Quantile Approximation of the Chi–square Distribution using the Quantile Mechanics

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***Abstract*— In the field of probability and statistics, the quantile function and the quantile density function which is the derivative of the quantile function are one of the important ways of characterizing probability distributions and as well, can serve as a viable alternative to the probability mass function or probability density function. The quantile function (QF) and the cumulative distribution function (CDF) of the chi-square distribution do not have closed form representations except at degrees of freedom equals to two and as such researchers devise some methods for their approximations. One of the available methods is the quantile mechanics approach. The paper is focused on using the quantile mechanics approach to obtain the quantile density function and their corresponding quartiles or percentage points. The outcome of the method is second order nonlinear ordinary differential equation (ODE) which was solved using the traditional power series method. The quantile density function was transformed to obtain the respective percentage points (quartiles) which were represented on a table. The results compared favorably with known results at high quartiles. A very clear application of this method will help in modeling and simulation of physical processes.**

***Index Terms*— Quantile, Quantile density function, Quantile mechanics, percentage points, Chi-square, approximation.**

I. Introduction

In statistics, In statistics, quantile function is very important in prescribing probability distributions. It is indispensable in determining the location and spread of any given distribution, especially the median which is resistant to extreme values or outliers [1] [2]. Quantile function is used extensively in the simulation of non-uniform random variables [3] and also can be seen as an alternative to the CDF in analysis of lifetime probability models with heavy tails. Details on and the use of the quantile function in modeling, statistical, reliability and survival analysis can be found in: [4], [5].

It should be noted that probability distributions whose statistical reliability measures do not have a close or explicit form can be conveniently represented through the QF. Chi square distribution is one of such distribution whose CDF does not have closed form.

The search for analytic expression of quantile functions has been a subject of intense research due to the importance of quantile functions. Several approximations are available in literature which can be categorized into four, namely functional approximations, series expansions; numerical algorithms and closed form written in terms of a quantile function of another probability distribution which can also be refer to quantile normalization.

The use of ordinary differential equations in approximating the quantile has been studied by Ulrich and Watson [6] and Leobacher and Pillichshammer [7]. The series solution to the ordinary differential equations used for the approximation of the quantile function was pioneered by Cornish and Fisher [8], Fisher and Cornish [9] and generalized as Quantile mechanics approach by Steinbrecher and Shaw [10]. The approach was inspired by the works of Hill and Davis [11].

Few researches done on the approximations of the quantile functions of Chi-square distribution were done by [12], [13], [14], [15], [16], [17], [18], [19], [20], [21], [22], [23].

II. Formulation

The probability density function of the chi-square distribution and the cumulative distribution function are given by;

$$f(x)=\frac{1}{2^{\frac{k}{2}}\Gamma(k/2)}x^{\frac{k}{2}-1}e^{-\frac{x}{2}},\ k>0,\ x\in[0,+\infty) \quad (1)$$

$$F(x,k)=\frac{\gamma\left(\frac{k}{2},\frac{x}{2}\right)}{\Gamma\left(\frac{k}{2}\right)}=P\left(\frac{k}{2},\frac{x}{2}\right) \quad (2)$$

where $\gamma(.,.)=$ incomplete gamma functions and $P(.,.)=$ regularized gamma function.

The quantile mechanics (QM) approach was used to obtain the second order nonlinear differential equation. QM is applied to distributions whose CDF is monotone increasing and absolutely continuous. Chi-square distribution is one of such distributions. That is;

$$Q(p)=F^{-1}(p) \quad (3)$$

Where the function $F^{-1}(p)$ is the compositional inverse of

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the CDF. Suppose the PDF f(x) is known and the differentiation exists. The first order quantile equation is obtained from the differentiation of equation (3) to obtain;

$$Q'(p) = \frac{1}{F'(F^{-1}(p))} = \frac{1}{f(Q(p))} \tag{4}$$

Since the probability density function is the derivative of the cumulative distribution function. The solution to equation (4) is often cumbersome as noted by Ulrich and Watson [6]. This is due to the nonlinearity of terms introduced by the density function f. Some algebraic operations are required to find the solution of equation (4).
Moreover, equation (4) can be written as;

