Classes of Ordinary Differential Equations Obtained for the Probability Functions of Exponentiated Generalized Exponential Distribution

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***Abstract*— In this paper, the differential calculus was used to obtain some classes of ordinary differential equations (ODE) for the probability density function, quantile function, survival function, inverse survival function, hazard function and reversed hazard function of the exponentiated generalized exponential distribution. The stated necessary conditions required for the existence of the ODEs are consistent with the various parameters that defined the distribution. Solutions of these ODEs by using numerous available methods are new ways of understanding the nature of the probability functions that characterize the distribution. The method can be extended to other probability distributions and can serve an alternative to approximation.**

***Index Terms*— Exponentiated, exponential distribution, reversed hazard function, calculus, differentiation.**

I. Introduction

THIS distribution was proposed by Oguntunde et al. [1] as a three parameter model that can be used as one of the generalizations of the exponential distribution. The proposed model has also the generalized exponential and exponentiated exponential distribution as its submodels. The distribution belongs to the exponentiated class of distributions which has seen a lot of research activities. Details on the general class of exponentiated distributions can be seen in [2-6].

In particular, some exponentiated distributions are available in scientific literature such as: exponentiated Gumbel type-2 distribution [7], exponentiated Weibull distribution [8-10], exponentiated generalized inverted exponential distribution [11], exponentiated generalized inverse Gaussian distribution [12], exponentiated generalized inverse Weibull distribution [13-14], gamma-exponentiated exponential distribution [15], exponentiated Gompertz distribution [16-17], beta Exponentiated Mukherjii-Islam Distribution [18], transmuted exponentiated Pareto-i distribution [19], gamma exponentiated exponential–Weibull distribution [20], exponentiated gamma distribution [21], exponentiated Gumbel distribution [22], exponentiated uniform distribution [23] and beta exponentiated Weibull distribution [24-25]. Others are: exponentiated log-logistic distribution [26], McDonald exponentiated gamma distribution [27], exponentiated Generalized Weibull Distribution [28], beta exponentiated gamma distribution [29], exponentiated gamma distribution [30], exponentiated Pareto distribution [31], exponentiated Kumaraswamy distribution [32], exponentiated modified Weibull extension distribution [33], exponentiated Weibull-Pareto distribution [34], exponentiated lognormal distribution [35], exponentiated Perks distribution [36] and Kumaraswamy-transmuted exponentiated modified Weibull distribution [37]. Also available are: exponentiated power Lindley–Poisson distribution [38], exponentiated Chen distribution [39], exponentiated reduced Kies distribution [40], exponentiated inverse Weibull geometric distribution [41], exponentiated geometric distribution [42-43], exponentiated Weibull geometric distribution [44], exponentiated transmuted Weibull geometric distribution [45], exponentiated half logistic distribution [46], transmuted exponentiated Gumbel distribution [47], exponentiated Kumaraswamy-power function distribution [48], exponentiated-log-logistic geometric distribution [49], bivariate exponentiated generalized Weibull-Gompertz distribution [50] and so on.

The aim of this research is to develop ordinary differential equations (ODE) for the probability density function (PDF), Quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of exponentiated generalized exponential distribution by the use of differential calculus. Calculus is a very key tool in the determination of mode of a given probability distribution and in estimation of parameters of probability distributions, amongst other uses. The research is an extension of the ODE to other probability functions other than the PDF. Similar works done where the PDF of probability distributions was expressed as ODE whose solution is the PDF are available. They include: Laplace distribution [51], beta distribution [52], raised cosine distribution [53], Lomax

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distribution [54], beta prime distribution or inverted beta distribution [55].

II. Probability Density Function

The probability density function of the exponentiated generalized exponential distribution is given as;

$$f(x)=\alpha\beta\,\mathrm{e}^{-\alpha\lambda x}[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-1}$$

(1) To obtain the first order ordinary differential equation for the probability density function of the exponentiated generalized exponential distribution, differentiate equation (1), to obtain;

$$f'(x)=\left\{\begin{array}{l}\dfrac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-2}}{[(1-\mathrm{e}^{-\lambda x})^{\alpha}]^{\beta-1}}\\ -\dfrac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda x}}{\mathrm{e}^{-\alpha\lambda x}}\end{array}\right\}f(x) \qquad (2)$$

The condition necessary for the existence of the equation is $x,\alpha,\beta,\lambda>0.$

$$f'(x)=\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}(1-\mathrm{e}^{-\lambda x})^{\alpha-1}}{(1-\mathrm{e}^{-\lambda x})^{\alpha}}\right\}f(x) \qquad (3)$$

$$f'(x)=\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}\right\}f(x) \qquad (4)$$

