Classes of Ordinary Differential Equations Obtained for the Probability Functions of Burr XII and Pareto Distributions

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of Burr XII and Pareto distributions. This was made easier since later distribution is a special case of the former. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distributions. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distributions.**

***Index Terms*— Burr XII distribution, differential calculus, probability density function, survival function, quantile function, Pareto distribution.**

I. Introduction

THE 2-parameter Burr XII distribution was considered in this research. It is a continuous distribution proposed by Burr [1] but was popularized through the work of [2], where they applied the distribution to model income distributions. Tadikamalla [3] reviewed the distribution and suggested possible relationships with other distributions while a detailed guide on the application was given by [4]. Al-Hussaini [5] proposed the detailed nature of the order statistics. Other aspects of the distribution available includes: Bayesian estimation [6],[7[,[8], parameter estimation in the presence of outliers [9], expected Fisher information [10], Loss function [11], maximum likelihood in the presence of censored samples [12], estimation of parameters using order statistics [13], estimation of parameters under progressive type II censoring [14], application of neural network in the estimation of the parameters [15], minimax estimation of the parameters [16], entropy based parameter estimation [17], explicit closed form for the characteristic function [18], reliability analysis under random censoring [19], estimation with middle censored samples [20], optimal b-robust estimator [21].

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Other variants, sub models, generalizations of the model have been studied by researchers such as: log Burr XII regression model [22], three parameter [23], beta Burr XII distribution [24], Kumaraswamy Burr XII distribution [25].

The Pareto distribution is a special case of Burr XII was also considered. Pareto distribution is hierarchal, skewed, heavy tailed distribution and characterized by scale and shape parameter. The distribution was famously used in the modeling of distribution of wealth. Recent applications include: modeling loss payment data [26], neurophysiology [27], volatility cluster analysis [28], network management [29], transportation [30], wage distribution [31[and modeling flood frequency [32].

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Burr XII and Pareto distributions by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [33], beta distribution [34], raised cosine distribution [35], Lomax distribution [36], beta prime distribution or inverted beta distribution [37].

II. Probability Density Function

The probability density function of the Burr XII distribution is given by;

$$f(x)=\frac{ckx^{c-1}}{(1+x^{c})^{k+1}} \tag{1}$$

When c = 1, the distribution reduces to the Pareto Distribution.

The probability density function can also be written as

$$f(x)=ckx^{c-1}(1+x^{c})^{-(k+1)} \tag{2}$$

Differentiate equation (2) to obtain;

$$f'(x)=ck\left\{\begin{matrix}-(k+1)c(x^{c-1})^{2}(1+x^{c})^{-(k+2)}\\ +(c-1)x^{c-2}(1+x^{c})^{-(k+1)}\end{matrix}\right\} \tag{3}$$

Simplify to obtain;

$$f'(x) = ck\left\{-\frac{(k+1)c(x^{c-1})^2}{(1+x^c)^{(k+2)}} + \frac{(c-1)x^{c-2}}{(1+x^c)^{(k+1)}}\right\} \quad (4)$$

The condition necessary for the existence of the equation is $c, k, x > 0$

When c = 1, equation (4) becomes;

$$f'(x) = -\frac{k(k+1)}{(1+x)^{(k+2)}} \quad (5)$$

The first order ordinary differential equation for the probability density function of the Pareto distribution is given as;

$$(1+x)^{(k+2)} f'(x) + k(k+1) = 0 \quad (6)$$

$$f(1) = \frac{k}{2^{k+1}} \quad (7)$$

When k = 1, 2, n; equation (6) becomes

$$(1+x)^3 f'(x) + 2 = 0 \quad (8)$$

$$(1+x)^4 f'(x) + 6 = 0 \quad (9)$$

$$(1+x)^{(n+2)} f'(x) + n(n+1) = 0 \quad (10)$$

Equation (4) can be simplified further to obtain;

$$f'(x) = f(x)\left\{-\frac{(k+1)cx^{c-1}}{(1+x^c)} + \frac{(c-1)}{x}\right\} \quad (11)$$

Let $$A(x) = \frac{(k+1)cx^{c-1}}{(1+x^c)} + \frac{(c-1)}{x} \quad (12)$$

The first order ordinary differential for the probability density function of the Burr XII distribution is given as;

$$f'(x) - A(x) f(x) = 0 \quad (13)$$

$$f(1) = \frac{ck}{2^{k+1}} \quad (14)$$

III. Quantile Function

The quantile function of the Burr XII distribution is derived from the cumulative distribution function given as:

$$F(x) = 1 - (1+x^c)^{-k} \quad (15)$$

$$Q(p) = \left[\left(\frac{1}{1-p}\right)^{\frac{1}{k}} - 1\right]^{\frac{1}{c}} \quad (16)$$

$$Q(p) = \left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}} \quad (17)$$

Differentiate equation (17), to obtain;

$$Q'(p) = \frac{1}{ck}\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-1\right)} (1-p)^{-\left(\frac{1}{k}+1\right)} \quad (18)$$

$$Q'(p) = \frac{1}{ck} \frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]} \frac{(1-p)^{-\frac{1}{k}}}{(1-p)} \quad (19)$$

The condition necessary for the existence of the equation is $c, k > 0, 0 \le p < 1$.

