Application of Semi-Analytical Technique for Solving Thirteenth Order Boundary Value Problem

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***Abstract*—This work considers the numerical solution of thirteenth order boundary value problems using the modified Adomian decomposition method (MADM). Some examples are considered to illustrate the efficiency of the method. It is demonstrated that MADM converges more rapidly to the exact solution than the existing methods in literature and it reduces the computational involvement**

***Index Terms*— Boundary value problem, modified Adomian decomposition method, series solution.**

I. INTRODUCTION

Research has shown that boundary value problems arise in various fields of studies ranging from physical, biological and chemical processes. Proposing a numerical solution to various boundary value problems has posed a great challenge in the past years because most of the numerical methods are computationally intensive. Some proposed numerical methods which have been applied to solve some of these boundary value problems include: Spline method [1-2], Exp-Function method [3], Generalized Differential Quadrature rule (GDQR) [4], Variational Iteration Technique [5-7], Homotopy Perturbation Method [8-9], Finite-difference method [10-11], Differential Transform method [12-13], Rung-kutta method [14].

In 1980s, George Adomian (1923-1996) introduced a powerful method for solving linear and nonlinear differential equations. Since then, this method is known as the Adomian Decomposition Method (ADM) [15-17]. Later, Wazwaz [18-19] developed the modified form of the Adomian decomposition method. The modified technique provides a qualitative improvement over standard Adomian method, although it introduces a slight change in the formulation of Adomian recursive relation. The reason for this improvement rests on the fact that the technique accelerates the convergence of the solution and facilitates the formulation of Adomian polynomials.

In this work, the objective is the application of MADM for the solution of thirteenth order boundary value problems. Some examples are presented to illustrate the efficiency of the method and its rapid convergence to the exact solution.

Manuscript received July 17, 2017; revised July 30, 2017. This work was supported by Centre for Research and Innovation, Covenant University, Ota, Nigeria.

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II. ANALYSIS OF ADOMIAN DECOMPOSITION METHOD.

Consider the generalized differential equation of the form

$$Ly + Ry + Ny = g \tag{1}$$

L is the highest order derivative (L is invertible), R is a linear differential operator, Ny is the nonlinear term and g is the source term.

Applying L^{-1} on both sides of equation (1), we have

$$y = L^{-1}(Ry) - L^{-1}(Ny) + L^{-1}(g) \tag{2}$$

We can then write equation (2) as

$$y = h - L^{-1}(Ry) - L^{-1}(Ny) \tag{3}$$

Note that h represents integral of the source term, ($L^{-1}(g)$) and boundary conditions.

Using the standard Adomian decomposition method, we identify the zeroth component as

$$y_0 = h \tag{4}$$

and the recursive relation is

$$y_{n+1} = -L^{-1}(Ry_n) - L^{-1}(Ny_n), n \geq 0 \tag{5}$$

$$y_1 = -L^{-1}(Ry_0) - L^{-1}(Ny_0) \tag{6}$$

$$y_2 = -L^{-1}(Ry_1) - L^{-1}(Ny_1) \tag{7}$$

$$y_3 = -L^{-1}(Ry_2) - L^{-1}(Ny_2) \tag{8}$$

Then the solution will be of the form

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{9}$$

The modification by Wazwaz [18] splits the function h into two parts say h_0 and h_1,

$$h = h_0 + h_1 \tag{10}$$

We will then have the zeroth component as

$$y_0 = h_0 \tag{11}$$

and the rest terms written as

$$y_1 = h_1 - L^{-1}(Ry_0) - L^{-1}(Ny_0) \tag{12}$$

$$y_2 = -L^{-1}(Ry_1) - L^{-1}(Ny_1) \tag{13}$$

The above modification will reduce the size of computations involved in the method and thereby enhance the rapidity of its convergence. The nonlinear term Ny can be determined by an infinite series of Adomian polynomials.

