j)} &= 0 && \text{for } j\notin\{i,g-i\},\\ \lambda^{(i)} &= \lambda_{B_1}\pi_{2*}(\sigma_2^2(B_2)) && \text{if } 3\leqslant i<g/2,\\ \lambda^{(g-i)} &= \lambda_{B_2}\pi_{1*}(\sigma_1^2(B_1)) && \text{if } 3\leqslant i<g/2,\\ \lambda^{(g/2)} &= \lambda_{B_1}\pi_{2*}(\sigma_2^2(B_2)) + \lambda_{B_2}\pi_{1*}(\sigma_1^2(B_1)) && \text{if } g=2i,\\ \lambda^{(j)} &= \lambda_{B_1}\delta_{j-i,1}|_{B_2} + \lambda_{B_2}\delta_{j-g+i,1}|_{B_1} && \text{for } j\notin\{i,g-i\}, \end{aligned}$$

where $\delta_{h,1}|_{B_1}\in\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{i,1})$ and similarly $\delta_{h,1}|_{B_2}\in\mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g-i,1})$.

Proof. Let $\nu\colon\widetilde{X}\to X$ be the normalization, where $\widetilde{X} := X_1\times B_2\cup B_1\times X_2$. Let $K_{X/B_1\times B_2} = c_1(\omega_{X/B_1\times B_2})$. We have

$$\kappa_2 = \pi_*(K^3_{X/B_1\times B_2}) = \pi_*\nu_*((\nu^* K_{X/B_1\times B_2})^3)$$

where we have used that ν is a proper morphism, hence the push-forward is well defined. One has

$$K_{\widetilde{X}/B_1\times B_2} = (K_{X_1/B_1}\times B_2)\oplus(B_1\times K_{X_2/B_2})$$

hence

$$\nu^*K_{X/B_1\times B_2} = ((K_{X_1/B_1}+\sigma_1(B_1))\times B_2)\oplus(B_1\times(K_{X_2/B_2}+\sigma_2(B_2))).$$

Finally

$$((K_{X_1/B_1}+\sigma_1(B_1))\times B_2)^3=(K_{X_1/B_1}+\sigma_1(B_1))^3\times B_2=0$$

since $K_{X_1/B_1}+\sigma_1(B_1)$ is a class on the surface X_1, and similarly for $B_1\times(K_{X_2/B_2}+\sigma_2(B_2))$, hence κ_2 is zero.

The statement about the classes $\omega^{(i)}$ and $\lambda^{(i)}$ follows almost by definition. For instance, since the divisor δ_i is

$$\delta_i=\pi_*(\sigma_1^2(B_1)\times B_2)+\pi_*(B_1\times\sigma_2^2(B_2))$$

we have

$$\omega^{(i)}=-\pi_{1*}(\sigma_1^2(B_1))\cdot\pi_{2*}(\sigma_2^2(B_2)).$$

The other equalities follow in a similar way. □

3. Enumerative geometry on the general curve

In order to construct admissible covers, we will often have to count pencils on general curves. Here we recall some well-known results in Brill–Noether theory.

Let C be a complex smooth projective curve of genus g and $l=(\mathscr{L},V)$ a linear series of type $\mathfrak{g}^r_d$ on C, that is $\mathscr{L}\in\operatorname{Pic}^d(C)$ and $V\subset H^0(C,\mathscr{L})$ is a subspace of vector-space dimension $r+1$. The *vanishing sequence* $a^l(p):0\leqslant a_0<\cdots<a_r\leqslant d$ of l at a point $p\in C$ is defined as the sequence of distinct order of vanishing of sections in V at p, and the *ramification sequence* $\alpha^l(p):0\leqslant\alpha_0\leqslant\cdots\leqslant\alpha_r\leqslant d-r$ as $\alpha_i:=a_i-i$, for $i=0,\dots,r$. The *weight* $w^l(p)$ will be the sum of the quantities α_i.

Given an n-pointed curve $(C,p_1,\dots,p_n)$ of genus g and l a $\mathfrak{g}^r_d$ on C, the *adjusted Brill–Noether number* is

$$\rho(C,p_1,\dots p_n)=\rho(g,r,d,\alpha^l(p_1),\dots,\alpha^l(p_n)):=g-(r+1)(g-d+r)-\sum_{i,j}\alpha^l_j(p_i).$$

3.1 Fixing two general points

Let (C,p,q) be a general 2-pointed curve of genus $g\geqslant 1$ and let $\alpha=(\alpha_0,\dots,\alpha_r)$ and $\beta=(\beta_0,\dots,\beta_r)$ be Schubert indices of type r,d (that is $0\leqslant\alpha_0\leqslant\cdots\leqslant\alpha_r\leqslant d-r$ and similarly for β) such that $\rho(g,r,d,\alpha,\beta)=0$. The number of linear series $\mathfrak{g}^r_d$ having ramification sequence α at the point p and β at the point q is counted by the *adjusted Castelnuovo number*

$$g!\,\det\left(\frac{1}{[\alpha_i+i+\beta_{r-j}+r-j+g-d]!}\right)_{0\leqslant i,j\leqslant r}$$

where $1/[\alpha_i+i+\beta_{r-j}+r-j+g-d]!$ is taken to be zero when the denominator is negative (see [Far09, Proof of Proposition 2.2] and [Ful98, Example 14.7.11(v)]). Note that the above expression may be zero, that is the set of desired linear series may be empty.

When $r=1$ let us denote the above expression by $N_{g,d,\alpha,\beta}$. If $\alpha_0=\beta_0=0$ then

$$N_{g,d,\alpha,\beta}=g!\left(\frac{1}{(\beta_1+1+g-d)!(\alpha_1+1+g-d)!}-\frac{1}{(g-d)!(\alpha_1+\beta_1+2+g-d)!}\right).$$

Subtracting the base locus $\alpha_0p+\beta_0q$, one can reduce the count to the case $\alpha_0=\beta_0=0$, hence $N_{g,d,\alpha,\beta}=N_{g,d-\alpha_0-\beta_0,(0,\alpha_1-\alpha_0),(0,\beta_1-\beta_0)}$.

In the following we will also use the abbreviation $N_{g,d,\alpha}$ when β is zero, that is $N_{g,d,\alpha}$ counts the linear series with the only condition of ramification sequence α at a single general point.

3.2 A moving point

Let C be a general curve of genus $g>1$ and $\alpha=(\alpha_0,\alpha_1)$ be a Schubert index of type $1,d$ (that is $0\leqslant\alpha_0\leqslant\alpha_1\leqslant d-1$). When $\rho(g,1,d,\alpha)=-1$, there is a finite number $n_{g,d,\alpha}$ of $(x,l_C)\in C\times W^1_d(C)$ such that $\alpha^{l_C}(x)=\alpha$. (Necessarily $\rho(g,1,d)\geqslant 0$ since the curve is general.) Assuming $\alpha_0=0$, one has $\alpha_1=2d-g-1$ and

$$n_{g,d,\alpha}=(2d-g-1)(2d-g)(2d-g+1)\frac{g!}{d!(g-d)!}.$$

If $\alpha_0>0$ then $n_{g,d,\alpha}=n_{g,d-\alpha_0,(0,\alpha_1-\alpha_0)}$. Each $\tilde{l}_C:=l_C(-\alpha_0 x)$ satisfies $h^0(\tilde{l}_C)=2$, is generated by global sections, and $H^0(C,\tilde{l}_C)$ gives a covering of $\mathbb{P}^1$ with ordinary branch points except for a $(\alpha_1-\alpha_0)$-fold branch point, all lying over distinct points of $\mathbb{P}^1$. Moreover, since for general C the above points x are distinct, one can suppose that fixing one of them, the l_C is unique. See [HM82, Theorem B and p. 78]. Clearly α in the lower indexes of the numbers n is redundant in our notation, but for our purposes it is useful to keep track of it.

3.3 Two moving points

Let C be a general curve of genus $g>1$ and $\alpha=(\alpha_0,\alpha_1)$ be a Schubert index of type $1,d$ (that is $0\leqslant\alpha_0\leqslant\alpha_1\leqslant d-1$). When $\rho(g,1,d,\alpha,(0,1))=-2$ (and $\rho(g,1,d)\geqslant 0$), there is a finite number $m_{g,d,\alpha}$ of $(x,y,l_C)\in C\times C\times G^1_d(C)$ such that $\alpha^{l_C}(x)=\alpha$ and $\alpha^{l_C}(y)=(0,1)$. Subtracting the base locus as usual, one can always reduce to the case $\alpha_0=0$.

Lemma 4. *Assuming $\alpha_0=0$, one has*

$$m_{g,d,\alpha}=n_{g,d,\alpha}\cdot(3g-1).$$

Proof. Since $\rho(g,1,d,\alpha)=-1$, one can first compute the number of points of type x, and then, fixing one of these, use the Riemann–Hurwitz formula to find the number of points of type y. □

4. Compactified Hurwitz scheme

Let $H_{k,b}$ be the Hurwitz scheme parametrizing coverings $\pi\colon C\to\mathbb{P}^1$ of degree k with b ordinary branch points and C a smooth irreducible curve of genus g. By considering only the source curve C, $H_{k,b}$ admits a map to $\mathcal{M}_g$

$$\sigma\colon H_{k,b}\to\mathcal{M}_g.$$

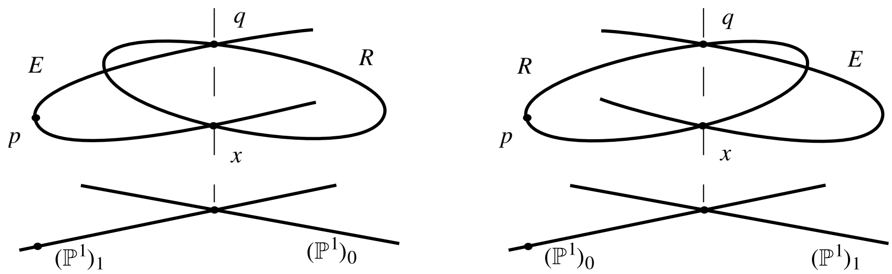
In the following, we will use the compactification $\overline{H}_{k,b}$ of $H_{k,b}$ by the space of admissible covers of degree k, introduced by Harris and Mumford in [HM82]. Given a semi-stable curve C of genus g and a stable b-pointed curve $(R,p_1,p_2,\ldots,p_b)$ of genus 0, an *admissible cover* is a regular map $\pi\colon C\to B$ such that the following hold: $\pi^{-1}(B_{\text{smooth}})=C_{\text{smooth}}$, $\pi|_{C_{\text{smooth}}}$ is simply branched over the points p_i and unramified elsewhere, $\pi^{-1}(B_{\text{singular}})=C_{\text{singular}}$ and if C_1 and C_2 are two branches of C meeting at a point p, then $\pi|_{C_1}$ and $\pi|_{C_2}$ have the same ramification index at p. Note that one may attach rational tails at C to cook up the degree of π.

The map σ extends to

$$\sigma\colon\overline{H}_{k,b}\to\overline{\mathcal{M}}_g.$$

In our case $g=2k$, the image of this map is $\overline{\mathcal{M}}^1_{2k,k}$. It is classically known that the Hurwitz scheme is connected and its image in $\mathcal{M}_g$ (that is, $\mathcal{M}^1_{2k,k}$ in our case) is irreducible (see for instance [Ful69]).

Similarly for a Schubert index $\alpha=(\alpha_0,\alpha_1)$ of type $1,k$ such that $\rho(g,1,k,\alpha)=-1$ (and $\rho(g,1,k)\geqslant 0$), the Hurwitz scheme $H_{k,b}(\alpha)$ (respectively $\overline{H}_{k,b}(\alpha)$) parameterizes k-sheeted



(a) The case $2p \equiv q + x$ (b) The case $x = p$

Figure 2. The admissible covers for the two fibers of the family $\overline{C}$ when $g = 2$.

(admissible) coverings $\pi\colon C \to \mathbb{P}^1$ with b ordinary branch points $p_1, \ldots, p_b$ and one point p with ramification profile described by α (see [Dia85, § 5]). By forgetting the covering and keeping only the pointed source curve (C, p), we obtain a map $\overline{H}_{k,b}(\alpha) \to \overline{\mathcal{M}}_{g,1}$ with image the pointed Brill–Noether divisor $\overline{\mathcal{M}}^1_{g,k}(\alpha)$.

Let us see these notions at work. Let (C, p, q) be a 2-pointed general curve of genus $g - 1 \geqslant 1$. In the following, we consider the curve $\overline{C}$ in $\overline{\mathcal{M}}_{g,1}$ obtained by identifying the point q with a moving point x in C. In order to construct this family of curves, one blows up $C \times C$ at (p, p) and (q, q) and identifies the proper transforms S_1 and S_2 of the diagonal Δ_C and $q \times C$. This is a family $\pi\colon X \to C$ with a section corresponding to the proper transform of $p \times C$, hence there exists a map $C \to \overline{\mathcal{M}}_{g,1}$. We denote by $\overline{C}$ the image of C in $\overline{\mathcal{M}}_{g,1}$.

Lemma 5. *Let $g = 2$ and let $\mathcal{W}$ be the closure of the Weierstrass divisor in $\overline{\mathcal{M}}_{2,1}$. We have*

$$\ell_{2,2} := \deg(\overline{C} \cdot \mathcal{W}) = 2.$$

Proof. There are two points in $\overline{C}$ with an admissible cover of degree 2 with simple ramification at the marked point, and such admissible covers contribute with multiplicity 1. Note that here C is an elliptic curve. One admissible cover is for the fiber over x such that $2p \equiv q + x$, and the other one for the fiber over $x = p$ (see Figure 2). In both cases the covering is determined by $|q + x|$ and there is a rational curve R meeting C in q and x.

When $2p \equiv q + x$, the situation is as in [HM82, Theorem 6(a)]. Let $C' \to P$ be the corresponding admissible covering. If

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & B & \end{tikzcd}$$

is a general deformation of $[C' \to P]$ in $\overline{H}_{2,b}(0,1)$, blowing down the curve R we obtain a family of curves $\widetilde{\mathcal{C}} \to B$ with one ordinary double point. That is, B meets Δ_0 with multiplicity 2. Considering the involution of $[C' \to P]$ obtained by interchanging the two ramification points of R, we see that the map $\overline{H}_{2,b}(0,1) \to \overline{\mathcal{M}}_{2,1}$ is ramified at $[C' \to P]$. Hence $[C']$ is a transverse point of intersection of $\mathcal{W}$ with Δ_0 and it follows that $\overline{C}$ and $\mathcal{W}$ meet transversally at $[C']$.

