Classes of Ordinary Differential Equations Obtained for the Probability Functions of Half-Cauchy and Power Cauchy Distributions

Hilary I. Okagbue, *Member, IAENG,* Muminu O. Adamu, Enahoro A. Owoloko and Sheila A. Bishop

Abstract— **In this paper, the differential calculus (product rule) was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the half-Cauchy and power Cauchy distributions. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distributions. The method can be extended to other probability distributions and can serve an alternative to approximation especially the cases of the quantile and inverse survival functions.**

Index Terms— **Power Cauchy, half-Cauchy, product rule, differentiation, survival function.**

I. Introduction

HALF-CAUCHY distribution is obtained by restricting the domain of the standard Cauchy distribution to only positive values or observations. Polson and Scott [1] argued for the replacement of inverse-gamma distribution with this distribution in Bayesian hierarchical models while Shaw [2] suggested that the distribution can be used in lieu of the exponential distribution in prediction and modeling. The distribution, according to [3], is one of few distributions related to the ratio of two folded normal distributions. The distribution is also related to the folded t distribution as proved by [4]. Half-Cauchy distribution is one of distributions that are self-decomposable [5] and infinitely divisible [6]. Details on the distribution can be found in [7]. Some modifications or proposed improved models of the distribution includes: Kumaraswamy half-Cauchy distribution [8], beta half-Cauchy distribution [9] and generalized odd half-Cauchy family of distributions [10]. Paradis et al. [11] applied the distribution in ecological modeling,

In an attempt to find a suitable distribution that effectively fits medical survival data, Power Cauchy Distribution was proposed by [12]. The distribution is a submodel of transformed beta family earlier proposed by [13]. Tahir et al. [14] proposed Poisson power Cauchy as an improved model over the power Cauchy distribution.

The aim of this paper is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of half-Cauchy and power Cauchy distributions by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [15], beta distribution [16], raised cosine distribution [17], Lomax distribution [18], beta prime distribution or inverted beta distribution [19].

I. Half-Cauchy Distribution

Probability Density Function

The probability density function of the half-Cauchy distribution is given by;

$$f(x)=\frac{2}{\pi\sigma\left[1+\left(\frac{x}{\sigma}\right)^{2}\right]} \tag{1}$$

To obtain the first order ordinary differential equation for the probability density function of the half-Cauchy distribution, differentiate equation (1);

$$f'(x)=-\frac{4x}{\pi\sigma^{3}\left[1+\left(\frac{x}{\sigma}\right)^{2}\right]^{2}} \tag{2}$$

The condition necessary for the existence of equation is $\sigma, x>0$.

Simplify;

$$\left(1+\left(\frac{x}{\sigma}\right)^{2}\right)f'(x)=-\frac{4x}{\pi\sigma^{3}\left[1+\left(\frac{x}{\sigma}\right)^{2}\right]} \tag{3}$$

Substitute equation (1) into equation (3);

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H. I. Okagbue, E.A. Owoloko and S. A. Bishop are with the Department of Mathematics, Covenant University, Ota, Nigeria.
hilary.okagbue@covenantuniversity.edu.ng
alfred.owoloko@covenantuniversity.edu.ng
sheila.bishop@covenantuniversity.edu.ng
M. O. Adamu is with the Department of Mathematics, University of Lagos, Akoka, Lagos, Nigeria.

$$\left(1+\left(\frac{x}{\sigma}\right)^2\right)f'(x)=-\frac{2x}{\sigma^2}f(x) \quad (4)$$

The first order ordinary differential for the probability density function of the half-Cauchy distribution is given as;

$$(\sigma^2+x^2)f'(x)+2xf(x)=0 \quad (5)$$

$$f(1)=\frac{2\sigma}{\pi(\sigma^2+1)} \quad (6)$$

Quantile Function

The Quantile function of the half-Cauchy distribution is given by;

$$Q(p)=\sigma\tan\left(\frac{\pi p}{2}\right) \quad (7)$$

To obtain the first order ordinary differential equation for the Quantile function of the half-Cauchy distribution, differentiate equation (7);

$$Q'(p)=\frac{\pi\sigma}{2}\sec^2\left(\frac{\pi p}{2}\right) \quad (8)$$

The condition necessary for the existence of equation is $\sigma>0, 0<p<1.$

Applying the trigonometric identity which is;

$$\sec^2\left(\frac{\pi p}{2}\right)=\tan^2\left(\frac{\pi p}{2}\right)+1$$

(9) $$Q'(p)=\frac{\pi\sigma}{2}\left(\tan^2\left(\frac{\pi p}{2}\right)+1\right)$$