Differentiate equation (4) to obtain;

$$f''(x)=\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}\right\}f'(x) -\left\{\frac{\alpha(\beta-1)\lambda^2(\mathrm{e}^{-\lambda x})^2}{(1-\mathrm{e}^{-\lambda x})^2}+\frac{\alpha(\beta-1)\lambda^2\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}\right\}f(x) \qquad (5)$$

The condition necessary for the existence of the equation is $x,\alpha,\beta,\lambda>0.$

The following equations obtained from equation (4) are needed to simplify equation (5);

$$-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}=\frac{f'(x)}{f(x)} \qquad (6)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}=\frac{f'(x)}{f(x)}+\alpha\lambda \qquad (7)$$

$$\left(\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}\right)^2=\left(\frac{f'(x)}{f(x)}+\alpha\lambda\right)^2 \qquad (8)$$

$$\frac{\alpha(\beta-1)(\lambda\,\mathrm{e}^{-\lambda x})^2}{(1-\mathrm{e}^{-\lambda x})^2}=\frac{1}{\alpha(\beta-1)}\left(\frac{f'(x)}{f(x)}+\alpha\lambda\right)^2 \qquad (9)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda x}}{(1-\mathrm{e}^{-\lambda x})}=\lambda\left(\frac{f'(x)}{f(x)}+\alpha\lambda\right) \qquad (10)$$

Substitute equations (6), (9) and (10) into equation (5);

$$f''(x)=\frac{f'^2(x)}{f(x)}-\left\{\begin{array}{l}\dfrac{1}{\alpha(\beta-1)}\left(\dfrac{f'(x)}{f(x)}+\alpha\lambda\right)^2\\ +\lambda\left(\dfrac{f'(x)}{f(x)}+\alpha\lambda\right)\end{array}\right\}f(x) \qquad (11)$$

The condition necessary for the existence of the equation is $x,\alpha,\lambda>0,\beta>1.$

The ordinary differential equations can be obtained for the particular values of the parameters.

III. Quantile Function

The Quantile function of the exponentiated generalized exponential distribution is given as;

$$Q(p)=\frac{1}{\alpha\lambda}\ln\left(\frac{1}{1-p^{\frac{1}{\beta}}}\right) \qquad (12)$$

To obtain the first order ordinary differential equation for the Quantile function of the exponentiated generalized exponential distribution, differentiate equation (12), to obtain;

$$Q'(p)=\frac{p^{\frac{1}{\beta}-1}}{\alpha\beta\lambda(1-p^{\frac{1}{\beta}})} \qquad (13)$$

The condition necessary for the existence of the equation is $\alpha,\beta,\lambda>0, 0<p<1.$

$$\alpha\beta\lambda(1-p^{\frac{1}{\beta}})Q'(p)-p^{\frac{1}{\beta}-1}=0 \qquad (14)$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered in shown in **Table 1.**

Table 1: Classes of differential equations obtained for the quantile function of exponentiated generalized exponential distribution for different parameters.

α	β	λ	Ordinary differential equation
1	1	1	$(1-p)Q'(p)-1=0$
1	1	2	$2(1-p)Q'(p)-1=0$
1	2	1	$2(1-p)Q'(p)-1=0$
1	2	2	$4(1-p)Q'(p)-1=0$
2	1	1	$2(\sqrt{p}-p)Q'(p)-1=0$
2	1	2	$4(\sqrt{p}-p)Q'(p)-1=0$
2	2	1	$4(\sqrt{p}-p)Q'(p)-1=0$
2	2	2	$8(\sqrt{p}-p)Q'(p)-1=0$

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IV. SURVIVAL FUNCTION

The survival function of the exponentiated generalized exponential distribution is given as;

$$S(t)=1-[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta} \quad (15)$$

To obtain the first order ordinary differential equation for the survival function of the exponentiated generalized exponential distribution, differentiate equation (15), to obtain;

$$S'(t)=-\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda x}[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta-1} \quad (16)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0.$

Equation (16) can be written as;

$$[1-\mathrm{e}^{-\alpha\lambda x}]^{\beta}=1-S(t) \quad (17)$$

Substitute equation (17) into equation (16) to obtain;

$$S'(t)=-\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda x}(1-S(t))}{[1-\mathrm{e}^{-\alpha\lambda x}]}$$

(18) From equation (17),

$$1-\mathrm{e}^{-\alpha\lambda x}=(1-S(t))^{\frac{1}{\beta}} \quad (19)$$

$$\mathrm{e}^{-\alpha\lambda x}=1-(1-S(t))^{\frac{1}{\beta}} \quad (20)$$

Substitute equations (19) and (20) into equation (18);

$$S'(t)=-\frac{\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))}{(1-S(t))^{\frac{1}{\beta}}} \quad (21)$$

$$S'(t)=-\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))^{1-\frac{1}{\beta}} \quad (22)$$

$$S'(t)+\alpha\beta\lambda(1-(1-S(t))^{\frac{1}{\beta}})(1-S(t))^{1-\frac{1}{\beta}}=0 \quad (23)$$

$$S(1)=1-[1-\mathrm{e}^{-\alpha\lambda}]^{\beta} \quad (24)$$

The ordinary differential equations can be obtained for the different values of the parameters.