When $x = p$, the situation is similar. In a general deformation in $\overline{H}_{2,b}(0,1)$

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & B & \end{tikzcd}$$

of the corresponding admissible covering $[C' \to P]$, one sees that C' is the only fiber of $\mathcal{C} \to B$ inside Δ_{00}, and at each of the two nodes of C', the space $\mathcal{C}$ has local equation $x \cdot y = t$. It follows that C' is a transverse point of intersection of $\mathcal{W}$ with Δ_{00}. Hence C' is a transverse point of intersection of $\overline{C}$ with $\mathcal{W}$. See also [Har84, § 3]. □

Lemma 6. *Let $g = 2k-2 > 2$. The intersection of $\overline{C}$ with the pointed Brill–Noether divisor $\overline{\mathcal{M}}^1_{2k-2,k}(0,1)$ is reduced and it has degree*

$$\ell_{g,k} := \deg(\overline{C} \cdot \overline{\mathcal{M}}^1_{2k-2,k}(0,1)) = 2\frac{(2k-3)!}{(k-2)!(k-1)!}.$$

Proof. Let us write the class of $\overline{\mathcal{M}}^1_{g,k}(0,1)$ as $a\lambda + c\psi - \sum b_i\delta_i \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}_{g,1})$. First we study the intersection of the curve $\overline{C}$ with the classes generating the Picard group. Let $\pi\colon \overline{\mathcal{M}}_{g,1} \to \overline{\mathcal{M}}_g$ be the map forgetting the marked point and $\sigma\colon \overline{\mathcal{M}}_g \to \overline{\mathcal{M}}_{g,1}$ the section given by the marked point. Note that on $\overline{C}$ we have $\deg \psi = -\deg \pi_*(\sigma^2) = 1$, since the marked point is generically fixed and is blown up in one fiber. Moreover, $\deg \delta_{g-1} = 1$, since only one fiber contains a disconnecting node and the family is smooth at this point. The intersection with δ_0 deserves more care. The family is indeed inside Δ_0: the generic fiber has one non-disconnecting node and moreover the fiber over $x = p$ has two non-disconnecting nodes. We have to use [HM98, Lemma 3.94]. Then

$$\deg \delta_0 = \deg S_1^2 + \deg S_2^2 + 1 = -2(g-1) - 1 + 1 = 2 - 2g. \tag{4.1}$$

All other generating classes restrict to zero. Then

$$\deg(\overline{C} \cdot [\overline{\mathcal{M}}^1_{g,k}(0,1)]) = c + (2g-2)b_0 - b_{g-1}.$$

On the other hand, one has an explicit expression for the class of $\overline{\mathcal{M}}^1_{g,k}(0,1)$:

$$\frac{(2k-4)!}{(k-2)!k!}\left(6(k+1)\lambda + 6(k-1)\psi - k\delta_0 + \sum_{i=1}^{g-1} 3(i+1)(2+i-2k)\delta_i\right)$$

(see [Log03, Theorem 4.5]), whence the first part of the statement is proved.

Finally, the intersection is reduced. Indeed, since the curve C is general, an admissible cover with the desired property for a fiber of the family over $\overline{C}$ is determined by a unique linear series (see [HM82, p. 75]). Moreover, reasoning as in the proof of the previous lemma, one sees that $\overline{C}$ and $\overline{\mathcal{M}}^1_{g,k}(0,1)$ always meet transversally. □

5. Limit linear series

The theory of limit linear series will be used. Let us quickly recall some notation and results. On a tree-like curve, a linear series or a limit linear series is called *generalized* if the line bundles involved are torsion-free (see [EH87, § 1]). For a tree-like curve $C = Y_1 \cup \cdots \cup Y_s$ of arithmetic genus g with disconnecting nodes at the points $\{p_{ij}\}_{ij}$, let $\{l_{Y_1}, \ldots, l_{Y_s}\}$ be a generalized limit linear series $\mathfrak{g}^r_d$ on C. Let $\{q_{ik}\}_k$ be smooth points on Y_i, $i = 1, \ldots, s$. In [EH86] a moduli space of such limit series is constructed as a disjoint union of schemes on which the vanishing sequences

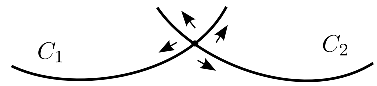


Figure 3. How the general fiber of a family in (S1) moves.

of the aspects l_{Y_i} at the nodes are specified. A key property is the additivity of the adjusted Brill–Noether number, that is

$$\rho(g, r, d, \{\alpha^{l_{Y_i}}(q_{ik})\}_{ik}) \geqslant \sum_i \rho(Y_i, \{p_{ij}\}_j, \{q_{ik}\}_k).$$

The smoothing result [EH86, Corollary 3.7] assures the smoothability of dimensionally proper limit series. The following facts will ease the computations. The adjusted Brill–Noether number for any $\mathfrak{g}^r_d$ on 1-pointed elliptic curves or on n-pointed rational curves is non-negative. For a general curve C of arbitrary genus g, the adjusted Brill–Noether number for any $\mathfrak{g}^r_d$ with respect to n general points is non-negative. Moreover, $\rho(C, y) \geqslant -1$ for any $y \in C$ and any $\mathfrak{g}^r_d$ (see [EH89]).

We will use the fact that if a curve of compact type has no limit linear series of type $\mathfrak{g}^r_d$, then it is not in the closure of the locus $\mathcal{M}^r_{g,d} \subset \mathcal{M}_g$ of smooth curves admitting a $\mathfrak{g}^r_d$.

6. Test surfaces

We are going to intersect both sides of (1.1) with several test surfaces. This will produce linear relations in the coefficients A.

The surfaces will be defined for arbitrary $g \geqslant 6$ (also odd values). Note that while the intersections of the surfaces with the generating classes (that is the left-hand sides of the relations we get) clearly depend solely on g, only the right-hand sides are specific to our problem of intersecting the test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$.

When the base of a family is the product of two curves $C_1 \times C_2$, we will denote the obvious projections by π_1 and π_2.

(S1) For $2 \leqslant i \leqslant \lfloor g/2 \rfloor$ consider the family of curves whose fibers are obtained by identifying a moving point on a general curve C_1 of genus i with a moving point on a general curve C_2 of genus $g - i$ (see Figure 3).

The base of the family is the surface $C_1 \times C_2$. In order to construct this family, consider $C_1 \times C_1 \times C_2$ and $C_1 \times C_2 \times C_2$ and identify $\Delta_{C_1} \times C_2$ with $C_1 \times \Delta_{C_2}$. Let us denote this family by $X \to C_1 \times C_2$.

One has

$$\delta_i = c_1(N_{(\Delta_{C_1} \times C_2)/X} \otimes N_{(C_1 \times \Delta_{C_2})/X}) = -\pi_1^*(K_{C_1}) - \pi_2^*(K_{C_2}).$$

Such surfaces are in the interior of the boundary of $\overline{\mathcal{M}}_g$. The only nonzero classes in codimension two are the ones considered in § 2.

We claim that the intersection of these test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$ has degree

$$T_i := \sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \rho(i,1,k,\alpha)=-1}} n_{i,k,\alpha} \cdot n_{g-i,k,(k-1-\alpha_1,k-1-\alpha_0)}$$

(in the sum, α is a Schubert index of type $1, k$). Indeed, by the remarks in § 5, if $\{l_{C_1}, l_{C_2}\}$ is a limit linear series of type $\mathfrak{g}^1_k$ on the fiber over some $(x, y) \in C_1 \times C_2$, then the only possibility

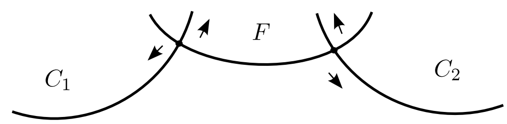


Figure 4. How the general fiber of a family in (S2) moves.

is $\rho(C_1, x) = \rho(C_2, y) = -1$. By § 3.2, there are exactly T_i points (x, y) with this property, the linear series l_{C_1}, l_{C_2} are uniquely determined and give an admissible cover of degree k. Thus to prove the claim we have to show that such points contribute with multiplicity 1.

Let us first assume that $i > 2$. Let $\pi\colon C' \to P$ be one of these admissible covers of degree k, that is, C' is stably equivalent to a certain fiber $C_1 \cup_{x\sim y} C_2$ of the family over $C_1 \times C_2$. Let us describe the admissible covering more precisely. Note that P is the union of two rational curves $P = (\mathbb{P}^1)_1 \cup (\mathbb{P}^1)_2$. Moreover, $\pi|_{C_1}\colon C_1 \to (\mathbb{P}^1)_1$ is the admissible covering of degree $k - \alpha_0$ defined by $l_{C_1}(-\alpha_0 x)$, $\pi|_{C_2}\colon C_2 \to (\mathbb{P}^1)_2$ is the admissible covering of degree $k - (k - 1 - \alpha_1) = \alpha_1 + 1$ defined by $l_{C_2}(-(k - 1 - \alpha_1)y)$, and π has ℓ-fold branching at $p := x \equiv y$ with $\ell := \alpha_1 + 1 - \alpha_0$. Finally there are α_0 copies of $\mathbb{P}^1$ over $(\mathbb{P}^1)_1$ and further $k - 1 - \alpha_1$ copies over $(\mathbb{P}^1)_2$.

Such a cover has no automorphisms, hence the corresponding point $[\pi\colon C' \to P]$ in the Hurwitz scheme $\overline{H}_{k,b}$ is smooth, and moreover such a point is not fixed by any $\sigma \in \Sigma_b$. Let us embed $\pi\colon C' \to P$ in a one-dimensional family of admissible coverings

$$\begin{tikzcd} \mathcal{C} \arrow[rr] \arrow[dr] & & \mathcal{P} \arrow[dl] \\ & B & \end{tikzcd}$$

where locally near the point p

$$\begin{aligned} &\mathcal{C} \text{ is } r \cdot s = t, \\ &\mathcal{P} \text{ is } u \cdot v = t^\ell, \\ &\pi \text{ is } u = r^\ell, \quad v = s^\ell \end{aligned}$$

and $B := \operatorname{Spec} \mathbb{C}[[t]]$. Now $\mathcal{C}$ is a smooth surface and after contracting the extra curves $\mathbb{P}^1$, we obtain a family $\mathcal{C} \to B$ in $\overline{\mathcal{M}}_g$ transverse to Δ_i at the point $[C']$. Hence (x, y) appears with multiplicity 1 in the intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with $C_1 \times C_2$.

Finally, if $i = 2$, then one has to take into account the automorphisms of the covers. To solve this, one has to work with the universal deformation space of the corresponding curve. The argument is similar (see [HM82, p. 80]).

For each i we deduce the following relation:

$$(2i-2)(2(g-i)-2)[2A_{\kappa_1^2} - A_{\omega^{(i)}} - A_{\omega^{(g-i)}}] = T_i.$$

Note that, if $i = g/2$, then $A_{\omega^{(i)}}$ and $A_{\omega^{(g-i)}}$ sum up.

(S2) Choose i, j such that $2 \leqslant i \leqslant j \leqslant g-3$ and $i + j \leqslant g - 1$. Take a general 2-pointed curve (F, p, q) of genus $g - i - j$ and attach at p a moving point on a general curve C_1 of genus i and at q a moving point on a general curve C_2 of genus j (see Figure 4).

The base of the family is $C_1 \times C_2$. To construct the family, consider $C_1 \times C_1 \times C_2$ and $C_1 \times C_2 \times C_2$ and identify $\Delta_{C_1} \times C_2$ and $C_1 \times \Delta_{C_2}$ with the general constant sections $p \times C_1 \times C_2$

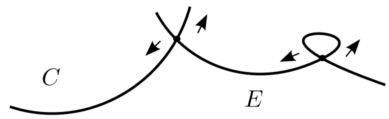


Figure 5. How the general fiber of a family in (S3) moves.

and $q \times C_1 \times C_2$ of $F \times C_1 \times C_2 \to C_1 \times C_2$. Denote this family by $X \to C_1 \times C_2$. Then

$$\delta_i = c_1(N_{(\Delta_{C_1} \times C_2)/X} \otimes N_{(p \times C_1 \times C_2)/X}) = -\pi_1^*(K_{C_1}),$$
$$\delta_j = c_1(N_{(C_1 \times \Delta_{C_2})/X} \otimes N_{(q \times C_1 \times C_2)/X}) = -\pi_2^*(K_{C_2})$$

and

$$\begin{aligned}\delta_{ij} &= c_1(N_{(\Delta_{C_1} \times C_2)/X} \otimes N_{(p \times C_1 \times C_2)/X}) \cdot c_1(N_{(C_1 \times \Delta_{C_2})/X} \otimes N_{(q \times C_1 \times C_2)/X}) \\ &= \pi_1^*(K_{C_1})\pi_2^*(K_{C_2}).\end{aligned}$$

We claim that the intersection of these test surfaces with $\overline{\mathcal{M}}^1_{2k,k}$ has degree

$$D_{ij} := \sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \beta=(\beta_0,\beta_1)\\ \rho(i,1,k,\alpha)=-1\\ \rho(j,1,k,\beta)=-1}} n_{i,k,\alpha}\, n_{j,k,\beta}\, N_{g-i-j,k,(k-1-\alpha_1,k-1-\alpha_0),(k-1-\beta_1,k-1-\beta_0)}$$

(in the sum, α and β are Schubert indices of type $1,k$). Indeed by § 5, if $\{l_{C_1}, l_F, l_{C_2}\}$ is a limit linear series of type $\mathfrak{g}^1_k$ on the fiber over some $(x,y) \in C_1 \times C_2$, then the only possibility is $\rho(C_1,x) = \rho(C_2,y) = -1$ while $\rho(F,p,q) = 0$. By §§ 3.1 and 3.2, there are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \beta=(\beta_0,\beta_1)\\ \rho(i,1,k,\alpha)=-1\\ \rho(j,1,k,\beta)=-1}} n_{i,k,\alpha}\, n_{j,k,\beta}$$

points (x,y) in $C_1 \times C_2$ with this property, the l_{C_1}, l_{C_2} are uniquely determined and there are

$$N := N_{g-i-j,k,(k-1-\alpha_1,k-1-\alpha_0),(k-1-\beta_1,k-1-\beta_0)}$$

choices for l_F. That is, there are N points of $\overline{H}_{k,b}/\Sigma_b$ over $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2] \in \overline{\mathcal{M}}^1_{2k,k}$ and $\overline{\mathcal{M}}^1_{2k,k}$ has N branches at $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2]$. The claim is thus equivalent to saying that each branch meets Δ_{ij} transversely at $[C_1 \cup_{x\sim p} F \cup_{y\sim q} C_2]$.

The argument is similar to the previous case. Let $\pi\colon C' \to D$ be an admissible cover of degree k with C' stably equivalent to a certain fiber of the family over $C_1 \times C_2$. The image of a general deformation of $[C' \to D]$ in $\overline{H}_{k,b}$ to the universal deformation space of C' meets Δ_{ij} only at $[C']$ and locally at the two nodes, the deformation space has equation $xy = t$. Hence $[C']$ is a transverse point of intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with Δ_{ij} and the surface $C_1 \times C_2$ and $\overline{\mathcal{M}}^1_{2k,k}$ meet transversally.