When c = 1, equation (19) becomes;

$$Q'(p) = \frac{1}{k} \frac{(1-p)^{-\frac{1}{k}}}{(1-p)} \quad (20)$$

The first order ordinary differential equation for the quantile function of the Pareto distribution is given as;

$$k(1-p)^{\frac{k+1}{k}} Q'(p) - 1 = 0 \quad (21)$$

$$Q(0) = 0 \quad (22)$$

Simplify equation (19) using equation (16), equation (16) becomes;

$$Q^c(p) = \left(\frac{1}{1-p}\right)^{\frac{1}{k}} - 1 \quad (23)$$

$$Q^c(p) + 1 = (1-p)^{-\frac{1}{k}} \quad (24)$$

Substitute equations (17), (23) and (24) into equation (19), to obtain;

$$Q'(p) = \frac{1}{ck}\left(\frac{Q(p)}{Q^c(p)}\right)\left(\frac{Q^c(p)+1}{1-p}\right) \quad (25)$$

$$ck(1-p)Q^c(p)Q'(p) = Q(p)(Q^c(p)+1) \quad (26)$$

The first order ordinary differential for the quantile function of the Burr XII distribution is given as;

$$ck(1-p)Q^c(p)Q'(p) - Q(p)(Q^c(p)+1) = 0 \quad (27)$$

$$Q(0) = 0 \quad (28)$$

To obtain the second order differential equation, differentiate equation (18) to obtain;

$$Q''(p) = \frac{1}{ck}\left\{\begin{array}{l} \left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-1\right)} \left(\frac{k+1}{k}\right)(1-p)^{-\left(\frac{1}{k}+2\right)} \\ +(1-p)^{-\left(\frac{1}{k}+1\right)} \left(\frac{1-c}{ck}\right)\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\left(\frac{1}{c}-2\right)} \\ (1-p)^{-\left(\frac{1}{k}+1\right)} \end{array}\right\} \quad (29)$$

$$Q''(p) = \frac{1}{ck}\left\{\begin{array}{l} \left(\frac{k+1}{k}\right) \frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]} \frac{(1-p)^{-\frac{1}{k}}}{(1-p)^2} \\ +\left(\frac{1-c}{ck}\right) \frac{\left[(1-p)^{-\frac{1}{k}} - 1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}} - 1\right]^2} \frac{(1-p)^{-\frac{2}{k}}}{(1-p)^2} \end{array}\right\} \quad (30)$$

The condition necessary for the existence of the equation is $c,k>0, 0\le p<1$.

When c = 1, equation (30) becomes;

$$Q''(p)=\frac{1}{k}\left\{\left(\frac{k+1}{k}\right)\frac{(1-p)^{-\frac{1}{k}}}{(1-p)^{2}}\right\} \tag{31}$$

Equation (31) is simplified to obtain the second order differential equation for the quantile function of the Pareto distribution, given as;

$$k^2(1-p)^{\frac{2k+1}{k}}Q''(p)-(k+1)=0 \tag{32}$$

$$Q'(0)=\frac{1}{k} \tag{33}$$

Simplifying equation (30) to obtain;

$$Q''(p)=\frac{1}{ck}\frac{\left[(1-p)^{-\frac{1}{k}}-1\right]^{\frac{1}{c}}}{\left[(1-p)^{-\frac{1}{k}}-1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)}$$

$$\left\{\left(\frac{k+1}{k}\right)\left(\frac{1}{1-p}\right)+\left(\frac{1-c}{ck}\right)\frac{1}{\left[(1-p)^{-\frac{1}{k}}-1\right]}\frac{(1-p)^{-\frac{1}{k}}}{(1-p)}\right\} \tag{34}$$