(10) Equation (7) can be written as;

$$\left(\frac{Q(p)}{\sigma}\right)^2=\tan^2\left(\frac{\pi p}{2}\right)$$

(11) Substitute equation (11) into equation (10);

$$Q'(p)=\frac{\pi\sigma}{2}\left(\frac{Q^2(p)}{\sigma^2}+1\right)$$

(12) $$Q'(p)=\frac{\pi}{2\sigma}\left(Q^2(p)+\sigma^2\right)$$

(13) The first order ordinary differential for the Quantile function of the half-Cauchy distribution is given as;

$$2\sigma Q'(p)-\pi(Q^2(p)+\sigma^2)=0$$

(14) $$Q(0.1)=0.1584\sigma$$

(15)

Survival Function

The survival function of the half-Cauchy distribution is given by;

$$S(t)=1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right) \quad (16)$$

To obtain the first order ordinary differential equation for the survival function of the half-Cauchy distribution, differentiate equation (16);

$$S'(t)=-\frac{2}{\pi\sigma\left[1+\left(\frac{t}{\sigma}\right)^2\right]} \quad (17)$$

The condition necessary for the existence of equation is $\sigma, x>0.$

$$S'(t)=-\frac{2\sigma}{\pi(\sigma^2+t^2)} \quad (18)$$

The first order ordinary differential for the survival function of the half-Cauchy distribution is given as;

$$\pi(\sigma^2+t^2)S'(t)+2\sigma=0$$

(19) $$S(1)=1-\frac{2}{\pi}\tan^{-1}\left(\frac{1}{\sigma}\right)$$

(20)

Inverse Survival Function

The inverse survival function of the half-Cauchy distribution is given by;

$$Q(p)=\sigma\tan\left(\frac{\pi(1-p)}{2}\right) \quad (21)$$

To obtain the first order ordinary differential equation for the inverse survival function of the half-Cauchy distribution, differentiate equation (21);

$$Q'(p)=-\frac{\pi\sigma}{2}\sec^2\left(\frac{\pi(1-p)}{2}\right) \quad (22)$$

The condition necessary for the existence of equation is $\sigma>0, 0<p<1.$

Applying the same technique as obtained from the Quantile function, to obtain that;

$$Q'(p)=-\frac{\pi\sigma}{2}\left(\frac{Q^2(p)}{\sigma^2}+1\right) \quad (23)$$

$$Q'(p)=-\frac{\pi}{2\sigma}\left(Q^2(p)+\sigma^2\right) \quad (24)$$

The first order ordinary differential for the inverse survival function of the half-Cauchy distribution is given as;

$$2\sigma Q'(p)+\pi(Q^2(p)+\sigma^2)=0 \quad (25)$$

$$Q(0.1)=6.31375\sigma \quad (26)$$

Hazard Function

The hazard function of the half-Cauchy distribution is given by;

$$h(t)=\frac{2}{\pi\sigma\left[1+\left(\frac{t}{\sigma}\right)^2\right]\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)\right]} \quad (27)$$

To obtain the first order ordinary differential equation for the hazard function of the half-Cauchy distribution, differentiate equation (27);

$$h'(t)=\left\{\begin{array}{l} -\dfrac{\left(\dfrac{2t}{\sigma^2}\right)\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-2}}{\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}} \\ +\dfrac{\dfrac{2}{\pi\sigma}\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}\left[1-\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]^{-2}}{\left[1-\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]^{-1}} \end{array}\right\}h(t) \quad (28)$$

$$h'(t)=\left\{-\dfrac{\left(\dfrac{2t}{\sigma^2}\right)}{\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]}+\dfrac{\dfrac{2}{\pi\sigma}\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}}{\left[1-\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]}\right\}h(t) \quad (29)$$

The condition necessary for the existence of equation is $\sigma, t > 0$.

$$h'(t)=\left(-\frac{2t}{\sigma^2+t^2}+h(t)\right)h(t) \quad (30)$$

The first order ordinary differential for the hazard function of the half-Cauchy distribution is given as;

$$(\sigma^2+t^2)h'(t)-(\sigma^2+t^2)h^2(t)+2th(t)=0$$

(31) $$h(1)=\frac{2\sigma}{\pi\left[\sigma^2+1\right]\left[1-\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{1}{\sigma}\right)\right]}$$