For i,j we obtain the following relation:

$$(2i-2)(2j-2)[2A_{\kappa_1^2} + A_{\delta_{ij}}] = D_{ij}.$$

(S3) Let (E,p,q) be a general 2-pointed elliptic curve. Identify the point q with a moving point x on E and identify the point p with a moving point on a general curve C of genus $g-2$ (see Figure 5).

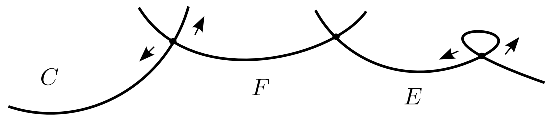


Figure 6. How the general fiber of a family in (S4) moves.

The base of the family is $E \times C$. To construct the family, let us start from the blow-up $\widetilde{E \times E}$ of $E \times E$ at the points (p,p) and (q,q). Denote by $\sigma_p, \sigma_q, \sigma_\Delta$ respectively the proper transforms of $p \times E$, $q \times E$, Δ_E. The family is the union of $\widetilde{E \times E} \times C$ and $E \times C \times C$ with $\sigma_q \times C$ identified with $\sigma_\Delta \times C$ and $\sigma_p \times C$ identified with $E \times \Delta_C$. We denote the family by $\pi\colon X \to E \times C$.

The study of the restriction of the generating classes in codimension one is similar to the case in the proof of Lemma 6. Namely

$$\delta_0 = -\pi_1^*(2q), \quad \delta_1 = \pi_1^*(q), \quad \delta_{g-2} = -\pi_1^*(p) - \pi_2^*(K_C).$$

Indeed, the family is entirely contained inside Δ_0: each fiber has a unique non-disconnecting node with the exception of the fibers over $p \times C$, which have two non-disconnecting nodes. Looking at the normalization of the family, fibers become smooth with the exception of the fibers over $p \times C$, which now have one non-disconnecting node, and the family is smooth at these points. It follows that $\delta_0 = \pi_*(\sigma_q \times C)^2 + \pi_*(\sigma_\Delta \times C)^2 + p \times C$. Only the fibers over $q \times C$ contain a node of type Δ_1, and the family is smooth at these points. Finally the family is entirely inside Δ_{g-2} and $\delta_{g-2} = \pi_*(\sigma_p \times C)^2 + \pi_*(E \times \Delta_C)^2$. We note the following

$$\delta_{1,g-2} = [\pi_1^*(q)][-\pi_2^*(K_C)], \quad \delta_{0,g-2} = [-\pi_1^*(2q)][-\pi_2^*(K_C)].$$

Let us study the intersection of this test surface with $\overline{\mathcal{M}}^1_{2k,k}$. Let $C' \to D$ be an admissible cover of degree k with C' stably equivalent to a certain fiber of the family. Clearly the only possibility is to map E and C to two different rational components of D with q and x in the same fiber, and have a 2-fold ramification at p. From Lemma 5 there are two possibilities for the point $x \in E$, and there are $n_{g-2,k,(0,1)}$ points in C where a degree k covering has a 2-fold ramification. In each case the covering is unique up to isomorphism. The combination of the two makes

$$2n_{g-2,k,(0,1)}$$

admissible coverings. We claim that they count with multiplicity 1.

The situation is similar to Lemma 5. The image of a general deformation of $[C' \to D]$ in $\overline{H}_{k,b}$ to the universal deformation space of C' meets $\Delta_{00} \cap \Delta_2$ only at $[C']$. Locally at the three nodes, the deformation space has equation $xy = t$. Hence $[C']$ is a transverse point of intersection of $\overline{\mathcal{M}}^1_{2k,k}$ with $\Delta_{00} \cap \Delta_2$ and counts with multiplicity 1 in the intersection of the surface $E \times C$ with $\overline{\mathcal{M}}^1_{2k,k}$.

We deduce the following relation:

$$(2(g-2)-2)[4A_{\kappa_1^2} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} - A_{\delta_{1,g-2}} + 2A_{\delta_{0,g-2}}] = 2n_{g-2,k,(0,1)}.$$

(S4) For $2 \leqslant i \leqslant g-3$, let (F, r, s) be a general 2-pointed curve of genus $g-i-2$. Let (E, p, q) be a general 2-pointed elliptic curve and, as above, identify the point q with a moving point x on E. Finally identify the point $p \in E$ with $r \in F$ and identify the point $s \in F$ with a moving point on a general curve C of genus i (see Figure 6).

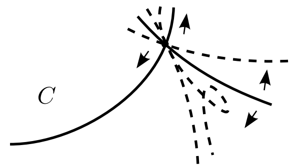


FIGURE 7. How the general fiber of a family in (S5) moves.

The base of the family is $E \times C$. Let $\widetilde{E \times E}, \sigma_p, \sigma_q, \sigma_\Delta$ be as above. Then the family is the union of $\widetilde{E \times E} \times C, E \times C \times C$ and $F \times E \times C$ with the following identifications. First, $\sigma_q \times C$ is identified with $\sigma_\Delta \times C$. Next, $\sigma_p \times E$ is identified with $r \times E \times C \subset F \times E \times C$, and $s \times E \times C \subset F \times E \times C$ with $E \times \Delta_C$.

The restriction of the generating classes in codimension one is

$$\delta_0 = -\pi_1^*(2q), \quad \delta_1 = \pi_1^*(q), \quad \delta_2 = -\pi_1^*(p), \quad \delta_i = -\pi_2^*(K_C)$$

and one has the following restrictions:

$$\begin{aligned} \delta_{1,i} &= [\pi_1^*(q)][-\pi_2^*(K_C)], \\ \delta_{0,i} &= [-\pi_1^*(2q)][-\pi_2^*(K_C)], \\ \delta_{2,i} &= [-\pi_1^*(p)][-\pi_2^*(K_C)]. \end{aligned}$$

Suppose $C' \to D$ is an admissible covering of degree k with C' stably equivalent to a certain fiber of this family. The only possibility is to map E, F, C to three different rational components of D, with a 2-fold ramification at r and ramification prescribed by $\alpha = (\alpha_0, \alpha_1)$ at s, such that $\rho(i, 1, k, \alpha) = -1$. The condition on α is equivalent to

$$\rho(g - i - 2, 1, k, (0, 1), (k - 1 - \alpha_1, k - 1 - \alpha_0)) = 0.$$

Moreover, q and x have to be in the same fiber of such a covering. There are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1) \\ \rho(i,1,k,\alpha)=-1}} 2n_{i,k,\alpha}$$

fibers which admit an admissible covering with such properties (in the sum, α is a Schubert index of type $1, k$). While the restriction of the covering to E and C is uniquely determined up to isomorphism, there are

$$N := N_{g-i-2,k,(0,1),(k-1-\alpha_1,k-1-\alpha_0)}$$

choices for the restriction to F up to isomorphism. As in (S2), this is equivalent to saying that $\overline{\mathcal{M}}^1_{2k,k}$ has N branches at $[C']$. Moreover, each branch meets the boundary transversally at $[C']$ (similarly to (S3)), hence $[C']$ counts with multiplicity 1 in the intersection of $E \times C$ with $\overline{\mathcal{M}}^1_{2k,k}$.

Finally, for each i we deduce the following relation:

$$(2i-2)[4A_{\kappa_1^2} - A_{\delta_{1,i}} + 2A_{\delta_{0,i}} + A_{\delta_{2,i}}] = \sum_{\substack{\alpha=(\alpha_0,\alpha_1) \\ \rho(i,1,k,\alpha)=-1}} 2N_{g-i-2,k,(0,1),(k-1-\alpha_1,k-1-\alpha_0)} \cdot n_{i,k,\alpha}.$$

(S5) Identify a base point of a generic pencil of plane cubic curves with a moving point on a general curve C of genus $g-1$ (see Figure 7).

The base of the family is $\mathbb{P}^1 \times C$. Let us construct this family. We start from an elliptic pencil $Y \to \mathbb{P}^1$ of degree 12 with zero section σ. To construct Y, blow up $\mathbb{P}^2$ in the nine points

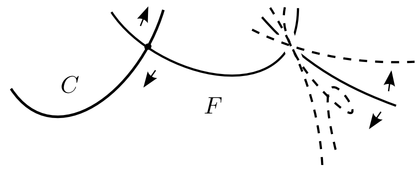


Figure 8. How the general fiber of a family in (S6) moves.

of intersection of two general cubics. Then consider $Y \times C$ and $\mathbb{P}^1 \times C \times C$ and identify $\sigma \times C$ with $\mathbb{P}^1 \times \Delta_C$. Let x be the class of a point in $\mathbb{P}^1$. Then

$$\lambda = \pi_1^*(x), \quad \delta_0 = 12\lambda, \quad \delta_1 = -\pi_1^*(x) - \pi_2^*(K_C).$$

Note that

$$\delta_{0,g-1} = [12\pi_1^*(x)][-\pi_2^*(K_C)].$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. Indeed, C has no linear series with adjusted Brill–Noether number less than -1 at some point, and an elliptic curve or a rational nodal curve has no (generalized) linear series with adjusted Brill–Noether number less than 0 at some point. Adding, we see that no fiber of the family has a linear series with Brill–Noether number less than -1, hence

$$(2(g-1)-2)[2A_{\kappa_1^2} - 12A_{\delta_{0,g-1}} + 2A_{\delta_1^2} - A_{\lambda\delta_1}] = 0.$$

(S6) For $3 \leqslant i \leqslant g-3$ take a general curve F of genus $i-1$ and attach at a general point p an elliptic tail varying in a pencil of degree 12 and at another general point a moving point on a general curve C of genus $g-i$ (see Figure 8).

The base of the family is $\mathbb{P}^1 \times C$. In order to construct the family, start from $Y \times C$ and $\mathbb{P}^1 \times C \times C$ and then identify $\sigma \times C$ and $\mathbb{P}^1 \times \Delta_C$ with two general constant sections of $F \times \mathbb{P}^1 \times C \to \mathbb{P}^1 \times C$. Here Y, σ are as above. Then

$$\lambda = \pi_1^*(x), \quad \delta_0 = 12\lambda, \quad \delta_1 = -\pi_1^*(x), \quad \delta_{g-i} = -\pi_2^*(K_C).$$

Note that

$$\delta_{1,g-i} = [-\pi_1^*(x)][-\pi_2^*(K_C)], \quad \delta_{0,g-i} = [12\pi_1^*(x)][-\pi_2^*(K_C)].$$

Again C has no linear series with adjusted Brill–Noether number less than -1 at some point, an elliptic curve or a rational nodal curve has no (generalized) linear series with adjusted Brill–Noether number less than 0 at some point and F has no linear series with adjusted Brill–Noether number less than 0 at some general points. Adding, we see that no fiber of the family has a linear series with Brill–Noether number less than -1, hence

$$(2(g-i)-2)[2A_{\kappa_1^2} - A_{\lambda^{(i)}} + A_{\delta_{1,g-i}} - 12A_{\delta_{0,g-i}}] = 0.$$

In case $i = g-2$ we have

$$2[2A_{\kappa_1^2} - A_{\lambda\delta_2} + A_{\delta_{1,2}} - 12A_{\delta_{0,2}}] = 0.$$

(S7) Let (E_1, p_1, q_1) and (E_2, p_2, q_2) be two general pointed elliptic curves. Identify the point q_i with a moving point x_i in E_i, for $i = 1, 2$. Then identify p_1 and p_2 with two general points r_1, r_2 on a general curve F of genus $g-4$ (see Figure 9).

The base of the family is $E_1 \times E_2$. For $i = 1, 2$, let $\widetilde{E_i \times E_i}$ be the blow-up of $E_i \times E_i$ at (p_i, p_i) and (q_i, q_i). Denote by $\sigma_{p_i}, \sigma_{q_i}, \sigma_{\Delta_{E_i}}$ the proper transforms of $p_i \times E_i, q_i \times E_i, \Delta_{E_i}$, respectively.

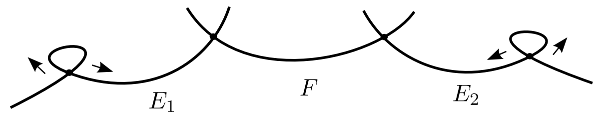


Figure 9. How the general fiber of a family in (S7) moves.

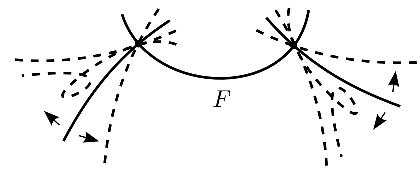


Figure 10. How the general fiber of a family in (S8) moves.

The family is the union of $\widetilde{E_1 \times E_1} \times E_2$, $E_1 \times \widetilde{E_2 \times E_2}$ and $F \times E_1 \times E_2$ with the following identifications. First, $\sigma_{q_1} \times E_2$ and $E_1 \times \sigma_{q_2}$ are identified with $\sigma_{\Delta_{E_1}} \times E_2$ and $E_1 \times \sigma_{\Delta_{E_2}}$, respectively. Then $\sigma_{p_1} \times E_2$ and $E_1 \times \sigma_{p_2}$ are identified with $r_1 \times E_1 \times E_2$ and $r_2 \times E_1 \times E_2$, respectively. We deduce that

$$\delta_0 = -\pi_1^*(2q_1) - \pi_2^*(2q_2),$$
$$\delta_1 = \pi_1^*(q_1) + \pi_2^*(q_2),$$
$$\delta_2 = -\pi_1^*(p_1) - \pi_2^*(p_2)$$

and we note that

$$\delta_{2,2} = \pi_1^*(p_1)\pi_2^*(p_2),$$
$$\delta_{1,2} = -\pi_1^*(q_1)\pi_2^*(p_2) - \pi_2^*(q_2)\pi_1^*(p_1),$$
$$\delta_{1,1} = \pi_1^*(q_1)\pi_2^*(q_2),$$
$$\delta_{00} = \pi_1^*(2q_1)\pi_2^*(2q_2),$$
$$\delta_{02} = \pi_1^*(2q_1)\pi_2^*(p_2) + \pi_2^*(2q_2)\pi_1^*(p_1),$$
$$\delta_{01} = -\pi_1^*(q_1)\pi_2^*(2q_2) - \pi_2^*(q_2)\pi_1^*(2q_1).$$

If a fiber of this family admits an admissible cover of degree k, then r_1 and r_2 have to be 2-fold ramification points, and q_i and x_i have to be in the same fiber, for $i = 1, 2$. From Lemma 5 there are only 4 fibers with this property, namely the fibers over (p_1, p_2), $(p_1, \overline{q}_2)$, $(\overline{q}_1, p_2)$ and $(\overline{q}_1, \overline{q}_2)$, where $\overline{q}_i$ is such that $2p_i \equiv q_i + \overline{q}_i$ for $i = 1, 2$.

