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Scattering amplitudes of regularized bosonic strings

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We compute scattering amplitudes of the regularized bosonic Nambu-Goto string in the mean-field approximation, disregarding fluctuations of the Lagrange multiplier and an independent metric about their mean values. We use the previously introduced Lilliputian scaling limit to recover the Regge behavior of the amplitudes with the usual linear Regge trajectory in space-time dimensions $d > 2$. We demonstrate a stability of this minimum of the effective action under fluctuations for $d < 26$.

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I. INTRODUCTION

String theory emerged in very early 1970's from dual resonance models which were introduced to explain linear Regge trajectories of hadrons. Canonical quantization of relativistic bosonic string is consistent only in $d = 26$ dimensions and on mass shell. These restrictions can be potentially overcome by an alternative path-integral string quantization [1], where the problem remains non-linear in $d \neq 26$ and/or off mass shell even after a gauge fixing. More precisely, these nonlinearities are due to the fact that the ultraviolet cutoff depends on the metric on the string world sheet, as is prescribed by diffeomorphism invariance. They are natural when one uses the proper time regularization, but they are hardly seen for the zeta-function regularization which is intimately linked to the mode expansion used in canonical quantization.

In the recent papers [2, 3] we have analyzed the bosonic Nambu-Goto string in the mean field approximation, where the world sheet metric can be substituted by its mean value. This approximation becomes exact at large d and is applicable for finite d . We considered the string with the Dirichlet boundary conditions and computed its ground state energy as a function of the string length. We showed that the results obtained by canonical quantization are reproduced if the bare string tension K_0 approaches its critical value K_* from above:

$$K_0 \rightarrow K_* + \frac{K_R^2}{2\Lambda^2 \sqrt{d^2 - 2d}}, \quad K_* = \left(d - 1 + \sqrt{d^2 - 2d} \right) \Lambda^2, \quad (1)$$

where K_R stands for the renormalized string tension. Associated with this scaling limit was a renormalization of length scales [2, 3]

$$L_R = \sqrt{\frac{d + \sqrt{d^2 - 2d}}{2K_R}} \Lambda L_0. \quad (2)$$

In the limit where the cutoff $\Lambda \rightarrow \infty$ and K_R and L_R stay

finite, the ground state energy also stays finite and agrees with the one obtained using canonical quantization. Such a limit is possible because the bare metric on the world-sheet becomes singular. We called the scaling limit ‘‘Lilliputian’’ since the bare length scales as $L_0 \sim 1/\Lambda$.

The goal of this Paper is to further understand the Lilliputian string world and, in particular, the meaning of the performed renormalization of the length scale by computing scattering amplitudes.

II. NAMBU-GOTO STRING IN THE MEAN-FIELD APPROXIMATION

Our starting point is the Nambu-Goto string whose action is rewritten in the standard way by introducing a Lagrange multiplier λ^{ab} and an independent metric ρ_{ab} as

$$S_{\text{NG}} = K_0 \int d^2z \sqrt{\det \partial_a X \cdot \partial_b X} = K_0 \int d^2z \sqrt{\det \rho_{ab}} + \frac{K_0}{2} \int d^2z \lambda^{ab} (\partial_a X \cdot \partial_b X - \rho_{ab}). \quad (3)$$

The path integration goes independently over real values of X^μ and ρ_{ab} and imaginary values of λ^{ab} . To obtain the effective action governing the fields λ^{ab} and ρ_{ab} , we split $X^\mu = X_{\text{cl}}^\mu + X_{\text{q}}^\mu$ and perform the Gaussian path integral over X_{q}^μ . Fixing the conformal gauge $\rho_{ab} = \rho \delta_{ab}$, we find

$$S_{\text{eff}} = K_0 \int d^2z \rho + \frac{K_0}{2} \int d^2z \lambda^{ab} (\partial_a X_{\text{cl}} \cdot \partial_b X_{\text{cl}} - \rho \delta_{ab}) + \frac{d}{2} \text{tr} \log \left[-\frac{1}{\rho} \partial_a \lambda^{ab} \partial_b \right] - \text{tr} \log \left[-\frac{1}{\rho} \partial^2 + \frac{1}{2\rho} (\partial^2 \log \rho) \right], \quad (4)$$

where the last term on the right-hand side comes from the ghost determinant.