$$Ny = \sum_{n=0}^{\infty} A_n \tag{14}$$

Where $An's$ are calculated by the relation

$$An = \frac{1}{n!}\frac{d^n}{dt^n}\left[N\left(\sum_{i=0}^{n} t^i y_i \right) \right]_{t=0}, n = 0,1,2\ldots \tag{15}$$

III. NUMERICAL EXAMPLES

We consider the following non-linear thirteenth order two-point boundary value problems

$$u^{13}(t) = e^{-t}u^2(t), \quad 0 \le t \le 1 \tag{16}$$

With the following boundary conditions

$$\begin{aligned}
&u(0)=1, && u(1)=e\\
&u'(0)=1, && u'(1)=e\\
&u''(0)=1, && u''(1)=e\\
&u'''(0)=1, && u'''(1)=e\\
&u^{iv}(0)=1, && u^{iv}(1)=e\\
&u^{v}(0)=1, && u^{v}(1)=e\\
&u^{vi}(0)=1
\end{aligned} \tag{17}$$

The exact solution of the boundary value problems is

$$u(t) = e^t \tag{18}$$

To solve the bvp by Adomian decomposition method, we express (16) in operator form

$$Lu = e^{-t}u^2 \tag{19}$$

We then apply L^{-1} to both sides of equation (19) and impose the boundary conditions (L^{-1} is a thirteenth-fold integral operator) to obtain

$$u(t) = 1 + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \frac{t^6}{6!} + \frac{At^7}{7!} + \frac{Bt^8}{8!} + \frac{Ct^9}{9!} + \frac{Dt^{10}}{10!} + \frac{Et^{11}}{11!} + \frac{Ft^{12}}{12!} + L^{-1}(e^{-t}\cdot A_n) \tag{20}$$

and the constants

$$A = u^{vii}(0),\quad B = u^{viii}(0),\quad C = u^{ix}(0),\quad D = u^{x}(0),\quad E = u^{xi}(0),\quad F = u^{xii}(0) \tag{21}$$

which will be determined later.

By modified Adomian decomposition method, we identify the zeroth component as

$$u_0(t) = 1 \tag{22}$$

$$u_1(t) = \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} - \frac{t^6}{6!} + \frac{At^7}{7!} + \frac{Bt^8}{8!} + \frac{Ct^9}{9!} + \frac{Dt^{10}}{10!} + \frac{Et^{11}}{11!} + \frac{Ft^{12}}{12!} + L^{-1}(e^{-t}\cdot y^2{}_0) \tag{23}$$

Other components are determined by the recursive relations below
(23)

$$u_{n+1}(t) = L^{-1}(e^{-t}\cdot A_n) \tag{24}$$

$$u_2 = L^{-1}(e^{-t}\cdot 2\cdot y_0 y_1) \tag{25}$$

The series solution of equation (16) is written as

$$u(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} + \frac{At^7}{5040} + \frac{Bt^8}{40320} + \frac{Ct^9}{362880} + \frac{Dt^{10}}{3628800} + \frac{Et^{11}}{39916800} + \frac{Ft^{12}}{479001600} + \cdots \tag{26}$$

To determine the constants, we apply the boundary conditions (17) at $t = 1$ to obtain the following set of equations

$$\begin{aligned}
&2.718055556 + \frac{A}{5040} + \frac{B}{40320} + \frac{C}{362880} + \frac{D}{3628800} + \frac{E}{39916800} + \frac{F}{479001600}\\
&2.716666669 + \frac{A}{720} + \frac{B}{5040} + \frac{C}{40320} + \frac{D}{362880} + \frac{E}{3628800} + \frac{F}{39916800}\\
&2.708333356 + \frac{A}{120} + \frac{B}{720} + \frac{C}{5040} + \frac{D}{40320} + \frac{E}{362880} + \frac{F}{3628800}\\
&2.666666920 + \frac{A}{24} + \frac{B}{120} + \frac{C}{720} + \frac{D}{5040} + \frac{E}{40320} + \frac{F}{362880}\\
&2.50000503 + \frac{A}{6} + \frac{B}{24} + \frac{C}{120} + \frac{D}{720} + \frac{E}{5040} + \frac{F}{40320}\\
&2.000022299 + \frac{A}{2} + \frac{B}{6} + \frac{C}{24} + \frac{D}{120} + \frac{E}{720} + \frac{F}{5040}
\end{aligned} \tag{27}$$