In these cases, the restriction of the covers to E_1, E_2 is uniquely determined up to isomorphism, while there are $N_{g-4,k,(0,1),(0,1)}$ choices for the restriction to F up to isomorphism. As for (S3), such covers contribute with multiplicity 1, hence we have the following relation:

$$8A_{\kappa_1^2} + A_{\delta_{2,2}} - 2A_{\delta_{1,2}} + A_{\delta_{1,1}} + 2A_{\delta_1^2} + 8A_{\delta_0^2} + 4A_{\delta_{00}} + 4A_{\delta_{02}} - 4A_{\delta_{01}} = 4N_{g-4,k,(0,1),(0,1)}.$$

(S8) Consider a general curve F of genus $g-2$ and, at two general points, attach elliptic tails varying in pencils of degree 12 (see Figure 10).

The base of the family is $\mathbb{P}^1 \times \mathbb{P}^1$. Let us construct the family. Let $Y \to \mathbb{P}^1$ and $Y' \to \mathbb{P}^1$ be two elliptic pencils of degree 12, and let σ and σ' be the respective zero sections. Consider $Y \times \mathbb{P}^1$

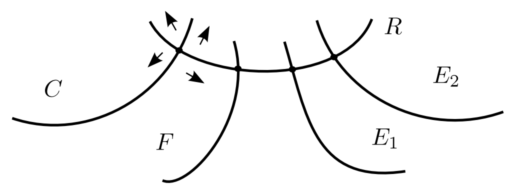


Figure 11. How the general fiber of a family in (S9) moves.

and $\mathbb{P}^1 \times Y'$ and identify $\sigma \times \mathbb{P}^1$ and $\mathbb{P}^1 \times \sigma'$ with two general constant sections of $F \times \mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^1$. If x is the class of a point in $\mathbb{P}^1$, then

$$\lambda = \pi_1^*(x) + \pi_2^*(x), \quad \delta_0 = 12\lambda, \quad \delta_1 = -\lambda.$$

Note that

$$\begin{aligned} \delta_{00} &= [12\pi_1^*(x)][12\pi_2^*(x)], \\ \delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(x)], \\ \delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(x)] + [-\pi_1^*(x)][12\pi_2^*(x)]. \end{aligned}$$

Studying the possibilities for the adjusted Brill–Noether numbers of the aspects of limit linear series on some fiber of this family, we see that this surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$2A_{\kappa_1^2} + 288A_{\delta_0^2} + 24A_{\lambda\delta_0} + 2A_{\delta_1^2} - 2A_{\lambda\delta_1} + 144A_{\delta_{00}} + A_{\delta_{1,1}} - 24A_{\delta_{01}} = 0.$$

(S9) For $2 \leqslant j \leqslant g-3$ let R be a smooth rational curve, attach at the point $\infty \in R$ a general curve F of genus $g-j-2$, attach at the points $0, 1 \in R$ two elliptic tails E_1, E_2 and identify a moving point in R with a moving point on a general curve C of genus j (see Figure 11).

The base of the family is $R \times C$. Let us start from a family $P \to R$ of 4-pointed rational curves. Construct P by blowing up $\mathbb{P}^1 \times \mathbb{P}^1$ at $(0,0)$, $(1,1)$ and (∞,∞), and consider the sections $\sigma_0, \sigma_1, \sigma_\infty$ and σ_Δ corresponding to the proper transforms of $0 \times \mathbb{P}^1$, $1 \times \mathbb{P}^1$, $\infty \times \mathbb{P}^1$ and $\Delta_{\mathbb{P}^1}$.

To construct the family over $R \times C$, consider $P \times C$ and $R \times C \times C$. Identify $\sigma_\Delta \times C$ with $R \times \Delta_C$. Finally identify $\sigma_0 \times C, \sigma_1 \times C$ and $\sigma_\infty \times C$ respectively with general constant sections of the families $E_1 \times R \times C, E_2 \times R \times C$ and $F \times R \times C$. Then

$$\begin{aligned} \delta_1 &= -\pi_1^*(0+1), \\ \delta_2 &= \pi_1^*(\infty), \\ \delta_j &= -\pi_1^*(K_{\mathbb{P}^1} + 0 + 1 + \infty) - \pi_2^*(K_C), \\ \delta_{g-j-2} &= -\pi_1^*(\infty), \\ \delta_{g-j-1} &= \pi_1^*(0+1). \end{aligned}$$

If for some value of j some of the above classes coincide (for instance, if $j = g-3$ then $\delta_1 \equiv \delta_{g-j-2}$), then one has to sum up the contributions. Note that

$$\begin{aligned} \delta_{1j} &= [-\pi_1^*(0+1)][-\pi_2^*(K_C)], \\ \delta_{j,g-j-2} &= [-\pi_1^*(\infty)][-\pi_2^*(K_C)], \\ \delta_{2,j} &= [\pi_1^*(\infty)][-\pi_2^*(K_C)], \\ \delta_{j,g-j-1} &= [\pi_1^*(0+1)][-\pi_2^*(K_C)]. \end{aligned}$$

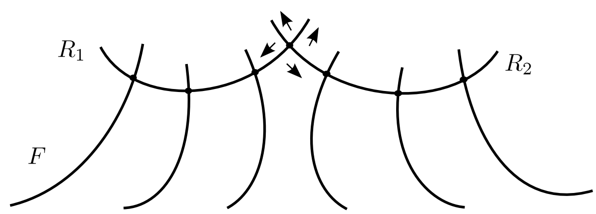


Figure 12. How the general fiber of a family in (S10) moves.

As for (S8), this surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$(2j-2)[2A_{\kappa_1^2}+2A_{\delta_{1j}}+A_{\delta_{j,g-j-2}}-A_{\delta_{2,j}}-2A_{\delta_{j,g-j-1}}-A_{\omega^{(j)}}-A_{\omega^{(g-j)}}]=0.$$

Again, let us remark that for some value of j, some terms add up.

(S10) Let $(R_1,0,1,\infty)$ and $(R_2,0,1,\infty)$ be two 3-pointed smooth rational curves, identify a moving point on R_1 with a moving point on R_2, attach a general pointed curve F of genus $g-5$ to $\infty \in R_2$ and attach elliptic tails to all the other marked points (see Figure 12).

The base of the family is $R_1 \times R_2$. First construct two families of 4-pointed rational curves $P_1 \to R_1$ and $P_2 \to R_2$ respectively with sections $\sigma_0,\sigma_1,\sigma_\infty,\sigma_\Delta$ and $\tau_0,\tau_1,\tau_\infty,\tau_\Delta$ as for the previous surface. Consider $P_1 \times R_2$ and $R_1 \times P_2$. Identify $\sigma_\Delta \times R_2$ with $R_1 \times \tau_\Delta$. Finally identify $R_1 \times \tau_\infty$ with a general constant section of $F \times R_1 \times R_2$ and identify $\sigma_0 \times R_2, \sigma_1 \times R_2, \sigma_\infty \times R_2, R_1 \times \tau_0, R_1 \times \tau_1$ with the respective zero sections of five constant elliptic fibrations over $R_1 \times R_2$.

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. For $g>8$

$$\begin{aligned}
\delta_1 &= -\pi_1^*(0+1+\infty)-\pi_2^*(0+1),\\
\delta_2 &= \pi_1^*(0+1+\infty)+\pi_2^*(\infty),\\
\delta_3 &= -\pi_1^*(K_{R_1}+0+1+\infty)-\pi_2^*(K_{R_2}+0+1+\infty),\\
\delta_{g-5} &= -\pi_2^*(\infty),\\
\delta_{g-4} &= \pi_2^*(0+1)
\end{aligned}$$

and note the restriction of the following classes

$$\begin{aligned}
\delta_{1,1} &= [-\pi_1^*(0+1+\infty)][-\pi_2^*(0+1)],\\
\delta_{1,g-5} &= [-\pi_1^*(0+1+\infty)][-\pi_2^*(\infty)],\\
\delta_{1,3} &= [-\pi_1^*(K_{R_1}+0+1+\infty)][-\pi_2^*(0+1)],\\
\delta_{3,g-5} &= [-\pi_1^*(K_{R_1}+0+1+\infty)][-\pi_2^*(\infty)],\\
\delta_{1,g-3} &= [-\pi_1^*(0+1+\infty)][-\pi_2^*(K_{R_2}+0+1+\infty)],\\
\delta_{2,g-3} &= [\pi_1^*(0+1+\infty)][-\pi_2^*(K_{R_2}+0+1+\infty)],\\
\delta_{2,g-5} &= [\pi_1^*(0+1+\infty)][-\pi_2^*(\infty)],\\
\delta_{1,2} &= [\pi_1^*(0+1+\infty)][-\pi_2^*(0+1)]+[-\pi_1^*(0+1+\infty)][\pi_2^*(\infty)],\\
\delta_{1,g-4} &= [-\pi_1^*(0+1+\infty)][\pi_2^*(0+1)],
\end{aligned}$$

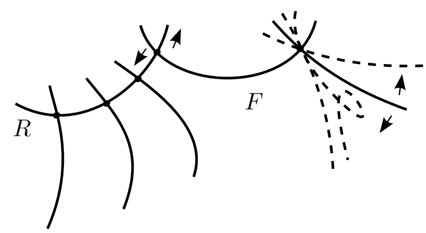


Figure 13. How the general fiber of a family in (S11) moves.

$$\begin{aligned}
\delta_{3,g-4} &= [-\pi_1^*(K_{R_1}+0+1+\infty)][\pi_2^*(0+1)],\\
\delta_{2,3} &= [-\pi_1^*(K_{R_1}+0+1+\infty)][\pi_2^*(\infty)],\\
\delta_{2,g-4} &= [\pi_1^*(0+1+\infty)][\pi_2^*(0+1)],\\
\delta_{2,2} &= [\pi_1^*(0+1+\infty)][\pi_2^*(\infty)].
\end{aligned}$$

It follows that

$$\begin{aligned}
&2A_{\kappa_1^2}+12A_{\delta_1^2}+6A_{\delta_{1,1}}+3A_{\delta_{1,g-5}}+2A_{\delta_{1,3}}+A_{\delta_{3,g-5}}+3A_{\delta_{1,g-3}}\\
&\quad -A_{\omega^{(3)}}-A_{\omega^{(g-3)}}-3(A_{\delta_{2,g-3}}+A_{\delta_{2,g-5}}+2A_{\delta_{1,2}})\\
&\quad -2(3A_{\delta_{1,g-4}}+A_{\delta_{3,g-4}})-(3A_{\delta_{1,2}}+A_{\delta_{2,3}})+6A_{\delta_{2,g-4}}+3A_{\delta_{2,2}}=0.
\end{aligned}$$

For $g=6$ the coefficient of $A_{\delta_1^2}$ is 18. When $g\in\{6,7,8\}$, note that some terms add up.

(S11) Consider a general curve F of genus $g-4$, attach at a general point an elliptic tail varying in a pencil of degree 12 and identify a second general point with a moving point on a rational 3-pointed curve $(R,0,1,\infty)$. Attach elliptic tails at the marked point on the rational curve (see Figure 13).

The base of the family is $\mathbb{P}^1\times R$. Consider the elliptic fibration Y over $\mathbb{P}^1$ with zero section σ as in (S5), and the family P over R with sections $\sigma_0,\sigma_1,\sigma_\infty,\sigma_\Delta$ as in (S9). Identify $\sigma\times R\subset Y\times R$ and $\mathbb{P}^1\times\sigma_\Delta\subset\mathbb{P}^1\times P$ with two general constant sections of $F\times\mathbb{P}^1\times R$. Finally identify $\mathbb{P}^1\times\sigma_0,\mathbb{P}^1\times\sigma_1,\mathbb{P}^1\times\sigma_\infty\subset\mathbb{P}^1\times P$ with the respective zero sections of three constant elliptic fibrations over $\mathbb{P}^1\times R$. Then

$$\lambda=\pi_1^*(x),$$
$$\delta_0=12\lambda,\quad \delta_1=-\pi_1^*(x)-\pi_2^*(0+1+\infty),$$
$$\delta_2=\pi_2^*(0+1+\infty),\quad \delta_3=-\pi_2^*(K_R+0+1+\infty).$$

Note the restriction of the following classes

$$\begin{aligned}
\delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(0+1+\infty)],\\
\delta_{1,3} &= [-\pi_1^*(x)][-\pi_2^*(K_R+0+1+\infty)],\\
\delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(0+1+\infty)],\\
\delta_{03} &= [12\pi_1^*(x)][-\pi_2^*(K_R+0+1+\infty)],\\
\delta_{02} &= [12\pi_1^*(x)][\pi_2^*(0+1+\infty)],\\
\delta_{1,2} &= [-\pi_1^*(x)][\pi_2^*(0+1+\infty)].
\end{aligned}$$

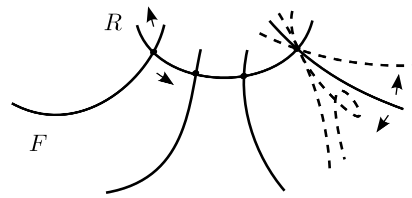


Figure 14. How the general fiber of a family in (S12) moves.

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$2A_{\kappa_1^2} - A_{\lambda^{(g-3)}} + 6A_{\delta_1^2} + 3A_{\delta_{1,1}} - 3A_{\lambda\delta_1} + A_{\delta_{1,3}} - 36A_{\delta_{01}} - 12A_{\delta_{03}} + 3[A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}] = 0.$$

(S12) Let R be a rational curve, attach two fixed elliptic tails at the points 0 and 1, attach at the point ∞ an elliptic tail moving in a pencil of degree 12 and identify a moving point in R with a general point on a general curve F of genus $g-3$ (see Figure 14).