The condition necessary for the existence of the equation is $c,k,0\le p<1$

Simplify using equations (16) and (19);

$$Q''(p)=Q'(p)\left\{\left(\frac{k+1}{k}\right)\left(\frac{1}{1-p}\right)+\frac{(1-c)Q'(p)}{Q(p)}\right\} \tag{35}$$

$$\begin{aligned}&k(1-p)Q(p)Q''(p)=(k+1)Q(p)Q'(p)\\&+k(1-c)(1-p)Q'^2(p)\end{aligned} \tag{36}$$

The second order ordinary differential for the quantile function of the Burr XII distribution is given as;

$$\begin{aligned}&k(1-p)Q(p)Q''(p)-(k+1)Q(p)Q'(p)\\&-k(1-c)(1-p)Q'^2(p)=0\end{aligned} \tag{37}$$

$$Q'(0)=0 \tag{38}$$

IV. SURVIVAL FUNCTION

The survival function of the Burr XII distribution is given as:

$$S(t)=(1+t^c)^{-k} \tag{39}$$

Differentiate equation (39), to obtain;

$$S'(t)=-ckt^{c-1}(1+t^c)^{-(k+1)} \tag{40}$$

Equation (40) can also be written as;

$$S'(t)=-ck\frac{t^c}{t}\frac{(1+t^c)^{-k}}{(1+t^c)} \tag{41}$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (41) becomes;

$$S'(t)=-k\frac{(1+t)^{-k}}{(1+t)} \tag{42}$$

$$(1+t)^{k+1}S'(t)=-k \tag{43}$$

The first order ordinary differential equation for the survival function of the Pareto distribution is given as;

$$(1+t)^{k+1}S'(t)+k=0 \tag{44}$$

$$S(1)=\frac{1}{2^k} \tag{45}$$

When k = 1, 2, n, equation (44) become,

$$(1+t)^2S'(t)+1=0 \tag{46}$$

$$(1+t)^3S'(t)+2=0 \tag{47}$$

$$(1+t)^{n+1}S'(t)+n=0 \tag{48}$$

Simplify equation (41) using equation (39), to obtain;

$$S'(t)=-ck\frac{t^c}{t}\frac{S(t)}{(1+t^c)} \tag{49}$$

$$S'(t)=-ckB(t)S(t) \tag{50}$$

Where $B(t)=\dfrac{ckt^c}{t(1+t^c)}$ (51)

The first order ordinary differential equation for the survival function of the Burr XII distribution is given as;

$$S'(t)+ckB(t)S(t)=0 \tag{52}$$

$$S(1)=\frac{1}{2^k} \tag{53}$$

To obtain the second order differential equation, differentiate equation (40) to obtain;

$$S''(t)=-ck\left\{\begin{aligned}&-(k+1)c(t^{c-1})^2(1+t^c)^{-(k+2)}\\&+(c-1)t^{c-2}(1+t^c)^{-(k+1)}\end{aligned}\right\} \tag{54}$$

$$S''(t)=-\left\{\begin{aligned}&-c(k+1)ck\left(\frac{t^c}{t}\right)^2\frac{(1+t^c)^{-k}}{(1+t^c)^2}\\&+(c-1)ck\frac{t^c}{t^2}\frac{(1+t^c)^{-k}}{(1+t^c)}\end{aligned}\right\} \tag{55}$$

The condition necessary for the existence of the equation is $c,k,t>0$.

When c = 1, equation (55) becomes;

$$S''(t)=(k+1)k\frac{(1+t)^{-k}}{(1+t)^2} \tag{56}$$

The second order ordinary differential equation for the survival function of the Pareto distribution is given as;

$$(1+t)^{k+2}S'(t)-k(k+1)=0 \tag{57}$$

$$S'(1)=-\frac{k}{2^{k+1}} \tag{58}$$

When k = 1, 2, n, equation (57) become,

$$(1+t)^3S'(t)-2=0 \tag{59}$$

$$(1+t)^4S'(t)-6=0 \tag{60}$$

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$$(1+t)^{n+2}S'(t)-n(n+1)=0 \quad (61)$$

Simplify using equation (55), to obtain;

$$S''(t)=-\left\{\frac{c(k+1)t^{c}S'(t)}{t(1+t^{c})}-\frac{(c-1)S'(t)}{t}\right\} \quad (62)$$

$$S''(t)=-\left\{\frac{c(k+1)t^{c}}{t(1+t^{c})}-\frac{(c-1)}{t}\right\}S'(t)=-D(t)S'(t) \quad (63)$$

Where $D(t)=\frac{c(k+1)t^{c}}{t(1+t^{c})}-\frac{(c-1)}{t}$ (64)