Solving the system of equations gives

ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

$$A = 0.9996458699, \quad B = 1.011394205,$$
$$C = 0.8300462688, D = 2.441964672, \qquad (28)$$
$$E = -5.862746736, \quad F = 15.51629883$$

Substituting (28) in (26), yields the series solution

$$u(t) = 1 + t + \frac{t^2}{2} + \frac{t^3}{6} + \frac{t^4}{24} + \frac{t^5}{120} + \frac{t^6}{720} +$$
$$0.00019834243 45t^7 + 0.00002508418167t^8 + \qquad (29)$$
$$2.287385000 \times 10^{-6} t^9 + \cdots$$

TABLE 1: NUMERICAL RESULTS FOR EXAMPLE 1

t	u_{EXACT}	u_{ADM}	ABS ERROR
0	1.000000	1.000000	0
0.1	1.105170	1.105170	5.107E-15
0.2	1.221402	1.221402	3.785E-13
0.3	1.349858	1.349858	3.986E-12
0.4	1.491824	1.491824	1.772E-11
0.5	1.648721	1.648721	4.829E-11
0.6	1.822118	1.822118	9.536E-11
0.7	2.013752	2.013752	1.506E-10
0.8	2.225540	2.225540	2.042E-10
0.9	2.459603	2.459603	2.517E-10
1	2.718281	2.718281	2.936E-10

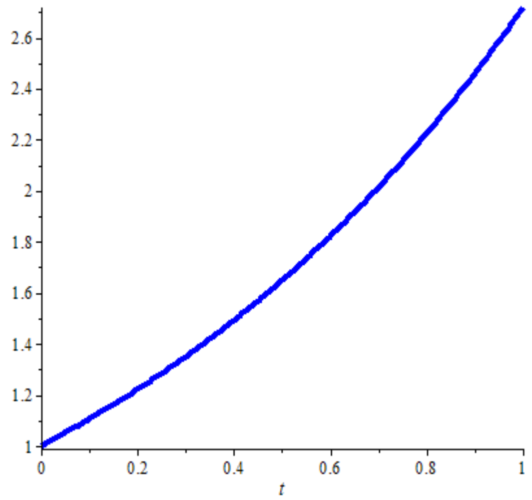

FIG1: PLOT OF NUMERICAL SOLUTION(ADM)

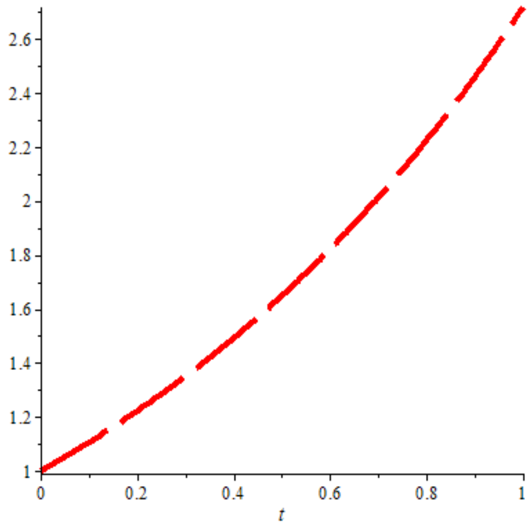

FIG 2: PLOT OF EXACT SOLUTION

IV. CONCLUSION

In the present work, solution of thirteenth order boundary value problem via modified Adomian decomposition method has been obtained. The MADM is applied without any form of transformation, linearization, perturbation or discretization. The approximate solution is compared with those obtained using variationational iteration and differential transform techniques, it is demonstrated that MADM reduces the computational involment and converges rapidly to the exact even with few terms.

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ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)

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ISBN: 978-988-14047-5-6
ISSN: 2078-0958 (Print); ISSN: 2078-0966 (Online)