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The global dimension of the algebras of polynomial integro-differential operators $\mathbb{I}_n$ and the Jacobian algebras $\mathbb{A}_n$

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Abstract

The aim of the paper is to prove two conjectures from the paper [8] that the (left and right) global dimension of the algebra $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \int_1, \ldots, \int_n \rangle$ of polynomial integro-differential operators and the Jacobian algebra $\mathbb{A}_n$ is equal to n (over a field of characteristic zero). The algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are neither left nor right Noetherian and $\mathbb{I}_n \subset \mathbb{A}_n$. Furthermore, they contain infinite direct sums of nonzero left/right ideals and are not domains. An analogue of Hilbert's Syzygy Theorem is proven for the algebras $\mathbb{I}_n, \mathbb{A}_n$ and their factor algebras. It is proven that the global dimension of all prime factor algebras of the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ is n and the weak global dimension of all the factor algebras of $\mathbb{I}_n$ and $\mathbb{A}_n$ is n.

Key Words: the algebra of polynomial integro-differential operators, the Jacobian algebra, the global dimension, the weak global dimension, the Weyl algebra, prime ideal, the projective dimension, the flat dimension, localization of a ring.

Mathematics subject classification 2010: 16E10, 16D25, 16S32, 16L30, 16S85, 16U20, 16S60, 16W50.

Contents

1 Introduction

Throughout, K is a field of characteristic zero and K^* is its group of units; algebra means an associative K-algebra with 1; module means a left module; $\mathbb{N} := \{0, 1, \ldots\}$ is the set of natural numbers; $P_n := K[x_1, \ldots, x_n]$ is a polynomial algebra over K; $\partial_1 := \frac{\partial}{\partial x_1}, \ldots, \partial_n := \frac{\partial}{\partial x_n}$ are the partial derivatives (K-linear derivations) of P_n; $\operatorname{End}_K(P_n)$ is the algebra of all K-linear maps from P_n to P_n; the subalgebra $A_n := K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n \rangle$ of $\operatorname{End}_K(P_n)$ is called the n'th *Weyl algebra*.

Definition, [7]. The *Jacobian algebra* $\mathbb{A}_n = K\langle x_1, \ldots, x_n, \partial_1, \ldots, \partial_n, H_1^{-1}, \ldots, H_n^{-1} \rangle$ is the subalgebra of $\operatorname{End}_K(P_n)$ generated by the Weyl algebra A_n and the elements $H_1^{-1}, \ldots, H_n^{-1} \in \operatorname{End}_K(P_n)$ where

$$H_1 := \partial_1 x_1, \ldots, H_n := \partial_n x_n.$$

The Jacobian algebras appeared in study of automorphisms of polynomial algebras as a very effective computational tool (which is not surprising as they contain the algebras of polynomial integro-differential operators $\mathbb{I}_n$, see below). Clearly, $\mathbb{A}_n = \bigotimes_{i=1}^n \mathbb{A}_1(i) \simeq \mathbb{A}_1^{\otimes n}$ where $\mathbb{A}_1(i) := K\langle x_i, \partial_i, H_i^{-1} \rangle \simeq \mathbb{A}_1$. The algebra $\mathbb{A}_n$ is neither a left nor right localization of the Weyl algebra A_n at the multiplicative set generated by the elements $H_1, \ldots, H_n$ since the algebra $\mathbb{A}_n$ is not

a domain but the Weyl algebra A_n is a domain. The algebra $\mathbb{A}_n$ contains all the integrations $\int_i : P_n \to P_n$, $p \mapsto \int p \, dx_i$ since

$$\int_i = x_i H_i^{-1} : \; x_1^{\alpha_1} \cdots x_n^{\alpha_n} \mapsto (\alpha_i + 1)^{-1} x_i x_1^{\alpha_1} \cdots x_n^{\alpha_n} \;\; \text{where} \;\; \alpha_1, \ldots, \alpha_n \in \mathbb{N}.$$

In particular, the algebra $\mathbb{A}_n$ contains the *algebra* $\mathbb{I}_n := K\langle x_1, \ldots, x_n, \, \partial_1, \ldots, \partial_n, \int_1, \ldots, \int_n \rangle$ of *polynomial integro-differential operators*. Notice that $\mathbb{I}_n = \bigotimes_{i=1}^{n} \mathbb{I}_1(i) \simeq \mathbb{I}_1^{\otimes n}$ where $\mathbb{I}_1(i) := K\langle x_i, \partial_i, \int_i \rangle$ and
$$P_n \subset A_n \subset \mathbb{I}_n \subset \mathbb{A}_n.$$

The algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are neither left nor right Noetherian and not domains. The algebras $\mathbb{I}_n$ were introduced and studied in detail in [8] and they have remarkable properties, name just a few (see [8] for details):

- $\mathbb{I}_n$ is a prime, catenary, central algebra of classical Krull dimension n and of Gelfand-Kirillov dimension $2n$, and there is a unique maximal ideal $\mathfrak{a}_n$ of the algebra $\mathbb{I}_n$.

- A canonical form is found for each element of $\mathbb{I}_n$ by showing that the algebra $\mathbb{I}_n$ is a generalized Weyl algebra.

- The lattice $\mathcal{J}(\mathbb{I}_n)$ of ideals of the algebra $\mathbb{I}_n$ is a distributive lattice.

- $\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$ and $\mathfrak{a}^2 = \mathfrak{a}$ for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{J}(\mathbb{I}_n)$.

- Classifications of all the ideals and the prime ideals of the algebra $\mathbb{I}_n$ are given.

- The set of ideals $\mathcal{J}(\mathbb{I}_n)$ of $\mathbb{I}_n$ is a finite set and the number $s_n := |\mathcal{J}(\mathbb{I}_n)|$ is equal to the *Dedekind number* $(2 - n + \sum_{i=1}^{n} 2^{\binom{n}{i}} \leq s_n \leq 2^{2^n})$.

- P_n is the only (up to isomorphism) faithful simple $\mathbb{I}_n$-module.

- Two sets of defining relations are given for the algebra $\mathbb{I}_n$.

- The factor algebra $\mathbb{I}_n/\mathfrak{a}$ is a Noetherian algebra iff the ideal $\mathfrak{a}$ is equal to the unique maximal ideal of $\mathbb{I}_n$.

- $\mathrm{GK}\,(\mathbb{I}_n/\mathfrak{a}) = 2n$ for all ideals $\mathfrak{a}$ of $\mathbb{I}_n$ distinct from $\mathbb{I}_n$ (where GK is the Gelfand-Kirillov dimension).

- The algebra $\mathbb{I}_n$ admits an involution $*$ given by the rule $\partial_i^* = \int_i$, $\int_i^* = \partial_i$, and $H_i^* = H_i$.

- $\mathfrak{a}^* = \mathfrak{a}$ for all ideals $\mathfrak{a}$ of the algebra $\mathbb{I}_n$.

- Each ideal of the algebra $\mathbb{I}_n$ is an essential left and right submodule of $\mathbb{I}_n$.

The fact that certain rings of differential operators are catenary was proven by Brown, Goodearl and Lenagan in [16].

The group of automorphisms of the algebra $\mathbb{I}_n$ was found in [9], it is an iterated semi-direct product of three obvious subgroups [9, Theorem 5.5.(1)].

- ([9, Theorem 5.5.(1)]) $\mathrm{Aut}_K(\mathbb{I}_n) = S_n \ltimes \mathbb{T}^n \ltimes \mathrm{Inn}(\mathbb{I}_n)$ *where S_n is the symmetric group (it permutes the tensor components of the algebra $\mathbb{I}_n = \mathbb{I}_1^{\otimes n}$), $\mathbb{T}^n$ is the n-dimensional algebraic torus and* $\mathrm{Inn}(\mathbb{I}_n)$ *is the group of inner automorphisms of the algebra $\mathbb{I}_n$.*

In [9], the group of units $\mathbb{I}_n^\times$ of the algebra $\mathbb{I}_n$ is described.

- ([10, Theorem 3.1.(2)]) $\mathbb{I}_n^\times = K^* \times (1 + \mathfrak{a}_n)^\times$ *where $\mathfrak{a}_n$ is the unique maximal ideal of the algebra $\mathbb{I}_n$ and $(1 + \mathfrak{a}_n)^\times$ is the group of units of the multiplicative monoid $1 + \mathfrak{a}_n$.*

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The group $\mathrm{Inn}(\mathbb{I}_n) \simeq (1 + \mathfrak{a}_n)^{\times}$ is huge. The groups of automorphisms of the polynomial algebras P_n (resp., the Weyl algebras A_n) are found only for $n = 1, 2$ (resp., $n = 1$, see [22]). The groups of automorphisms of the algebras P_2, A_1 and $\mathbb{I}_1$ have similar structure. One of the serious obstacles in finding the groups of automorphisms for the polynomial and Weyl algebras are the *Jacobian Conjecture* (for polynomials) and the *Dixmier Conjecture* (for the Weyl algebras) that are still open. The Dixmier Conjecture states that *every algebra endomorphism of the Weyl algebra A_n is an automorphism, [22].* In [10], it is shown that an analogue of the Dixmier Conjecture holds for the algebra $\mathbb{I}_1$:

- ([10, Theorem 1.1]) *Every algebra endomorphism of the algebra $\mathbb{I}_1$ is an automorphism.*

In [10], it was conjectured that the same is true for all algebras $\mathbb{I}_n$. The algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are closely related and have similar properties, see [7, 11].

The global dimension of the algebras $\mathbb{I}_n$, $\mathbb{A}_n$ and their prime factor algebras. In 1962, Rinehart found the global dimension of the first Weyl algebra A_1, [35]. In 1972, Roos proved that the global dimension of the Weyl algebra A_n is n, [36]. Goodearl obtained formulae for the global dimension of certain rings of differential operators [25], [26]. The weak global dimension of the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ is n, [8, Theorem 6.2, Theorem 7.2]. Since the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ admit an involution, they are *self-dual algebras* (i.e., they are isomorphic to their *opposite algebras*). As a result, for the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$, the left global dimension (l.gldim) coincides with the right global dimension (r.gldim) and we use the notation gldim for their common value.

- ([8, Proposition 6.7]) $n \leq \mathrm{gldim}(\mathbb{I}_n) \leq 2n$.

- ([8, Proposition 7.5]) $n \leq \mathrm{gldim}(\mathbb{A}_n) \leq 2n$.

In [8], it was conjectured that $\mathrm{gldim}(\mathbb{I}_n) = n$ and $\mathrm{gldim}(\mathbb{A}_n) = n$. The aim of the paper is to prove these conjecturers.

- (Theorem 3.4) $\mathrm{gldim}(\mathbb{I}_n) = n$.

- (Theorem 4.2.(2)) $\mathrm{gldim}(\mathbb{A}_n) = n$.

For a Noetherian algebra, its global dimension and the weak global dimension coincide. In general, it is much more easy to compute the weak global dimension than the global dimension as the former behaves better under many constructions (like localizations). It is a standard approach that in order to find the global dimension of a Noetherian algebra actually its weak global dimension is computed. The algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are not Noetherian algebras and they contain infinite direct sums of nonzero left and right ideals. These fact are major difficulty in computing the global dimension of $\mathbb{I}_n$ and $\mathbb{A}_n$ as not much technique is available in the non-Noetherian case (even on the level of examples).

Ideas of the proof of Theorem 3.4: To show that $\mathrm{gldim}(\mathbb{I}_n) = n$ we use an induction on n. The case $n = 1$ is turned out to be a challenging one. We spend entire Section 2 to prove that $\mathrm{gldim}(\mathbb{I}_1) = 1$. In Section 2, we recall some facts about the algebra $\mathbb{I}_1$ that are used in the proof of the case $n = 1$ and in the general case. Then we use localizations and properties of certain ideals of the algebra $\mathbb{I}_n$ and of the $\mathbb{I}_n$-module P_n (especially, that it is a projective $\mathbb{I}_n$-module) to tackle the inductive step. $\square$

The global dimension of prime factor algebras of $\mathbb{I}_n$ and $\mathbb{A}_n$. Using an explicit description of all the prime factor algebras of the algebra $\mathbb{I}_n$ obtained in [8] and the fact that $\mathrm{gldim}(\mathbb{I}_n) = \mathrm{gldim}(\mathbb{A}_n) = n$, the global dimension of prime factor algebras of the algebras $\mathbb{I}_n$ and $\mathbb{A}_n$ are found.

- (Theorem 3.5) $\mathrm{gldim}(A) = n$ *for all prime factor algebras A of $\mathbb{I}_n$.*

- (Theorem 4.3) $\mathrm{gldim}(A) = n$ *for all prime factor algebras A of $\mathbb{A}_n$.*

The weak global dimension of factor algebras of $\mathbb{I}_n$ and $\mathbb{A}_n$. In Section 5, a technique is developed of finding the weak global dimension of a ring via certain left/right localizations that satisfy some natural conditions (Theorem 5.5 and Theorem 5.6). It is applied to factor algebras of the algebra $\mathbb{I}_n$ and $\mathbb{A}_n$ to prove the following theorems.

- (Theorem 5.7.(2)) w.dim$(A) = n$ *for all factor algebras A of $\mathbb{I}_n$.*

- (Theorem 6.2.(2)) w.dim$(A) = n$ *for all factor algebras A of $\mathbb{A}_n$.*

Explicit descriptions of ideals of the algebras $\mathbb{A}_n$ and $\mathbb{I}_n$ (obtained in [7, 8]) are one of the crucial steps in the proofs of the theorems.

An analogue of Hilbert's Syzygy Theorem for the algebras $\mathbb{I}_n$, $\mathbb{A}_n$ and their prime factor algebras. Many classical algebras are tensor product of algebras (eg, $P_n = P_1^{\otimes n}$, $A_n = A_1^{\otimes n}$, $\mathbb{A}_n = \mathbb{A}_1^{\otimes n}$, $\mathbb{I}_n = \mathbb{I}_1^{\otimes n}$, etc). In general, it is difficult to find the left global dimension l.gldim$(A \otimes B)$ of the tensor product of two algebras (even to answer the question when it is finite). In [24], it was pointed out by Eilenberg, Rosenberg and Zelinsky that '*the questions concerning the dimension of the tensor product of two algebras have turned out to be surprisingly difficult.*' An answer is known if one of the algebras is a polynomial algebra:

Hilbert's Syzygy Theorem :   l.gldim$(P_n \otimes B) = $ l.gldim$(P_n) + $ l.gldim$(B) = n + $ l.gldim(B).

In [5, 4], an analogue of Hilbert's Syzygy Theorem was proven for certain *generalized Weyl algebras* A (eg, $A = A_n$, the Weyl algebra):

$$\text{l.gldim}(A \otimes B) = \text{l.gldim}(A) + \text{l.gldim}(B)$$

for all left Noetherian finitely generated algebras B (K is an algebraically closed uncountable field of characteristic zero). In this paper, the same result is proven for the algebras $\mathbb{I}_n$, $\mathbb{A}_n$ and for all their prime factor algebras (Theorem 3.6 and Theorem 4.4).

- (Theorem 3.6.(2)) *Let K be an algebraically closed uncountable field of characteristic zero and B be a left Noetherian finitely generated algebra over K. Then* l.gldim$(\mathbb{I}_n \otimes B) = $ l.gldim$(\mathbb{I}_n) + $ l.gldim$(B) = n + $ l.gldim(B).