The second order ordinary differential equation for the survival function of the Burr XII distribution is given as;

$$S''(t)+D(t)S'(t)=0 \quad (65)$$

$$S(1)=\frac{1}{2^{k}} \quad (66)$$

$$S'(1)=-\frac{ck}{2^{k+1}} \quad (67)$$

V. Inverse Survival Function

The inverse survival function of the Burr XII distribution is given as;

$$Q(p)=\left[p^{-\frac{1}{k}}-1\right]^{\frac{1}{c}} \quad (68)$$

Differentiate equation (68), o obtain;

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}}{ck}\left[p^{-\frac{1}{k}}-1\right]^{\left(\frac{1}{c}-1\right)} \quad (69)$$

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}(p^{-\frac{1}{k}}-1)^{\frac{1}{c}}}{ck(p^{-\frac{1}{k}}-1)} \quad (70)$$

The condition necessary for the existence of the equation is $c,k,0\le p<1$.

Substitute equation (68) into equation (70) to obtain;

$$Q'(p)=-\frac{p^{-\left(\frac{1}{k}+1\right)}Q(p)}{ck(p^{-\frac{1}{k}}-1)} \quad (71)$$

$$Q'(p)=-\frac{Q(p)}{ckp^{\left(\frac{1}{k}+1\right)}(p^{-\frac{1}{k}}-1)} \quad (72)$$

$$Q'(p)=-\frac{Q(p)}{ckp(1-p^{\frac{1}{k}})} \quad (73)$$

Equation (68) can be further simplify as;

$$Q^{c}(p)=p^{-\frac{1}{k}}-1 \quad (74)$$

$$p^{-\frac{1}{k}}=Q^{c}(p)+1 \quad (75)$$

$$p^{\frac{1}{k}}=(Q^{c}(p)+1)^{-1} \quad (76)$$

Substitute equation(76) into equation (73);

$$Q'(p)=-\frac{Q(p)}{ckp\left(1-\frac{1}{Q^{c}(p)+1}\right)}=-\frac{Q(p)(Q^{c}(p)+1)}{ckpQ^{c}(p)} \quad (77)$$

$$Q'(p)=-\frac{Q^{1-c}(p)(Q^{c}(p)+1)}{ckp} \quad (78)$$

$$Q'(p)=-\left(\frac{Q(p)+Q^{1-c}(p)}{ckp}\right) \quad (79)$$

The first order ordinary differential equation for the inverse survival function of the Burr XII distribution is given as;

$$ckpQ'(p)+Q(p)+Q^{1-c}(p)=0 \quad (80)$$

$$Q(0)=0 \quad (81)$$

When c = 1, equation (80) becomes;

$$kpQ'(p)+Q(p)+1=0 \quad (82)$$

VI. Hazard Function

The hazard function of the Burr XII distribution is given as:

$$h(t)=\frac{ckt^{c-1}}{1+t^{c}} \quad (83)$$

Differentiate equation (83) to obtain;

$$h'(t)=ck[-c(t^{c-1})^{2}(1+t^{c})^{-2}+(c-1)t^{c-2}(1+t^{c})^{-1}] \quad (84)$$

$$h'(t)=ck\left[-\frac{c(t^{c-1})^{2}}{(1+t^{c})^{2}}+\frac{(c-1)t^{c-2}}{(1+t^{c})}\right] \quad (85)$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (85) becomes;

$$h'(t)=k\left[-\frac{1}{(1+t)^{2}}\right] \quad (86)$$

The first order ordinary differential equation for the hazard function of the Pareto distribution is given as;

$$(1+t)^{2}h'(t)+k=0 \quad (87)$$

$$h(1)=\frac{k}{2} \quad (88)$$

Simplify equation (85) to obtain;

$$h'(t)=\left[-\frac{ct^{c-1}}{(1+t^{c})}+\frac{(c-1)}{t}\right]h(t) \quad (89)$$

$$h'(t)=\left[-\frac{h(t)}{k}+\frac{(c-1)}{t}\right]h(t) \quad (90)$$

$$kth'(t)=-th^{2}(t)+(c-1)kh(t) \quad (91)$$

The first order ordinary differential equation for the hazard function of the Burr XII distribution is given as;

$$kth'(t)+th^{2}(t)-(c-1)kh(t)=0 \quad (92)$$

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$$h(1)=\frac{ck}{2} \quad (93)$$