The base of the family is $\mathbb{P}^1 \times R$. Let Y, σ and $P, \sigma_0, \sigma_1, \sigma_\infty, \sigma_\Delta$ be as above. Identify $\sigma \times R \subset Y \times R$ with $\mathbb{P}^1 \times \sigma_\infty \subset \mathbb{P}^1 \times P$, and $\mathbb{P}^1 \times \sigma_\Delta \subset \mathbb{P}^1 \times P$ with a general constant section of $F \times \mathbb{P}^1 \times R$. Finally identify $\mathbb{P}^1 \times \sigma_0, \mathbb{P}^1 \times \sigma_1$ with the zero sections of two constant elliptic fibrations over $\mathbb{P}^1 \times R$. Then

$$\begin{gathered} \lambda = \pi_1^*(x), \\ \delta_0 = 12\lambda, \quad \delta_1 = -\pi_1^*(x) - \pi_2^*(\infty + 0 + 1), \\ \delta_2 = \pi_2^*(\infty + 0 + 1), \quad \delta_3 = -\pi_2^*(K_{\mathbb{P}^1} + 0 + 1 + \infty). \end{gathered}$$

Let us note the following restrictions

$$\begin{aligned} \delta_{01} &= [12\pi_1^*(x)][-\pi_2^*(0+1)], \\ \delta_{0,g-3} &= [12\pi_1^*(x)][-\pi_2^*(K_{\mathbb{P}^1} + 0 + 1 + \infty)], \\ \delta_{0,g-1} &= [12\pi_1^*(x)][-\pi_2^*(\infty)], \\ \delta_{1,1} &= [-\pi_1^*(x)][-\pi_2^*(0+1)], \\ \delta_{1,g-3} &= [-\pi_1^*(x)][-\pi_2^*(K_{\mathbb{P}^1} + 0 + 1 + \infty)], \\ \delta_{0,g-2} &= [12\pi_1^*(x)][\pi_2^*(0+1)], \\ \delta_{1,g-2} &= [-\pi_1^*(x)][\pi_2^*(0+1)], \\ \delta_{02} &= [12\pi_1^*(x)][\pi_2^*(\infty)], \\ \delta_{1,2} &= [-\pi_1^*(x)][\pi_2^*(\infty)]. \end{aligned}$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$2A_{\kappa_1^2} - 3A_{\lambda\delta_1} - 24A_{\delta_{01}} - 12A_{\delta_{0,g-3}} - 12A_{\delta_{0,g-1}} + 6A_{\delta_1^2} + 2A_{\delta_{1,1}} + A_{\delta_{1,g-3}} - A_{\lambda^{(3)}} + 2(A_{\lambda\delta_2} + 12A_{\delta_{0,g-2}} - A_{\delta_{1,g-2}}) + (A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}) = 0.$$

(S13) Let (C, p, q) be a general 2-pointed curve of genus $g-3$ and identify the point q with a moving point x on C. Let (E, r, s) be a general 2-pointed elliptic curve and identify the point s with a moving point y on E. Finally identify the points p and r (see Figure 15).

The base of the family is $C \times E$. Let $\widetilde{C \times C}$ (respectively $\widetilde{E \times E}$) be the blow-up of $C \times C$ at (p, p) and (q, q) (respectively of $E \times E$ at (r, r) and (s, s)). Let $\tau_p, \tau_q, \tau_\Delta$ (respectively $\sigma_r, \sigma_s, \sigma_\Delta$)

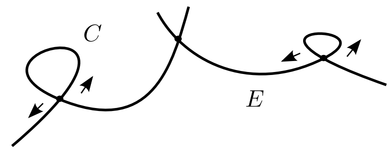


Figure 15. How the general fiber of a family in (S13) moves.

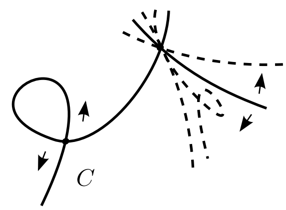


Figure 16. How the general fiber of a family in (S14) moves.

be the proper transform of $p \times C, q \times C, \Delta_C$ (respectively $r \times E, s \times E, \Delta_E$) and identify τ_q with τ_Δ (respectively σ_s with σ_Δ). Finally identify $\tau_p \times E$ with $C \times \sigma_r$. Then from the proof of Lemma 6, we have

$$\begin{aligned}\delta_0 &= -\pi_1^*(K_C + 2q) - \pi_2^*(2s),\\ \delta_1 &= \pi_1^*(q) + \pi_2^*(s),\\ \delta_2 &= -\pi_1^*(p) - \pi_2^*(r)\end{aligned}$$

and note that

$$\begin{aligned}\delta_{00} &= [-\pi_1^*(K_C + 2q)][-\pi_2^*(2s)],\\ \delta_{02} &= [-\pi_1^*(K_C + 2q)][-\pi_2^*(r)],\\ \delta_{0,g-2} &= [-\pi_2^*(2s)][-\pi_1^*(p)],\\ \delta_{01} &= [-\pi_1^*(K_C + 2q)][\pi_2^*(s)] + [-\pi_2^*(2s)][\pi_1^*(q)],\\ \delta_{1,g-2} &= [-\pi_1^*(p)][\pi_2^*(s)],\\ \delta_{1,2} &= [\pi_1^*(q)][-\pi_2^*(r)],\\ \delta_{1,1} &= [\pi_1^*(q)][\pi_2^*(s)].\end{aligned}$$

If a fiber of this family admits an admissible covering of degree k, then such a covering has a 2-fold ramification at the point $p \sim r$, q is in the same fiber as x, and s is in the same fiber as y. By Lemmas 5 and 6 there are two points in E and $\ell_{g-2,k}$ points in C with such a property, and the cover is unique up to isomorphism. Reasoning as in (S3), one shows that each cover contributes with multiplicity 1. It follows that

$$\begin{aligned}2(g-3)[4A_{\kappa_1^2} + 2A_{\delta_{00}} + 4A_{\delta_0^2} + A_{\delta_{02}}] + 2A_{\delta_{0,g-2}} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}}\\ - [2(g-3)A_{\delta_{01}} + A_{\delta_{1,g-2}}] - [2A_{\delta_{01}} + A_{\delta_{1,2}}] + [A_{\delta_{1,1}} + 2A_{\delta_1^2}] = 2 \cdot \ell_{g-2,k}.\end{aligned}$$

(S14) Let (C, p, q) be a general 2-pointed curve of genus $g-2$, attach at p an elliptic tail moving in a pencil of degree 12 and identify q with a moving point on C (see Figure 16).

The base of this family is $C \times \mathbb{P}^1$. Let $\widetilde{C \times C}$ be the blow-up of $C \times C$ at the points (p, p) and (q, q). Let $\tau_p, \tau_q, \tau_\Delta$ be the proper transform of $p \times C, q \times C, \Delta$ and identify τ_q with τ_Δ.

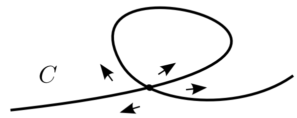


Figure 17. How the general fiber of a family in (S15) moves.

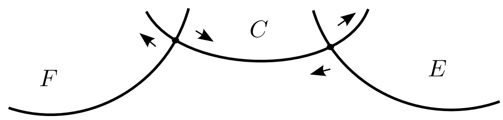


Figure 18. How the general fiber of a family in (S16) moves.

Then consider Y, σ as in (S5) and identify $C \times \sigma$ with $\tau_p \times \mathbb{P}^1$. Then

$$\lambda = \pi_2^*(x), \quad \delta_0 = 12\lambda - \pi_1^*(K_C + 2q), \quad \delta_1 = \pi_1^*(q) - \pi_1^*(p) - \lambda.$$

Note that

$$\begin{aligned} \delta_{00} &= [12\pi_2^*(x)][-\pi_1^*(K_C + 2q)], \\ \delta_{01} &= [\pi_1^*(q)][12\pi_2^*(x)] + [-\pi_1^*(K_C + 2q)][-\pi_2^*(x)], \\ \delta_{0,g-1} &= [-\pi_1^*(p)][12\pi_2^*(x)], \\ \delta_{1,1} &= [\pi_1^*(q)][-\pi_2^*(x)]. \end{aligned}$$

This surface is disjoint from $\overline{\mathcal{M}}^1_{2k,k}$, hence

$$(2g-4)[2A_{\kappa_1^2} - A_{\lambda\delta_0} - 24A_{\delta_0^2} - 12A_{\delta_{00}} + A_{\delta_{01}}] - 12A_{\delta_{0,g-1}} + (12A_{\delta_{01}} - A_{\delta_{1,1}}) = 0.$$

(S15) Let C be a general curve of genus $g-1$ and consider the surface $C \times C$ with fiber $C/(p \sim q)$ over (p, q) (see Figure 17).

To construct the family, start from $p_{2,3}\colon C \times C \times C \to C \times C$, blow up the diagonal $\Delta \subset C \times C \times C$ and then identify the proper transform of $\Delta_{1,2} := p_{1,2}^*(\Delta)$ with the proper transform of $\Delta_{1,3} := p_{1,3}^*(\Delta)$. Then

$$\delta_0 = -(\pi_1^* K_C + \pi_2^* K_C + 2\Delta), \quad \delta_1 = \Delta.$$

The class κ_2 has been computed in [Fab90a, § 2.1(1)]. The curve C has no generalized linear series with Brill–Noether number less than zero, hence

$$(8g^2 - 26g + 20)A_{\kappa_1^2} + (2g-4)A_{\kappa_2} + (4-2g)A_{\delta_1^2} + 8(g-1)(g-2)A_{\delta_0^2} = 0.$$

(S16) For $\lfloor g/2 \rfloor \leqslant i \leqslant g-2$, take a general curve C of genus i and attach an elliptic curve E and a general pointed curve F of genus $g-i-1$ at two varying points in C (see Figure 18).

To construct the family, blow up the diagonal Δ in $C \times C \times C$ as before, and then identify the proper transform of $\Delta_{1,2}$ with the zero section of a constant elliptic fibration over $C \times C$, and identify the proper transform of $\Delta_{1,3}$ with a general constant section of $F \times C \times C$. For $i < g-2$

$$\delta_1 = -\pi_1^* K_C - \Delta, \quad \delta_{g-i-1} = -\pi_2^* K_C - \Delta, \quad \delta_i = \Delta$$

while for $i = g-2$ the δ_1 is the sum of the above δ_1 and δ_{g-i-1}.

Note that replacing the tail of genus $g-i-1$ with an elliptic tail does not affect the computation of the class κ_2, hence we can use the count from [Fab90b, § 3(γ)], that is $\kappa_2 = 2i - 2$.

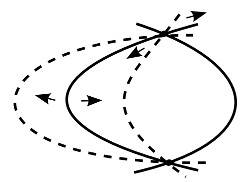


Figure 19. How the general fiber of a family in (S17) moves.

Regarding the ω classes, on these test surfaces one has $\omega^{(i)} = -\delta_i^2$ and $\omega^{(i+1)} = -\delta_{i+1}^2 = -\delta_{g-i-1}^2$. Finally note that $\delta_{1,g-i-1}$ is the product of the classes c_1 coming from the two nodes, that is, $\delta_{1,g-i-1} = \delta_1 \delta_{g-i-1}$.

If a fiber of this family has a $\mathfrak{g}^1_k$ limit linear series $\{l_E, l_C, l_F\}$, then necessarily the adjusted Brill–Noether number has to be zero on F and E, and -2 on C. Note that in any case $l_E = |2 \cdot 0_E|$. From § 3.3 there are

$$\sum_{\substack{\alpha=(\alpha_0,\alpha_1)\\ \rho(i,1,k,\alpha)=-1}} m_{i,k,\alpha}$$

pairs in C with such a property, l_C is also uniquely determined and there are

$$N_{g-i-1,d,(d-1-\alpha_1,d-1-\alpha_0)}$$

choices for l_F. With a similar argument to (S2), such pairs contribute with multiplicity 1.

All in all for $i < g-2$

$$\begin{aligned}(2i-2)&[(4i-1)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(i)}} - A_{\omega^{(i+1)}} + A_{\delta_1^2} + (2i-1)A_{\delta_{1,g-i-1}}]\\ &= \sum_{\substack{0\leqslant\alpha_0\leqslant\alpha_1\leqslant k-1\\ \alpha_0+\alpha_1=g-i-1}} m_{i,k,(\alpha_0,\alpha_1)} \cdot N_{g-i-1,k,(k-1-\alpha_1,k-1-\alpha_0)},\end{aligned}$$

while for $i = g-2$

$$(2g-6)[(4g-9)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(g-2)}} + (4g-8)A_{\delta_1^2} + (2g-5)A_{\delta_{1,1}}] = m_{g-2,k,(0,1)}.$$

(S17) Consider a general element in θ_1, vary the elliptic curve in a pencil of degree 12 and vary one point on the elliptic curve (see Figure 19).

The base of this family is the blow-up of $\mathbb{P}^2$ in the nine points of intersection of two general cubic curves. Let us denote by H the pull-back of an hyperplane section in $\mathbb{P}^2$, by Σ the sum of the nine exceptional divisors and by E_0 one of the exceptional divisors. We have

$$\lambda = 3H - \Sigma, \quad \delta_0 = 30H - 10\Sigma - 2E_0, \quad \delta_1 = E_0$$

(see also [Fab89, § 2(9)]). Replacing the component of genus $g-2$ with a curve of genus 2, we obtain a surface in $\overline{\mathcal{M}}_4$. The computation of the class κ_2 remains unaltered, that is $\kappa_2 = 1$ (see [Fab90b, § 3(ι)]). Similarly for δ_{00} and θ_1, while $\delta_{0,g-1}$ corresponds to the value of δ_{01a} on the surface in $\overline{\mathcal{M}}_4$.

Let us study the intersection with $\overline{\mathcal{M}}^1_{2k,k}$. An admissible cover for some fiber of this family would necessarily have the two nodes in the same fiber, which is impossible, since the two points are general on the component of genus $g-2$. We deduce the following relation:

$$3A_{\kappa_1^2} + A_{\kappa_2} - 2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} - A_{\delta_1^2} + 12A_{\delta_{0,g-1}} - 12A_{\delta_{00}} + A_{\theta_1} = 0.$$

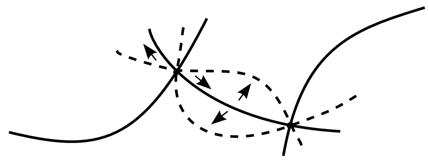


Figure 20. How the general fiber of a family in (S18) moves.

(S18) For $2 \leqslant i \leqslant \lfloor (g+1)/2 \rfloor$ we consider a general curve of type $\delta_{i-1,g-i}$ and we vary the central elliptic curve E in a pencil of degree 12 and one of the points on E (see Figure 20).

The base of this family is the same surface as in (S17). For $i \geqslant 3$ we have

$$\lambda = 3H - \Sigma,$$
$$\delta_0 = 12\lambda, \quad \delta_1 = E_0,$$
$$\delta_{i-1} = -3H + \Sigma - E_0, \quad \delta_{g-i} = -3H + \Sigma - E_0$$

while for $i = 2$ the δ_1 is the sum of the above δ_1 and δ_{i-1}, that is $\delta_1 = -3H + \Sigma$ (see also [Fab90b, § 3(λ)]).

Note that replacing the two tails of genus $i-1$ and $g-i$ with tails of genus 1 and 2, we obtain a surface in $\overline{\mathcal{M}}_4$. The computation of the class κ_2 remains unaltered, that is $\kappa_2 = 1$ (see [Fab90b, § 3(λ)]). Moreover, on these test surfaces $\omega^{(i)} = -\delta_i^2 = -\delta_{g-i}^2$ and for $i \geqslant 3$, $\omega^{(g-i+1)} = -\delta_{g-i+1}^2 = -\delta_{i-1}^2$ holds, while $\lambda^{(i)} = \lambda\delta_i = \lambda\delta_{g-i}$ for $i \geqslant 3$ and $\lambda^{(g-i+1)} = \lambda\delta_{g-i+1} = \lambda\delta_{i-1}$ for $i \geqslant 4$. All fibers are in $\delta_{i-1,g-i}$, hence $\delta_{i-1,g-i}$ is the product of the classes c_1 of the two nodes, that is, $\delta_{i-1,g-i} = \delta_{i-1} \cdot \delta_{g-i}$. Note that on these surfaces, $\delta_{0,i-1} = \delta_0\delta_{i-1}$ and $\delta_{0,g-i} = \delta_0\delta_{g-i}$. There are exactly 12 fibers which contribute to θ_{i-1}, namely when the elliptic curve degenerates into a rational nodal curve and the moving point hits the non-disconnecting node. Similarly, there are 12 fibers which contribute to $\delta_{0,g-1}$, namely when the elliptic curve degenerates into a rational nodal curve and the moving point hits the disconnecting node.