For definiteness we can use the proper-time regularization of the traces in Eq. (4)

$$\text{tr} \log \mathcal{O} = - \int_{a^2}^{\infty} \frac{d\tau}{\tau} \int d^2\omega \langle \omega | e^{-\tau \mathcal{O}} | \omega \rangle, \quad (5)$$

where the proper-time cutoff a^2 is related to the momentum cutoff Λ^2 by

$$a^2 = \frac{1}{4\pi\Lambda^2}, \quad (6)$$

but the results will not depend on the regularization used.

In the mean-field approximation, which becomes exact at large d , we can disregard fluctuations of λ^{ab} and ρ about their saddle-point values, i.e. simply substitute them by the mean values. This is analogous to the study of the N -component sigma-model at large N , where we can disregard quantum fluctuations of the Lagrange multiplier.

III. MEAN FIELD FOR SCATTERING AMPLITUDE

To compute the scattering amplitude, we introduce a piecewise constant momentum loop

$$P(\tau) = \sum_k p_k \theta(\tau - \tau_k), \quad \sum_k p_k = 0 \quad (7)$$

and consider

$$\mathcal{A}[P(\cdot)] = \left\langle e^{i \int P_\mu dx^\mu} \right\rangle = \int \mathcal{D}X^\mu e^{-S_{NG} + i \int P_\mu dx^\mu}, \quad (8)$$

where $x^\mu(\tau)$ is the value of X^μ at the boundary modulo a reparametrization of the boundary. For the upper half-plane coordinates $z = \sigma + iy$ the boundary is along the real axis, $y = 0$, and we have explicitly $X^\mu(\sigma, 0) = x^\mu(\tau(\sigma))$ with the nonnegative derivative $d\tau/d\sigma \geq 0$.

Integrating by parts in the exponent in Eq. (8) we obtain

$$\int P_\mu dx^\mu = - \sum_k p_k \cdot x_k, \quad x_k = x(\tau_k), \quad (9)$$

which reproduces the string vertex operators for scalars. The averaging in Eq. (8) can be also represented as the path integral over λ^{ab} and ρ with the effective action (4). We shall return to this issue in Sect. V.

Let us regularize (7) as

$$P(\tau) = \frac{1}{\pi} \sum_k p_k \arctan \left(\frac{\tau - \tau_k}{\sqrt{\varepsilon_k}} \right), \quad \sum_k p_k = 0, \quad (10)$$

where

$$\varepsilon_k = \varepsilon \frac{(\tau_{k+1} - \tau_k)(\tau_k - \tau_{k-1})}{(\tau_{k+1} - \tau_{k-1})} \quad (11)$$

to comply with diffeomorphism invariance. Then it becomes clear that (10) is a singular parametrization of a polygonal momentum loop. This is like the Wilson-loop/scattering-amplitude duality in the $\mathcal{N} = 4$ supersymmetric Yang-Mills theory [4] which was extended to QCD string in [5].

For the scattering amplitude it is convenient to use the upper half-plane coordinates, where the boundary of the string world sheet is parametrized by the real axis, resulting in the Koba-Nielsen variables. Repeating the technique of Sect. II for integrating over X^μ , now with the Neumann boundary conditions, we obtain for constant $\bar{\lambda}$ (to be justified below) the effective action

$$S_{\text{eff}} = \frac{1}{2K_0\bar{\lambda}} \sum_{i,j} p_i G_\varepsilon \left(\frac{\sigma_i - \sigma_j}{\sqrt{\bar{\lambda}}} \right) p_j + K_0 A (1 - \bar{\lambda}) + \frac{d}{2} \text{tr} \log \left[-\frac{\bar{\lambda}}{\rho} \partial^2 \right] - \text{tr} \log \left[-\frac{1}{\rho} \partial^2 + \frac{1}{2\rho} (\partial^2 \log \rho) \right]. \quad (12)$$