$$f(Q(p))Q'(p) = 1 \tag{5}$$

Applying the traditional product rule of differentiation to obtain;

$$Q''(p) = V(Q(p))(Q(p))^2 \tag{6}$$

Where the nonlinear term;

$$V(x) = -\frac{d}{dx}(\ln f(x)) \tag{7}$$

These were the results of [10].
It can be deduced that the further differentiation enables researchers to apply some known techniques to finding the solution of equation (6).
The reciprocal of the probability density function of the chi-square distribution is transformed as a function of the quantile function.

$$\frac{dQ(p)}{dp} = 2^{\frac{k}{2}}(\Gamma(k/2))Q(p)^{1-\frac{k}{2}} e^{\frac{Q(p)}{2}} \tag{8}$$

Differentiate again to obtain;

$$\frac{d^2Q(p)}{dp^2} = 2^{\frac{k}{2}}(\Gamma(k/2)) \begin{bmatrix} Q(p)^{1-\frac{k}{2}} e^{\frac{Q(p)}{2}} \frac{dQ(p)}{2dp} + \\ \left(1-\frac{k}{2}\right) Q(p)^{-\frac{k}{2}} e^{\frac{Q(p)}{2}} \frac{dQ(p)}{dp} \end{bmatrix} \tag{9}$$

Factorization is carried out;

$$\frac{d^2Q(p)}{dp^2} = 2^{\frac{k}{2}}(\Gamma(k/2)) \begin{bmatrix} Q(p)^{1-\frac{k}{2}} e^{\frac{Q(p)}{2}} \frac{dQ(p)}{2dp} \\ + \left(\frac{2-k}{2}\right) \frac{Q(p)^{1-\frac{k}{2}}}{Q(p)^{1-\frac{k}{2}}} Q(p)^{-\frac{k}{2}} e^{\frac{Q(p)}{2}} \frac{dQ(p)}{dp} \end{bmatrix} \tag{10}$$

$$\frac{d^2Q(p)}{dp^2} = \frac{1}{2}\left(\frac{dQ(p)}{dp}\right)^2 + \left(\frac{2-k}{2Q(p)}\right)\left(\frac{dQ(p)}{dp}\right)^2 \tag{11}$$

The second order nonlinear ordinary differential equations is given as;

$$\frac{d^2Q(p)}{dp^2} = \left(\frac{1}{2} + \frac{2-k}{2Q(p)}\right)\left(\frac{dQ(p)}{dp}\right)^2 \tag{12}$$

With the boundary conditions; $Q(0) = 0,\ Q'(0) = 1$.

III. Power Series Solution

The cumulative distribution function and its inverse (quantile function) of the chi- square distribution do not have closed form. The power series method was used to find the solution of the Chi-square quantile differential equation (equation (12)) for different degrees of freedom. It was observed that the series solution takes the form of equation (13)
The equations formed a series which can be used to predict p for any given degree of freedom k.

$$Q(p) \approx p + \frac{1}{4(k-1)} p^2, \quad k > 1 \tag{13}$$

For very large k,

$$Q(p) \approx p \tag{14}$$

In order to get a very close convergence approximations of the probability p, equation (13) is used for all the degrees of freedom. For examples the values of degrees of freedom from one to twelve is given in **Tables 1a and 1b.**

Table 1a: Quantile density function table for the Chi-square distribution for degrees of freedom from 1 to 6.