These surfaces are disjoint from $\overline{\mathcal{M}}^1_{2k,k}$. Indeed the two tails of genus $i-1$ and $g-i$ have no linear series with adjusted Brill–Noether number less than zero at general points. Moreover, an elliptic curve has no $\mathfrak{g}^1_k$ with adjusted Brill–Noether number less than -1 at two arbitrary points. Finally a rational nodal curve has no generalized linear series with adjusted Brill–Noether number less than zero at arbitrary points.

It follows that for $i \geqslant 4$ we have

$$\begin{aligned} &3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(i)}} - A_{\omega^{(g-i+1)}} - A_{\delta_1^2} + A_{\delta_{i-1,g-i}} - A_{\lambda^{(i)}} - A_{\lambda^{(g-i+1)}} \\ &\quad + A_{\lambda\delta_1} - 12A_{\delta_{0,i-1}} - 12A_{\delta_{0,g-i}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_{i-1}} = 0, \end{aligned}$$

when $i = 3$

$$\begin{aligned} &3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(3)}} - A_{\omega^{(g-2)}} - A_{\delta_1^2} + A_{\delta_{2,g-3}} - A_{\lambda^{(3)}} - A_{\lambda\delta_2} \\ &\quad + A_{\lambda\delta_1} - 12A_{\delta_{0,2}} - 12A_{\delta_{0,g-3}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_2} = 0, \end{aligned}$$

and when $i = 2$

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(2)}} + A_{\delta_{1,g-2}} - A_{\lambda\delta_2} - 12A_{\delta_{0,1}} - 12A_{\delta_{0,g-2}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_1} = 0.$$

7. Non-singularity

In (S1)–(S18) we have constructed

$$\lfloor (g^2-1)/4 \rfloor + 3g - 1$$

linear relations in the coefficients A. Let us collect here all the relations. For $2 \leqslant i \leqslant \lfloor g/2 \rfloor$ from (S1) we obtain

$$2A_{\kappa_1^2} - A_{\omega^{(i)}} - A_{\omega^{(g-i)}} = \frac{T_i}{(2i-2)(2(g-i)-2)},$$

from (S2) for $2 \leqslant i \leqslant j \leqslant g-3$ and $i+j \leqslant g-1$

$$2A_{\kappa_1^2} + A_{\delta_{ij}} = \frac{D_{ij}}{(2i-2)(2j-2)},$$

from (S3)

$$4A_{\kappa_1^2} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} - A_{\delta_{1,g-2}} + 2A_{\delta_{0,g-2}} = \frac{n_{g-2,k,(0,1)}}{g-3},$$

from (S4) for $2 \leqslant i \leqslant g-3$

$$4A_{\kappa_1^2} - A_{\delta_{1,i}} + 2A_{\delta_{0,i}} + A_{\delta_{2,i}} = \frac{D_{2,i}}{6(i-1)},$$

from (S5)

$$2A_{\kappa_1^2} - 12A_{\delta_{0,g-1}} + 2A_{\delta_1^2} - A_{\lambda\delta_1} = 0,$$

from (S6) for $3 \leqslant i \leqslant g-3$

$$2A_{\kappa_1^2} - A_{\lambda^{(i)}} + A_{\delta_{1,g-i}} - 12A_{\delta_{0,g-i}} = 0$$

and

$$2A_{\kappa_1^2} - A_{\lambda\delta_2} + A_{\delta_{1,2}} - 12A_{\delta_{0,2}} = 0,$$

from (S7)

$$8A_{\kappa_1^2} + A_{\delta_{2,2}} - 2A_{\delta_{1,2}} + A_{\delta_{1,1}} + 2A_{\delta_1^2} + 8A_{\delta_0^2} + 4A_{\delta_{00}} + 4A_{\delta_{02}} - 4A_{\delta_{01}} = 4N_{g-4,k,(0,1),(0,1)},$$

from (S8)

$$2A_{\kappa_1^2} + 288A_{\delta_0^2} + 24A_{\lambda\delta_0} + 2A_{\delta_1^2} - 2A_{\lambda\delta_1} + 144A_{\delta_{00}} + A_{\delta_{1,1}} - 24A_{\delta_{01}} = 0,$$

from (S9) for $2 \leqslant j \leqslant g-3$

$$2A_{\kappa_1^2} + 2A_{\delta_{1j}} + A_{\delta_{j,g-j-2}} - A_{\delta_{2,j}} - 2A_{\delta_{j,g-j-1}} - A_{\omega^{(j)}} - A_{\omega^{(g-j)}} = 0,$$

from (S10) for $g > 6$

$$\begin{aligned}
&2A_{\kappa_1^2} + 12A_{\delta_1^2} + 6A_{\delta_{1,1}} + 3A_{\delta_{1,g-5}} + 2A_{\delta_{1,3}} + A_{\delta_{3,g-5}} + 3A_{\delta_{1,g-3}} \\
&\quad - A_{\omega^{(3)}} - A_{\omega^{(g-3)}} - 3(A_{\delta_{2,g-3}} + A_{\delta_{2,g-5}} + 2A_{\delta_{1,2}}) \\
&\quad - 2(3A_{\delta_{1,g-4}} + A_{\delta_{3,g-4}}) - (3A_{\delta_{1,2}} + A_{\delta_{2,3}}) + 6A_{\delta_{2,g-4}} + 3A_{\delta_{2,2}} = 0
\end{aligned}$$

while for $g = 6$

$$\begin{aligned}
&2A_{\kappa_1^2} + 18A_{\delta_1^2} + 6A_{\delta_{1,1}} + 3A_{\delta_{1,g-5}} + 2A_{\delta_{1,3}} + A_{\delta_{3,g-5}} + 3A_{\delta_{1,g-3}} \\
&\quad - A_{\omega^{(3)}} - A_{\omega^{(g-3)}} - 3(A_{\delta_{2,g-3}} + A_{\delta_{2,g-5}} + 2A_{\delta_{1,2}}) \\
&\quad - 2(3A_{\delta_{1,g-4}} + A_{\delta_{3,g-4}}) - (3A_{\delta_{1,2}} + A_{\delta_{2,3}}) + 6A_{\delta_{2,g-4}} + 3A_{\delta_{2,2}} = 0,
\end{aligned}$$

from (S11)

$$2A_{\kappa_1^2} - A_{\lambda^{(g-3)}} + 6A_{\delta_1^2} + 3A_{\delta_{1,1}} - 3A_{\lambda\delta_1} + A_{\delta_{1,3}} - 36A_{\delta_{01}} - 12A_{\delta_{03}} + 3[A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}] = 0,$$

from (S12)

$$2A_{\kappa_1^2} - 3A_{\lambda\delta_1} - 24A_{\delta_{01}} - 12A_{\delta_{0,g-3}} - 12A_{\delta_{0,g-1}} + 6A_{\delta_1^2} + 2A_{\delta_{1,1}} + A_{\delta_{1,g-3}} - A_{\lambda^{(3)}} + 2(A_{\lambda\delta_2} + 12A_{\delta_{0,g-2}} - A_{\delta_{1,g-2}}) + (A_{\lambda\delta_2} + 12A_{\delta_{02}} - A_{\delta_{1,2}}) = 0,$$

from (S13)

$$2(g-3)[4A_{\kappa_1^2} + 2A_{\delta_{00}} + 4A_{\delta_0^2} + A_{\delta_{02}}] + 2A_{\delta_{0,g-2}} - A_{\omega^{(2)}} - A_{\omega^{(g-2)}} - [2(g-3)A_{\delta_{01}} + A_{\delta_{1,g-2}}] - [2A_{\delta_{01}} + A_{\delta_{1,2}}] + [A_{\delta_{1,1}} + 2A_{\delta_1^2}] = 2 \cdot \ell_{g-2,k},$$

from (S14)

$$(2g-4)[2A_{\kappa_1^2} - A_{\lambda\delta_0} - 24A_{\delta_0^2} - 12A_{\delta_{00}} + A_{\delta_{01}}] - 12A_{\delta_{0,g-1}} + (12A_{\delta_{01}} - A_{\delta_{1,1}}) = 0,$$

from (S15)

$$(8g^2 - 26g + 20)A_{\kappa_1^2} + (2g-4)A_{\kappa_2} + (4-2g)A_{\delta_1^2} + 8(g-1)(g-2)A_{\delta_0^2} = 0,$$

from (S16) for $\lfloor g/2 \rfloor \leqslant i \leqslant g-3$

$$(4i-1)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(i)}} - A_{\omega^{(i+1)}} + A_{\delta_1^2} + (2i-1)A_{\delta_{1,g-i-1}} = \frac{1}{2i-2} \sum_{\substack{0 \leqslant \alpha_0 \leqslant \alpha_1 \leqslant k-1 \\ \alpha_0 + \alpha_1 = g-i-1}} m_{i,k,(\alpha_0,\alpha_1)} \cdot N_{g-i-1,k,(k-1-\alpha_1,k-1-\alpha_0)}$$

and

$$(4g-9)A_{\kappa_1^2} + A_{\kappa_2} + A_{\omega^{(g-2)}} + (4g-8)A_{\delta_1^2} + (2g-5)A_{\delta_{1,1}} = \frac{m_{g-2,k,(0,1)}}{2g-6},$$

from (S17)

$$3A_{\kappa_1^2} + A_{\kappa_2} - 2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} - A_{\delta_1^2} + 12A_{\delta_{0,g-1}} - 12A_{\delta_{00}} + A_{\theta_1} = 0,$$

from (S18) for $4 \leqslant i \leqslant \lfloor (g+1)/2 \rfloor$

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(i)}} - A_{\omega^{(g-i+1)}} - A_{\delta_1^2} + A_{\delta_{i-1,g-i}} - A_{\lambda^{(i)}} - A_{\lambda^{(g-i+1)}} + A_{\lambda\delta_1} - 12A_{\delta_{0,i-1}} - 12A_{\delta_{0,g-i}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_{i-1}} = 0,$$

and

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(3)}} - A_{\omega^{(g-2)}} - A_{\delta_1^2} + A_{\delta_{2,g-3}} - A_{\lambda^{(3)}} - A_{\lambda\delta_2} + A_{\lambda\delta_1} - 12A_{\delta_{0,2}} - 12A_{\delta_{0,g-3}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_2} = 0,$$

$$3A_{\kappa_1^2} + A_{\kappa_2} - A_{\omega^{(2)}} + A_{\delta_{1,g-2}} - A_{\lambda\delta_2} - 12A_{\delta_{0,1}} - 12A_{\delta_{0,g-2}} + 12A_{\delta_{0,g-1}} + 12A_{\theta_1} = 0.$$

Our aim is to show that the above linear relations yield a non-degenerate linear system. Let

$$\{e_{\kappa_1^2}, e_{\kappa_2}, e_{\delta_0^2}, e_{\lambda\delta_0}, e_{\delta_1^2}, e_{\lambda\delta_1}, e_{\lambda\delta_2}, \ldots, e_{\omega^{(i)}}, \ldots, e_{\lambda^{(j)}}, \ldots, e_{\delta_{k,l}}, \ldots, e_{\theta_1}, \ldots, e_{\theta_{\lfloor (g-1)/2 \rfloor}}\}$$

be the canonical basis of $\mathbb{Q}^{\lfloor (g^2-1)/4 \rfloor + 3g - 1}$ indexed by the tautological codimension-two generating classes from § 1. Let Q_g be the square matrix of order $\lfloor (g^2-1)/4 \rfloor + 3g - 1$ associated to the linear system given by the above relations. As we have already noted, since the test surfaces in (S1)–(S18) are also defined for odd values of $g \geqslant 6$, the matrix Q_g is also defined for g odd, $g \geqslant 7$.

For each $g \geqslant 6$ we construct a square matrix T_g of order $\lfloor (g^2-1)/4 \rfloor + 3g - 1$ such that $Q_g \cdot T_g$ is lower-triangular with nonzero diagonal coefficients.

We describe the columns of T_g dividing them into 18 groups, similarly to the description of the relations that yield the rows of Q_g.

(**T1**) For $2 \leqslant i \leqslant \lfloor g/2 \rfloor$ consider the column $e_{\omega^{(i)}}$.

(**T2**) For i, j such that $2 \leqslant i \leqslant j \leqslant g-3$ and $i+j \leqslant g-1$ consider $e_{\delta_{i,j}}$.

(**T3**) Consider the column $e_{\delta_{1,g-2}}$.

(**T4**) For $2 \leqslant i \leqslant g-3$ consider $e_{\delta_{1,i}}$.

(**T5**) Take $e_{\delta_{0,g-1}}$.

(**T6**) For $3 \leqslant i \leqslant g-3$ consider $e_{\lambda^{(i)}}$. Moreover, consider the column $e_{\lambda\delta_2}$.

(**T7**) Consider the column $e_{\delta_{1,1}}$.

(**T8**) Consider $e_{\lambda\delta_0}$.