Here we have denoted $\sigma_i = \sigma(\tau_i)$, G_ε is a regularized Green function of the type

$$G_\varepsilon(\sigma) = -\frac{1}{2\pi} \log(\sigma^2 + \varepsilon(\sigma)), \quad (13)$$

and we have used the notation

$$A = \int d^2z \bar{\rho}. \quad (14)$$

For the $2 \rightarrow 2$ amplitude we have four σ_k 's ($k = 1, \dots, 4$), but the amplitude depends only on the projective-invariant ratio

$$r = \frac{(\sigma_4 - \sigma_3)(\sigma_2 - \sigma_1)}{(\sigma_4 - \sigma_2)(\sigma_3 - \sigma_1)}. \quad (15)$$

From the conformal mapping of the upper half-plane onto a $\omega_L \times \omega_R$ rectangle we have

$$\frac{\omega_L}{\omega_R} = \frac{K(\sqrt{r})}{K(\sqrt{1-r})}, \quad (16)$$

where K is the complete elliptic integral.

Noting that

$$\sum_{i,j=1}^4 p_i G_\varepsilon \left(\frac{\sigma_i - \sigma_j}{\sqrt{\bar{\lambda}}} \right) p_j = -s \log r - t \log(1-r) - \frac{\log \varepsilon}{2\pi} \sum_{i=1}^4 p_i^2 \quad (17)$$

with $s = -(p_1 + p_2)^2$, $t = -(p_1 + p_4)^2$ being Mandelstam's variables and dropping the last term on the right-hand side of Eq. (17) for lightlike momenta $p_i^2 = 0$, we find

$$S_{\text{eff}} = \frac{1}{2\pi K_0 \bar{\lambda}} [s \log r + t \log(1-r)] + K_0 A (1 - \bar{\lambda}) - \frac{d\Lambda^2 A}{2\lambda} + \Lambda^2 A + \frac{d-2}{24} \log[r(1-r)], \quad (18)$$

while the (ordered) integration over σ_i 's is inherited from reparametrizations of the boundary. It gives the volume of the projective group times the integration over r , which is equivalent to the integration over the ratio ω_R/ω_L . The boundary terms are negligible at least for lightlike momenta as we show below.

It is worth noting that our procedure of the smearing (10) is similar to the one introduced in [6] for computing the Lüscher term for a rectangular Wilson loop. Actually, the last term in Eq. (18), coming from the determinants, for $s \gg t$ makes sense of the momentum Lüscher term [7, 8]. However, it is *exact* for arbitrary r in view of the identity [9]

$$\begin{aligned} & \frac{1}{2(K(\sqrt{1-r}))^{1/2}} \eta\left(i \frac{K(\sqrt{1-r})}{K(\sqrt{r})}\right) \\ &= \frac{1}{2^{5/6}\pi^{1/2}} [r(1-r)]^{1/12} \end{aligned} \quad (19)$$

for the Dedekind η -function:

$$\prod_{m,n=1}^{\infty} \left(\frac{\pi m^2}{\omega_R^2} + \frac{\pi n^2}{\omega_L^2} \right) = \frac{1}{\sqrt{2\omega_L}} \eta\left(i \frac{\omega_R}{\omega_L}\right). \quad (20)$$

Minimizing (18) with respect to A , we find

$$\bar{\lambda} = \frac{1}{2} + \frac{\Lambda^2}{2K_0} + \sqrt{\frac{1}{4} \left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{d\Lambda^2}{2K_0}} \quad (21)$$

which is the same value as found in [3]. Minimizing (18) with respect to $\bar{\lambda}$, we obtain

$$A = \frac{1}{2\pi K_0^2} \frac{1}{(2\bar{\lambda} - 1 - \Lambda^2/K_0)\bar{\lambda}} [s \log r + t \log(1-r)]. \quad (22)$$