p	k = 1	k= 2	k= 3
0.001	0.001001	0.00100025	0.001000125
0.01	0.0101	0.010025	0.0100125
0.025	0.025625	0.02515625	0.025078125
0.05	0.0525	0.050625	0.0503125
0.10	0.11	0.1025	0.10125
0.25	0.3125	0.265625	0.2578125
0.50	0.75	0.5625	0.53125
0.75	1.3125	0.890625	0.8203125
0.90	1.71	1.1025	1.00125
0.95	1.8525	1.175625	1.0628125
0.975	1.925625	1.21265625	1.093828125
p	k= 4	k = 5	k= 6
0.001	0.001000083	0.001000063	0.00100005
0.01	0.010008333	0.01000625	0.010005
0.025	0.025052083	0.025039063	0.02503125
0.05	0.050208333	0.05015625	0.050125
0.10	0.100833333	0.100625	0.1005
0.25	0.255208333	0.25390625	0.253125
0.50	0.520833333	0.515625	0.5125
0.75	0.796875	0.78515625	0.778125
0.90	0.9675	0.950625	0.9405
0.95	1.025208333	1.00640625	0.995125
0.975	1.05421875	1.034414063	1.02253125

Table 1b: Quantile density function table for the Chi-square distribution for degrees of freedom from 7 to 12.

P	k= 7	k= 8	k = 9
0.001	0.001000042	0.001000036	0.001000031
0.01	0.010004167	0.010003571	0.010003125
0.025	0.025026042	0.025022321	0.025019531
0.05	0.050104167	0.050089286	0.050078125
0.10	0.100416667	0.100357143	0.1003125
0.25	0.252604167	0.252232143	0.251953125
0.50	0.510416667	0.508928571	0.5078125
0.75	0.7734375	0.770089286	0.767578125
0.90	0.93375	0.928928571	0.9253125
0.95	0.987604167	0.982232143	0.978203125
0.975	1.014609375	1.008950893	1.004707031
P	k= 10	k= 11	k= 12
0.001	0.001000028	0.001000025	0.001000023
0.01	0.010002778	0.0100025	0.010002273
0.025	0.025017361	0.025015625	0.025014205
0.05	0.050069444	0.0500625	0.050056818
0.10	0.100277778	0.10025	0.100227273
0.25	0.251736111	0.2515625	0.251420455
0.50	0.506944444	0.50625	0.505681818
0.75	0.765625	0.7640625	0.762784091
0.90	0.9225	0.92025	0.918409091
0.95	0.975069444	0.9725625	0.970511364
0.975	1.00140625	0.998765625	0.996605114

These values are the extent to which the Quantile Mechanics was able to approach the probability.

IV. Transformation and Comparison

Transformation to the percentage points and comparison with the exact was done here.
The probability p obtained is transformed using the definition.

Definition
Given a probability p which lies between 0 and 1, the percentage points or quartiles or quantile of the chi-square distribution with the non-negative k degrees of freedom is the value $\chi^2_{1-p}(k)$ such that the area under the curve and to the right of $\chi^2_{1-p}(k)$ is equals to the value 1 – p. The quantile in **Table 1** are computed and compared with the exact values. The readers are refer the r software given as for example

$> qchisq(0.95,3)$

$[1]7.814728$

$> qchisq(0.95,4)$

$[2]9.48773$

The comparisons are presented in **Tables 2** for degrees of freedom ranges from 1 to 12. The Quantile mechanics method compares favorably at the following: low probability, high percentage points and higher degrees of freedom. However the methods perform fairly well at the following: high probability, low percentage points and low degrees of freedom.

V. Percentage Points For The Chi-square Distribution

The final table for the percentage points or quantile of the chi-square distribution is shown on **Table 3.** The table of the quantile (percentage points) is quite similar to the one summarized by Goldberg and Levine [24], which includes the results of Fisher [25], Wilson and Hilferty [26], Peiser [27] and Cornish and Fisher [8]. In addition, the result is similar to the works of Thompson [28], Hoaglin [29], Zar [30], Johnson et al. [31] [32] and Ittrich et al. [33].

The same outcome was obtained when compared with the result of Severo and Zelen [15]. This can be seen in **Table 4.**

In particular, the QM method performs better at higher percentiles and degrees of freedom when compared with others. The summary is in **Table 5.**

VI. Concluding Remarks

The quantile mechanics has been used to obtain the approximations of the percentage points of the chi-square distribution. The method is very efficient at high degrees of freedom, higher percentage points and lower probabilities. However the method performed fairly in the lower degrees of freedom, lower percentiles and high probabilities. This was a part of points noted by [34] that approximation efficiency decreases with the degrees of freedom.