- (Theorem 4.4.(2)) *Let K be an algebraically closed uncountable field of characteristic zero and B be a left Noetherian finitely generated algebra over K. Then* l.gldim$(\mathbb{A}_n \otimes B) = $ l.gldim$(\mathbb{A}_n) + $ l.gldim$(B) = n + $ l.gldim(B).

In case of w.dim, we prove even stronger results.

- (Theorem 5.8.(2)) *Let K be an algebraically closed uncountable field of characteristic zero and B be a left Noetherian finitely generated algebra over K. Then* w.dim$(A \otimes B) = $ w.dim$(A) + $ w.dim$(B) = n + $ w.dim(B) *for all factor algebras A of $\mathbb{I}_n$.*

- (Theorem 6.3.(2)) *Let K be an algebraically closed uncountable field of characteristic zero and B be a left Noetherian finitely generated algebra over K. Then* w.dim$(A \otimes B) = $ w.dim$(A) + $ w.dim$(B) = n + $ w.dim(B) *for all factor algebras A of $\mathbb{A}_n$.*

The global dimension and the weak global dimension of the (largest) left quotient ring of $\mathbb{I}_1$. Let R be a ring and $\mathcal{C}_R$ be the set of regular elements of R (the set of non-zero-divisors). The ring $Q_{l,cl}(R) := \mathcal{C}_R^{-1}R$ (resp., $Q_{r,cl}(R) = R\mathcal{C}_R^{-1}$) is called the *classical left* (resp., *right*) *quotient ring* of R. The left and right classical quotient rings not always exist. For the algebra $\mathbb{I}_1$ neither the left nor right classical quotient ring exists [12]. For an arbitrary ring R, there exists the *largest left* (resp., *right*) *Ore set* of R that consists of regular elements of R, it is denoted by $S_l(R)$ (resp., $S_r(R)$) and the ring $Q_l(R) := S_l(R)^{-1}R$ (resp., $Q_r(R) = RS_r(R)^{-1}$) is called the *(largest) left* (resp., *right*) *quotient ring* of R, [12, 14]. If the ring $Q_{l,cl}(R)$ (resp., $Q_{r,cl}(R)$) exists then $Q_{l,cl}(R) = Q_l(R)$ (resp., $Q_{r,cl}(R) = Q_r(R)$). For the algebra $\mathbb{I}_1$, the sets $S_l(\mathbb{I}_1)$, $S_r(\mathbb{I}_1)$ and the rings $Q_l(\mathbb{I}_1)$, $Q_r(\mathbb{I}_1)$ are explicitly described in [12, Theorem 9.7]. In particular, $S_l(\mathbb{I}_1) \neq S_r(\mathbb{I}_1)$

and the rings $Q_l(\mathbb{I}_1)$ and $Q_r(\mathbb{I}_1)$ are not isomorphic. For the Weyl algebra A_1, $Q_{l,cl}(A_1) = Q_l(A_1)$ is a skew field (a division ring), and so $\mathrm{gldim}(Q_l(A_1)) = 0$. This is not the case for the algebra $\mathbb{I}_1$ as the next theorem shows.

Theorem 1.1   *1.* $\mathrm{l.gldim}(Q_l(\mathbb{I}_1)) = 1$.

  2. For all regular left Ore sets S of the algebra $\mathbb{I}_1$, $\mathrm{l.gldim}(S^{-1}\mathbb{I}_1)) = 1$.

For left Noetherian rings the left global dimension coincides with the weak global dimension. For not left Noetherian rings this is not true, in general, and the next result demonstrates this phenomenon.

Theorem 1.2 $\mathrm{w.dim}(Q_l(\mathbb{I}_1)) = 0$ *and* $\mathrm{w.dim}(Q_r(\mathbb{I}_1)) = 0$.

The algebras of integro-differential operators have connections to the *Rota-Baxter* algebras. The latter appeared in the work of Baxter [15] and further explored by Rota [37, 38], Cartier [17], and Atkinson [2], and more recently by many others: Aguiar, Moreira [1]; Cassidy, Guo, Keigher, Sit, Ebrahimi-Fard [18], [23]; Connes, Kreimer, Marcoli [19], [20], Guo, Regensburger, Rosenkranz and Middeke [34, 28], name just a few.

2   The global dimension of $\mathbb{I}_1$ is 1

The aim of this section is to prove that the global dimension of the algebra $\mathbb{I}_1$ is 1 (Theorem 2.8). For reader's convenience we split the proof into several steps Lemma 2.1–Lemma 2.7. Many of the steps are interesting on their own (like Theorem 2.6). We start this section by recalling some necessary facts about the algebra $\mathbb{I}_1$, see [8, 12] for details.

The algebra $\mathbb{I}_1$ is generated by the elements ∂, $H := \partial x$ and $\int$ (since $x = \int H$) that satisfy the defining relations (Proposition 2.2, [8]):

$$\partial \int = 1, \quad [H, \int] = \int, \quad [H, \partial] = -\partial, \quad H(1 - \int \partial) = (1 - \int \partial)H = 1 - \int \partial.$$

The elements of the algebra $\mathbb{I}_1$,

$$e_{ij} := \int^i \partial^j - \int^{i+1} \partial^{j+1}, \quad i, j \in \mathbb{N}, \tag{1}$$

satisfy the relations $e_{ij}e_{kl} = \delta_{jk}e_{il}$ where δ_{jk} is the Kronecker delta function. Notice that $e_{ij} = \int^i e_{00}\partial^j$. The matrices of the linear maps $e_{ij} \in \mathrm{End}_K(K[x])$ with respect to the basis $\{x^{[s]} := \frac{x^s}{s!}\}_{s \in \mathbb{N}}$ of the polynomial algebra $K[x]$ are the elementary matrices, i.e., $e_{ij}(x^{[s]}) = \delta_{js}x^{[i]}$. Let $E_{ij} \in \mathrm{End}_K(K[x])$ be the usual matrix units, i.e., $E_{ij}(x^s) = \delta_{js}x^i$ for all $i, j, s \in \mathbb{N}$. Then

$$e_{ij} = \frac{j!}{i!}E_{ij}, \tag{2}$$

$Ke_{ij} = KE_{ij}$, and $F := \bigoplus_{i,j \geq 0} Ke_{ij} = \bigoplus_{i,j \geq 0} KE_{ij} \simeq M_\infty(K)$, the algebra (without 1) of infinite dimensional matrices. Using induction on i and the fact that $\int^j e_{kk}\partial^j = e_{k+j,k+j}$, we can easily prove that

$$\int^i \partial^i = 1 - e_{00} - e_{11} - \cdots - e_{i-1,i-1} = 1 - E_{00} - E_{11} - \cdots - E_{i-1,i-1}, \quad i \geq 1. \tag{3}$$

$\mathbb{Z}$-grading on the algebra $\mathbb{I}_1$ and the canonical form of an integro-differential operator. The algebra $\mathbb{I}_1 = \bigoplus_{i \in \mathbb{Z}} \mathbb{I}_{1,i}$ is a $\mathbb{Z}$-graded algebra ($\mathbb{I}_{1,i}\mathbb{I}_{1,j} \subseteq \mathbb{I}_{1,i+j}$ for all $i, j \in \mathbb{Z}$) where

$$\mathbb{I}_{1,i} = \begin{cases} D_1 \int^i = \int^i D_1 & \text{if } i > 0, \\ D_1 & \text{if } i = 0, \\ \partial^{|i|}D_1 = D_1\partial^{|i|} & \text{if } i < 0, \end{cases}$$

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the algebra $D_1 := K[H] \oplus \oplus_{i \in \mathbb{N}} K e_{ii}$ is a commutative non-Noetherian subalgebra of $\mathbb{I}_1$, $H e_{ii} = e_{ii} H = (i+1) e_{ii}$ for $i \in \mathbb{N}$; $(\int^i D_1)_{D_1} \simeq D_1$, $\int^i d \mapsto d$; $_{D_1}(D_1 \partial^i) \simeq D_1$, $d \partial^i \mapsto d$, for all $i \geq 0$ since $\partial^i \int^i = 1$. Notice that the maps $\cdot \int^i : D_1 \to D_1 \int^i$, $d \mapsto d \int^i$, and $\partial^i \cdot : D_1 \to \partial^i D_1$, $d \mapsto \partial^i d$, have the same kernel $\bigoplus_{j=0}^{i-1} K e_{jj}$. Each element a of the algebra $\mathbb{I}_1$ is the unique finite sum

$$a = \sum_{i>0} a_{-i} \partial^i + a_0 + \sum_{i>0} \int^i a_i + \sum_{i,j \in \mathbb{N}} \lambda_{ij} e_{ij} \tag{4}$$

where $a_k \in K[H]$ and $\lambda_{ij} \in K$. This is the *canonical form* of the polynomial integro-differential operator [8]. So, the set $\{H^j \partial^i, H^j, \int^i H^j, e_{st} \,|\, i \geq 1; j, s, t \geq 0\}$ is a K-basis for the algebra $\mathbb{I}_1$. The multiplication in the algebra $\mathbb{I}_1$ is given by the rule:

$$\int H = (H-1) \int, \;\; H \partial = \partial(H-1), \;\; \int e_{ij} = e_{i+1,j}, \;\; e_{ij} \int = e_{i,j-1}, \;\; \partial e_{ij} = e_{i-1,j} \;\; e_{ij} \partial = \partial e_{i,j+1}.$$

$$H e_{ii} = e_{ii} H = (i+1) e_{ii}, \;\; i \in \mathbb{N},$$

The ideal F of compact operators of $\mathbb{I}_1$. Let V be an infinite dimensional vector space over a field K. A linear map $\varphi \in \mathrm{End}_K(V)$ is called a *compact* linear map/operator if $\dim_K(\mathrm{im}(\varphi)) < \infty$. The set $\mathcal{C} = \mathcal{C}(V)$ of all compact operators is a (two-sided) ideal of the algebra $\mathrm{End}_K(V)$. The algebra $\mathbb{I}_1$ has the only proper ideal $F = \bigoplus_{i,j \in \mathbb{N}} K e_{ij} \simeq M_\infty(K)$, the ideal of compact operators in $\mathbb{I}_1$, $F = \mathbb{I}_1 \cap \mathcal{C}(K[x])$, $F^2 = F$, [8]. The factor algebra $\mathbb{I}_1/F$ is canonically isomorphic to the skew Laurent polynomial algebra

$$B_1 := K[H][\partial, \partial^{-1}; \tau], \;\; \tau(H) = H+1, \;\; \text{via } \partial \mapsto \partial, \;\; \int \mapsto \partial^{-1}, \;\; H \mapsto H$$

where $\partial^{\pm 1} \alpha = \tau^{\pm 1}(\alpha) \partial^{\pm 1}$ for all elements $\alpha \in K[H]$. The algebra B_1 is canonically isomorphic to the (left and right) localization $A_{1,\partial}$ of the Weyl algebra A_1 at the powers of the element ∂ (notice that $x = \partial^{-1} H$). Therefore, they have the common skew field of fractions, $\mathrm{Frac}(A_1) = \mathrm{Frac}(B_1)$, the *first Weyl skew field*. The algebra B_1 is a subalgebra of the skew Laurent polynomial algebra

$$\mathcal{B}_1 := K(H)[\partial, \partial^{-1}; \tau]$$

where $K(H)$ is the field of rational functions over the field K in H. The algebra $\mathcal{B}_1 = S^{-1} B_1$ is the left and right localization of the algebra B_1 at the multiplicative set $S = K[H] \backslash \{0\}$. The algebra $\mathcal{B}_1$ is a *noncommutative Euclidean domain*, i.e., the left and right division algorithms with remainder hold with respect to the length function l on B_1: $l(\alpha_m \partial^m + \alpha_{m+1} \partial^{m+1} + \cdots + \alpha_n \partial^n) = n - m$ where $\alpha_i \in K(H)$, $\alpha_m \neq 0$, $\alpha_n \neq 0$, and $m < \cdots < n$. In particular, the algebra $\mathcal{B}_1$ is a principal left and right ideal domain. A $\mathcal{B}_1$-module M is simple iff $M \simeq \mathcal{B}_1/\mathcal{B}_1 b$ for some irreducible element $b \in \mathcal{B}_1$, and $\mathcal{B}_1/\mathcal{B}_1 b \simeq \mathcal{B}_1/\mathcal{B}_1 c$ iff the elements b and c are *similar* (that is, there exists an element $d \in \mathcal{B}_1$ such that 1 is the greatest common right divisor of c and d, and bd is the least common left multiple of c and d).

The involution $*$ on the algebra $\mathbb{I}_1$. The algebra $\mathbb{I}_1$ admits the involution $*$ over the field K: $\partial^* = \int$, $\int^* = \partial$ and $H^* = H$, i.e., it is a K-algebra *anti-isomorphism* $((ab)^* = b^* a^*)$ such that $a^{**} = a$. Therefore, the algebra $\mathbb{I}_1$ is *self-dual*, i.e., it is isomorphic to its *opposite algebra* $\mathbb{I}_1^{op}$. As a result, the left and right properties of the algebra $\mathbb{I}_1$ are the same. Clearly, $e_{ij}^* = e_{ji}$ for all $i, j \in \mathbb{N}$, and so $F^* = F$.

Classification of $K[H]$-torsion simple $\mathbb{I}_1$-modules. In [12], a classification of simple $\mathbb{I}_1$-modules is given. In the proof of Theorem 2.8 we use only $K[H]$-torsion simple $\mathbb{I}_1$-modules. Let us recall their classification. Since the field K has characteristic zero, the group $\langle \tau \rangle \simeq \mathbb{Z}$ acts freely on the set $\mathrm{Max}(K[H])$ of maximal ideals of the polynomial algebra $K[H]$. That is, for each maximal ideal $\mathfrak{p} \in \mathrm{Max}(K[H])$, its orbit $\mathcal{O}(\mathfrak{p}) := \{\tau^i(\mathfrak{p}) \,|\, i \in \mathbb{Z}\}$ contains infinitely many elements. For two elements $\tau^i(\mathfrak{p})$ and $\tau^j(\mathfrak{p})$ of the orbit $\mathcal{O}(\mathfrak{p})$ we write $\tau^i(\mathfrak{p}) < \tau^j(\mathfrak{p})$ if $i < j$. Let $\mathrm{Max}(K[H])/\langle \tau \rangle$ be the set of all $\langle \tau \rangle$-orbits in $\mathrm{Max}(K[H])$. If K is an algebraically closed field then $\mathfrak{p} = (H - \lambda)$, for some scalar $\lambda \in K$; $\mathrm{Max}(K[H]) \simeq K$, $(H - \lambda) \leftrightarrow \lambda$; and $\mathrm{Max}(K[H])/\langle \tau \rangle \simeq K/\mathbb{Z}$.

For any algebra A, we denote by $\widehat{A}$ the set of isomorphism classes of simple A-modules and, for a simple A-module M, $[M] \in \widehat{A}$ stands for its isomorphism class. For an isomorphism invariant property $\mathcal{P}$ of simple A-modules, let $\widehat{A}(\mathcal{P}) := \{[M] \in \widehat{A} \mid M \text{ has property } \mathcal{P}\}$. The *socle* $\operatorname{soc}(M)$ of a module M is the sum of all the simple submodules if they exist and zero otherwise. Since the algebra $\mathbb{I}_1$ contains the Weyl algebra A_1, which is a simple infinite dimensional algebra, each nonzero $\mathbb{I}_1$-module is necessarily an *infinite dimensional* module.