For the saddle-point value of the effective action we thus have

$$\begin{aligned} S_{\text{eff}} &= \left(\frac{1}{2\pi K_0 \bar{\lambda}} s + \frac{d-2}{24} \right) \log r \\ &+ \left(\frac{1}{2\pi K_0 \bar{\lambda}} t + \frac{d-2}{24} \right) \log(1-r). \end{aligned} \quad (23)$$

If we integrate $e^{-S_{\text{eff}}}$ over r as is prescribed by the path integral over reparametrizations, we get the Veneziano amplitude $\mathcal{B}(-\alpha(s), -\alpha(t))$ with

$$\alpha(s) = \frac{1}{2\pi K_0 \bar{\lambda}} s + \frac{d-2}{24}, \quad \alpha(t) = \frac{1}{2\pi K_0 \bar{\lambda}} t + \frac{d-2}{24} \quad (24)$$

to *all orders*, not only semiclassically like in textbooks. This is a remarkable consequence of the identity (19).

Minimizing (23) with respect to r , we find

$$r_* = \frac{\alpha(s)}{\alpha(s) + \alpha(t)} \quad (25)$$

and

$$S_{\text{eff}} = \alpha(s) \log \frac{\alpha(s)}{\alpha(s) + \alpha(t)} + \alpha(t) \log \frac{\alpha(t)}{\alpha(s) + \alpha(t)}. \quad (26)$$

For $s \gg t$ this results in the Regge behavior

$$\mathcal{A} \sim s^{\alpha(t)} \quad (27)$$

with the linear Regge trajectory (24).

IV. SCALING LIMIT AND RENORMALIZATION

As mentioned in the introduction, the Lilliputian scaling regime was previously defined for a string with the Dirichlet boundary conditions by the following renormalization of string tension and the length scale [2, 3]

$$K_R = K_0 \sqrt{\left(1 + \frac{\Lambda^2}{K_0}\right)^2 - \frac{2d\Lambda^2}{K_0}} = K_0 \left(2\bar{\lambda} - 1 - \frac{\Lambda^2}{K_0}\right), \quad (28)$$

$$L_R = \sqrt{\frac{\bar{\lambda}}{2\bar{\lambda} - 1 - \frac{\Lambda^2}{K_0}}} L_0. \quad (29)$$

Equations (28) and (29) are just slight rewritings of (1) and (2) and the scaling limit is $\Lambda \rightarrow \infty$ while L_R and K_R stay finite.

Motivated by the length-scale renormalization (29), we write

$$s = \frac{\bar{\lambda}}{2\bar{\lambda} - 1 - \frac{\Lambda^2}{K_0}} s_R, \quad t = \frac{\bar{\lambda}}{2\bar{\lambda} - 1 - \frac{\Lambda^2}{K_0}} t_R \quad (30)$$

in the scaling regime (1). Accounting for the renormalization of the string tension (28), we have

$$\begin{aligned} S_{\text{eff}} &= \left(\frac{1}{2\pi K_R} s_R + \frac{d-2}{24} \right) \log r \\ &+ \left(\frac{1}{2\pi K_R} t_R + \frac{d-2}{24} \right) \log(1-r), \end{aligned} \quad (31)$$

resulting in the linear renormalized Regge trajectory

$$\alpha(t) = \alpha'_R t_R + \frac{d-2}{24}, \quad \alpha'_R = \frac{1}{2\pi K_R} \quad (32)$$

with a finite slope α'_R .

V. METRIC AND THE BOUNDARY TERM

Equation (22) represents the mean area of fluctuating surfaces, while for the computation of the boundary term in the effective action we need the metric $\bar{\rho}(z)$ itself. It can be computed as an average of the induced metric

$$\bar{\rho}_{ab}(z) = \langle \partial_a X(z) \cdot \partial_b X(z) \rangle. \quad (33)$$

To understand the structure of the boundary term, we shall compute (33) at the tree level, i.e. in the classical approximation.