(**T9**) Consider

$$2e_{\delta_{1,2}} + e_{\delta_{0,2}} - 10 e_{\lambda\delta_2}.$$

Moreover for $3 \leqslant j \leqslant g-3$ consider

$$2e_{\delta_{1,j}} + e_{\delta_{0,j}} - 10 e_{\lambda^{(g-j)}}.$$

Take the following:

(**T10**)
$$60 e_{\lambda\delta_1} + 12 e_{\delta_1^2} - 3 e_{\delta_{0,g-1}} + 8 e_{\delta_{0,1}} + 2 e_{\delta_{00}};$$

(**T11**)
$$12 e_{\lambda\delta_1} + e_{\lambda\delta_0} - e_{\delta_{0,g-1}};$$

(**T12**)
$$e_{\delta_{0,g-2}} + 2 e_{\delta_{1,g-2}};$$

(**T13**)
$$12 e_{\lambda\delta_1} + 6 e_{\lambda\delta_0} - e_{\delta_{0,g-1}} - e_{\delta_{0,1}} - e_{\delta_{00}};$$

(**T14**)
$$\begin{aligned} &6\Bigg(e_{\kappa_1^2} + e_{\omega^{(\lfloor g/2 \rfloor)}} + 2 \sum_{2 \leqslant s < g/2} e_{\omega^{(s)}} + 12 \Bigg(e_{\lambda\delta_0} + 2 e_{\lambda\delta_1} + 2 e_{\lambda\delta_2} + 2 \sum_{3 \leqslant s \leqslant g-3} e_{\lambda^{(s)}} \Bigg) \\ &\quad - 2 \sum_{(i,j) \neq (0,g-1)} e_{\delta_{ij}} \Bigg) - 11 e_{\delta_{0,g-1}}; \end{aligned}$$

(**T15**)
$$e_{\kappa_2}.$$

(**T16**) For $\lfloor g/2 \rfloor \leqslant i \leqslant g-3$ consider $e_{\omega^{(i+1)}} - e_{\omega^{(g-i-1)}}$. Furthermore consider

$$(g-1)\Bigg(\sum_{2 \leqslant s \leqslant g/2} 12 \bigg(\frac{g}{2} - s \bigg) (e_{\omega^{(g-s)}} - e_{\omega^{(s)}}) + 12 e_{\delta_1^2} - 24 e_{\delta_{1,1}} + 2 e_{\delta_{0,g-1}} \Bigg) + 3 e_{\delta_0^2} - 6 e_{\delta_{00}}.$$

(**T17**) Consider

$$\begin{aligned} &6g\, e_{\kappa_2} + \sum_{2 \leqslant s \leqslant g/2} 12 \bigg(\frac{g}{2} - s \bigg) (e_{\omega^{(g-s)}} - e_{\omega^{(s)}}) + 12 \bigg(1 - \frac{g}{2} \bigg) e_{\delta_1^2} \\ &\quad + 12(g-2) e_{\delta_{1,1}} - 3 e_{\delta_0^2} + (2-g) e_{\delta_{0,g-1}} + 6 e_{\delta_{00}}. \end{aligned}$$

(**T18**) For $4 \leqslant i \leqslant \lfloor (g+1)/2 \rfloor$ consider $e_{\theta_{i-1}}$. Moreover consider e_{θ_2} and finally the column

$$\begin{aligned} &-6g\, e_{\kappa_2} + \sum_{2 \leqslant s \leqslant g/2} 12 \bigg(\frac{g}{2} - s \bigg) (e_{\omega^{(s)}} - e_{\omega^{(g-s)}}) + 12 \bigg(\frac{g}{2} - 1 \bigg) e_{\delta_1^2} \\ &\quad + 12(2-g) e_{\delta_{1,1}} + 3 e_{\delta_0^2} + (g-2) e_{\delta_{0,g-1}} - 6 e_{\delta_{00}} + 72 e_{\theta_1}. \end{aligned}$$

One checks that $Q_g \cdot T_g$ is an lower-triangular matrix with all the coefficients on the main diagonal different from zero. It follows that $\det(Q_g) \neq 0$ for all $g \geqslant 6$. In particular, when $g = 2k$, we are able to solve the system and find the coefficients A.

Theorem 7. *For $k \geqslant 3$, the class of the locus $\overline{\mathcal{M}}^1_{2k,k} \subset \overline{\mathcal{M}}_{2k}$ is*

$$[\overline{\mathcal{M}}^1_{2k,k}]_Q = c\bigg[A_{\kappa_1^2}\kappa_1^2 + A_{\kappa_2}\kappa_2 + A_{\delta_0^2}\delta_0^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\delta_1^2}\delta_1^2 + A_{\lambda\delta_1}\lambda\delta_1 + A_{\lambda\delta_2}\lambda\delta_2 + \sum_{i=2}^{2k-2} A_{\omega^{(i)}}\omega^{(i)} + \sum_{i=3}^{2k-3} A_{\lambda^{(i)}}\lambda^{(i)} + \sum_{i,j} A_{\delta_{ij}}\delta_{ij} + \sum_{i=1}^{\lfloor (2k-1)/2\rfloor} A_{\theta_i}\theta_i\bigg]$$

in $R^2(\overline{\mathcal{M}}_{2k}, \mathbb{Q})$, where

$$\begin{aligned}
c &= \frac{2^{k-6}(2k-7)!!}{3(k!)},\\
A_{\kappa_1^2} = -A_{\delta_0^2} &= 3k^2 + 3k + 5,\\
A_{\kappa_2} &= -24k(k+5),\\
A_{\delta_1^2} &= -(3k(9k+41)+5),\\
A_{\lambda\delta_0} &= -24(3(k-1)k-5),\\
A_{\lambda\delta_1} &= 24(-33k^2 + 39k + 65),\\
A_{\lambda\delta_2} &= 24(3(37-23k)k + 185),\\
A_{\omega^{(i)}} &= -180i^4 + 120i^3(6k+1) - 36i^2(20k^2 + 24k - 5)\\
&\quad + 24i(52k^2 - 16k - 5) + 27k^2 + 123k + 5,\\
A_{\lambda^{(i)}} &= 24[6i^2(3k+5) - 6i(6k^2 + 23k + 5) + 159k^2 + 63k + 5],\\
A_{\theta^{(i)}} &= -12i[5i^3 + i^2(10-20k) + i(20k^2 - 8k - 5) - 24k^2 + 32k - 10],\\
A_{\delta_{1,1}} &= 48(19k^2 - 49k + 30),\\
A_{\delta_{1,2k-2}} &= \tfrac{2}{5}(3k(859k - 2453) + 2135),\\
A_{\delta_{00}} &= 24k(k-1),\\
A_{\delta_{0,2k-2}} &= \tfrac{2}{5}(3k(187k - 389) - 745),\\
A_{\delta_{0,2k-1}} &= 2(k(31k-49) - 65)
\end{aligned}$$

and for $i \geqslant 1$ and $2 \leqslant j \leqslant 2k-3$

$$A_{\delta_{ij}} = 2[3k^2(144ij - 1) - 3k(72ij(i+j+4)+1) + 180i(i+1)j(j+1) - 5]$$

while

$$A_{\delta_{0j}} = 2(-3(12j^2 + 36j + 1)k + (72j-3)k^2 - 5)$$

for $1 \leqslant j \leqslant 2k-3$.

As usual, for a positive integer n, the symbol $(2n+1)!!$ denotes

$$\frac{(2n+1)!}{2^n \cdot n!},$$

while $(-1)!! = 1$.

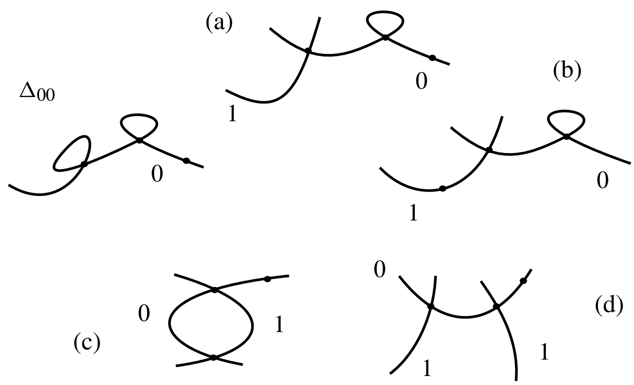


Figure 21. Loci in $\overline{\mathcal{M}}_{2,1}$.

8. Pull-back to $\overline{\mathcal{M}}_{2,1}$

As a check, in this section we obtain four more relations for the coefficients A considering the pull-back of $\overline{\mathcal{M}}^1_{2k,k}$ to $\overline{\mathcal{M}}_{2,1}$. Let $j\colon \overline{\mathcal{M}}_{2,1} \to \overline{\mathcal{M}}_g$ be the map obtained by attaching a fixed general pointed curve of genus $g-2$ at elements (D,p) in $\overline{\mathcal{M}}_{2,1}$. This produces a map $j^*\colon A^2(\overline{\mathcal{M}}_g) \to A^2(\overline{\mathcal{M}}_{2,1})$.

In [Fab88, Ch. 3, § 1] it is shown that $A^2(\overline{\mathcal{M}}_{2,1})$ has rank 5 and is generated by the classes of the loci composed by curves of type Δ_{00}, (a), (b), (c) and (d) as in Figure 21.

We have the following pull-backs:

$$
\begin{aligned}
j^*(\delta_{0,1}) &= [(a)]_Q, \quad j^*(\delta_{0,g-1}) = [(b)]_Q,\\
j^*(\theta_1) &= [(c)]_Q,\\
j^*(\delta_{1,1}) &= [(d)]_Q, \quad j^*(\delta_{00}) = [\Delta_{00}]_Q,\\
j^*(\delta_0^2) &= \tfrac{5}{3}[\Delta_{00}]_Q - 2[(a)]_Q - 2[(b)]_Q,\\
j^*(\delta_1^2) &= -\tfrac{1}{12}([(a)]_Q + [(b)]_Q),\\
j^*(\lambda\delta_0) &= \tfrac{1}{6}[\Delta_{00}]_Q,\\
j^*(\lambda\delta_1) &= \tfrac{1}{12}([(a)]_Q + [(b)]_Q),\\
j^*(\lambda\delta_2) &= -\lambda\psi\\
&= \tfrac{1}{60}(-[\Delta_{00}]_Q - 7[(a)]_Q - 12[(c)]_Q - 24[(d)]_Q),\\
j^*(\kappa_1^2) &= (\tfrac{1}{5}\delta_0 + \tfrac{7}{5}\delta_1 + \psi)^2\\
&= \tfrac{1}{120}(17[\Delta_{00}]_Q + 127[(a)]_Q + 37[(b)]_Q + 120[(c)]_Q + 840[(d)]_Q),\\
j^*(\kappa_2) &= \lambda(\lambda + \delta_1) + \psi^2\\
&= \tfrac{1}{120}(3[\Delta_{00}]_Q + 25[(a)]_Q + 11[(b)]_Q + 24[(c)]_Q + 168[(d)]_Q),\\
j^*(\delta_{1,g-2}) &= -\delta_1\psi\\
&= -\tfrac{1}{12}[(a)]_Q - 2[(d)]_Q,
\end{aligned}
$$

$$\begin{aligned} j^*(\delta_{0,g-2}) &= -\delta_0\psi \\ &= -\tfrac{1}{6}[\Delta_{00}]_Q - [(a)]_Q - 2[(c)]_Q, \\ j^*(\omega^{(2)}) &= -\psi^2 \\ &= -\tfrac{1}{120}([\Delta_{00}]_Q + 13[(a)]_Q - [(b)]_Q + 24[(c)]_Q + 168[(d)]_Q). \end{aligned}$$

For this, see relations in [Fab88, Ch. 3, § 1] and [Mum83, §§ 8–10]. We have used the fact that on $\overline{\mathcal{M}}_{g,1}$ one has

$$\kappa_i = \kappa_i|_{\overline{\mathcal{M}}_g} + \psi^i$$

(see [AC96, 1.10]).

All the other classes have zero pull-back. Finally, $j^*(\overline{\mathcal{M}}^1_{2k,k})$ is supported at most on the locus (c). Indeed, general elements in the loci Δ_{00}, (a), (b) and (d) do not admit any linear series $\mathfrak{g}^1_k$ with adjusted Brill–Noether number less than -1 (see also [Edi93, Lemma 5.1]). Since the restriction of $\overline{\mathcal{M}}^1_{2k,k}$ to $j(\overline{\mathcal{M}}_{2,1})$ is supported in codimension two, then $j(\overline{\mathcal{M}}_{2,1}\backslash(c)) = 0$. Hence, looking at the coefficients of $[\Delta_{00}]_Q$, $[(a)]_Q$, $[(b)]_Q$ and $[(d)]_Q$ in $j^*(\overline{\mathcal{M}}^1_{2k,k})$, we obtain the following relations:

$$\begin{aligned} &A_{\delta_{00}} + \tfrac{5}{3}A_{\delta_0^2} + \tfrac{1}{6}A_{\lambda\delta_0} - \tfrac{1}{60}A_{\lambda\delta_2} + \tfrac{17}{120}A_{\kappa_1^2} + \tfrac{1}{40}A_{\kappa_2} - \tfrac{1}{6}A_{\delta_{0,g-2}} - \tfrac{1}{120}A_{\omega^{(2)}} = 0, \\ &A_{\delta_{01}} - 2A_{\delta_0^2} - \tfrac{1}{12}A_{\delta_1^2} + \tfrac{1}{12}A_{\lambda\delta_1} - \tfrac{7}{60}A_{\lambda\delta_2} + \tfrac{127}{120}A_{\kappa_1^2} \\ &\quad + \tfrac{5}{24}A_{\kappa_2} - \tfrac{1}{12}A_{\delta_{1,g-2}} - A_{\delta_{0,g-2}} - \tfrac{13}{120}A_{\omega^{(2)}} = 0, \\ &A_{\delta_{0,g-1}} - 2A_{\delta_0^2} - \tfrac{1}{12}A_{\delta_1^2} + \tfrac{1}{12}A_{\lambda\delta_1} + \tfrac{37}{120}A_{\kappa_1^2} + \tfrac{11}{120}A_{\kappa_2} + \tfrac{1}{120}A_{\omega^{(2)}} = 0, \\ &A_{\delta_{1,1}} - \tfrac{2}{5}A_{\lambda\delta_2} + 7A_{\kappa_1^2} + \tfrac{7}{5}A_{\kappa_2} - 2A_{\delta_{1,g-2}} - \tfrac{7}{5}A_{\omega^{(2)}} = 0. \end{aligned}$$

The coefficients A shown in Theorem 7 satisfy these relations.

9. Further relations

In this section we will show how to get further relations for the coefficients A that can be used to produce more tests for our result.

9.1 The coefficients of κ_1^2 and κ_2

One can compute the class of $\mathcal{M}^1_{2k,k}$ in the open $\mathcal{M}_{2k}$ by the methods described by Faber in [Fab99]. Let $\mathcal{C}^k_{2k}$ be the k-fold fiber product of the universal curve over $\mathcal{M}_{2k}$ and let $\pi_i\colon \mathcal{C}^k_{2k} \to \mathcal{C}_{2k}$ be the map forgetting all but the ith point, for $i = 1, \ldots, k$. We define the following tautological classes on $\mathcal{C}^k_{2k}$: K_i is the class of $\pi_i^*(\omega)$, where ω is the relative dualizing sheaf of the map $\mathcal{C}_{2k} \to \mathcal{M}_{2k}$, and $\Delta_{i,j}$ is the class of the locus of curves with k points $(C, x_1, \ldots, x_k)$ such that $x_i = x_j$, for $1 \leqslant i, j \leqslant k$.