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References

[1] P.J. Huber, "Robust estimation of a location parameter," *Ann. Math. Stat.*, vol. 35, no. 1, pp. 73-101, 1964.
[2] F.R. Hampel, "The influence curve and its role in robust estimation," *J. Amer. Stat. Assoc.*, vol. 69, no. 346, pp. 383-393, 1974.
[3] W.J. Padgett, "A kernel-type estimator of a quantile function from right-censored data," *J. Amer. Stat. Assoc.*, vol. 81, no. 393, pp. 215-222, 1986.
[4] E. Parzen, "Nonparametric statistical data modeling," *J. Amer. Stat. Assoc.*, vol. 74, pp. 105-131. 1979.
[5] N. Reid, "Estimating the median survival time," *Biometrika*, vol. 68, pp. 601-608, 1981.
[6] G. Ulrich and L.T. Watson, "A method for computer generation of variates from arbitrary continuous distributions," *SIAM J. Scientific Comp.*, vol. 8, no. 2, pp. 185–197, 1987.
[7] G. Leobacher and F. Pillichshammer, "A Method for approximate inversion of the hyperbolic CDF," *Computing*, vol. 69, no. 4, pp. 291–303, 2002.

Table 2: Comparison between the exact and quantile mechanics for degrees of freedom from 1 to 12

p	k = 1		k = 2		k = 3		k= 4	
	Exact	QM	Exact	QM	Exact	QM	Exact	QM
0.001	10.82757	10.82572	13.81551	13.81501	16.26624	16.26597	18.46683	18.46664
0.01	6.63490	6.61717	9.21034	9.20535	11.34487	11.34216	13.27670	13.27479
0.025	5.02389	4.98115	7.3776	7.36530	9.34840	9.34155	11.14329	11.14132
0.05	3.84146	3.75976	5.99146	5.96662	7.81473	7.80082	9.48773	9.47766
0.10	2.70554	2.55422	4.60517	4.55578	6.25139	6.22302	7.77944	7.75857
0.25	1.32330	1.02008	2.77259	2.65134	4.10835	4.03403	5.38527	5.32863
0.50	0.45494	0.101531	1.38629	1.15073	2.36597	2.20355	3.35669	3.22545
0.75	0.10153	-	0.57536	0.23166	1.21253	0.92119	1.92256	1.66605
0.90	0.005	-	0.2000	-	0.58437	-	1.06362	0.55908
0.95	0.004	-	0.103	-	0.35185	-	0.71072	-
0.975	0.001	-	0.051	-	0.21580	-	0.48442	-
p	k = 5	-	k= 6		k = 7		k= 8	
	Exact	QM	Exact	QM	Exact	QM	Exact	QM
0.001	20.51501	20.51486	22.45774	22.45763	24.32189	24.32178	26.12448	26.12439
0.01	15.08627	15.08476	16.81189	16.81063	18.47531	18.47421	20.09024	20.08926
0.025	12.83250	12.82860	14.44938	14.44609	16.01276	16.00990	17.53455	17.53200
0.05	11.07050	11.06242	12.59159	12.58475	14.06714	14.06117	15.50731	15.50196
0.10	9.23636	9.21944	10.64464	10.63021	12.01704	12.00435	13.36157	13.35013
0.25	6.62568	6.57868	7.84080	7.80000	9.03715	9.00072	10.21885	10.18572
0.50	4.35146	4.23842	5.34812	5.24737	6.34581	6.25407	7.34412	7.25934
0.75	2.67460	2.44232	3.45460	3.24040	4.25485	4.05486	5.07064	4.88220
0.90	1.61031	1.13866	2.20413	1.75870	2.83311	2.40959	3.48954	3.08473
0.95	1.14548	-	1.63538	0.66954	2.16735	1.33055	2.73264	1.95937
0.975	0.83121	-	1.23734	-	1.68987	-	2.17973	-
p	k = 9		k= 10		k = 11		k= 12	
	Exact	QM	Exact	QM	Exact	QM	Exact	QM
0.001	27.87716	27.87708	29.58830	29.58822	31.26413	31.26407	32.90949	32.90943
0.01	21.66599	21.66511	23.20925	23.20845	24.72497	24.72423	26.21697	26.21627
0.025	19.02277	19.02046	20.48318	20.48105	21.92005	21.91808	23.33666	23.33482
0.05	16.91898	16.91411	18.30704	18.30255	19.67514	19.67097	21.02607	21.02216
0.10	14.68366	14.67321	15.98718	15.97755	17.27501	17.26600	18.54935	18.54088
0.25	11.38875	11.35819	12.54886	12.52040	13.70069	13.67396	14.84540	14.82014
0.50	8.34283	8.26363	9.34182	9.26728	10.34100	10.27030	11.34032	11.27299
0.75	5.89883	5.72004	6.73720	6.56664	7.58414	7.42072	8.43842	8.28129
0.90	4.16816	3.77957	4.86518	4.49085	5.57778	5.21611	6.30380	5.95366
0.95	3.32511	2.59553	3.94030	3.24454	4.57481	3.90687	5.22603	4.58180
0.975	2.70039	-	3.24697	-	3.81575	1.91767	4.40379	2.83518