Since the algebra $B_1 = \mathbb{I}_1/F$ is a factor algebra of $\mathbb{I}_1$, there is the tautological embedding

$$\widehat{B_1} \to \widehat{\mathbb{I}_1}, \ \ [M] \mapsto [M].$$

Therefore, $\widehat{B_1} \subseteq \widehat{\mathbb{I}_1}$. In [12, Theorem 2.1.(2)], it is proven that *the map*

$$\operatorname{Max}(K[H])/\langle\tau\rangle \to \widehat{B_1}(K[H] - \text{torsion}), \ \ [\mathfrak{p}] \mapsto [B_1/B_1\mathfrak{p}], \tag{5}$$

is a bijection with the inverse map $[N] \mapsto \operatorname{supp}(N) := \{\mathfrak{p} \in \operatorname{Max}(K[H]) \mid \mathfrak{p} \cdot n_{\mathfrak{p}} = 0 \text{ for some } 0 \neq n_{\mathfrak{p}} \in N\}$, *and*

$$\widehat{\mathbb{I}_1}(K[H] - \text{torsion}) = \{[K[x]]\} \coprod \widehat{B_1}(K[H] - \text{torsion}). \tag{6}$$

The left $\mathbb{I}_1$-module $F = \bigoplus_{i \in \mathbb{N}} E_{\mathbb{N},i}$ is a semi-simple $\mathbb{I}_1$-module where $_{\mathbb{I}_1}E_{\mathbb{N},i} := \mathbb{I}_1 e_{ii} = \bigoplus_{j \in \mathbb{N}} K e_{ji} \simeq \mathbb{I}_1/\mathbb{I}_1\partial \simeq _{\mathbb{I}_1}K[x]$ is a simple $\mathbb{I}_1$-module. Let $e_n := e_{00} + e_{11} + \cdots + e_{nn}$ for $n \in \mathbb{N}$. Then $\mathbb{I}_1 e_n = \bigoplus_{i=0}^{n} E_{\mathbb{N},i}$ is a semi-simple $\mathbb{I}_1$-module of length $n+1$. The left $\mathbb{I}_1$-module F has the ascending filtration $F = \bigcup_{n \in \mathbb{N}} \mathbb{I}_1 e_n$, $\mathbb{I}_1 e_0 \subset \mathbb{I}_1 e_1 \subset \cdots \subset \mathbb{I}_1 e_n \subset \cdots$.

Lemma 2.1 *For all $n \geq 1$,* $_{\mathbb{I}_1}\mathbb{I}_1 = \mathbb{I}_1\partial^n \oplus \mathbb{I}_1 e_{00} \oplus \cdots \oplus \mathbb{I}_1 e_{n-1,n-1}$ *and* $\mathbb{I}_{1\mathbb{I}_1} = \int^n \mathbb{I}_1 \oplus e_{00}\mathbb{I}_1 \oplus \cdots \oplus e_{n-1,n-1}\mathbb{I}_1$.

Proof. It suffices to prove only that the first equality holds since then the second is obtained from the first by applying the involution $*$ of the algebra $\mathbb{I}_1$. Notice that $\mathbb{I}_1 e_{00} \oplus \cdots \oplus \mathbb{I}_1 e_{n-1,n-1} = \mathbb{I}_1 e_{n-1}$ where $e_{n-1} := e_{00} + e_{11} + \cdots + e_{n-1,n-1}$ since $\mathbb{I}_1 e_{0i} = \mathbb{I}_1 e_{ii}$. Using the equality $\int^n \partial^n = 1 - e_{n-1}$ (see (3)), we see that $\mathbb{I}_1 = \mathbb{I}_1\partial^n + \mathbb{I}_1 e_{n-1}$. Since $\partial^n e_{n-1} = 0$ and $e_{n-1}^2 = e_{n-1}$, we have $\mathbb{I}_1\partial^n \cap \mathbb{I}_1 e_{n-1} = (\mathbb{I}_1\partial^n \cap \mathbb{I}_1 e_{n-1})e_{n-1} \subseteq \mathbb{I}_1\partial^n e_{n-1} = 0$. Therefore, $_{\mathbb{I}_1}\mathbb{I}_1 = \mathbb{I}_1\partial^n \oplus \mathbb{I}_1 e_{n-1}$. $\square$

It follows from Lemma 2.1 that the simple $\mathbb{I}_1$-module $_{\mathbb{I}_1}K[x] \simeq \mathbb{I}_1/\mathbb{I}_1\partial \simeq \mathbb{I}_1 e_{00}$ is a *projective* $\mathbb{I}_1$-module. For an algebra A and its element a, let $a\cdot$ and $\cdot a$ be the left and right multiplications by the element a in A.

Lemma 2.2   *1. For all elements $a \in \mathbb{I}_1$, the left ideal $\mathbb{I}_1 a$ of $\mathbb{I}_1$ is a projective left $\mathbb{I}_1$-module.*

2. For all elements $a \in \mathbb{I}_1 \backslash F$, the left ideal $\ker_{\mathbb{I}_1}(\cdot a)$ of $\mathbb{I}_1$ is a projective left $\mathbb{I}_1$-module.

Proof. If $a \in F$ then $\mathbb{I}_1 a \subseteq F$. Since $_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ is a semi-simple projective $\mathbb{I}_1$-module so is $\mathbb{I}_1 a$. Suppose that $a \notin F$. In [12], it is proved that $\mathcal{K} := \ker_{\mathbb{I}_1}(\cdot a)$ is a finitely generated $\mathbb{I}_1$-module which is contained in F, and so $\mathcal{K} \subseteq \mathbb{I}_1 e_{n-1}$ for some $n \geq 1$. The $\mathbb{I}_1$-module $\mathbb{I}_1 e_{n-1}$ is a semi-simple, hence $\mathbb{I}_1 e_{n-1} = \mathcal{K} \oplus L$ for some $\mathbb{I}_1$-submodule L of $\mathbb{I}_1 e_{n-1}$. By Lemma 2.1,

$$\mathbb{I}_1 = \mathbb{I}_1\partial^n \oplus \mathbb{I}_1 e_{n-1} = \mathbb{I}_1\partial^n \oplus \mathcal{K} \oplus L.$$

Therefore, the $\mathbb{I}_1$-modules $\mathcal{K}$ and $\mathbb{I}_1\partial^n \oplus L \simeq \mathbb{I}_1/\mathcal{K} \simeq \mathbb{I}_1 a$ are projective. $\square$

Lemma 2.3 *Let V be a left ideal of $\mathbb{I}_1$ such that $V \subseteq F$ and $a \in \mathbb{I}_1$. Then the left ideal $V + \mathbb{I}_1 a$ of the algebra $\mathbb{I}_1$ is a projective $\mathbb{I}_1$-module. In particular, the left ideal $F + \mathbb{I}_1 a$ of the algebra $\mathbb{I}_1$ is a projective $\mathbb{I}_1$-module.*

Proof. The left $\mathbb{I}_1$-module F is a projective semi-simple module, hence so is its submodule V. There is the short exact sequence of $\mathbb{I}_1$-modules

$$0 \to U := V \cap \mathbb{I}_1 a \xrightarrow{\varphi} V \oplus \mathbb{I}_1 a \xrightarrow{\psi} V + \mathbb{I}_1 a \to 0$$

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where $\varphi(u) := (u, -u)$ and $\psi(v, w) := v + w$. Then $V = U \oplus W$ for some, necessarily, projective $\mathbb{I}_1$-submodule W of V. By Lemma 2.2, the $\mathbb{I}_1$-module $V \oplus \mathbb{I}_1 a$ is projective. It follows from the explicit form of the map φ that $_{\mathbb{I}_1} V \oplus \mathbb{I}_1 a = \operatorname{im}(\varphi) \oplus W \oplus \mathbb{I}_1 a$. Therefore, the $\mathbb{I}_1$-module $V + \mathbb{I}_1 a \simeq (V \oplus \mathbb{I}_1 a)/\operatorname{im}(\varphi) \simeq W \oplus \mathbb{I}_1 a$ is projective. $\square$

Lemma 2.4 *Let I be a left ideal of the algebra $\mathbb{I}_1$. Then the left $\mathbb{I}_1$-module I is projective iff the left ideal $F + I$ of $\mathbb{I}_1$ is a projective $\mathbb{I}_1$-module.*

Proof. The short exact sequence of $\mathbb{I}_1$-modules

$$0 \to I \to F + I \to (F + I)/I \simeq F/F \cap I \to 0$$

splits since the $\mathbb{I}_1$-module $F/F \cap I$ is projective ($_{\mathbb{I}_1} F$ is a projective semi-simple module), and the result follows. $\square$

Lemma 2.5 *Let $\alpha(H) \in K[H]$ be a nonzero polynomial. Then $\ker_{\mathbb{I}_1}(\cdot \alpha(H)) = \bigoplus_{\{i \in \mathbb{N}: \alpha(i+1)=0\}} E_{\mathbb{N}, i}$ is a projective semi-simple left $\mathbb{I}_1$-module of finite length $|\{i \in \mathbb{N}: \alpha(i+1) = 0\}|$.*

Proof. Since $\alpha \notin F$ and $\mathbb{I}_1/F$ is a domain, $\ker_{\mathbb{I}_1}(\cdot \alpha) = \ker_F(\cdot \alpha)$. Since $_{\mathbb{I}_1} F_{\mathbb{I}_1} \simeq (_{\mathbb{I}_1} K[x]) \otimes (K[\partial]_{\mathbb{I}_1})$ where $K[\partial]_{\mathbb{I}_1} := \mathbb{I}_1/\int \mathbb{I}_1 = (\int \mathbb{I}_1 \oplus e_{00}\mathbb{I}_1)/\int \mathbb{I}_1 \simeq e_{00}\mathbb{I}_1 = \bigoplus_{i \in \mathbb{N}} K e_{0i} =: E_{0, \mathbb{N}}$ (Lemma 2.1) and $e_{0i} \cdot \alpha(H) = \alpha(i+1)$, the result is obvious. $\square$

Theorem 2.6 *Let $a \in \mathbb{I}_1 \backslash F$ and I be a nonzero ideal of the algebra $\mathbb{I}_1$.*

1. $_{\mathbb{I}_1} \mathbb{I}_1 \simeq \ker_{\mathbb{I}_1}(\cdot a) \oplus \mathbb{I}_1 a$.

2. *The left $\mathbb{I}_1$-module I is projective iff so is Ia.*

3. *Suppose that the left ideal $I \cap F$ is finitely generated. Then*

    (a) $I = I \cap F \oplus I_1$ *for a left ideal I_1 of $\mathbb{I}_1$; and $I_1 \simeq I/I \cap F$.*
    (b) *The $\mathbb{I}_1$-module I is projective iff the $\mathbb{I}_1$-module I_1 is so.*

Proof. 1. Since $a \in \mathbb{I}_1 \backslash F$, by [12, Theorem 4.2.(1)], the kernel $\mathcal{K} := \ker_{\mathbb{I}_1}(\cdot a)$ is a finitely generated (necessarily, semi-simple) submodule of the semi-simple $\mathbb{I}_1$-module $F = \bigoplus_{i \in \mathbb{N}} E_{\mathbb{N}, i}$ and so $\mathcal{K} \subseteq \bigoplus_{i=0}^{n} E_{\mathbb{N}, i}$ for some n. By Lemma 2.1, $_{\mathbb{I}_1} \mathbb{I}_1 = \mathcal{K} \oplus J$ for some left ideal J of $\mathbb{I}_1$. Since $J \simeq \mathbb{I}_1/\mathcal{K} \simeq \mathbb{I}_1 a$, statement 1 follows.

2. The intersection $\mathcal{K}' = \mathcal{K} \cap I$ is a finitely generated submodule of F (see the proof of statement 1). We keep the notation of the proof of statement 1. Then $\mathbb{I}_1 = \mathcal{K}' \oplus \mathbb{J}$ for some left ideal $\mathbb{J}$ of $\mathbb{I}_1$. Hence, $I = I \cap \mathbb{I}_1 = I \cap (\mathcal{K}' \oplus \mathbb{J}) = \mathcal{K}' \oplus I'$ where $I' = I \cap \mathbb{J}$. There is a short exact sequence of $\mathbb{I}_1$-modules, $0 \to \mathcal{K}' \to I \xrightarrow{\cdot a} Ia \to 0$. In particular, $Ia \simeq I/\mathcal{K}' \simeq I'$. So, $I \simeq \mathcal{K}' \oplus Ia$, and statement 2 follows.

3(a). Since the left ideal $I_0 := I \cap F$ is finitely generated, there is a natural number $n \geq 1$ such that $I_0 \subseteq V := \bigoplus_{i=0}^{n-1} \mathbb{I}_1 e_{ii}$. The module V is a projective semisimple $\mathbb{I}_1$-module, hence $V = I_0 \oplus U$ for some projective semisimple $\mathbb{I}_1$-module U. By Lemma 2.1, $\mathbb{I}_1 = V \oplus \mathbb{I}_1 \partial^n$, and so $I = I_0 \oplus I_1$ where $I_1 = I \cap (U \oplus \mathbb{I}_1 \partial^n)$.

3(b). The statement (b) follows from the statement (a). $\square$

Lemma 2.7 *Each simple $K[H]$-torsion $\mathbb{I}_1$-module has projective dimension ≤ 1. In more detail, $\operatorname{pd}_{\mathbb{I}_1}(K[x]) = 0$ and $\operatorname{pd}_{\mathbb{I}_1}(M) = 1$ for all $M \in \widehat{B}_1(K[H] - \text{torsion})$.*

Proof. By (Theorem 2.1, [12]), each simple $K[H]$-torsion $\mathbb{I}_1$-module M is isomorphic either to $_{\mathbb{I}_1} K[x] = \mathbb{I}_1/\mathbb{I}_1 \partial = (\mathbb{I}_1 \partial \oplus \mathbb{I}_1 e_{00})/\mathbb{I}_1 \partial \simeq \mathbb{I}_1 e_{00} = \bigoplus_{i \in \mathbb{N}} K e_{i0}$ or to $B_1/B_1 p = \mathbb{I}_1/(F + \mathbb{I}_1 p)$ for some irreducible polynomial $p \in K[H]$. Clearly, $\operatorname{pd}_{\mathbb{I}_1}(K[x]) = 0$ and $\operatorname{pd}_{\mathbb{I}_1}(F + \mathbb{I}_1 p) = \operatorname{pd}_{\mathbb{I}_1}(\mathbb{I}_1 p) = \operatorname{pd}_{\mathbb{I}_1}(\mathbb{I}_1) = 0$, by Lemma 2.4 and Theorem 2.6.(2). The ideal F is an essential left $\mathbb{I}_1$-submodule of $_{\mathbb{I}_1} \mathbb{I}_1$ ([8, Lemma 5.2.(2)]), then so is the left ideal $F + \mathbb{I}_1 p$, i.e., the projective resolution for the $\mathbb{I}_1$-module $\mathbb{I}_1/(F + \mathbb{I}_1 p)$,

$$0 \to F + \mathbb{I}_1 p \to \mathbb{I}_1 \to \mathbb{I}_1/(F + \mathbb{I}_1 p) \to 0,$$

does not split. Then $\operatorname{pd}_{\mathbb{I}_1}(\mathbb{I}_1/(F + \mathbb{I}_1 p)) = 1$. $\square$

By Lemma 2.7, $\operatorname{l.gldim}(\mathbb{I}_1) \geq 1$. The next theorem shows that $\operatorname{l.gldim}(\mathbb{I}_1) = 1$.