(32)

Reversed Hazard Function

The reversed hazard function of the half-Cauchy distribution is given by;

$$j(t)=\frac{2}{\pi\sigma\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]\left[\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]} \quad (33)$$

To obtain the first order ordinary differential equation for the probability density function of the half-Cauchy distribution, differentiate equation (33);

$$j'(t)=-\frac{\left(\dfrac{2t}{\sigma^2}\right)\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-2}}{\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}}j(t) -\frac{\dfrac{2}{\pi\sigma}\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}\left[\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]^{-2}}{\left[\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]^{-1}}j(t) \quad (34)$$

$$j'(t)=-\left\{\dfrac{\left(\dfrac{2t}{\sigma^2}\right)}{\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]}+\dfrac{\dfrac{2}{\pi\sigma}\left[1+\left(\dfrac{t}{\sigma}\right)^2\right]^{-1}}{\left[\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{t}{\sigma}\right)\right]}\right\}j(t) \quad (35)$$

The condition necessary for the existence of equation is $\sigma, t > 0$.

$$j'(t)=-\left(\frac{2t}{\sigma^2+t^2}+j(t)\right)j(t) \quad (36)$$

The first order ordinary differential for the reversed hazard function of the half-Cauchy distribution is given as;

$$(\sigma^2+t^2)j'(t)+(\sigma^2+t^2)j^2(t)+2tj(t)=0 \quad (37)$$

$$j(1)=\frac{2\sigma}{\pi\left[\sigma^2+1\right]\left[\dfrac{2}{\pi}\tan^{-1}\left(\dfrac{1}{\sigma}\right)\right]} \quad (38)$$

II. Power Cauchy Distribution

Probability Density Function

The probability density function of the power Cauchy distribution is given by;

$$f(x)=\frac{\dfrac{2\alpha}{\pi\sigma}\left(\dfrac{x}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\dfrac{x}{\sigma}\right)^{2\alpha}\right]} \quad (39)$$

To obtain the first order ordinary differential equation for the probability density function of the power Cauchy distribution, differentiate equation (39);

$$f'(x)=\left\{\dfrac{\dfrac{\alpha-1}{\sigma}\left(\dfrac{x}{\sigma}\right)^{\alpha-2}}{\left(\dfrac{x}{\sigma}\right)^{\alpha-1}}-\dfrac{\dfrac{2\alpha}{\sigma}\left(\dfrac{x}{\sigma}\right)^{2\alpha-1}\left[1+\left(\dfrac{x}{\sigma}\right)^{2\alpha}\right]^{-2}}{\left[1+\left(\dfrac{x}{\sigma}\right)^{2\alpha}\right]^{-1}}\right\}f(x) \quad (40)$$

The condition necessary for the existence of equation is $\alpha, \sigma, x > 0$.

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$$f'(x)=\left\{\frac{\alpha-1}{x}-\frac{\frac{2\alpha}{\sigma}\left(\frac{x}{\sigma}\right)^{\alpha-1}\left(\frac{x}{\sigma}\right)^{\alpha}}{\left[1+\left(\frac{x}{\sigma}\right)^{2\alpha}\right]}\right\}f(x) \quad (41)$$

$$f'(x)=\left(\frac{\alpha-1}{x}-\pi f(x)\left(\frac{x}{\sigma}\right)^{\alpha}\right)f(x) \quad (42)$$

Simply to obtain the first order ordinary differential equation which is dependent on the powers of the parameters that defined the probability density function.

$$\sigma^{\alpha}xf'(x)-\sigma^{\alpha}(\alpha-1)f(x)+\pi x^{\alpha+1}f^{2}(x)=0 \quad (43)$$

Some cases are considered;

When $\alpha=1$, equation (43) becomes;

$$\sigma xf'(x)+\pi x^{2}f^{2}(x)=0 \quad (44)$$

$$\sigma f'(x)+\pi xf^{2}(x)=0 \quad (45)$$

When $\alpha=2$, equation (43) becomes;

$$\sigma^{2}xf'(x)-\sigma^{2}f(x)+\pi x^{3}f^{2}(x)=0 \quad (46)$$

When $\alpha=3$, equation (43) becomes;

$$\sigma^{3}xf'(x)-2\sigma^{3}f(x)+\pi x^{4}f^{2}(x)=0 \quad (47)$$

QUANTILE FUNCTION

The Quantile function of the power Cauchy distribution is given by;

$$Q(p)=\sigma\left(\tan\left(\frac{\pi p}{2}\right)\right)^{\frac{1}{\alpha}} \quad (48)$$