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Table 3: The percentage points of the Chi-square Distribution

%ile / k	2.5	5	10	25	50	75	90	95	97.5	99	99.99
1	-	-	-	-	0.101531	1.02008	2.55422	3.75976	4.98115	6.61717	10.82572
2	-	-	-	0.23166	1.15073	2.65134	4.55578	5.96662	7.36530	9.20535	13.81501
3	-	-	-	0.92119	2.20355	4.03403	6.22302	7.80082	9.34155	11.34216	16.26597
4	-	-	0.55908	1.66605	3.22545	5.32863	7.75857	9.47766	11.14132	13.27479	18.46664
5	-	-	1.13866	2.44232	4.23842	6.57868	9.21944	11.06242	12.82860	15.08476	20.51486
6	-	0.66954	1.75870	3.24040	5.24737	7.80000	10.63021	12.58475	14.44609	16.81063	22.45763
7	-	1.33055	2.40959	4.05486	6.25407	9.00072	12.00435	14.06117	16.00990	18.47421	24.32178
8	-	1.95937	3.08473	4.88220	7.25934	10.18572	13.35013	15.50196	17.53200	20.08926	26.12439
9	-	2.59553	3.77957	5.72004	8.26363	11.35819	14.67321	16.91411	19.02046	21.66511	27.87708
10	-	3.24454	4.49085	6.56664	9.26728	12.52040	15.97755	18.30255	20.48105	23.20845	29.58822
11	1.91767	3.90687	5.21611	7.42072	10.27030	13.67396	17.26600	19.67097	21.91808	24.72423	31.26407
12	2.83518	4.58180	5.95366	8.28129	11.27299	14.82014	18.54088	21.02216	23.33482	26.21627	32.90943
13	3.59246	5.26830	6.70144	9.14744	12.27531	15.95990	19.80393	22.35835	24.73387	27.68760	34.52812
14	4.31155	5.96541	7.45880	10.01867	13.27739	17.09402	21.05654	23.68130	26.11731	29.14062	36.12322
15	5.01771	6.67220	8.22456	10.89439	14.27925	18.22314	22.29988	24.99247	27.48684	30.57733	37.69725
16	5.72045	7.38784	8.99790	11.77415	15.28094	19.34778	23.53489	26.29306	28.84387	31.99937	39.25230
17	6.42400	8.11161	9.77811	12.65759	16.28247	20.46836	24.76237	27.58407	30.18959	33.40813	40.79017
18	7.13041	8.84287	10.56460	13.54439	17.28387	21.58527	25.98301	28.86638	31.52502	34.80480	42.31235
19	7.84071	9.58106	11.35686	14.43427	18.28516	22.69882	27.19738	30.14071	32.85102	36.19038	43.82015
20	8.55540	10.32567	12.15443	15.32699	19.28635	23.80928	28.40600	31.40772	34.16834	37.56576	45.31471
21	9.27470	11.07625	12.95693	16.22234	20.28745	24.91690	29.60929	32.66794	35.47765	38.93172	46.79700
22	9.99865	11.83241	13.76401	17.12014	21.28848	26.02187	30.80766	33.92189	36.77953	40.28892	48.26790