When $\beta=1,$ equations (23) and (24) become;

$$S'(t)+\alpha\lambda S(t)=0 \quad (25)$$

$$S(1)=\mathrm{e}^{-\alpha\lambda} \quad (26)$$

When $\beta=2,$ equations (23) and (24) become;

$$(\sqrt{1-S(t)})S'(t)+2\alpha\lambda(1-\sqrt{1-S(t)})=0 \quad (27)$$

$$S(1)=1-[1-\mathrm{e}^{-\alpha\lambda}]^{2} \quad (28)$$

V. INVERSE SURVIVAL FUNCTION

The inverse survival function of the exponentiated generalized exponential distribution is given as;

$$Q(p)=\frac{1}{\alpha\lambda}\ln\left(\frac{1}{1-(1-p)^{\frac{1}{\beta}}}\right)$$

(29) To obtain the first order ordinary differential equation for the inverse survival function of the exponentiated generalized exponential distribution, differentiate equation (29), to obtain;

$$Q'(p)=-\frac{p^{\frac{1}{\beta}-1}}{\alpha\beta\lambda(1-(1-p)^{\frac{1}{\beta}})} \quad (30)$$

The condition necessary for the existence of the equation is $\alpha,\beta,\lambda>0, 0<p<1$.

$$\alpha\beta\lambda(1-(1-p)^{\frac{1}{\beta}})Q'(p)+p^{\frac{1}{\beta}-1}=0 \quad (31)$$

The ordinary differential equations can be obtained for given values of the parameters. Some cases are considered and shown in **Table 2**.

Table 2: Classes of differential equations obtained for the inverse survival function of exponentiated generalized exponential distribution for different parameters

α	β	λ	Ordinary differential equation
1	1	1	$pQ'(p)+1=0$
1	1	2	$2pQ'(p)+1=0$
1	2	1	$2pQ'(p)+1=0$
1	2	2	$4pQ'(p)+1=0$
2	1	1	$2\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	1	2	$4\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	2	1	$4\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$
2	2	2	$8\sqrt{p}(1-\sqrt{1-p})Q'(p)+1=0$

The complexity of the ODE increases as the value of the parameters changes.

VI. HAZARD FUNCTION

The hazard function of the exponentiated generalized exponential distribution is given as;

$$h(t)=\frac{\alpha\beta\,\mathrm{e}^{-\alpha\lambda t}[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-1}}{1-[1-\mathrm{e}^{-\alpha\lambda t}]^{\beta}} \quad (32)$$

To obtain the first order ordinary differential equation for the hazard function of the exponentiated generalized exponential distribution, differentiate equation (32), to obtain;

$$h'(t)=-\frac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda t}}{\mathrm{e}^{-\alpha\lambda t}}h(t)+$$
$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t})^{\alpha-1}[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-2}}{[(1-\mathrm{e}^{-\lambda t})^{\alpha}]^{\beta-1}}h(t) \quad (33)$$
$$+\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}(1-\mathrm{e}^{-\alpha\lambda t})^{\beta-1}[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]^{-2}}{[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]^{-1}}h(t)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0.$

$$h'(t)=\left\{\begin{aligned}&\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}(1-\mathrm{e}^{-\lambda t})^{\alpha-1}}{(1-\mathrm{e}^{-\lambda t})^{\alpha}}\\&+\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}(1-\mathrm{e}^{-\alpha\lambda t})^{\beta-1}}{[1-(1-\mathrm{e}^{-\alpha\lambda t})^{\beta}]}-\alpha\lambda\end{aligned}\right\}h(t) \quad (34)$$

$$h'(t)=\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)\right\}h(t) \quad (35)$$

Differentiate equation (35) to obtain;

$$h''(t)=\left\{\begin{aligned}&h'(t)-\frac{\alpha(\beta-1)\lambda^2(\mathrm{e}^{-\lambda t})^2}{(1-\mathrm{e}^{-\lambda t})^2}\\&+\frac{\alpha(\beta-1)\lambda^2\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}\end{aligned}\right\}h(t)+\left\{-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)\right\}h'(t) \quad (36)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0$.