The harmonic function in the upper half-plane with the boundary conditions (10) at the real axis is

$$P(x, y) = \frac{1}{\pi} \sum_k p_k \arctan\left(\frac{x - \sigma_k}{y + \sqrt{\epsilon_k}}\right), \quad \sum_k p_k = 0. \quad (34)$$

By T-duality the computation is the same as for for the

world sheet

$$X_{\text{cl}}^\mu = \frac{1}{K_0 \lambda} P^\mu. \quad (35)$$

Using (34), (35) we obtain for the classical induced metric

$$\partial_1 X_{\text{cl}} \cdot \partial_1 X_{\text{cl}} = \frac{1}{\pi^2 K_0^2 \lambda^2} \sum_{i,j} \frac{p_i \cdot p_j (y + \sqrt{\epsilon_i})(y + \sqrt{\epsilon_j})}{[(x - \sigma_i)^2 + (y + \sqrt{\epsilon_i})^2][(x - \sigma_j)^2 + (y + \sqrt{\epsilon_j})^2]}, \quad (36a)$$

$$\partial_1 X_{\text{cl}} \cdot \partial_2 X_{\text{cl}} = -\frac{1}{\pi^2 K_0^2 \lambda^2} \sum_{i,j} \frac{p_i \cdot p_j (x - \sigma_i)(y + \sqrt{\epsilon_j})}{[(x - \sigma_i)^2 + (y + \sqrt{\epsilon_i})^2][(x - \sigma_j)^2 + (y + \sqrt{\epsilon_j})^2]}, \quad (36b)$$

$$\partial_2 X_{\text{cl}} \cdot \partial_2 X_{\text{cl}} = \frac{1}{\pi^2 K_0^2 \lambda^2} \sum_{i,j} \frac{p_i \cdot p_j (x - \sigma_i)(x - \sigma_j)}{[(x - \sigma_i)^2 + (y + \sqrt{\epsilon_i})^2][(x - \sigma_j)^2 + (y + \sqrt{\epsilon_j})^2]}. \quad (36c)$$

As is shown in [8], (36b) vanishes and (36a) coincides with (36c), provided σ_i 's satisfy

$$\sum_{j \neq i} \frac{p_i \cdot p_j}{\sigma_i - \sigma_j} = 0 \quad (37)$$

for lightlike momenta. Then the metric becomes conformal (i.e. $\bar{\rho}_{ab} = \bar{\rho} \delta_{ab}$).

Equation (37) is the same condition as the recently advocated tree-level scattering equation [10]. Because of the projective symmetry three equations in Eq. (37) are not independent, so for the case of four particles there is only one independent equation. This equation is nothing but the tree-level approximation of Eq. (25). Notice that Eq. (25) itself sums up all loops and guarantees that $\bar{\rho}_{ab}$ given by (33) is conformal in the mean-field approximation.

The classical metric at the boundary

$$\begin{aligned} \partial_1 X_{\text{cl}} \cdot \partial_1 X_{\text{cl}}|_B &= \frac{1}{\pi^2 K_0^2 \lambda^2} \sum_{i,j} \frac{p_i \cdot p_j \sqrt{\epsilon_i} \sqrt{\epsilon_j}}{[(x - \sigma_i)^2 + \epsilon_i][(x - \sigma_j)^2 + \epsilon_j]} \quad (38) \end{aligned}$$

vanishes except near the points $x = \sigma_k$ associated with edges of the polygon. The integration along the boundary

$$\int_{-\infty}^{+\infty} dx \sqrt{\partial_1 X_{\text{cl}} \cdot \partial_1 X_{\text{cl}}|_B} = \frac{1}{K_0 \lambda} \sum_k \sqrt{p_k^2} \quad (39)$$

reproduces the length. Thus the boundary term in the effective action is proportional to (39). It vanishes for lightlike momenta $p_k^2 = 0$. Otherwise its contribution has to be analyzed and remains finite in the scaling limit (1).