To obtain the first order ordinary differential equation for the Quantile function of the power Cauchy distribution, differentiate equation (48);

$$Q'(p)=\frac{\sigma}{\alpha}\left(\tan\left(\frac{\pi p}{2}\right)\right)^{\frac{1}{\alpha}-1}\frac{\pi}{2}\sec^{2}\left(\frac{\pi p}{2}\right) \quad (49)$$

The condition necessary for the existence of equation is $\alpha,\sigma>0, 0<p<1$.

$$Q'(p)=\frac{\frac{\sigma}{\alpha}\left(\tan\left(\frac{\pi p}{2}\right)\right)^{\frac{1}{\alpha}}\frac{\pi}{2}\sec^{2}\left(\frac{\pi p}{2}\right)}{\tan\left(\frac{\pi p}{2}\right)} \quad (50)$$

Substitute equation (48) into equation (50);

$$Q'(p)=\frac{\frac{\pi}{2}\sec^{2}\left(\frac{\pi p}{2}\right)Q(p)}{\alpha\tan\left(\frac{\pi p}{2}\right)} \quad (51)$$

Equation (48) can be written as;

$$\left(\frac{Q(p)}{\sigma}\right)^{\alpha}=\tan\left(\frac{\pi p}{2}\right) \quad (52)$$

Substitute equation (52) into equation (51);

$$Q'(p)=\frac{\frac{\pi\sigma^{\alpha}}{2}\sec^{2}\left(\frac{\pi p}{2}\right)Q(p)}{\alpha Q^{\alpha}(p)} \quad (53)$$

$$Q'(p)=\frac{\pi\sigma^{\alpha}\sec^{2}\left(\frac{\pi p}{2}\right)Q^{1-\alpha}(p)}{2\alpha} \quad (54)$$

Applying the trigonometric identity which is;

$$\sec^{2}\left(\frac{\pi p}{2}\right)=\tan^{2}\left(\frac{\pi p}{2}\right)+1$$

(55) Square the both sides of equation (52)

$$\left(\frac{Q(p)}{\sigma}\right)^{2\alpha}=\tan^{2}\left(\frac{\pi p}{2}\right)$$

(56) Substitute equation (56) into equation (55), to obtain;

$$\sec^{2}\left(\frac{\pi p}{2}\right)=\left(\frac{Q(p)}{\sigma}\right)^{2\alpha}+1$$

(57) Substitute equation (57) into equation (54);

$$Q'(p)=\frac{\pi\sigma^{\alpha}\left(\left(\frac{Q(p)}{\sigma}\right)^{2\alpha}+1\right)Q^{1-\alpha}(p)}{2\alpha}$$

(58) $Q'(p)=\dfrac{\pi\left(Q^{2\alpha}(p)+\sigma^{2\alpha}\right)Q^{1-\alpha}(p)}{2\alpha\sigma^{\alpha}}$

(59) $Q'(p)=\dfrac{\pi\left(Q^{1+\alpha}(p)+\sigma^{2\alpha}Q^{1-\alpha}(p)\right)}{2\alpha\sigma^{\alpha}}$

(60) $2\alpha\sigma^{\alpha}Q'(p)-\pi\left(Q^{1+\alpha}(p)+\sigma^{2\alpha}Q^{1-\alpha}(p)\right)=0$

(61) Some cases are considered;

When $\alpha=1$, equation (61) becomes;

$$2\sigma Q'(p)-\pi\left(Q^{2}(p)+\sigma^{2}\right)=0 \quad (62)$$

When $\alpha=2$, equation (62) becomes;

$$4\sigma^{2}Q(p)Q'(p)-\pi\left(Q^{4}(p)+\sigma^{4}\right)=0 \quad (63)$$

SURVIVAL FUNCTION

The survival function of the power Cauchy distribution is given by;

$$S(t)=1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha} \quad (64)$$

To obtain the first order ordinary differential equation for the survival function of the power Cauchy distribution, differentiate equation (64);

$$S'(t)=-\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]} \quad (65)$$

The condition necessary for the existence of equation is $\alpha,\sigma,t>0$.