To obtain the second order differential equation, differentiate equation (84) to obtain;

$$h''(t)=ck\left\{\begin{matrix} 2c(t^{c-1})^3(1+t^c)^{-3}-2c(c-1)(t^{c-1})(t^{c-2}) \\ (1+t^c)^{-2}-c(c-1)(t^{c-1})(t^{c-2})(1+t^c)^{-2} \\ +(c-1)(c-2)t^{c-3}(1+t^c)^{-1} \end{matrix}\right\} \quad (94)$$

$$h''(t)=ck\left\{\begin{matrix} \frac{2c(t^{c-1})^3}{(1+t^c)^3}-\frac{2c(c-1)(t^{c-1})(t^{c-2})}{(1+t^c)^2} \\ -\frac{c(c-1)(t^{c-1})(t^{c-2})}{(1+t^c)^2}+\frac{(c-1)(c-2)t^{c-3}}{(1+t^c)} \end{matrix}\right\} \quad (95)$$

The condition necessary for the existence of the equation is $c,k,t>0$

When c = 1, equation (95) becomes;

$$h''(t)=k\left\{\frac{2}{(1+t)^3}\right\} \quad (96)$$

The second order ordinary differential equation for the hazard function of the Pareto distribution is given as;

$$(1+t)^3h''(t)-2k=0$$

(97) $h'(1)=-\frac{k}{2^2}$

(98) Simplify equation (95) to obtain;

$$h''(t)=\frac{ckt^{c-1}}{1+t^c}\left\{\begin{matrix} \frac{2c(t^{c-1})^2}{(1+t^c)^2}-\frac{2c(c-1)(t^{c-2})}{(1+t^c)} \\ -\frac{c(c-1)(t^{c-2})}{(1+t^c)}+\frac{(c-1)(c-2)}{t^2} \end{matrix}\right\} \quad (99)$$

$$h''(t)=h(t)\left\{\begin{matrix} -2c\left[-\frac{c(t^{c-1})^2}{(1+t^c)^2}-\frac{(c-1)(t^{c-2})}{(1+t^c)}\right] \\ -\frac{c(c-1)(t^{c-2})}{(1+t^c)}+\frac{(c-1)(c-2)}{t^2} \end{matrix}\right\} \quad (100)$$

$$h''(t)=h(t)\left\{-2c\left(\frac{h'(t)}{ck}\right)-(c-1)\left[\frac{c(t^{c-2})}{(1+t^c)}+\frac{(c-2)}{t^2}\right]\right\} \quad (101)$$

$$h''(t)=-h(t)\left\{\frac{2h'(t)}{k}+(c-1)\left[\frac{h(t)}{kt}+\frac{(c-2)}{t^2}\right]\right\} \quad (102)$$

The second order ordinary differential equation for the hazard function of the Burr XII distribution is given as;

$$kt^2h''(t)+h(t)\left[2t^2h'(t)+(c-1)(th(t)+k(c-2)\right]=0$$

(103)

$$h'(1)=ck\left[\frac{c-1}{2}-\frac{c}{4}\right] \quad (104)$$

VII. Reversed Hazard Function

The reversed hazard function of the Burr XII distribution is given as:

$$j(t)=\frac{ckt^{c-1}}{(1+t^c)[(1+t^c)^k-1]} \quad (105)$$

Differentiate equation (105) to obtain;

$$j'(t)=\left\{\begin{matrix} \frac{(c-1)t^{c-2}}{t^{c-1}}-\frac{ct^{c-1}(1+t^c)^{-2}}{(1+t^c)^{-1}} \\ -\frac{ckt^{c-1}(1+t^c)^{k-1}[(1+t^c)^k-1]^{-2}}{[(1+t^c)^k-1]^{-1}} \end{matrix}\right\}j(t) \quad (106)$$

The condition necessary for the existence of the equation is $c,k,t>0$.

$$j'(t)=\left\{\frac{(c-1)}{t}-\frac{ct^{c-1}}{(1+t^c)}-\frac{ckt^{c-1}(1+t^c)^k}{(1+t^c)((1+t^c)^k-1)}\right\}j(t) \quad (107)$$

$$j'(t)=\left\{\frac{(c-1)}{t}-\frac{ct^{c-1}}{(1+t^c)}-(1+t^c)^k j(t)\right\}j(t) \quad (108)$$

The ordinary differential equations can be obtained for particular values of the parameters.

When c = 1, equation (108) becomes;

$$j'(t)=-\left\{\frac{1}{1+t}+(1+t)^k j(t)\right\}j(t) \quad (109)$$

The first order ordinary differential equation for the reverse hazard function of the Pareto distribution is given as;

$$(1+t)j'(t)+(1+t)^{k+1}j^2(t)+j(t)=0$$

(110) $j(1)=\frac{k}{2(2^k-1)}$

(111)

The ODEs can be obtained for the particular values of the distribution which will require further classifications and analysis. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [38-49]. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

VIII. Concluding Remarks

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of Burr XII and Pareto distributions. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs.

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