Theorem 2.8 gldim($\mathbb{I}_1$) = 1.

Proof. The algebra $\mathbb{I}_1$ is self-dual, so it suffices to prove that l.gldim($\mathbb{I}_1$) = 1, i.e., every nonzero left ideal I of the algebra $\mathbb{I}_1$ is a projective $\mathbb{I}_1$-module. By Lemma 2.4, we may assume that $F \subseteq I$. Since ${}_{\mathbb{I}_1}F \simeq K[x]^{(\mathbb{N})}$ is a projective $\mathbb{I}_1$-module, we may assume that $F \subsetneq I$. Then its image $\pi(I)$ under the algebra epimorphism $\pi : \mathbb{I}_1 \to \mathbb{I}_1/F = B_1$, $a \mapsto \overline{a} := a + F$ is a nonzero left ideal of the algebra B_1. Since the algebra $\mathcal{B}_1 = S^{-1}B_1$, where $S := K[H]\backslash\{0\}$, is a principal left ideal domain, $\mathcal{B}_1\pi(I) = \mathcal{B}_1 b$ for some nonzero element $b \in K[H][\partial; \tau] \subseteq B_1$. Notice that the algebras B_1 and $\mathcal{B}_1$ are left and right Noetherian algebras, $B_1 b \subseteq \pi(I)$ and $S^{-1}B_1 b = \mathcal{B}_1 b = S^{-1}\pi(I)$. So, the B_1-module $\pi(I)/B_1 b$ is a finitely generated $K[H]$-torsion B_1-module. Therefore,

$$\pi(I) = B_1 b + \sum_{i=1}^{s} B_1 \alpha_i^{-1} a_i b$$

for some elements $\alpha_i \in S$ and $a_i \in K[H][\partial; \tau] \subseteq B_1$ such that $\alpha_i^{-1}a_i b \in B_1$. Clearly, $\alpha_i^{-1}a_i b \in K[H][\partial; \tau]$. Since $\mathcal{B}_1 = S^{-1}B_1 = B_1 S^{-1}$, there exists elements $c_i \in K[H][\partial; \tau]$ and $\alpha \in S$ such that $\alpha_i^{-1}a_i = c_i\alpha^{-1}$.

$$I = F + \mathbb{I}_1 b + \sum_{i=1}^{s} \mathbb{I}_1 c_i\alpha^{-1}b$$

where all elements $c_i\alpha^{-1}b = \alpha_i^{-1}a_i b_i \in K[H][\partial; \tau] \subseteq \mathbb{I}_1$. Fix an element $\beta \in S$ such that $\alpha^{-1}b = d\beta^{-1}$ for some $d \in K[H][\partial; \tau]$. Notice that $\mathbb{I}_1 b = \mathbb{I}_1\alpha\alpha^{-1}b = \mathbb{I}_1\alpha d\beta^{-1}$. By Theorem 2.6, ${}_{\mathbb{I}_1}I$ is projective iff ${}_{\mathbb{I}_1}I\beta = F\beta + \mathbb{I}_1\alpha d + \sum_{i=1}^{s} \mathbb{I}_1 c_i d$ is projective iff $(\mathbb{I}_1\alpha + \sum_{i=1}^{s} \mathbb{I}_1 c_i)d$ is projective $\mathbb{I}_1$-module (Theorem 2.4) iff $\mathbb{I}_1\alpha + \sum_{i=1}^{s} \mathbb{I}_1 c_i$ is a projective $\mathbb{I}_1$-module (Theorem 2.6.(2)) iff $J := F + \mathbb{I}_1\alpha + \sum_{i=1}^{s} \mathbb{I}_1 c_i$ is a projective $\mathbb{I}_1$-module (Lemma 2.4) iff $\mathrm{pd}_{\mathbb{I}_1}(M) \leq 1$ where $M := \mathbb{I}_1/J$. If $M = 0$, i.e., $J = \mathbb{I}_1$, we are done. So, we may assume that $M \neq 0$. The nonzero B_1-module $N := \mathbb{I}_1/(F + \mathbb{I}_1\alpha) \simeq B_1/B_1\alpha$ is a $K[H]$-torsion B_1-module of *finite length* (it is well known that any proper factor module of B_1 has finite length). The module M is an epimorphic image of the B_1-module N. Therefore, ${}_{B_1}M$ is $K[H]$-torsion of finite length. By Lemma 2.7, $\mathrm{pd}_{\mathbb{I}_1}(M) \leq 1$. Therefore, l.gldim($\mathbb{I}_1$) = 1. $\square$

The global dimension of localizations of $\mathbb{I}_1$ is 1.

Proof of Theorem 1.1. 1. Since l.gldim($\mathbb{I}_1$) = 1, we must have l.gldim($Q_l(\mathbb{I}_1)$) ≤ l.gldim($\mathbb{I}_1$) = 1. By [8, Corollary 3.3.(8)] and [12, Corollary 8.5], the unique proper ideal $\mathcal{I}$ of the algebra $Q_l(\mathbb{I}_1)$ is an essential left ideal, hence $\mathrm{pd}_{Q_l(\mathbb{I}_1)}(Q_l(\mathbb{I}_1)/\mathcal{I}) \geq 1$, and so l.gldim($Q_l(\mathbb{I}_1)$) ≥ 1. Therefore, l.gldim($Q_l(\mathbb{I}_1)$) = 1.

2. Let S be a regular left Ore set of the algebra $\mathbb{I}_1$. Then $S \subseteq S_l(\mathbb{I}_1)$, and so $S^{-1}\mathbb{I}_1 \subseteq S_l(R)^{-1}R = Q_l(R)$. Now,

$$1 = \text{l.gldim}(Q_l(\mathbb{I}_1)) \leq \text{l.gldim}(S^{-1}\mathbb{I}_1) \leq \text{l.gldim}(\mathbb{I}_1) = 1,$$

and so l.gldim($S^{-1}\mathbb{I}_1$) = 1. $\square$

Corollary 2.9   *1.* r.gldim($Q_r(\mathbb{I}_1)$) = 1.

2. For all regular right Ore sets T of the algebra $\mathbb{I}_1$, r.gldim($\mathbb{I}_1 T^{-1}$) = 1.

Proof. 1. r.gldim($Q_r(\mathbb{I}_1)$) = l.gldim($Q_r(\mathbb{I}_1)^*$) = l.gldim($Q_l(\mathbb{I}_1)$) = 1, by Theorem 1.1.(1).

2. r.gldim($\mathbb{I}_1 T^{-1}$) = l.gldim($(\mathbb{I}_1 T^{-1})^*$) = l.gldim($(T^*)^{-1}\mathbb{I}_1$) = 1, by Theorem 1.1.(2) since T^* is a regular left Ore set of $\mathbb{I}_1$. $\square$

Proof of Theorem 1.2. By [5, Corollary 4.5],

$$\text{w.dim}(Q_l(\mathbb{I}_1)) = \max\{\text{w.dim}(S_\partial^{-1}Q_l(\mathbb{I}_1)), \nu\} \text{ where } \nu = \sup\{\mathrm{fd}_{\mathbb{I}_1}(M) \,|\, S_\partial^{-1}M = 0\}.$$

Since the algebra $S_\partial^{-1}Q_l(\mathbb{I}_1) \simeq Q_{l,cl}(A_1)$ is a division ring, by [12, Corollary 8.5.(7b)], its weak global dimension is 0. The $\mathbb{I}_1$-module $K[x]$ is projective, hence so is the $Q_l(\mathbb{I}_1)$-module $S_l(\mathbb{I}_1)^{-1}K[x]$. In particular, the flat dimension of the $Q_l(\mathbb{I}_1)$-module $S_l(\mathbb{I}_1)^{-1}K[x]$ is 0. Every nonzero S_∂-torsion $Q_l(\mathbb{I}_1)$-module is a direct sum of copies of the projective $Q_l(\mathbb{I}_1)$-module $S_l(\mathbb{I}_1)^{-1}K[x]$. Hence, $\nu = 0$, and so w.dim($Q_l(\mathbb{I}_1)$) = max\{0, 0\} = 0.

Now, w.dim($Q_r(\mathbb{I}_1)$) = w.dim($Q_l(\mathbb{I}_1)^*$) = w.dim($Q_l(\mathbb{I}_1)$) = 0. $\square$

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3 The global dimension of the algebra $\mathbb{I}_n$

In this section, we prove that the global dimension of the algebra $\mathbb{I}_n$ and of all its prime factor algebras is n (Theorem 3.4). An analogue of Hilbert's Syzygy Theorem is proven for the algebra $\mathbb{I}_n$ and for all its prime factor algebras (Theorem 3.6).

Classification of ideals of the algebra $\mathbb{I}_n$. Theorem 3.1 describes all the ideals of the algebra $\mathbb{I}_n$ and shows that the ideal theory of $\mathbb{I}_n$ is 'very arithmetic' (it is the best possible and 'simple' ideal theory one can imagine). Let $\mathcal{B}_n$ be the set of all functions $f : \{1, 2, \ldots, n\} \to \{0, 1\}$. For each function $f \in \mathcal{B}_n$, $I_f := I_{f(1)} \otimes \cdots \otimes I_{f(n)}$ is the ideal of $\mathbb{I}_n$ where $I_0 := F$ and $I_1 := \mathbb{I}_1$. Let $\mathcal{C}_n$ be the set of all subsets of $\mathcal{B}_n$ all distinct elements of which are incomparable (two distinct elements f and g of $\mathcal{B}_n$ are *incomparable* if neither $f(i) \leq g(i)$ nor $f(i) \geq g(i)$ for all i). For each $C \in \mathcal{C}_n$, let $I_C := \sum_{f \in C} I_f$, the ideal of $\mathbb{I}_n$. The number $\mathfrak{d}_n$ of elements in the set $\mathcal{C}_n$ is called the *Dedekind number*. It appeared in the paper of Dedekind [21]. An asymptotic of the Dedekind numbers was found by Korshunov [31].

Theorem 3.1 ([8, Corollary 3.3].)

1. *The algebra $\mathbb{I}_n$ is a prime algebra.*

2. *The set of height one prime ideals of the algebra $\mathbb{I}_n$ is $\{\mathfrak{p}_1 := F \otimes \mathbb{I}_{n-1}, \mathfrak{p}_1 := \mathbb{I}_1 \otimes F \otimes \mathbb{I}_{n-2}, \ldots, \mathfrak{p}_n := \mathbb{I}_{n-1} \otimes F\}$.*

3. *Each ideal of the algebra $\mathbb{I}_n$ is an idempotent ideal ($\mathfrak{a}^2 = \mathfrak{a}$).*

4. *The ideals of the algebra $\mathbb{I}_n$ commute ($\mathfrak{a}\mathfrak{b} = \mathfrak{b}\mathfrak{a}$).*

5. *The lattice $\mathcal{J}(\mathbb{I}_n)$ of ideals of the algebra $\mathbb{I}_n$ is distributive.*

6. *The classical Krull dimension $\mathrm{cl.Kdim}(\mathbb{I}_n)$ of the algebra $\mathbb{I}_n$ is n.*

7. *$\mathfrak{a}\mathfrak{b} = \mathfrak{a} \cap \mathfrak{b}$ for all ideals $\mathfrak{a}$ and $\mathfrak{b}$ of the algebra $\mathbb{I}_n$.*

8. *The ideal $\mathfrak{a}_n := \mathfrak{p}_1 + \cdots + \mathfrak{p}_n$ is the largest (hence, the only maximal) ideal of $\mathbb{I}_n$ distinct from $\mathbb{I}_n$, and $F_n = F^{\otimes n} = \bigcap_{i=1}^{n} \mathfrak{p}_i$ is the smallest nonzero ideal of $\mathbb{I}_n$.*

9. *(A classification of ideals of $\mathbb{I}_n$) The map $\mathcal{C}_n \to \mathcal{J}(\mathbb{I}_n)$, $C \mapsto I_C := \sum_{f \in C} I_f$ is a bijection where $I_\emptyset := 0$. The number of ideals of $\mathbb{I}_n$ is the Dedekind number $\mathfrak{d}_n$. Moreover, $2 - n + \sum_{i=1}^{n} 2^{\binom{n}{i}} \leq \mathfrak{d}_n \leq 2^{2^n}$. For $n = 1$, F is the unique proper ideal of the algebra $\mathbb{I}_1$.*

10. *(A classification of prime ideals of $\mathbb{I}_n$) Let Sub_n be the set of all subsets of $\{1, \ldots, n\}$. The map $\mathrm{Sub}_n \to \mathrm{Spec}(\mathbb{I}_n)$, $I \mapsto \mathfrak{p}_I := \sum_{i \in I} \mathfrak{p}_i$, $\emptyset \mapsto 0$, is a bijection, i.e., any nonzero prime ideal of $\mathbb{I}_n$ is a unique sum of primes of height 1; $|\mathrm{Spec}(\mathbb{I}_n)| = 2^n$; the height of $\mathfrak{p}_I$ is $|I|$; and*

11. *$\mathfrak{p}_I \subset \mathfrak{p}_J$ iff $I \subset J$.*

The weak dimension of the algebra $\mathbb{S}_n$. Let S be a non-empty multiplicatively closed subset of a ring R, and let $\mathrm{ass}(S) := \{r \in R \,|\, sr = 0 \text{ for some } s \in S\}$. Then a *left quotient ring* of R with respect to S is a ring Q together with a homomorphism $\varphi : R \to Q$ such that
(i) for all $s \in S$, $\varphi(s)$ is a unit in Q,
(ii) for all $q \in Q$, $q = \varphi(s)^{-1}\varphi(r)$ for some $r \in R$ and $s \in S$, and
(iii) $\ker(\varphi) = \mathrm{ass}(S)$.

If there exists a left quotient ring Q of R with respect to S then it is unique up to isomorphism, and it is denoted by $S^{-1}R$. It is also said that the ring Q is the *left localization* of the ring R at S. The ring $S^{-1}R$ exists iff S is a *left denominator set* of the ring R, that is S is a *left Ore set* such that $rs = 0$ for some elements $r \in R$ and $s \in S$ implies $s'r = 0$ for some element $s' \in S$ (S is a left Ore set if $Sr \cap Rs \neq \emptyset$ for all elements $r \in R$ and $s \in S$). If M is a left R-module then the set $\mathrm{tor}_S(M) := \{m \in M \,|\, sm = 0 \text{ for some } s \in S\}$ is a submodule of M which is called the *S-torsion submodule* of M. For a left denominator set S of R, $\mathrm{ass}(S) = \mathrm{ass}_{l,R}(S) := \{r \in R \,|\, sr = 0 \text{ for}$

some element $s \in S$} is an ideal. In a similar way, a right denominator set of R is defined. If T is a right denominator set of R then the ring $RT^{-1} = \{rt^{-1} \,|\, r \in R, t \in T\}$ is a localization of R at T on the right and the set $\mathrm{ass}_{r,R}(T) := \{r \in R \,|\, rt = 0$ for some element $t \in T\}$ is an ideal of R.

Example 1. Let $S := S_\partial := \{\partial^i, i \geq 0\}$ and $R = \mathbb{I}_1$. Then $\mathrm{ass}(S) = F$, $\mathbb{I}_1/\mathrm{ass}(S) = B_1$ and the conditions (i)-(iii) hold where $Q = B_1$. This means that the ring $B_1 = \mathbb{I}_1/F$ is the left quotient ring of $\mathbb{I}_1$ at S, $B_1 \simeq S_\partial^{-1}\mathbb{I}_1$.