Simplify;

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$$\left(1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right)S'(t)=-\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1} \tag{66}$$

$$\left(\sigma^{2\alpha}+t^{2\alpha}\right)S'(t)=-\frac{2\alpha\sigma^{\alpha}t^{\alpha-1}}{\pi} \tag{67}$$

$$\pi\left(\sigma^{2\alpha}+t^{2\alpha}\right)S'(t)+2\alpha\sigma^{\alpha}t^{\alpha-1}=0 \tag{68}$$

The first order ordinary differential equations can be obtained for the particular values of the parameters. Some cases are considered and shown in **Table 1.**

Table 1: Classes of differential equations obtained for the survival function of the power Cauchy distribution for different parameters.

α	σ	π	ordinary differential equations
1	1	1	$(1+t^2)S'(t)+2=0$
1	1	2	$(1+t^2)S'(t)+1=0$
1	2	1	$(4+t^2)S'(t)+4=0$
1	2	2	$(4+t^2)S'(t)+2=0$
2	1	1	$(1+t^4)S'(t)+4t=0$
2	1	2	$(1+t^4)S'(t)+2t=0$
2	2	1	$(16+t^4)S'(t)+16t=0$
2	2	2	$(16+t^4)S'(t)+8t=0$

INVERSE SURVIVAL FUNCTION

The inverse survival function of the power Cauchy distribution is given by;

$$Q(p)=\sigma\left(\tan\left(\frac{\pi(1-p)}{2}\right)\right)^{\frac{1}{\alpha}} \tag{69}$$

To obtain the first order ordinary differential equation for the inverse survival function of the power Cauchy distribution, differentiate equation (69);

$$Q'(p)=\frac{\sigma}{\alpha}\left(\tan\left(\frac{\pi(1-p)}{2}\right)\right)^{\frac{1}{\alpha}-1}\left(-\frac{\pi}{2}\sec^2\left(\frac{\pi(1-p)}{2}\right)\right) \tag{70}$$

The condition necessary for the existence of equation is $\alpha,\sigma>0, 0<p<1$.

$$Q'(p)=-\frac{\frac{\sigma}{\alpha}\left(\tan\left(\frac{\pi(1-p)}{2}\right)\right)^{\frac{1}{\alpha}}\frac{\pi}{2}\sec^2\left(\frac{\pi(1-p)}{2}\right)}{\tan\left(\frac{\pi(1-p)}{2}\right)} \tag{71}$$

Substitute equation (69) into equation (71);

$$Q'(p)=-\frac{\frac{\pi}{2}\sec^2\left(\frac{\pi(1-p)}{2}\right)Q(p)}{\alpha\tan\left(\frac{\pi(1-p)}{2}\right)} \tag{72}$$

Repeating the same simplification done for the Quantile function;

$$Q'(p)=-\frac{\pi\left(Q^{2\alpha}(p)+\sigma^{2\alpha}\right)Q^{1-\alpha}(p)}{2\alpha\sigma^{\alpha}} \tag{73}$$

$$2\alpha\sigma^{\alpha}Q'(p)+\pi\left(Q^{1+\alpha}(p)+\sigma^{2\alpha}Q^{1-\alpha}(p)\right)=0 \tag{74}$$

The first order ordinary differential equations can be obtained for the particular values of the parameters. Some cases are considered and shown in **Table 2**.

Table 2: Classes of differential equations obtained for the inverse survival function of the power Cauchy distribution for different parameters

α	σ	π	ordinary differential equations
1	1	1	$2Q'(p)+Q^2(p)+1=0$
1	1	2	$Q'(p)+Q^2(p)+1=0$
1	2	1	$4Q'(p)+Q^2(p)+4=0$
1	2	2	$2Q'(p)+Q^2(p)+4=0$
2	1	1	$4Q(p)Q'(p)+Q^4(p)+1=0$
2	1	2	$2Q(p)Q'(p)+Q^4(p)+1=0$
2	2	1	$16Q(p)Q'(p)+Q^4(p)+16=0$
2	2	2	$8Q(p)Q'(p)+Q^4(p)+16=0$