23	10.72722	12.59380	14.57536	18.02021	22.28944	27.12440	32.00143	35.16999	38.07448	41.63797	49.72820
24	11.46031	13.36008	15.39070	18.92242	23.29033	28.22463	33.19092	36.41262	39.36296	42.97941	51.17856
25	12.19779	14.13098	16.20980	19.82663	24.29118	29.32272	34.37640	37.65014	40.64538	44.31370	52.61962
26	12.93953	14.90623	17.03243	20.73273	25.29197	30.41880	35.55811	38.88286	41.92211	45.64129	54.05193
27	13.68537	15.68559	17.85839	21.64060	26.29273	31.51299	36.73628	40.11105	43.19348	46.96256	55.47599
28	14.43517	16.46884	18.68749	22.55015	27.29344	32.60540	37.91109	41.33497	44.45979	48.27786	56.89225
29	15.18877	17.25578	19.51958	23.46129	28.29411	33.69611	39.08275	42.55484	45.72130	49.58752	58.30114
30	15.94604	18.04624	20.35450	24.37394	29.29475	34.78524	40.25140	43.77089	46.97828	50.89183	59.70303
40	23.69227	26.11237	28.83341	33.56952	39.29978	45.60370	51.80118	55.75675	59.34091	63.69045	73.40193
50	31.68651	34.40245	37.49117	42.86025	49.30322	56.32274	63.16373	67.50330	71.41950	76.15364	86.66079
60	39.86265	42.85288	46.27634	52.21867	59.30577	66.97163	74.39395	79.08059	83.29706	88.37919	99.60721
70	48.17900	51.42548	55.15825	61.62842	69.30776	77.56762	85.52425	90.52999	95.02262	100.42498	112.31691
80	56.60758	60.09517	64.11690	71.07886	79.30937	88.12186	96.57562	101.87834	106.62805	112.32860	124.83921
90	65.12859	68.84444	73.13833	80.56257	89.31071	98.64205	107.56259	113.14421	118.13541	124.11614	137.20834
100	73.72743	77.66051	82.21238	90.07415	99.31184	109.13379	118.49573	124.34111	129.56074	135.80656	149.44924

Table 4: Comparison with known results A

Probability		0.250	0.050	0.005		0.250	0.050	0.005
Percentage points	k	75	95	99.95	k	75	95	99.95
Exact Value	10	12.549	18.307	25.188	40	45.616	55.758	66.766
Severo and Zelen		12.550	18.313	25.178		45.722	55.473	65.712
Quantile Mechanics		12.520	18.302	25.186		45.604	55.757	66.766
Exact Value	20	23.828	31.410	39.997	50	56.334	67.505	78.488
Severo and Zelen		23.827	31.415	40.002		56.439	67.219	78.447
Quantile Mechanics		23.809	31.408	39.997		56.323	67.503	78.488
Exact Value	30	34.908	43.787	52.603	100	109.141	124.342	140.169
Severo and Zelen		34.799	43.772	52.665		109.242	124.056	139.154
Quantile Mechanics		34.785	43.771	52.603		109.138	124.341	140.169