Example 2. Let $S := S_{\partial_1,\ldots,\partial_n} := \{\partial^\alpha, \alpha \in \mathbb{N}^n\}$ and $R = \mathbb{I}_n$. Then

$$S_{\partial_1,\ldots,\partial_n}^{-1}\mathbb{I}_n \simeq \bigotimes_{i=1}^n S_{\partial_i}^{-1}\mathbb{I}_1(i) = \bigotimes_{i=1}^n B_1(i) =: B_n \tag{7}$$

where $B_i = K[H_i][\partial_i, \partial_i^{-1}; \tau_i]$ is a skew Laurent polynomial ring where $\tau_i(H_i) = H_i + 1$. Furthermore, $\mathrm{ass}(S_{\partial_1,\ldots,\partial_n}) = \mathfrak{a}_n$ and $\mathbb{I}_n/\mathfrak{a}_n = B_n$, see [8] for details (where $\mathfrak{a}_n$ is the unique maximal ideal of the algebra $\mathbb{I}_n$). The right localization $\mathbb{I}_n S_{\partial_1,\ldots,\partial_n}^{-1}$ of $\mathbb{I}_n$ at $S_{\partial_1,\ldots,\partial_n}$ does not exist, [8]. By applying the involution $*$ to (7), we see that

$$\mathbb{I}_n S_{\int_1,\ldots,\int_n}^{-1} \simeq B_n. \tag{8}$$

So, the algebra B_n is the right localization of $\mathbb{I}_n$ at the multiplicatively closed set $S_{\int_1,\ldots,\int_n} := \{\int^\alpha := \int_1^{\alpha_1} \cdots \int_n^{\alpha_n} \,|\, \alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{N}^n\}$.

Given a ring R and modules $_RM$ and N_R, we denote by $\mathrm{pd}(_RM)$ and $\mathrm{pd}(N_R)$ their projective dimensions. Let us recall two results which will be used in proofs later.

Proposition 3.2 *([3].) Let M be a module over an algebra A, I a non-empty well-ordered set, $\{M_i\}_{i \in I}$ be a family of submodules of M such that if $i, j \in I$ and $i \leq j$ then $M_i \subseteq M_j$. If $M = \bigcup_{i \in I} M_i$ and $\mathrm{pd}_A(M_i/M_{<i}) \leq n$ for all $i \in I$ where $M_{<i} := \bigcup_{j<i} M_j$ then $\mathrm{pd}_A(M) \leq n$.*

Let $V \subseteq U \subseteq W$ be modules. Then the factor module U/V is called a *sub-factor* of the module W. Let w.dim and fd denote the *weak global dimension* and the *flat* dimension, respectively.

It is obvious that $P_n \simeq A_n/\sum_{i=1}^n A_n \partial_i$ where A_n is the Weyl algebra. A similar result is true for the $\mathbb{I}_n$-module P_n (Proposition 3.3.(2)). Notice that $\mathrm{pd}_{A_n}(P_n) = n$ but $\mathrm{pd}_{\mathbb{I}_n}(P_n) = 0$ (Proposition 3.3.(3)).

Proposition 3.3 *([8, Proposition 6.1].)*

1. $\mathbb{I}_1 = \mathbb{I}_1 \partial \bigoplus \mathbb{I}_1 e_{00}$ and $\mathbb{I}_1 = \int \mathbb{I}_1 \bigoplus e_{00}\mathbb{I}_1$.

2. $_{\mathbb{I}_n}P_n \simeq \mathbb{I}_n/\sum_{i=1}^n \mathbb{I}_n \partial_i$.

3. The $\mathbb{I}_n$-module P_n is projective.

4. $F_n = F^{\otimes n}$ is a left and right projective $\mathbb{I}_n$-module.

5. The projective dimension of the left and right $\mathbb{I}_n$-module $\mathbb{I}_n/F_n$ is 1.

6. For each element $\alpha \in \mathbb{N}^n$, the $\mathbb{I}_n$-module $\mathbb{I}_n/\mathbb{I}_n \partial^\alpha$ is projective.

Let A be an algebra, M be an A-module and S be a left denominator set of A. The set of S-torsion elements in M, $\mathrm{tor}_S(M) := \{m \in M \,|\, sm$ for some $s \in S\}$, is an A-submodule of M.

Theorem 3.4 $\mathrm{gldim}(\mathbb{I}_n) = n$.

Proof. The algebra $\mathbb{I}_n$ is self-dual. Therefore, $\mathrm{l.gldim}(\mathbb{I}_n) = \mathrm{r.gldim}(\mathbb{I}_n)$. To prove that $\mathrm{l.gldim}(\mathbb{I}_n) = n$ we use induction on n. The case $n = 1$ holds (Theorem 2.8). So, we assume that $n > 1$ and the equality is true for all natural numbers $n' < n$. By [8, Proposition 6.7],

$$n \leq \mathrm{l.gldim}(\mathbb{I}_n) \leq 2n.$$

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Let I be a left ideal of the algebra $\mathbb{I}_n$. We have to show that

$$\mathrm{pd}_{\mathbb{I}_n}(I) \leq n - 1$$

(since then l.gldim($\mathbb{I}_n$) $\leq n$, and so l.gldim($\mathbb{I}_n$) $= n$). The algebra $\mathbb{I}_n$ is a tensor product of algebras $\mathbb{I}_1 \otimes \mathbb{I}_{n-1}$. By applying the exact functor $- \otimes \mathbb{I}_{n-1}$ to the short exact sequence of $\mathbb{I}_1$-modules $0 \to F \to \mathbb{I}_1 \to B_1 \to 0$ we obtain the short exact sequence of $\mathbb{I}_n$-modules $0 \to F \otimes \mathbb{I}_{n-1} \to \mathbb{I}_n \to B_1 \otimes \mathbb{I}_{n-1} \to 0$. As a result we have the short exact sequence of left $\mathbb{I}_n$-modules

$$0 \to I_0 := I \cap F \otimes \mathbb{I}_{n-1} \to I \to \overline{I} := I/I_0 \to 0. \tag{9}$$

Notice that $\mathrm{tor}_{S_{\partial_1}}(I) = I_0$ (where $S_{\partial_1} := \{\partial_1^i \,|\, i \in \mathbb{N}\}$) and $S_{\partial_1}^{-1}I = S_{\partial_1}^{-1}\overline{I}$. Therefore,

$$0 \to \overline{I} \to B_1 \otimes \mathbb{I}_{n-1} \to J := B_1 \otimes \mathbb{I}_{n-1}/\overline{I} \to 0 \tag{10}$$

is an exact sequence of $\mathbb{I}_n$-modules since $B_1 \simeq S_{\partial_1}^{-1}\mathbb{I}_1$.

   Claim. $\mathrm{pd}_{\mathbb{I}_n}(I_0) \leq n - 1$ *and* $\mathrm{pd}_{\mathbb{I}_n}(J) \leq n - 1$: The $\mathbb{I}_n$-modules I_0 and J are S_{∂_1}-torsion. By [8, Corollary 6.3], each S_{∂_1}-torsion $\mathbb{I}_n$-module M admits a family $\{T_\lambda\}_{\lambda \in \Lambda}$ of $\mathbb{I}_n$-modules such that $M = \cup_{\lambda \in \Lambda} T_\lambda$ where $(\Lambda, \leq)$ is a well-ordered set such that $\lambda \leq \mu$ implies $T_\lambda \subseteq T_\mu$ and $T_\lambda / \cup_{\mu < \lambda} T_\mu \simeq K[x_1] \otimes \mathcal{T}_\lambda$ for some $\mathbb{I}_{n-1}$-module $\mathcal{T}_\lambda$. The $\mathbb{I}_1$-module $K[x_1]$ is projective (Proposition 3.3.(3)). By Proposition 3.2,

$$\mathrm{pd}_{\mathbb{I}_n}(M) \leq \max\{\mathrm{pd}_{\mathbb{I}_n}(K[x] \otimes \mathcal{T}_\lambda\}_{\lambda \in \Lambda} \leq \text{l.gldim}(\mathbb{I}_{n-1}) = n - 1,$$

by induction. The proof of the Claim is complete.

   Suppose that $J = 0$. Then, by (9) and (10),

$$\mathrm{pd}_{\mathbb{I}_n}(I) \leq \max\{\mathrm{pd}_{\mathbb{I}_n}(I_0), \mathrm{pd}_{\mathbb{I}_n}(B_1 \otimes \mathbb{I}_{n-1})\} \leq \max\{n - 1, 1\} = n - 1$$

since $\mathrm{pd}_{\mathbb{I}_n}(I_0) \leq n - 1$ (by the Claim) and $\mathrm{pd}_{\mathbb{I}_n}(B_1 \otimes \mathbb{I}_{n-1}) \leq \mathrm{pd}_{\mathbb{I}_1}(B_1) \leq \text{l.gldim}(\mathbb{I}_1) = 1$.

   Suppose that $\overline{I} = 0$. Then, by (9), $\mathrm{pd}_{\mathbb{I}_n}(I) = \mathrm{pd}_{\mathbb{I}_n}(I_0) \leq n - 1$, by the Claim.

   So, we may assume that $\overline{I} \neq 0$ and $J \neq 0$. Then the $\mathbb{I}_n$-module $\overline{I}$ is an essential submodule of $B_1 \otimes \mathbb{I}_{n-1}$, and so the short exact sequence (10) does not split since $J \neq 0$. In particular, $\mathrm{pd}_{\mathbb{I}_n}(J) \geq 1$. Since l.gldim($\mathbb{I}_1$) $= 1$, $\mathrm{pd}_{\mathbb{I}_n}(B_1 \otimes \mathbb{I}_{n-1}) \leq \mathrm{pd}_{\mathbb{I}_1}(B_1) \leq 1$. Therefore,

$$\mathrm{pd}_{\mathbb{I}_n}(\overline{I}) \leq \max\{\mathrm{pd}_{\mathbb{I}_1}(B_1 \otimes \mathbb{I}_{n-1}), \mathrm{pd}_{\mathbb{I}_1}(J)\} \leq \max\{1, \mathrm{pd}_{\mathbb{I}_1}(J)\} \leq \mathrm{pd}_{\mathbb{I}_n}(J).$$

By (9) and (10) and the Claim,

$$\mathrm{pd}_{\mathbb{I}_n}(I) \leq \max\{\mathrm{pd}_{\mathbb{I}_n}(I_0), \mathrm{pd}_{\mathbb{I}_n}(\overline{I})\} \leq \max\{\mathrm{pd}_{\mathbb{I}_n}(I_0), \mathrm{pd}_{\mathbb{I}_n}(J)\} \leq n - 1, \tag{11}$$

as required. $\square$

Theorem 3.5 *Let* $\mathbb{I}_{n,m} := B_{n-m} \otimes \mathbb{I}_m$ *where* $m = 0, 1, \ldots, n$ *and* $\mathbb{I}_0 = B_0 := K$. *Then*

   1. gldim($\mathbb{I}_{n,m}$) $= n$ *for all* $m = 0, 1, \ldots, n$.

   2. *For all prime factor algebras* A *of* $\mathbb{I}_n$, gldim(A) $= n$.

   Proof. 1. The algebras B_{n-m} and $\mathbb{I}_n$ are self-dual, hence so is their tensor product $\mathbb{I}_{n,m}$. So, the left and right global dimensions of the algebra $\mathbb{I}_{n,m}$ coincide and they are denoted by gldim($\mathbb{I}_{n,m}$). Recall that $\mathbb{I}_{n,m} = S_{\partial_1, \ldots, \partial_{n-m}}^{-1} \mathbb{I}_n$ and gldim(B_k) $= k$ for all k. Hence,

$$
\begin{aligned}
n &= n - m + m = \text{gldim}(B_{n-m}) + \text{gldim}(\mathbb{I}_m) \leq \text{gldim}(B_{n-m} \otimes \mathbb{I}_m) \\
&\leq \text{l.gldim}(S_{\partial_1, \ldots, \partial_{n-m}}^{-1} \mathbb{I}_{n,m}) \leq \text{gldim}(\mathbb{I}_{n,m}) = n.
\end{aligned}
$$

   2. By [8, Corollary 3.3.(10)], the algebra A is isomorphic to $\mathbb{I}_{n,m}$, and statement 2 follows from statement 1. $\square$

   The next theorem is an analogue of Hilbert's Syzygy Theorem for the algebra $\mathbb{I}_n$ and its prime factor algebras.

Theorem 3.6 *Let K be an algebraically closed uncountable field of characteristic zero. Let A be a prime factor algebra of $\mathbb{I}_n$ (i.e., $A \simeq \mathbb{I}_{n,m}$) and B be a left Noetherian finitely generated algebra over K. Then*

1. $\mathrm{l.gldim}(A \otimes B) = \mathrm{l.gldim}(A) + \mathrm{l.gldim}(B) = n + \mathrm{l.gldim}(B)$.

2. $\mathrm{l.gldim}(\mathbb{I}_n \otimes B) = \mathrm{l.gldim}(\mathbb{I}_n) + \mathrm{l.gldim}(B) = n + \mathrm{l.gldim}(B)$.

Proof. 1. Recall that $A \simeq \mathbb{I}_{n,m}$ for some $m \in \{0, 1, \ldots, n\}$ and $\mathrm{l.gldim}(\mathbb{I}_{n,m}) = n$ (Theorem 3.4). The case $m = 0$, i.e., $A = B_n$, is well-known [5, Corollary 1.5]. Suppose that $m > 0$ and we assume that the result is true for all $m' < m$. Let $l = \mathrm{l.gldim}(B)$. By Theorem 3.5,

$$1 \leq n + l = \mathrm{l.gldim}(\mathbb{I}_{n,m}) + \mathrm{l.gldim}(B) \leq \mathrm{l.gldim}(\mathbb{I}_{n,m} \otimes B).$$

It suffices to show that $\mathrm{l.gldim}(\mathbb{I}_{n,m} \otimes B) \leq n + \mathrm{l.gldim}(B)$ or, equivalently, that $\mathrm{pd}_C(I) \leq n + l - 1$ for all left ideals I of the algebra $C := \mathbb{I}_{n,m} \otimes B$. Using the localization at S_{∂_1} and repeating the arguments of the proof of Theorem 3.4 we obtain the inequality

$$\mathrm{pd}_C(I) \leq \max\{\mathrm{pd}_C(I_0), \mathrm{pd}_C(J)\}$$

where $I_0 = \mathrm{tor}_{S_{\partial_1}}(I)$, $J = S_{\partial_1}^{-1}C/\overline{I}$ and $\overline{I} = C/I_0$. The C-modules I_0 and J are S_{∂_1}-torsion, hence their projective dimensions are $\leq \mathrm{l.gldim}(S_{\partial_1}^{-1}C) - 1 = \mathrm{l.gldim}(\mathbb{I}_{n,m-1}) - 1 = n + l - 1$ (by repeating the proof of the Claim of Theorem 3.4). Hence, $\mathrm{pd}_C(I) \leq n + l - 1$.

2. Statement 2 is a particular case of statement 1 since $\mathbb{I}_n = \mathbb{I}_{n,n}$. $\square$

4 The global dimension of the Jacobian algebra $\mathbb{A}_n$

In this section, we prove that the global dimension of the Jacobian algebra $\mathbb{A}_n$ and of all its prime factor algebras is n (Theorem 4.2, Corollary 4.3). An analogue of Hilbert's Syzygy Theorem is proven for the Jacobian algebras $\mathbb{A}_n$ and for all its prime factor algebras (Theorem 4.4).