HAZARD FUNCTION

The hazard function of the power Cauchy distribution is given by;

$$h(t)=\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]}$$

(75) To obtain the first order ordinary differential equation for the hazard function of the power Cauchy distribution, differentiate equation (75);

$$h'(t)=\frac{\frac{\alpha-1}{\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-2}}{\left(\frac{t}{\sigma}\right)^{\alpha-1}}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]^{-2}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]^{-1}}h(t)$$

$$+\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-2}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-1}}h(t) \tag{76}$$

$$h'(t)=\left\{\begin{array}{l}\frac{\alpha-1}{t}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]}\\+\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[1-\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-1}}\end{array}\right\}h(t) \tag{77}$$

The condition necessary for the existence of equation is $\alpha,\sigma,t>0$.

$$h'(t)=\left\{\frac{\alpha-1}{t}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]}+h(t)\right\}h(t) \tag{78}$$

$$h'(t)=\left\{\frac{\alpha-1}{t}-\frac{2\alpha t^{2\alpha-1}}{\sigma^{2\alpha}+t^{2\alpha}}+h(t)\right\}h(t) \tag{79}$$

The first order ordinary differential equations can be obtained for the particular values of the parameters. Some cases are considered.

When $\alpha=1$, equation (79) becomes;

$$h'(t)=\left(h(t)-\frac{2t}{\sigma^2+t^2}\right)h(t) \tag{80}$$

$$(\sigma^2+t^2)h'(t)-(\sigma^2+t^2)h^2(t)+2th(t)=0 \tag{81}$$

When $\alpha=2$, equation (79) becomes;

$$h'(t)=\left\{\frac{1}{t}-\frac{4t^3}{\sigma^4+t^4}+h(t)\right\}h(t) \tag{82}$$

$$(\sigma^4+t^4)th'(t)+(3t^4-\sigma^4)h(t)-(\sigma^4+t^4)th^2(t)=0 \tag{83}$$

REVERSED HAZARD FUNCTION

The reversed hazard function of the power Cauchy distribution is given by;

$$j(t)=\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]} \tag{84}$$

To obtain the first order ordinary differential equation for the reversed hazard function of the power Cauchy distribution, differentiate equation (84);

$$j'(t)=\left\{\begin{array}{l}\frac{\frac{\alpha-1}{\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-2}}{\left(\frac{t}{\sigma}\right)^{\alpha-1}}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]^{-2}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]^{-1}}\\-\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}\left[\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-2}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-1}}\end{array}\right\}j(t) \tag{85}$$

$$j'(t)=\left\{\begin{array}{l}\frac{\alpha-1}{t}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]}\\-\frac{\frac{2\alpha}{\pi\sigma}\left(\frac{t}{\sigma}\right)^{\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]\left[\frac{2}{\pi}\tan^{-1}\left(\frac{t}{\sigma}\right)^{\alpha}\right]^{-1}}\end{array}\right\}j(t) \tag{86}$$

The condition necessary for the existence of equation is $\alpha,\sigma,t>0$.

$$j'(t)=\left\{\frac{\alpha-1}{t}-\frac{\frac{2\alpha}{\sigma}\left(\frac{t}{\sigma}\right)^{2\alpha-1}}{\left[1+\left(\frac{t}{\sigma}\right)^{2\alpha}\right]}-j(t)\right\}j(t) \tag{87}$$

$$j'(t)=\left\{\frac{\alpha-1}{t}-\frac{2\alpha t^{2\alpha-1}}{\sigma^{2\alpha}+t^{2\alpha}}-j(t)\right\}j(t) \tag{88}$$

The first order ordinary differential equations can be obtained for the particular values of the parameters. Some cases are considered.

When $\alpha=1$, equation (88) becomes;

$$j'(t)=-\left(j(t)+\frac{2t}{\sigma^2+t^2}\right)j(t) \tag{89}$$

$$(\sigma^2+t^2)j'(t)+(\sigma^2+t^2)j^2(t)+2tj(t)=0 \tag{90}$$

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When $\alpha = 2,$ equation (88) becomes;

$$j'(t) = \left\{ \frac{1}{t} - \frac{4t^3}{\sigma^4 + t^4} - j(t) \right\} j(t) \qquad (91)$$

$$(\sigma^4 + t^4) t j'(t) + (3t^4 - \sigma^4) j(t) + (\sigma^4 + t^4) t j^2(t) = 0 \qquad (92)$$

The ordinary differential equations can be obtained for the particular values of the parameters or higher orders.

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [20-33]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

III. Concluding Remarks

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the half-Cauchy and power Cauchy distributions. The work was simplified by the application of simple algebraic procedures. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. The varying degrees of the effects of the parameters in the construction of the ODEs are subject of further research.

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