We do not expect this boundary term (like the last term on the right-hand side of Eq. (17)) to have any

effect on the Regge trajectory (24) because it depends on the masses $\sqrt{p_k^2}$ of colliding particles, while the Regge trajectory does not.

To avoid the problem with the boundary terms, it is tempting to consider the case of closed string scattering, when the boundary conditions are periodic along both axes (the torus topology). Then the boundary terms in the determinant are missing and what remains is four times larger than for the disk. This would change $(d - 2)/24$ to $(d - 2)/6$ in the above formulas as usual.

VI. STABILITY OF FLUCTUATIONS

We can check if fluctuations about the mean values of λ^{ab} and ρ are stable in the quadratic approximation, following the considerations in [3]. The only difference from [3] is that $\bar{\rho}(z)$ is now coordinate-dependent, a dependence given by the right-hand side of Eq. (36a).

Let us first consider the divergent part of the effective action to quadratic order in fluctuations for the nondiagonal element of λ^{ab} . It is given by Eq. (25) of [3]:

$$\begin{aligned} S_{\text{div}} &= \int d^2 z \left[\frac{K_0}{2} \lambda^{ab} \partial_a X_{\text{cl}} \cdot \partial_b X_{\text{cl}} + K_0 \rho \left(1 - \frac{1}{2} \lambda^{aa} \right) \right. \\ &\quad \left. - \frac{d\Lambda^2}{2} \frac{\rho}{\sqrt{\det \lambda}} + \Lambda^2 \rho \right], \quad \lambda^{aa} = \lambda^{11} + \lambda^{22}. \quad (40) \end{aligned}$$

Here X_{cl} is given by Eq. (35) for the present case.

Expanding to quadratic order

$$\begin{aligned} \sqrt{\det(\bar{\lambda} \delta^{ab} + \delta \lambda^{ab})} &= \lambda + \frac{1}{2} \delta \lambda^{aa} - \delta \lambda_2 + \mathcal{O}((\delta \lambda)^3), \\ \delta \lambda_2 &= \frac{1}{8\lambda} (\delta \lambda_{11} - \delta \lambda_{22})^2 + \frac{1}{2\lambda} (\delta \lambda_{12})^2, \quad (41) \end{aligned}$$

we find from (40) for constant $\bar{\lambda}^{ab} = \bar{\lambda}\delta^{ab}$

$$S_{\text{div}}^{(2)} = -\frac{d\Lambda^2}{2\bar{\lambda}} \int d^2z \bar{\rho} \delta\lambda_2 - \left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right) \int d^2z \delta\rho \frac{\delta\lambda^{aa}}{2} - \frac{d\Lambda^2}{2\bar{\lambda}^3} \int d^2z \bar{\rho} \left(\frac{\delta\lambda^{aa}}{2}\right)^2. \quad (42)$$

The first term on the right-hand side of Eq. (42) plays

a very important role for dynamics of quadratic fluctuations. Because the path integral over λ^{ab} goes parallel to imaginary axis, i.e. $\delta\lambda^{ab}$ is pure imaginary, the first term is always *positive*. Moreover, its exponential plays the role of a (functional) delta-function as $\Lambda \rightarrow \infty$, forcing $\delta\lambda^{ab} = \delta\lambda \delta^{ab}$.

For $s \gg t$ we keep only the bulk term to get the effective action to quadratic order in fluctuations

$$\begin{aligned} \delta S_2 = & -\left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right) \int d^2z \delta\rho \delta\lambda - \frac{d\Lambda^2}{2\bar{\lambda}} \int d^2z \bar{\rho} \left(\frac{\delta\lambda}{\bar{\lambda}}\right)^2 + \frac{(26-d)}{96\pi} \int d^2z \left(\frac{\partial_a \delta\rho}{\bar{\rho}}\right)^2 \\ & - \frac{d}{24\pi} \int d^2z \left(\frac{\partial_a \delta\rho}{\bar{\rho}}\right) \left(\frac{\partial_a \delta\lambda}{\bar{\lambda}}\right) + \frac{d}{32\pi} \int \frac{d^2p}{(2\pi)^2} \left(\frac{\delta\lambda(p)}{\bar{\lambda}}\right) \left(\frac{\delta\lambda(-p)}{\bar{\lambda}}\right) p^2 \log\left(\frac{1}{\epsilon p^2}\right) \end{aligned} \quad (43)$$

for a certain $\epsilon \sim 1/\Lambda^2$. Notice the last term on the right-hand side is normal (and therefore regularization dependent) rather than anomalous as the third and fourth terms.