The involution θ on $\mathbb{A}_n$, [7]. The Weyl algebra A_n admits the *involution*

$$\theta : A_n \to A_n, \ \ x_i \mapsto \partial_i, \ \ \partial_i \mapsto x_i, \ \ \text{for } i = 1, \ldots, n,$$

i.e., it is a K-algebra anti-isomorphism ($\theta(ab) = \theta(b)\theta(a)$) such that $\theta^2 = \mathrm{id}_{A_n}$. The involution θ can be uniquely extended to the involution of $\mathbb{A}_n$ by the rule

$$\theta : \mathbb{A}_n \to \mathbb{A}_n, \ \ x_i \mapsto \partial_i, \ \ \partial_i \mapsto x_i, \ \ \theta(H_i^{-1}) = H_i^{-1}, \ \ i = 1, \ldots, n. \tag{12}$$

The algebras $\mathcal{A}_n$. The polynomial algebra $P_n = K[H_1, \ldots, H_n]$ admits a set $\{\sigma_1, \ldots, \sigma_n\}$ of commuting automorphisms where $\sigma_i(H_i) = H_i - 1$ and $\sigma_i(H_j) = H_j$, for $i \neq j$. The multiplicative submonoid $\mathcal{S}_n$ of P_n generated by the elements $H_i + j$ (where $i = 1, \ldots, n$ and $j \in \mathbb{Z}$) is an Ore set of the Weyl algebra A_n, and the (two-sided) localization $\mathcal{A}_n := \mathcal{S}_n^{-1}A_n \simeq A_n\mathcal{S}_n^{-1}$ of the Weyl algebra A_n at $\mathcal{S}_n$ is the *skew Laurent polynomial ring*

$$\mathcal{A}_n = \mathcal{S}_n^{-1}P_n[x_1^{\pm 1}, \ldots, x_n^{\pm 1}; \sigma_1, \ldots, \sigma_n] \tag{13}$$

with coefficients from the localization of P_n at $\mathcal{S}_n$,

$$\mathcal{S}_n^{-1}P_n = K[H_1^{\pm 1}, (H_1 \pm 1)^{-1}, (H_1 \pm 2)^{-1}, \ldots, H_n^{\pm 1}, (H_n \pm 1)^{-1}, (H_n \pm 2)^{-1}, \ldots].$$

We identify the Weyl algebra A_n with the subalgebra of $\mathcal{A}_n$ via the monomorphism,

$$A_n \to \mathcal{A}_n, \ \ x_i \mapsto x_i, \ \ \partial_i \mapsto H_i x_i^{-1}, \ \ i = 1, \ldots, n.$$

Let $k_n = Q_{l,cl}(A_n) = Q_{r,cl}(A_n)$ be the n'th *Weyl skew field* (it exists by Goldie's Theorem since the Weyl algebra A_n is a Noetherian domain). Then the algebra $\mathcal{A}_n$ is a K-subalgebra of k_n generated by the elements x_i, x_i^{-1}, H_i and H_i^{-1}, $i = 1, \ldots, n$ since, for all natural numbers j,

$$(H_i \mp j)^{-1} = x_i^{\pm j} H_i^{-1} x_i^{\mp j}, \ \ i = 1, \ldots, n.$$

Clearly, $\mathcal{A}_n \simeq \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1$ (n times).

The algebra $\mathbb{A}_n$ contains a unique maximal ideal $\mathfrak{a}_n$ (see [7, Corollary 2.7.(4)]) and $\mathbb{A}_n/\mathfrak{a}_n \simeq \mathcal{A}_n$ (see [7, Eq. (22)]). Since $\theta(\mathfrak{a}_n) = \mathfrak{a}_n$, the algebra $\mathcal{A}_n$ inherits the involution θ of the algebra $\mathbb{A}_n$ which is given by the rule $\theta(a + \mathfrak{a}_n) = \theta(a) + \mathfrak{a}_n$ for all elements $a \in \mathbb{A}_n$. In particular, the algebras $\mathbb{A}_n$ and $\mathcal{A}_n$ are self-dual.

A K-algebra R has the *endomorphism property over* K if, for each simple R-module M, $\mathrm{End}_R(M)$ is algebraic over K.

Theorem 4.1 ([6].) *Let K be a field of characteristic zero.*

1. *The algebra $\mathcal{A}_n$ is a simple, affine, Noetherian domain.*

2. *The Gelfand-Kirillov dimension* $\mathrm{GK}(\mathcal{A}_n) = 3n$ ($\neq 2n = \mathrm{GK}(A_n)$).

3. *The (left and right) global dimension* $\mathrm{gl.dim}(\mathcal{A}_n) = n$.

4. *The (left and right) Krull dimension* $\mathrm{K.dim}(\mathcal{A}_n) = n$.

5. *Let* $\mathrm{d} = \mathrm{gl.dim}$ *or* $\mathrm{d} = \mathrm{K.dim}$. *Let R be a Noetherian K-algebra with $\mathrm{d}(R) < \infty$ such that $R[t]$, the polynomial ring in a central indeterminate, has the endomorphism property over K. Then $\mathrm{d}(\mathcal{A}_1 \otimes R) = \mathrm{d}(R) + 1$. If, in addition, the field K is algebraically closed and uncountable, and the algebra R is affine, then $\mathrm{d}(\mathcal{A}_n \otimes R) = \mathrm{d}(R) + n$.*

$\mathrm{GK}(\mathcal{A}_1) = 3$ is due to A. Joseph [30], p. 336; see also [32], Example 4.11, p. 45.

The Jacobian algebra $\mathbb{A}_n$ is a localization of the algebra $\mathbb{I}_n$. The algebra $\mathbb{A}_n$ is a (left and right) localization of the algebra $\mathbb{I}_n$, [8, Eq. (31)],

$$\mathbb{A}_n = S^{-1}\mathbb{I}_n = \mathbb{I}_n S^{-1}, \tag{14}$$

at the (left and right) Ore set $S := \{\prod_{i=1}^{n}(H_i + \alpha_i)_*^{n_i} \,|\, (\alpha_i) \in \mathbb{Z}^n, (n_i) \in \mathbb{N}^n\}$ of $\mathbb{I}_n$ that consists of regular elements of $\mathbb{I}_n$ where

$$(H_i + \alpha_i)_* := \begin{cases} H_i + \alpha_i & \text{if } \alpha_i \geq 0, \\ (H_i + \alpha_i)_1 & \text{if } \alpha_i < 0, \end{cases} \text{ and } (H_i + \alpha_i)_1 := H_i + \alpha_i - 1 + x_i H_i^{-1}\partial_i.$$

The left (resp. right) localization of the Jacobian algebra

$$\mathbb{A}_n = K\langle y_1, \ldots, y_n, H_1^{\pm 1}, \ldots, H_n^{\pm 1}, x_1, \ldots, x_n\rangle \text{ (where } y_i := H_i^{-1}x_i)$$

at the left denominator set $S_{y_1,\ldots,y_n} := \{y^\alpha \,|\, \alpha \in \mathbb{N}^n\}$ (resp., the right denominator set $S_{x_1,\ldots,x_n} := \{x^\alpha \,|\, \alpha \in \mathbb{N}^n\}$) is the algebra, [8, Eq. (33)],

$$\mathcal{A}_n \simeq S_{y_1,\ldots,y_n}^{-1}\mathbb{A}_n \simeq \mathbb{A}_n S_{x_1,\ldots,x_n}^{-1}. \tag{15}$$

The algebras $\mathbb{A}_n$ and $\mathcal{A}_n$ are self-dual, hence so are the algebras $\mathbb{A}_{n,m} := \mathcal{A}_{n-m} \otimes \mathbb{A}_m$ where $m = 0, 1, \ldots, n$ and $\mathbb{A}_0 = \mathcal{A}_0 := K$, and so $\mathrm{l.gldim}(\mathbb{A}_{n,m}) = \mathrm{r.gldim}(\mathbb{A}_{n,m}) := \mathrm{gldim}(\mathbb{A}_{n,m})$.

Theorem 4.2 *Let $\mathbb{A}_{n,m} := \mathcal{A}_{n-m} \otimes \mathbb{A}_m$ where $m = 0, 1, \ldots, n$ and $\mathbb{A}_0 = \mathcal{A}_0 := K$. Then*

1. $\mathrm{gldim}(\mathbb{A}_{n,m}) = n$ *for all $m = 0, 1, \ldots, n$.*

2. $\mathrm{gldim}(\mathbb{A}_n) = n$.

Proof. 1. Recall that $\mathrm{l.gldim}(\mathbb{A}_{n,m}) = \mathrm{r.gldim}(\mathbb{A}_{n,m})$. By Theorem 4.1.(1,3),

$$
\begin{aligned}
n &= \mathrm{l.gldim}(\mathcal{A}_n) = \mathrm{l.gldim}(S_{y_1,\ldots,y_n}^{-1}\mathbb{A}_n) \leq \mathrm{l.gldim}(\mathbb{A}_{n,m}) \leq \mathrm{l.gldim}(\mathbb{A}_n) \\
&= \mathrm{l.gldim}(S^{-1}\mathbb{I}_n) \leq \mathrm{l.gldim}(\mathbb{I}_n) = n \quad \text{(Theorem 3.4)}.
\end{aligned}
$$

Therefore, $\mathrm{l.gldim}(\mathbb{A}_{n,m}) = n$ for all n and m.

2. Statement 2 is a particular case of statement 1 when $n = m$. $\square$

Corollary 4.3 *Let A be a prime factor algebra of the algebra $\mathbb{A}_n$. Then $\mathrm{l.gldim}(A) = \mathrm{r.gldim}(A) = n$.*

Proof. By [7, Corollary 3.5], the algebra A is isomorphic to the algebra $\mathbb{A}_{n,m}$ for some m. Now, the corollary follows from Theorem 4.2. $\square$

The next theorem is an analogue of Hilbert's Syzygy Theorem for the Jacobian algebras and their prime factor algebras.

Theorem 4.4 *Let K be an algebraically closed uncountable field of characteristic zero. Let A be a prime factor algebra of $\mathbb{A}_n$ and B be a left Noetherian finitely generated algebra over K. Then*

1. *$\mathrm{l.gldim}(A \otimes B) = \mathrm{l.gldim}(A) + \mathrm{l.gldim}(B) = n + \mathrm{l.gldim}(B)$.*

2. *$\mathrm{l.gldim}(\mathbb{A}_n \otimes B) = \mathrm{l.gldim}(\mathbb{A}_n) + \mathrm{l.gldim}(B) = n + \mathrm{l.gldim}(B)$.*

Proof. 1. Recall that $A \simeq \mathbb{A}_{n,m}$ for some $m \in \{0, 1, \ldots, n\}$ and $\mathrm{l.gldim}(\mathbb{A}_{n,m}) = n$ (Theorem 4.2). Since

$$
\begin{aligned}
n + \mathrm{l.gldim}(B) &= \mathrm{l.gldim}(\mathbb{A}_{n,m}) + \mathrm{l.gldim}(B) \leq \mathrm{l.gldim}(\mathbb{A}_{n,m} \otimes B) \\
&\leq \mathrm{l.gldim}(S^{-1}\mathbb{I}_n \otimes B) \\
&\leq \mathrm{l.gldim}(\mathbb{I}_n \otimes B) = n + \mathrm{l.gldim}(B) \ \ \text{(by Theorem 3.6)}.
\end{aligned}
$$

Therefore, $\mathrm{l.gldim}(\mathbb{A}_{n,m} \otimes B) = n + \mathrm{l.gldim}(B)$. The proof of statement 1 is complete.

2. Statement 2 is a particular case of statement 1 when $A = \mathbb{A}_n$. $\square$

5 The weak global dimension of factor algebras of $\mathbb{I}_n$

The aim of this section is to show that the weak global dimension of all factor algebras of the algebra $\mathbb{I}_n$ is n (Theorem 5.7) and that an analogue of Hilbert's Syzygy Theorem holds for them (Theorem 5.8).

Let I be an ideal of the algebra $\mathbb{I}_n$ and $A = \mathbb{I}_n/I$. The ideal I is the intersection of the minimal primes over the ideal I,

$$
I = \bigcap_{\mathfrak{q} \in \min(I)} \mathfrak{q} \ \ \text{(see } [8, \text{Corollary } 3.4.(4)]). \tag{16}
$$

The map

$$
f : A = \mathbb{I}_n/I \to \widetilde{A} := \prod_{\mathfrak{q} \in \min(I)} \mathbb{I}_n/\mathfrak{q}, \ \ a \mapsto (\ldots, a + \mathfrak{q}, \ldots)_{\mathfrak{q} \in \min(I)} \tag{17}
$$

is an algebra monomorphism.

For a ring R, an element $r \in R$ is called a *left* (resp., *right*) *regular* if the map $\cdot r : R \to R$, $s \mapsto sr$ (resp., $r \cdot : R \to R$, $s \mapsto rs$) is an injection. The set of all left (resp., right) regular elements of the ring R is denoted by $'\mathcal{C}_R$ (resp., $\mathcal{C}'_R$). An element $r \in R$ is called a *regular element* if it is left and right regular. By the very definition, the set $\mathcal{C}_R$ of regular elements of R is equal to the intersection $'\mathcal{C}_R \cap \mathcal{C}'_R$.

Proposition 5.1 *Let I be an ideal of the algebra $\mathbb{I}_n$ and $A = \mathbb{I}_n/I$.*

1. *The involution $*$ on the algebra $\mathbb{I}_n$ induces the involution $*$ on the algebra A by the rule $(a + I)^* = a^* + I$ for all elements $a \in \mathbb{I}_n$.*

2. *$\partial_1, \ldots, \partial_n \in '\mathcal{C}_A$.*

3. *$\int_1, \ldots, \int_n \in \mathcal{C}'_A$.*

4. Let S (resp., T) be the multiplicative monoid generated by the elements $\partial_{i_1}, \ldots, \partial_{i_m}$ (resp., $\int_{i_1}, \ldots, \int_{i_m}$). Then S (resp., T) is a left (resp., right) denominator set of the algebra A, $\operatorname{ass}_{l,A}(S) = \operatorname{ass}_{r,A}(T) = (\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m} + I)/I$ and $S^{-1}A \simeq \mathbb{I}_n/(\mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m} + I) \simeq AT^{-1}$.

Proof. 1. By [8, Lemma 5.1.(1)], all the ideals of the algebra $\mathbb{I}_n$ are $*$-invariant, and so statement 1 follows.

2. For every prime ideal $\mathfrak{p}$ of the algebra $\mathbb{I}_n$, the factor algebra $\mathbb{I}_n/\mathfrak{p}$ is isomorphic to the algebra $\mathbb{I}_{n,m}$ for some $m = m(\mathfrak{p})$. Since the element ∂_i is a left regular element of the ring $\mathbb{I}_1(i)$ and $B_1(i) = \mathbb{I}_1(i)/F(i)$. The elements $\partial_1, \ldots, \partial_n$ are left regular elements of the ring $\mathbb{I}_{n,m}$ for all $m = 0, 1, \ldots, n$. Now, by (17), the elements $\partial_1, \ldots, \partial_n$ are left regular elements of the algebra A.