Let $\mathbb{E}$ be the pull-back to $\mathcal{C}^k_{2k}$ of the Hodge bundle of rank $2k$ and let $\mathbb{F}_k$ be the bundle on $\mathcal{C}^k_{2k}$ of rank k whose fiber over $(C, x_1, \ldots, x_k)$ is

$$H^0(C, K_C/K_C(-x_1 \cdots - x_k)).$$

We consider the locus X in $\mathcal{C}^k_{2k}$ where the evaluation map

$$\varphi_k\colon \mathbb{E} \to \mathbb{F}_k$$

has rank at most $k-1$. Equivalently, X parameterizes curves with k points $(C, x_1, \ldots, x_k)$ such that $H^0(C, K_C(-x_1 \cdots - x_k)) \geqslant k+1$ or, in other terms, $H^0(C, x_1 + \cdots + x_k) \geqslant 2$. By Porteous formula, the class of X is

$$[X] = \begin{bmatrix} e_1 & e_2 & e_3 & \cdots & e_{k+1} \\ 1 & e_1 & e_2 & \cdots & e_k \\ 0 & 1 & e_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & e_2 \\ 0 & \cdots & 0 & 1 & e_1 \end{bmatrix}$$

where the classes e_i are the Chern classes of $\mathbb{F}_k - \mathbb{E}$. The total Chern class of $\mathbb{F}_k - \mathbb{E}$ is

$$(1+K_1)(1+K_2-\Delta_{1,2})\cdots(1+K_k-\Delta_{1,k}\cdots-\Delta_{k-1,k})(1-\lambda_1+\lambda_2-\lambda_3+\cdots+\lambda_{2k}).$$

Intersecting the class of X with $\Delta_{1,2}$ we obtain a class that pushes forward via $\pi := \pi_1\pi_2\cdots\pi_k$ to the class of $\mathcal{M}^1_{2k,k}$ with multiplicity $(k-2)!(6k-2)$. We refer the reader to [Fab99, § 4] for formulae for computing the push-forward π_*.

For instance, when $k=3$ one constructs a degeneracy locus X on the 3-fold fiber product of the universal curve over $\mathcal{M}_6$. The class of X is

$$[X] = e_1^4 - 3e_1^2e_2 + e_2^2 + 2e_1e_3 - e_4$$

where the classes e_i are determined by the following total Chern class

$$(1+K_1)(1+K_2-\Delta_{1,2})(1+K_3-\Delta_{1,3}-\Delta_{2,3})(1-\lambda_1+\lambda_2-\lambda_3+\cdots+\lambda_6).$$

Upon intersecting the class of X with $\Delta_{1,2}$ and using the following identities

$$\begin{gathered} \Delta_{1,3}\Delta_{2,3} = \Delta_{1,2}\Delta_{1,3}, \\ \Delta_{1,2}^2 = -K_1\Delta_{1,2}, \quad \Delta_{1,3}^2 = -K_1\Delta_{1,3}, \quad \Delta_{2,3}^2 = -K_2\Delta_{2,3}, \\ K_2\Delta_{1,2} = K_1\Delta_{1,2}K_3, \quad \Delta_{1,3} = K_1\Delta_{1,3}, \quad K_3\Delta_{2,3} = K_2\Delta_{2,3}, \end{gathered}$$

one obtains

$$\begin{aligned} [X]\cdot\Delta_{1,2} = {} & K_3^4\Delta_{1,2} - 3K_3^3\Delta_{1,2}^2 + 7K_3^2\Delta_{1,2}^3 - 15K_3\Delta_{1,2}^4 + 31\Delta_{1,2}^5 \\ & + 72\Delta_{1,2}\Delta_{2,3}^4 + 172\Delta_{1,3}\Delta_{2,3}^4 - K_3^3\Delta_{1,2}\lambda_1 + 3K_3^2\Delta_{1,2}^2\lambda_1 \\ & - 7K_3\Delta_{1,2}^3\lambda_1 + 15\Delta_{1,2}^4\lambda_1 + 23\Delta_{1,2}\Delta_{2,3}^3\lambda_1 + 41\Delta_{1,3}\Delta_{2,3}^3\lambda_1 \\ & + K_3^2\Delta_{1,2}\lambda_1^2 - 3K_3\Delta_{1,2}^2\lambda_1^2 + 7\Delta_{1,2}^3\lambda_1^2 + 6\Delta_{1,2}\Delta_{2,3}^2\lambda_1^2 \\ & + 8\Delta_{1,3}\Delta_{2,3}^2\lambda_1^2 - K_3\Delta_{1,2}\lambda_1^3 + 3\Delta_{1,2}^2\lambda_1^3 + \Delta_{1,2}\Delta_{2,3}\lambda_1^3 \\ & + \Delta_{1,3}\Delta_{2,3}\lambda_1^3 + \Delta_{1,2}\lambda_1^4 - K_3^2\Delta_{1,2}\lambda_2 + 3K_3\Delta_{1,2}^2\lambda_2 - 7\Delta_{1,2}^3\lambda_2 \\ & - 6\Delta_{1,2}\Delta_{2,3}^2\lambda_2 - 8\Delta_{1,3}\Delta_{2,3}^2\lambda_2 + 2K_3\Delta_{1,2}\lambda_1\lambda_2 - 6\Delta_{1,2}^2\lambda_1\lambda_2 \\ & - 2\Delta_{1,2}\Delta_{2,3}\lambda_1\lambda_2 - 2\Delta_{1,3}\Delta_{2,3}\lambda_1\lambda_2 - 3\Delta_{1,2}\lambda_1^2\lambda_2 + \Delta_{1,2}\lambda_2^2 \\ & - K_3\Delta_{1,2}\lambda_3 + 3\Delta_{1,2}^2\lambda_3 + \Delta_{1,2}\Delta_{2,3}\lambda_3 + \Delta_{1,3}\Delta_{2,3}\lambda_3 \\ & + 2\Delta_{1,2}\lambda_1\lambda_3 - \Delta_{1,2}\lambda_4. \end{aligned}$$

Computing the push-forward to $\mathcal{M}_6$ of the above class, one has

$$\begin{aligned} [\mathcal{M}^1_{6,3}]_Q = \tfrac{1}{16}((18\kappa_0 - 244)\kappa_2 + 7\kappa_1^2 + (64 - 10\kappa_0)\kappa_1\lambda_1 \\ + (3\kappa_0^2 - 14\kappa_0)\lambda_1^2 + (14\kappa_0 - 3\kappa_0^2)\lambda_2). \end{aligned}$$

Note that $\kappa_0 = 2g-2 = 10$, $12\lambda_1 = \kappa_1$ and $2\lambda_2 = \lambda_1^2$, hence we recover

$$[\mathcal{M}^1_{6,3}]_Q = \tfrac{41}{144}\kappa_1^2 - 4\kappa_2.$$

Remark 8. As a corollary one obtains the class of the Maroni locus in $\mathcal{M}_6$. The trigonal locus in $\mathcal{M}_{2k}$ has a divisor known as the Maroni locus (see [Mar46, MS86]). While the general trigonal curve of even genus admits an embedding in $\mathbb{P}^1 \times \mathbb{P}^1$ or in $\mathbb{P}^2$ blown up in one point, the trigonal curves admitting an embedding to other kinds of ruled surfaces constitute a subvariety of codimension one inside the trigonal locus.

The class of the Maroni locus in the Picard group of the trigonal locus in $\overline{\mathcal{M}}_{2k}$ has been studied in [Sta00]. For $k=3$, one has that the class of the Maroni locus is $8\lambda \in \mathrm{Pic}_{\mathbb{Q}}(\overline{\mathcal{M}}^1_{6,3})$. Knowing the class of the trigonal locus in $\mathcal{M}_6$, one has that the class of the Maroni locus in $\mathcal{M}_6$ is

$$8\lambda(\tfrac{41}{144}\kappa_1^2 - 4\kappa_2).$$

9.2 More test surfaces

One could also consider more test surfaces. For instance one can easily adapt the test surfaces of type (ε) and (κ) from [Fab90b, § 3]. They are all disjoint from the locus $\overline{\mathcal{M}}^1_{2k,k}$ and produce relations compatible with the ones we have shown.

9.3 The relations for $g = 5$

As an example, let us consider the case $g=5$. We know that the tautological ring of $\mathcal{M}_5$ is generated by λ, that is, there is a non-trivial relation among κ_1^2 and κ_2 (see [Fab99]). The square matrix Q_5 from § 7 expressing the restriction of the generating classes in $\overline{\mathcal{M}}_5$ to the test surfaces (S1)–(S18) (we have to exclude the relation from (S10) which is defined only for $g \geqslant 6$), has rank 19, showing that the class κ_1^2 (or the class κ_2) and the 18 boundary classes in codimension two in $\overline{\mathcal{M}}_5$ are independent.

10. The hyperelliptic locus in $\overline{\mathcal{M}}_4$

The class of the hyperelliptic locus in $\overline{\mathcal{M}}_4$ has been computed in [FP05, Proposition 5]. In this section we will recover the formula by the means of the techniques used so far.

The class will be expressed as a linear combination of the 14 generators for $R^2(\overline{\mathcal{M}}_4)$ from [Fab90b]: κ_2, λ^2, $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, δ_0^2, $\delta_0\delta_1$, δ_1^2, $\delta_1\delta_2$, δ_2^2, δ_{00}, γ_1, δ_{01a} and $\delta_{1,1}$. Remember that there exists one unique relation among these classes, namely

$$60\kappa_2 - 810\lambda^2 + 156\lambda\delta_0 + 252\lambda\delta_1 - 3\delta_0^2 - 24\delta_0\delta_1 + 24\delta_1^2 - 9\delta_{00} + 7\delta_{01a} - 12\gamma_1 - 84\delta_{1,1} = 0,$$

hence $R^2(\overline{\mathcal{M}}_4)$ has rank 13. Write $[\overline{\mathcal{M}}^1_{4,2}]_Q$ as

$$\begin{aligned}[\overline{\mathcal{M}}^1_{4,2}]_Q = {} & A_{\kappa_2}\kappa_2 + A_{\lambda^2}\lambda^2 + A_{\lambda\delta_0}\lambda\delta_0 + A_{\lambda\delta_1}\lambda\delta_1 + A_{\lambda\delta_2}\lambda\delta_2 + A_{\delta_0^2}\delta_0^2 + A_{\delta_0\delta_1}\delta_0\delta_1 \\ & + A_{\delta_1^2}\delta_1^2 + A_{\delta_1\delta_2}\delta_1\delta_2 + A_{\delta_2^2}\delta_2^2 + A_{\delta_{00}}\delta_{00} + A_{\gamma_1}\gamma_1 + A_{\delta_{01a}}\delta_{01a} + A_{\delta_{1,1}}\delta_{1,1}.\end{aligned}$$

Let us construct 13 independent relations among the coefficients A.

The surfaces (S1), (S3), (S5), (S6), (S8), (S12)–(S18) from § 6 give respectively the following 12 independent relations

$$
\begin{aligned}
8A_{\delta_2^2} &= 36,\\
4A_{\delta_2^2} - 2A_{\delta_1\delta_2} &= 12,\\
-4A_{\lambda\delta_1} - 48A_{\delta_0\delta_1} + 8A_{\delta_1^2} - 48A_{\delta_{01a}} &= 0,\\
A_{\lambda\delta_2} - A_{\delta_1\delta_2} &= 0,\\
2A_{\lambda^2} + 24A_{\lambda\delta_0} - 2A_{\lambda\delta_1} + 288A_{\delta_0^2} - 24A_{\delta_0\delta_1} + 2A_{\delta_1^2} + 144A_{\delta_{00}} + A_{\delta_{1,1}} &= 0,\\
-4A_{\lambda\delta_1} + 3A_{\lambda\delta_2} - 48A_{\delta_0\delta_1} + 8A_{\delta_1^2} - 3A_{\delta_1\delta_2} - 12A_{\delta_{01a}} + 3A_{\delta_{1,1}} &= 0,\\
8A_{\delta_0^2} - 4A_{\delta_0\delta_1} + 2A_{\delta_1^2} - 2A_{\delta_1\delta_2} + 2A_{\delta_2^2} + 4A_{\delta_{0,0}} + A_{\delta_{1,1}} &= 4,\\
-4A_{\lambda\delta_0} - 96A_{\delta_0^2} + 4A_{\delta_0\delta_1} - 48A_{\delta_{00}} - A_{\delta_{1,1}} - 12A_{\delta_{01a}} &= 0,\\
48A_{\delta_0^2} - 4A_{\delta_1^2} + 4A_{\kappa_2} &= 0,\\
16A_{\delta_1^2} - 2A_{\delta_2^2} + 2A_{\kappa_2} + 6A_{\delta_{1,1}} &= 30,\\
-2A_{\lambda\delta_0} + A_{\lambda\delta_1} - 44A_{\delta_0^2} + 12A_{\delta_0\delta_1} - A_{\delta_1^2} + A_{\kappa_2} - 12A_{\delta_{00}} + 12A_{\delta_{01a}} + A_{\gamma_1} &= 0,\\
A_{\delta_1\delta_2} - A_{\lambda\delta_2} + A_{\delta_2^2} + A_{\kappa_2} + 12A_{\delta_{01a}} + 12A_{\gamma_1} &= 0.
\end{aligned}
$$

Next we look at the pull-back to $\overline{\mathcal{M}}_{2,1}$. The pull-back of the classes κ_2, $\lambda\delta_0$, $\lambda\delta_1$, $\lambda\delta_2$, δ_0^2, δ_1^2, δ_{00}, $\gamma_1 = \theta_1$, $\delta_{01a} = \delta_{0,g-1}$ and $\delta_{1,1}$ have been computed in § 8. Moreover,

$$
\begin{aligned}
j^*(\lambda^2) &= \tfrac{1}{60}([\Delta_{00}]_Q + [(a)]_Q + [(b)]_Q),\\
j^*(\delta_0\delta_1) &= [(a)]_Q + [(b)]_Q,\\
j^*(\delta_1\delta_2) &= -\delta_1\psi\\
&= -\tfrac{1}{12}[(a)]_Q - 2[(d)]_Q,\\
j^*(\delta_2^2) &= \psi^2\\
&= \tfrac{1}{120}([\Delta_{00}]_Q + 13[(a)]_Q - [(b)]_Q + 24[(c)]_Q + 168[(d)]_Q).
\end{aligned}
$$

Considering the coefficient of $[\Delta_{00}]_Q$ yields the following relation

$$A_{\delta_{00}} + \tfrac{5}{3}A_{\delta_0^2} + \tfrac{1}{6}A_{\lambda\delta_0} - \tfrac{1}{60}A_{\lambda\delta_2} + \tfrac{1}{40}A_{\kappa_2} + \tfrac{1}{60}A_{\lambda^2} + \tfrac{1}{120}A_{\delta_2^2} = 0.$$

All in all we get 13 independent relations, and the class of $\overline{\mathcal{M}}^1_{4,2}$ follows

$$
\begin{aligned}
2[\overline{\mathcal{M}}^1_{4,2}]_Q = {}& 27\kappa_2 - 339\lambda^2 + 64\lambda\delta_0 + 90\lambda\delta_1 + 6\lambda\delta_2 - \delta_0^2 - 8\delta_0\delta_1\\
&+ 15\delta_1^2 + 6\delta_1\delta_2 + 9\delta_2^2 - 4\delta_{00} - 6\gamma_1 + 3\delta_{01a} - 36\delta_{1,1}.
\end{aligned}
$$

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