From Eq. (43) for the effective action to the second order in fluctuations we find the quadratic form

$$\delta S_2 = \int d^2z \left[\delta(\log \rho) A_{\rho\rho} \delta(\log \rho) + 2 \delta(\log \rho) A_{\rho\lambda} \delta(\log \lambda) + \delta(\log \lambda) A_{\lambda\lambda} \delta(\log \lambda) \right]$$

with

$$A_{ij}(p) = \begin{bmatrix} \frac{(26-d)p^2}{96\pi} & -\frac{1}{2} \left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right) \bar{\rho}(p)\bar{\lambda} - \frac{dp^2}{48\pi} \\ -\frac{1}{2} \left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right) \bar{\rho}(p)\bar{\lambda} - \frac{dp^2}{48\pi} & -A(p) \end{bmatrix}, \quad (44)$$

where

$$A(p) = \frac{d\Lambda^2 \bar{\rho}(p)}{2\bar{\lambda}} + \frac{dp^2}{32\pi} \log(\epsilon p^2) \quad (45)$$

is always positive.

Since $\delta\lambda(z)$ is pure imaginary, i.e. $\delta\lambda(-p) = -\delta\lambda^*(p)$, we find for the determinant associated with the matrix in Eq. (44)

$$D = \left[\frac{1}{2} \left(K_0 - \frac{d\Lambda^2}{2\bar{\lambda}^2}\right) \bar{\rho}\bar{\lambda} + \frac{dp^2}{48\pi} \right]^2 + \frac{(26-d)p^2}{96\pi} A \quad (46)$$

and the propagators corresponding to the action (44) are given by

$$\langle \phi_i^*(p) \phi_j(p) \rangle = \frac{A_{ij}}{D}, \quad \phi_i = [\delta(\log \rho), \delta(\log \lambda)]. \quad (47)$$

The situation with stability of fluctuations is just the same as described in [3]: they are unconditionally stable for $2 < d < 26$. For $d > 26$ they are stable in the regularized case, where Λ is large but finite, because the first term on the right-hand side of Eq. (46) then dominates. In the scaling regime (1), where K_R is finite as $\Lambda \rightarrow \infty$, we have the situation where the first term on the right-hand side of Eq. (46) is finite after the renormalization of $\bar{\rho}$, so the second term dominates. The action (44) thus becomes unstable for $d > 26$ in the Lilliputian scaling limit.

VII. CONCLUSION

We have computed scattering amplitudes of the regularized bosonic Nambu-Goto string in the mean-field approximation, disregarding fluctuations of the Lagrange multiplier and an independent metric about their mean values. We have recovered the Regge behavior of the amplitudes in the Regge regime with the usual linear Regge trajectory in space-time dimensions $d > 2$. We have considered the Lilliputian scaling limit, previously introduced [2] to recover the results of canonical quantization for the spectrum the Dirichlet string, and showed that it is the one where the slope of the Regge trajectory scales to a finite value. We have also demonstrated that the effective action indeed has a stable minimum given by the mean field and fluctuations about the mean values of λ^{ab} and ρ increase the effective action for $2 < d < 26$.

The fact that the mean-field approximation reproduces canonical quantization in $d = 26$ is not surprising and is well understood. However, canonical quantization is not consistent in $2 < d < 26$, where the effective action depends nonlinearly on the world sheet metric ρ . An open interesting question is as to whether or not corrections to the mean field will change the above formulas in $2 < d < 26$.

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