3. Statement 2 follows from statement 1 in view of the involution $*$ and the fact that $\partial_1^* = \int_1, \ldots, \partial_n^* = \int_n$.

4. (i) *S is a left Ore set of A*: Recall that $S_{\partial_1, \ldots, \partial_n}^{-1}\mathbb{I}_n \simeq B_n$ and $\operatorname{ass}_{\mathbb{I}_n}(S_{\partial_1, \ldots, \partial_n}) = \mathfrak{a}_n$. Clearly, $I \subseteq \mathfrak{a}_n$ since the ideal $\mathfrak{a}_n$ is the largest proper ideal of the algebra $\mathbb{I}_n$. The multiplicative set S is a left Ore set of $\mathbb{I}_n$. Hence, its image in the factor algebra $A = \mathbb{I}_n/I$ is a left Ore set provided $S \cap I = \emptyset$ but this is obvious ($\emptyset = S_{\partial_1, \ldots, \partial_n} \cap \mathfrak{a}_n \supseteq S \cap I$).

(ii) *S is a left denominator set of A*: The statement (ii) follows from the statement (i) and statement 2.

(iii) *T is a right denominator set of A*: The statement (iii) follows from the statement (ii), statement 1 and the fact that $T = S^*$.

Let $\mathfrak{p} = \mathfrak{p}_{i_1} + \cdots + \mathfrak{p}_{i_m}$.

(iv) $\operatorname{ass}_{l,A}(S) = (\mathfrak{p} + I)/I$ and $S^{-1}A \simeq A/\operatorname{ass}_{l,A}(S) \simeq \mathbb{I}_n/(\mathfrak{p} + I)$: The short exact sequence of $\mathbb{I}_n$-modules $0 \to I \to \mathbb{I}_n \to A \to 0$ yields the short exact sequence of $S^{-1}\mathbb{I}_n$-modules $0 \to S^{-1}I \to S^{-1}\mathbb{I}_n \xrightarrow{\varphi} S^{-1}A \to 0$. By the statement (ii), $S^{-1}A$ is an algebra and φ is an algebra homomorphism (since $\varphi(s^{-1}a) = s^{-1}a$). Since $S^{-1}\mathbb{I}_n = \mathbb{I}_n/\mathfrak{p}$ (the operation of localization at S on the left is equal to the operation of taking modulo the ideal $\mathfrak{p}$) , $S^{-1}A \simeq \mathbb{I}_n/(\mathfrak{p} + I)$ and $\operatorname{ass}_{l,A}(S) = (\mathfrak{p} + I)/I$.

(v) $\operatorname{ass}_{r,A}(T) = (\mathfrak{p} + I)/I$ and $AT^{-1} \simeq A/\operatorname{ass}_{r,A}(T) \simeq \mathbb{I}_n/(\mathfrak{p} + I) \simeq S^{-1}A$: Since $T^* = S$, we see that $AT^{-1} = (S^{-1}A)^* \simeq (\mathbb{I}_n/(\mathfrak{p} + I))^* = \mathbb{I}_n/(\mathfrak{p}^* + I^*) = \mathbb{I}_n/(\mathfrak{p} + I) \simeq = S^{-1}A$ since all the ideals of the algebra $\mathbb{I}_n$ are $*$-invariant; and

$$\operatorname{ass}_{r,A}(T) = (\operatorname{ass}_{r,A}(T))^{**} = (\operatorname{ass}_{l,A^*}(T^*))^* = (\operatorname{ass}_{l,A}(S))^* \overset{\text{st. 1}}{=} \operatorname{ass}_{l,A}(S) \overset{(iv)}{=} (\mathfrak{p} + I)/I. \quad \square$$

A ring homomorphism $A \to B$ is called an *extension* of A. An extension $A \to B$ is called a *left* (resp., *right*) *flat* if $_AB$ (resp., B_A) is a flat A-module. An extension is called a *flat extension* if it is left and right flat. If S is a left denominator set of A then the extension $A \to S^{-1}A$ is right flat. If T is a right denominator set of A then the extension $A \to T^{-1}A$ is left flat. If, in addition, $S^{-1}A \simeq AT^{-1}$, then the extension $A \to S^{-1}A$ is flat. In particular, if S is a left and right denominator set of A then the extension $A \to S^{-1}A \simeq AS^{-1}$ is flat. A left (resp., right; left and right) extension $A \to B$ is called a *left* (resp., *right; left and right*) *faithfully flat extension* if the functor $- \otimes_A B$ (resp., $B \otimes_A -$; $- \otimes_A B$ and $B \otimes_A -$) is faithful, i.e., a nonzero module is mapped in a nonzero module.

Proposition 5.2 *Let I be a nonzero ideal of the algebra $\mathbb{I}_n$, $A = \mathbb{I}_n/I$ and $A_i = S_{\partial_i}^{-1}A \simeq AS_{\int_i}^{-1}$ for $i = 1, \ldots, n$ (Proposition 5.1.(4)). Then*

1. *The set $\{A \to A_i\}_{i=1,\ldots,n}$ is a set of (left and right) flat extensions such that the extension $A \to A^{ff} := \prod_{i=1}^{n} A_i$ is a (left and right) faithfully flat extension.*

2. *For each $i = 1, \ldots, n$, $\mathbb{I}_n = \mathbb{I}_1(i) \otimes \mathbb{I}_{n-1}[i]$, where $\mathbb{I}_{n-1}[i] := \bigotimes_{j \neq i} \mathbb{I}_1(i)$, and $A_i \simeq B_1(i) \otimes (\mathbb{I}_{n-1}[i]/I_i)$ for some ideal I_i of $\mathbb{I}_{n-1}[i]$.*

Proof. 1. Since $A_i = S_{\partial_i}^{-1}A \simeq AS_{\int_i}^{-1}$ (Proposition 5.1.(4)), the set $\{A \to A_i\}_{i=1,\ldots,n}$ is a set of (left and right) flat extensions.

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Since $A^* = A$ and $A_i^* \simeq A_i$ for $i = 1, \ldots, n$, it suffices to show that the extension $A \to A^{ff}$ is right faithfully flat, i.e., if $A^{ff} \otimes_A M = 0$ for some A-module M then $M = 0$. Clearly, $A^{ff} \otimes_A M = 0$ iff $S_{\partial_i}^{-1} M = 0$ for all $i = 1, \ldots, m$. Suppose that $M \neq 0$, then we can choose a nonzero element m of M such that $\partial_1 m = 0, \ldots, \partial_m m = 0$. Since the simple $\mathbb{I}_n$-module P_n is isomorphic to the $\mathbb{I}_n$-module $\mathbb{I}_n / \mathbb{I}_n(\partial_1, \ldots, \partial_n)$ ([8, Proposition 3.8]), the $\mathbb{I}_n$-module $\mathbb{I}_n m$ is isomorphic to the $\mathbb{I}_n$-module P_n. Since M is an A-module, it is also an $\mathbb{I}_n$-module such that $IM = 0$. In particular, $I\mathbb{I}_n m = 0$ but the $\mathbb{I}_n$-module P_n is faithful ([8, Proposition 3.8]), a contradiction. Therefore, $M = 0$.

2. The algebra $B_1(i)$ is a central simple algebra and the algebra A_i is an epimorphic image of the algebra $B_1(i) \otimes \mathbb{I}_{n-1}[i]$. Therefore, $A_i \simeq B_1(i) \otimes (\mathbb{I}_{n-1}[i]/I_i)$ for some ideal I_i of the algebra $\mathbb{I}_{n-1}[i]$. $\square$

Let I be an nonzero ideal of the algebra $\mathbb{I}_n$, By Theorem 3.1.(8), each minimal prime $\mathfrak{q}$ over I is a unique sum $\mathfrak{q} = \sum_{i \in I(\mathfrak{q})} \mathfrak{p}_i$ for a unique non-empty subset $I(\mathfrak{q})$ of $\{1, \ldots, n\}$ and the set $\{I(\mathfrak{q}) \,|\, \mathfrak{q} \in \min(I)\}$ consists of incomparable elements (for all distinct $\mathfrak{q}, \mathfrak{q}' \in \min(I)$, $\mathfrak{q} \nsubseteq \mathfrak{q}'$ and $\mathfrak{q} \nsupseteq \mathfrak{q}'$). Let $S(\mathfrak{q})$ (resp., $T(\mathfrak{q})$) be the monoid generated by the elements $\{\partial_i \,|\, i \in I(\mathfrak{q})\}$ (resp., $\{\int_i \,|\, i \in I(\mathfrak{q})\}$). By Proposition 5.1.(4), the set $S(\mathfrak{q})$ (resp., $T(\mathfrak{q})$) is a left (resp., right) denominator set of the algebras $\mathbb{I}_n$ and A such that $S(\mathfrak{q})^* = T(\mathfrak{q})$, $\mathrm{ass}_{l,A} S(\mathfrak{q})) = \mathfrak{q}/I = \mathrm{ass}_{r,A}(T(\mathfrak{q}))$ and $S(\mathfrak{q})^{-1} A \simeq A T(\mathfrak{q})^{-1} \simeq A/\mathfrak{q}$. Therefore, the extension (17) is a (left and right) flat.

Proposition 5.3 *Let I be a nonzero ideal of the algebra $\mathbb{I}_n$ and $A = \mathbb{I}_n/I$. Then the extension $A \to \widetilde{A} = \prod_{\mathfrak{q} \in \min(I)} \mathbb{I}_n/\mathfrak{q}$ is a (left and right) faithfully flat extension.*

Proof. We have seen above that the extension is (left and right) flat. Since $f(a^*) = f(a)^*$ for all elements $a \in A$, to finish the proof it suffices to show that the extension is left faithful. By [33, Proposirion 2.3], we have to show that $J\widetilde{A} \neq \widetilde{A}$ for all proper right ideals J of A. Suppose that $J\widetilde{A} = \widetilde{A}$ for some proper right ideal J of A, we seek a contradiction. Then for all ideals $\mathfrak{q} \in \min(I)$, $J(\mathbb{I}_n/\mathfrak{q}) = \mathbb{I}_n/\mathfrak{q}$, and so $J + \mathfrak{q} = \mathbb{I}_n$ for all ideals $\mathfrak{q} \in \min(I)$. For each ideal $\mathfrak{q} \in \min(I)$, there are elements $j(\mathfrak{q}) \in J$ and $q(\mathfrak{q}) \in \mathfrak{q}$ such that $j(\mathfrak{q}) + q(\mathfrak{q}) = 1$. In A, $0 = \prod_{\mathfrak{q} \in \min(I)} q(\mathfrak{q}) = \prod_{\mathfrak{q} \in \min(I)} (1 - j(\mathfrak{q})) = 1 - j$ for some element $j \in J$ (the order of the elements in the products above is an arbitrary but fixed) and so $1 = j \in J$, a contradiction. $\square$

Theorem 5.4 ([5, Theorem 1.2].) *Suppose that $\{A \to A_\alpha \,|\, \alpha \in I\}$ is a set of flat extensions of a ring A such that $\mathrm{w.dim}(A) < \infty$, $\nu = \sup\{\mathrm{w.dim}(A_\alpha) \,|\, \alpha \in I\} < \infty$, $\mu = \sup\{\mathrm{fd}(_A M) \,|\, M$ is a cyclic A-module such that $A_\alpha \otimes_A M = 0$ for all $\alpha \in I\}$; the set I is either a finite set or the ring A is right coherent. Then*

1. *either $\mathrm{w.dim}(A) = \mu$ or $\mu < \mathrm{w.dim}(A) \leq \nu$.*

2. *If, additionally, $\nu \leq \mathrm{w.dim}(A)$ (for example, all the rings A_α are two-sided localizations of A), then $\mathrm{w.dim}(A) = \max\{\mu, \nu\}$.*

Theorem 5.5 and Theorem 5.6 are generalizations of Theorem 5.4. They are used in the proof of Theorem 5.7.

Theorem 5.5 *Let A be a ring, $S_1, \ldots S_n$ be left denominator sets of the ring A such that the rings $A_i := S_i^{-1} A$ $(i = 1, \ldots, n)$ are flat left A-modules. Then $\mathrm{w.dim}(A) = \max\{\nu, \mu\}$ where $\nu = \max\{\mathrm{w.dim}(A_1), \ldots, \mathrm{w.dim}(A_n)\}$ and $\mu = \sup\{\mathrm{fd}(_A L) \,|\, L$ is a cyclic S_i-torsion A-module for $i = 1, \ldots, n\}$.*

Proof. To prove the theorem we use induction on n. The case $n = 1$ follows from [4, Theorem 2.5]. So, let $n > 1$ and we assume that the equality holds for all $n' < n$. Clearly, $\nu \leq \max\{\nu, \mu\} \leq \mathrm{w.dim}(A)$. So, if $\max\{\nu, \mu\} = \infty$ there is nothing to prove. So, we assume that $\max\{\nu, \mu\} < \infty$. It suffices to show that $\mathrm{w.dim}(A) < \infty$ since then the equality would follows from Theorem 5.4.(2). By induction on n, $\mathrm{w.dim}(A) = \sup\{\nu_{n-1}, \mu_{n-1}\}$ where

$\nu_{n-1} = \max\{\text{w.dim}(A_1), \ldots, \text{w.dim}(A_{n-1})\}$ and $\mu_{n-1} = \sup\{\text{fd}(_A L) \mid L$ is a cyclic S_i-torsion A-module for $i = 1, \ldots, n-1\}$. Let L be an S_i-torsion left A-module for $i = 1, \ldots, n-1$, and $L_0 = \text{tor}_{S_n}(L)$. We have short exact sequences of A-modules

$$0 \to L_0 \to L \to \bar{L} \to 0, \tag{18}$$

$$0 \to \bar{L} \to S_n^{-1} L \to L_1 \to 0. \tag{19}$$

Since $A \to A_n$ is a flat extension, $\text{fd}(_A S_n^{-1} L) \leq \text{fd}(_{A_n} S_n^{-1} L) \leq \text{w.dim}(A_n) < \infty$ (since for all modules M_A and $_{A_n} N$ and $m \geq 0$, $\text{Tor}_m^A(M, N) = \text{Tor}_m^A(M, A_n \otimes_{A_n} N) = \text{Tor}_m^{A_n}(M \otimes_A A_n, N)$). Clearly, the A-modules L_0 and L_1 are S_i-torsion for $i = 1, \ldots, n$. Therefore, $\text{fd}(L_j) \leq \mu < \infty$ for $j = 0, 1$. By (19),

$$\text{fd}(_A \bar{L}) \leq \max\{\text{fd}(_A S_n^{-1} L), \text{fd}(_A L_1)\} \leq \max\{\text{w.dim}(A_n), \mu\} < \infty.$$

Then, by (18), $\text{fd}(_A L) \leq \max\{\text{fd}(_A L_0), \text{fd}(_A \bar{L})\} \leq \max\{\text{w.dim}(A_n), \mu\} < \infty$, as required. $\square$

Theorem 5.6 *Let A be a ring. For each $i = 1, \ldots, n$, S_i be left denominator set and T_i be a right denominator set of A such that $A_i := S_i^{-1} A \simeq A T_i^{-1}$. Then $\text{w.dim}(A) = \max\{\nu, \mu\} = \max\{\nu, \mu'\}$ where $\nu = \max\{\text{w.dim}(A_1), \ldots, \text{w.dim}(A_n)\}$, $\mu = \sup\{\text{fd}(_A L) \mid L$ is a cyclic S_i-torsion left A-module for $i = 1, \ldots, n\}$ and $\mu' = \sup\{\text{fd}(N_A) \mid N$ is a cyclic T_i-torsion right A-module for $i = 1, \ldots, n\}$. If, in addition, the flat extension $A \to \prod_{i=1}^n A_i$ is a left or right faithfully flat extension then $\text{w.dim}(A) = \nu$.*