The following equations obtained from equation (35) are needed to simplify equation (36);

$$-\alpha\lambda+\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}+h(t)=\frac{h'(t)}{h(t)} \quad (37)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}=\frac{h'(t)}{h(t)}+\alpha\lambda-h(t) \quad (38)$$

$$\left(\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}\right)^2=\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^2 \quad (39)$$

$$\frac{\alpha(\beta-1)(\lambda\,\mathrm{e}^{-\lambda t})^2}{(1-\mathrm{e}^{-\lambda t})^2}=\frac{1}{\alpha(\beta-1)}\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^2 \quad (40)$$

$$\frac{\alpha(\beta-1)\lambda\,\mathrm{e}^{-\lambda t}}{(1-\mathrm{e}^{-\lambda t})}=\lambda\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right) \quad (41)$$

Substitute equations (37), (40) and (41) into equation (36);

$$h''(t)=\frac{h'^2(t)}{h(t)}-\left\{\begin{aligned}&\frac{1}{\alpha(\beta-1)}\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)^2\\&+\lambda\left(\frac{h'(t)}{h(t)}+\alpha\lambda-h(t)\right)-h(t)\end{aligned}\right\}h(t) \quad (42)$$

The condition necessary for the existence of the equation is $t,\alpha,\lambda>0,\beta>1$.

The ordinary differential equations can be obtained for the particular values of the parameters.

VII. Reversed Hazard Function

The reversed hazard function of the exponentiated generalized exponential distribution is given as;

$$j(t)=\frac{\alpha\beta\lambda\,\mathrm{e}^{-\alpha\lambda t}}{[1-\mathrm{e}^{-\lambda t}]^{\alpha}}$$

(43) To obtain the first order ordinary differential equation for the reversed hazard function of the exponentiated generalized exponential distribution, differentiate equation (43), to obtain;

$$j'(t)=\left\{-\frac{\alpha\lambda\,\mathrm{e}^{-\alpha\lambda t}}{\mathrm{e}^{-\alpha\lambda t}}-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}[1-\mathrm{e}^{-\lambda t}]^{-(\alpha+1)}}{[1-\mathrm{e}^{-\lambda t}]^{-\alpha}}\right\}j(t) \quad (44)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0$.

$$j'(t)=-\left\{\alpha\lambda+\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j(t) \quad (45)$$

Differentiate equation (45);

$$j''(t)=\left\{-\alpha\lambda-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j'(t)+\left\{\frac{\alpha(\lambda\,\mathrm{e}^{-\lambda t})^2}{(1-\mathrm{e}^{-\lambda t})^2}+\frac{\alpha\lambda^2\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right\}j(t) \quad (46)$$

The condition necessary for the existence of the equation is $t,\alpha,\beta,\lambda>0$.

The following equations obtained from equation (45) are needed to simplify equation (46);

$$\frac{j'(t)}{j(t)}=-\alpha\lambda-\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}} \quad (47)$$

$$\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}=-\frac{j'(t)}{j(t)}-\alpha\lambda \quad (48)$$

$$\left(\frac{\alpha\lambda\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}\right)^2=\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^2 \quad (49)$$

$$\frac{\alpha(\lambda\,\mathrm{e}^{-\lambda t})^2}{(1-\mathrm{e}^{-\lambda t})^2}=\frac{1}{\alpha}\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^2 \quad (50)$$

$$\frac{\alpha\lambda^2\,\mathrm{e}^{-\lambda t}}{1-\mathrm{e}^{-\lambda t}}=-\lambda\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right) \quad (51)$$

Substitute equations (47), (50) and (51) into equation (46);

$$j''(t)=\frac{j'^2(t)}{j(t)}+\left\{\frac{1}{\alpha}\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)^2-\lambda\left(\frac{j'(t)}{j(t)}+\alpha\lambda\right)\right\}j(t) \quad (52)$$

The ordinary differential equations can be obtained for the particular values of the parameters.

The ODEs of all the probability functions considered can be obtained for the particular values of the distribution. Several analytic, semi-analytic and numerical methods can be applied to obtain the solutions of the respective differential equations [56-69], especially for the cases of the quantile and inverse survival functions. Also comparison with two or more solution methods is useful in understanding the link between ODEs and the probability distributions.

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VIII. Concluding Remarks

In this paper, differentiation was used to obtain some classes of ordinary differential equations for the probability density function (PDF), quantile function (QF), survival function (SF), inverse survival function (ISF), hazard function (HF) and reversed hazard function (RHF) of the exponentiated generalized exponential distribution. In all, the parameters that define the distribution determine the nature of the respective ODEs and the range determines the existence of the ODEs. Furthermore, the complexity of the ODEs depends greatly on the values of the parameters.

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