Table 5: Comparison with known results B

Percentage Points	K	Exact Value	Cornish-Fisher	Peiser	Wilson and Hilferty	Fisher	Quantile Mechanics
75	1	1.3233	1.2730	1.2437	1.3156	1.4020	1.0201
90		2.7055	2.6857	2.7012	2.6390	2.6027	2.5542
95		3.8415	3.8632	3.9082	3.7468	3.4976	3.7598
99		6.6349	6.8106	6.9409	6.5858	5.5323	6.6172
99.95		7.8794	8.1457	8.3255	7.9048	6.3933	7.8704
75	2	2.7726	2.7595	2.7403	2.7628	2.8957	2.6513
90		4.6052	4.6018	4.6099	4.5590	4.5409	4.5558
95		5.9915	6.0004	6.0343	5.9369	5.7017	5.9666
99		9.2103	9.2632	9.3887	9.2205	8.2353	9.2054
99.95		10.5966	10.6749	10.8560	10.6729	9.2789	10.5941
75	10	12.5489	12.5484	12.5434	12.5386	12.6675	12.5204
90		15.9871	15.9872	15.9889	15.9677	15.9073	15.9776
95		18.3070	18.3077	18.3175	18.2918	18.0225	18.3024
99		23.2093	23.2120	23.2532	23.2304	22.3463	23.2085
99.95		25.1882	25.1921	25.2527	25.2523	24.0452	25.1878
75	20	23.8277	23.8276	23.8249	23.8194	23.9397	23.8093
90		28.4120	28.4120	28.4129	28.3989	28.3245	28.4060
95		31.4104	31.4106	31.4159	31.4017	31.1249	31.4077
99		37.5662	37.5670	37.5895	37.5914	36.7340	37.5658
99.95		39.9968	40.0309	40.0641	40.0461	38.9035	39.9966
75	40	45.6160	45.6160	45.6146	45.6097	45.7225	45.6037
90		51.8050	51.8051	51.8055	51.7963	51.7119	51.8012
95		55.7585	55.7585	55.7613	55.7534	55.4726	55.7568
99		63.6907	63.6909	63.7029	63.7104	62.8830	63.6905
99.95		66.7659	66.7896	66.8072	66.8024	65.7119	66.7660
75	60	66.9814	66.9814	66.9805	66.9762	67.0853	66.9716
90		74.3970	74.3970	74.3973	74.3900	74.3013	74.3940
95		79.0819	79.0820	79.0838	79.0782	78.7960	79.0806
99		88.3794	88.3795	88.3877	88.3961	88.5834	88.3792
99.5		91.9517	91.9709	91.9829	91.9820	90.9164	91.9516
75	80	88.1303	88.1303	88.1295	88.1256	88.2325	88.1219
90		96.5782	96.5782	96.5784	96.5723	96.4809	96.5756
95		101.879	101.879	101.881	101.876	101.594	101.878
99		112.329	112.329	112.335	112.344	111.540	112.329
99.5		116.321	116.338	116.347	116.348	115.297	116.321
75	100	109.141	109.141	109.141	109.137	109.242	109.138
90		118.498	118.498	118.498	118.493	118.400	118.496
95		124.342	124.342	124.343	124.340	124.056	124.341
99		135.807	135.807	135.812	135.820	135.023	135.807
99.5		140.169	140.184	140.192	140.193	139.154	140.169

[8] E.A. Cornish and R.A. Fisher, “Moments and cumulants in the Specification of Distributions,” *Rev. Inter. Stat. Inst.*, vol. 5, no. 4, pp. 307–320, 1938.

[9] R.A. Fisher and E.A. Cornish, “The percentile points of distributions having known cumulants,” *Technometrics*, vol. 2, no. 2, pp. 209–225, 1960.

[10] G. Steinbrecher and W.T. Shaw, “Quantile mechanics,” *Euro. J. Appl. Math.*, vol. 19, no. 2, pp. 87-112, 2008.

[11] G.W. Hill and A.W. Davis, “Generalized asymptotic expansions of Cornish-Fisher type,” *Ann. Math. Stat.*, vol. 39, no. 4, pp. 1264–1273, 1968.