Proof. The theorem is a particular case of Theorem 5.5, hence $\text{w.dim}(A) = \max\{\nu, \mu\}$. Then the second equality, $\text{w.dim}(A) = \max\{\nu, \mu'\}$, follows from the first one in view of symmetry of the weak global dimension with respect left and right modules. If, in addition, the flat extension $A \to \prod_{i=1}^n A_i$ is a left (resp., right) faithfully flat extension then $\mu' = 0$ (resp., $\mu = 0$) and so $\text{w.dim}(A) = \nu$. $\square$

Theorem 5.7 1. *Let $\mathcal{A}$ be a factor algebra of the algebra $\mathbb{I}_{n,m} = B_{n-m} \otimes \mathbb{I}_m$ where $0 \leq m \leq n$. Then $\text{w.dim}(\mathcal{A}) = n$.*

2. *Let A be a factor algebra of $\mathbb{I}_n$. Then $\text{w.dim}(A) = n$.*

Proof. 1. To prove the equality we use induction on n. For $n = 1$, there are only two options for the algebra $\mathcal{A}$. Namely, either $A = \mathbb{I}_1$ or $A = B_1$. By Theorem [8, Theorem 6.2], $\text{w.dim}(\mathbb{I}_1) = 1$. It is a well-known fact that the algebra B_1 is a Noetherian algebra of global dimension 1. In particular, $\text{w.dim}(B_1) = 1$. Suppose that $n > 1$ and the result holds for all $n' < n$. Now, we use the second induction on $m = 0, 1, \ldots, n$. The initial case $m = 0$ is obvious since the algebra $\mathbb{I}_{n,0} = B_n$ is a simple algebra of global dimension n. So, we assume that $m > 0$ and the result holds for all m' such that $0 \leq m' < m$. The algebras B_k ($k \geq 0$) are central simple algebras. So, the factor algebra $\mathcal{A}$ of the algebra $\mathbb{I}_{n,m} = B_{n-m,m} \otimes \mathbb{I}_m$ is isomorphic to the algebra $B_{n-m} \otimes (\mathbb{I}_m / I_m)$ for some ideal I_m of $\mathbb{I}_m$. If $I_m = 0$ then $\mathcal{A} = \mathbb{I}_{n,m}$ and $\text{w.dim}(\mathcal{A}) = \text{w.dim}(\mathbb{I}_{n,m}) = n$ ([8, Theorem 6.2]). We can assume that $I_m \neq 0$. Then I_m contains the least nonzero ideal $F^{\otimes m}$ of the algebra $\mathbb{I}_m$ (Theorem 3.1.(8)). By Proposition 5.1.(4), for each $i = n - m + 1, \ldots, n$, consider the ring extension

$$\mathcal{A} \to \mathcal{A}_i := S_{\partial_i}^{-1} \mathcal{A} \simeq \mathcal{A} S_{\int_i}^{-1} \simeq B_{n-m} \otimes B_1 \otimes (\mathbb{I}_{m-1} / J_i) \tag{20}$$

for some ideal J_i of the algebra $\mathbb{I}_{m-1}$. By induction, $\text{w.dim}(\mathcal{A}_i) = n$ for all $i = n - m + 1, \ldots, n$. By Theorem 5.6, $\text{w.dim}(\mathcal{A}) = \max\{\nu, \mu\}$ where $\nu = \max\{\text{w.dim}(\mathcal{A}_i) \mid i = n - m + 1, \ldots, n\} = n$ and $\mu = \sup\{\text{fd}(_A L) \mid L \in \mathbb{E}\} = 0$ since $\mathbb{E} := \{L \mid L$ is a cyclic S_i-torsion left $\mathcal{A}$-module for $i = n - m + 1, \ldots, n\} = \{0\}$, by Proposition 5.2 as $I_m \neq 0$. Therefore, $\text{w.dim}(\mathcal{A}) = n$.

2. Statement 2 is a particular case of statement 1 since $\mathbb{I}_n = \mathbb{I}_{n,n}$. $\square$

Theorem 5.8 is an analogue of Hilbert's Syzygy Theorem for all factor algebras of the algebra $\mathbb{I}_n$ in case of the weak global dimension.

Theorem 5.8 *Let K be an algebraically closed uncountable field of characteristic zero, $\mathcal{A}$ be a factor algebra of $\mathbb{I}_{n,m} = B_{n-m} \otimes \mathbb{I}_m$, where $0 \le m \le n$, and B be a left Noetherian finitely generated algebra over K. Then*

1. $\mathrm{w.dim}(\mathcal{A} \otimes B) = \mathrm{w.dim}(\mathcal{A}) + \mathrm{w.dim}(B) = n + \mathrm{w.dim}(B)$.

2. $\mathrm{w.dim}(A \otimes B) = \mathrm{w.dim}(A) + \mathrm{w.dim}(B) = n + \mathrm{w.dim}(B)$ *for all factor algebras A of $\mathbb{I}_n$.*

Proof. 1. To prove the theorem we use induction on n. For $n = 1$, there are only two options for the algebra $\mathcal{A}$. Namely, either $\mathcal{A} = \mathbb{I}_1$ or $\mathcal{A} = B_1$. In both cases, the result is a particular case of [8, Theorem 6.5]. Suppose that $n > 1$ and we assume that the equality holds for all $n' < n$. Now, we use the second induction on $m = 0, 1, \ldots, n$. In the initial case when $m = 0$, $\mathcal{A} = B_n$ (since $\mathbb{I}_{n,0} = B_n$ is a simple Noetherian finitely generated algebra) and the result is known (see [8, Theorem 6.5]). So, we may assume that $m > 0$ and the equality holds for all $m' < m$. Notice that $\mathbb{I}_{n,m} = B_{n-m} \otimes \mathbb{I}_m$ and the algebra B_{n-m} is a central simple algebra. Therefore, $\mathcal{A} \simeq B_{n-m} \otimes (\mathbb{I}_m / I_m)$ for some ideal I_m of the algebra $\mathbb{I}_m$. If $I_m = 0$ then the result is precisely [8, Theorem 6.5]. So, we assume that $I_m \ne 0$. By Proposition 5.1.(4), for each $i = n - m + 1, \ldots, n$, consider the extension $\mathcal{A} \to \mathcal{A}_i$, see (20). By Proposition 5.2.(1), the extension $\mathcal{A} \to \widetilde{A} := \prod_{i=n-m+1}^{n} \mathcal{A}_i$ is faithfully flat, hence so is the extension $A \otimes B \to \widetilde{A} \otimes B$. By (20) and the induction on m, for all i,

$$\mathrm{w.dim}(\mathcal{A}_i \otimes B) = \mathrm{w.dim}(\mathcal{A}_i) + \mathrm{w.dim}(B) = n + \mathrm{w.dim}(B)$$

(since $\mathrm{w.dim}(\mathcal{A}_i) = n$, by Theorem 5.7). Now, by Theorem 5.6 and Theorem 5.7, $\mathrm{w.dim}(\mathcal{A} \otimes B) = \max\{\mathrm{w.dim}(\mathcal{A}_1 \otimes B) \mid i = n - m + 1, \ldots, n\} = n + \mathrm{w.dim}(B) = \mathrm{w.dim}(\mathcal{A}) + \mathrm{w.dim}(B)$.

2. Statement 2 is a particular case of statement 1 since $\mathbb{I}_n = \mathbb{I}_{n,n}$. $\square$

6 The weak global dimension of factor algebras of $\mathbb{A}_n$

The aim of this section is to show that the weak global dimension of all factor algebras of the Jacobian algebra $\mathbb{A}_n$ is n (Theorem 6.2) and that an analogue of Hilbert's Syzygy Theorem holds for them (Theorem 6.3).

A classification of ideals of the Jacobian algebra $\mathbb{A}_n$ is given in [7].

Definition. Let A and B be algebras, and let $\mathcal{J}(A)$ and $\mathcal{J}(B)$ be their lattices of ideals. We say that a bijection $f : \mathcal{J}(A) \to \mathcal{J}(B)$ is an *isomorphism* if $f(\mathfrak{a} * \mathfrak{b}) = f(\mathfrak{a}) * f(\mathfrak{b})$ for $* \in \{+, \cdot, \cap\}$, and in this case we say that the algebras A and B are *ideal equivalent*. The ideal equivalence is an equivalence relation on the class of algebras.

The next theorem shows that the algebras $\mathbb{A}_n$ and $\mathbb{I}_n$ are ideal equivalent.

Theorem 6.1 ([7, Theorem 3.1].) *The restriction map $\mathcal{J}(\mathbb{A}_n) \to \mathcal{J}(\mathbb{I}_n)$, $\mathfrak{a} \mapsto \mathfrak{a}^r := \mathfrak{a} \cap \mathbb{I}_n$, is an isomorphism (i.e., $(\mathfrak{a}_1 * \mathfrak{a}_2)^r = \mathfrak{a}_1^r * \mathfrak{a}_2^r$ for $* \in \{+, \cdot, \cap\}$) and its inverse is the extension map $\mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{A}_n \mathfrak{b} \mathbb{A}_n$.*

It follows from the explicit description of ideals of the algebra $\mathbb{I}_n$ (Theorem 3.1.(9)) and $\mathbb{A}_n$ ([7, Theorem 3.1]) that the lattice isomorphism

$$\mathcal{J}(\mathbb{I}_n) \to \mathcal{J}(\mathbb{A}_n), \quad \mathfrak{b} \mapsto \mathfrak{b}^e := \mathbb{A}_n \mathfrak{b} \mathbb{A}_n = S^{-1}\mathfrak{b} = \mathfrak{b}S^{-1}, \tag{21}$$

can be written via the localizations at S, see (14).

Theorem 6.2   1. *Let $\mathcal{A}$ be a factor algebra of the algebra $\mathbb{A}_{n,m} = \mathcal{A}_{n-m} \otimes \mathbb{A}_m$ where $0 \le m \le n$. Then $\mathrm{w.dim}(\mathcal{A}) = n$.*

2. *Let A be a factor algebra of $\mathbb{A}_n$. Then $\mathrm{w.dim}(A) = n$.*

Proof. 1. Since the algebra $\mathcal{A}_{n-m}$ is a central simple algebra, $\mathcal{A} \simeq \mathcal{A}_{n-m} \otimes (\mathbb{A}_m/J)$ for some ideal J of the algebra $\mathbb{A}_m$. By (15),

$$S_{y_{n-m+1,\ldots y_n}}^{-1}\mathcal{A} \simeq \mathcal{A}S_{x_{n-m+1,\ldots x_n}}^{-1} \simeq \mathcal{A}_{n-m} \otimes \mathcal{A}_m \simeq \mathcal{A}_n \tag{22}$$

since $\mathrm{ass}_{l,\mathbb{A}_m}(S_{y_{n-m+1,\ldots y_n}}) = \mathrm{ass}_{r,\mathbb{A}_m}(S_{x_{n-m+1,\ldots x_n}}) = \mathfrak{a}_m$ is the largest proper ideal of the algebra $\mathbb{A}_m$ (it follows from the inclusion $J \subseteq \mathfrak{a}_m$ that $S_{y_{n-m+1,\ldots y_n}}^{-1}(\mathbb{A}_m/J) \simeq S_{y_{n-m+1,\ldots y_n}}^{-1}\mathbb{A}_m \simeq \mathcal{A}_m$ and $(\mathbb{A}_m/J)S_{x_{n-m+1,\ldots x_n}}^{-1} \simeq \mathbb{A}_m S_{x_{n-m+1,\ldots x_n}}^{-1} \simeq \mathcal{A}_m$). By Theorem 4.1.(1,3) and (22),

$$n = \mathrm{gldim}(\mathcal{A}_n) = \mathrm{w.dim}(\mathcal{A}_n) = \mathrm{w.dim}(S_{y_{n-m+1,\ldots y_n}}^{-1}\mathcal{A}) \leq \mathrm{w.dim}(\mathcal{A}).$$

So, by (21), $\mathbb{A}_m/J \simeq S^{-1}(\mathbb{I}_m/I)$ for some ideal I of $\mathbb{I}_m$. So,

$$\mathcal{A} \simeq \mathcal{A}_{n-m} \otimes (\mathbb{A}_m/J) \simeq (S_{y_1,\ldots,y_{n-m}}^{-1}\mathbb{I}_{n-m}) \otimes S^{-1}(\mathbb{I}_m/I). \tag{23}$$

Now,

$$\mathrm{w.dim}(\mathcal{A}) \overset{(23)}{\leq} \mathrm{w.dim}(\mathbb{I}_{n-m} \otimes (\mathbb{I}_m/I)) = n, \quad \text{(by Theorem 5.7.(1))}.$$

Therefore, $\mathrm{w.dim}(\mathcal{A}) = n$.

2. Statement 2 is a particular case of statement 1 when $m = n$ since $\mathbb{A}_{n,n} = \mathbb{A}_n$. $\square$

Theorem 6.3 *Let K be an algebraically closed uncountable field of characteristic zero, $\mathcal{A}$ be a factor algebra of $\mathbb{A}_{n,m} = \mathcal{A}_{n-m} \otimes \mathbb{A}_m$, where $0 \leq m \leq n$, and B be a left Noetherian finitely generated algebra over K. Then*

1. $\mathrm{w.dim}(\mathcal{A} \otimes B) = \mathrm{w.dim}(\mathcal{A}) + \mathrm{w.dim}(B) = n + \mathrm{w.dim}(B)$.

2. $\mathrm{w.dim}(A \otimes B) = \mathrm{w.dim}(A) + \mathrm{w.dim}(B) = n + \mathrm{w.dim}(B)$ *for all factor algebras A of $\mathbb{A}_n$.*

Proof. 1. The algebra B is a left Noetherian algebra, hence so is the algebra $\mathcal{A}_n \otimes B$. By Theorem 4.1.(5),

$$\mathrm{w.dim}(\mathcal{A}_n \otimes B) = \mathrm{l.gldim}(\mathcal{A}_n \otimes B) = \mathrm{l.gldim}(\mathcal{A}_n) + \mathrm{l.gldim}(B) = n + \mathrm{w.dim}(B).$$

We keep the notation of the proof of Theorem 6.2.

$$\begin{aligned}
\mathrm{w.dim}(\mathcal{A}_n \otimes B) \overset{(22)}{=} & \ \mathrm{w.dim}(S_{y_{n-m+1,\ldots,y_n}}^{-1}\mathcal{A} \otimes B) \leq \mathrm{w.dim}(\mathcal{A} \otimes B) \overset{(23)}{\leq} \mathrm{w.dim}(\mathbb{I}_{n-m} \otimes \mathbb{I}_m/I \otimes B) \\
= & \ n + \mathrm{w.dim}(B), \quad \text{(by Theorem 5.8.(2))} \\
= & \ \mathrm{w.dim}(\mathcal{A}) + \mathrm{w.dim}(B), \quad \text{(by Theorem 6.2.(2))}.
\end{aligned}$$

2. Statement 2 is a particular case of statement 1 when $m = n$ since $\mathbb{A}_{n,n} = \mathbb{A}_n$. $\square$

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