[12] M. Merrington, “Numerical approximations to the percentage points of the χ^2 distribution,” *Biometrika*, vol. 32, pp. 200-202, 1941.

[13] L.A. Aroian, “A new approximation to the level of significance of the chi-square distribution,” *Ann. Math. Stat.*, vol. 14, pp. 93-95, 1943.

[14] S.H. Abdel-Aty, “Approximate formulae for the percentage points and the probability integral of the non-central χ 2 distribution,” *Biometrika*, vol. 41, no. 3/4, pp. 538-540, 1954.

[15] N.C. Severo and M. Zelen, “Normal approximation to the Chi-square and non-central F probability functions,” *Biometrika*, vol. 47, pp. 411-416, 1960.

[16] M. Sankaran, “Approximations to the non-central chi-square distribution,” *Biometrika*, vol. 50, no. 1/2, pp. 199-204, 1963.

[17] H.L. Harter, “A new Table of percentage points of the chi-square distribution,” *Biometrika*, vol. 51, pp. 231-239, 1964.

[18] R.B. Goldstein, “Algorithm 451: Chi-square Quantiles,” *Comm. ACM.* Vol. 16, pp. 483-485, 1973.

[19] D.J. Best and O.E. Roberts, “Algorithm AS 91: The percentage points of the χ2 distribution,” *Appl. Stat.*, vol. 24, pp. 385-388, 1975.

[20] M.R. Heyworth, “Approximation to chi-square,” *Amer. Statist.*, vol. 30, pp. 204, 1976.

[21] M.R. Chernick and V.K. Murthy, “Chi-square percentiles: old and new approximations with applications to sample size determination,” *Amer. J. Math. Magt. Sci.*, vol. 3, no. 2, pp. 145-161, 1983.

[22] J.T. Lin, “Approximating the cumulative chi-square distribution and its inverse,” *J. Roy. Stat. Soc. Ser. D.*, vol. 37, no. 1, pp. 3-5, 1988.

[23] J.T. Lin, “New Approximations for the percentage points of the Chi-square distribution,” *Prob. Engine. Inform. Sci.*, vol. 8, no. 1, pp. 135-146, 1994.

[24] H. Goldberg and H. Levine, H. (1946). Approximate formulas for the percentage points and normalization of t and χ^2. *Ann. Math. Stat.*, 17(2), 216-225, 1943.
[25] R.A. Fisher, Statistical methods for research workers, Oliver and Boyd, Edinburgh, 1925.
[26] E.B. Wilson and M.M. Hilferly, "The distribution of chi-square" *Proc. Nat. Acad. Sci.*, vol. 17, pp. 684-688, 1931.
[27] A.M. Peiser, "Asymptotic formulae for significance levels of certain distributions," *Ann. Math. Stat.*, vol. 14, pp. 56-62, 1943.
[28] C.M. Thompson, "Table of percentage points of the χ2 distribution," *Biometrika,* vol. 32, pp. 188-189, 1941.
[29] D.C. Hoaglin, "Direct approximation for chi-square percentage points," *J. Amer. Stat. Assoc.*, vol. 72, pp. 508-515, 1977.
[30] J.H. Zar, "Approximations for the percentage points of the chi-square distribution," *J. Roy. Stat. Stat. Soc. Ser. C.*, vol. 27, no. 3, pp. 280-290, 1978.
[31] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Vol 1, Wiley, ISBN: 978-0-471-58495-7, 1994.
[32] N.L. Johnson, S. Kotz and N. Balakrishnan, Continuous Univariate Distributions, Vol 2, Wiley, ISBN: 978-0-471-58494-0, 1995.
[33] C. Ittrich, D. Krause and W.D. Richter, "Probabilities and large quantiles of non-central chi-square distribution," *Statistics*, vol. 34, pp. 53-101, 2000.
[34] R.M. Kozelka, "Approximate upper percentage points for extreme values in multinomial sampling". *Ann. Math. Stat.*, vol. 27, no. 2, pp. 507-512, 1956.