Jurnal
Teknologi

Full Paper

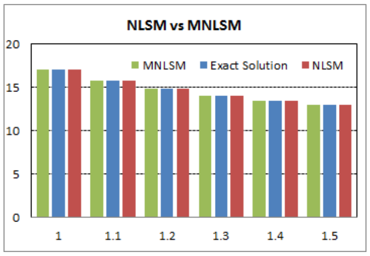
PERFORMANCE OF MODIFIED NON-LINEAR SHOOTING METHOD FOR SIMULATION OF 2ND ORDER TWO-POINT BVPS

Article history
Received
15 November 2015
Received in revised form
08 March 2016
Accepted
06 March 2016

Norma Alias[a], Abdul Manaf[b]*, Mustafa Habib[c]

[a]Center for Sustainable Nanomaterials (CSNano), Ibnu Sina Institute for Scientific and Industrial Research, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor, Malaysia
[b]Ibnu Sina Institute, Department of Science Mathematical, Faculty of Science, Universiti Teknologi Malaysia, 81310 UTM Johor Bahru, Johor Malaysia
[c]Department of Mathematics, University of Engineering and Technology, Lahore, Pakistan.

*Corresponding author
mabdul3@live.utm.my



Abstract

In this research article, numerical solution of nonlinear 2nd order two-point boundary value problems (TPBVPs) is discussed by the help of nonlinear shooting method (NLSM), and through the modified nonlinear shooting method (MNLSM). In MNLSM, fourth order Runge-Kutta method for systems is replaced by Adams Bashforth Moulton method which is a predictor-corrector scheme. Results acquired numerically through NLSM and MNLSM of TPBVPs are discussed and analyzed. Results of the tested problems obtained numerically indicate that the performance of MNLSM is rapid and provided desirable results of TPBVPs, meanwhile MNLSM required less time to implement as comparable to the NLSM for the solution of TPBVPs.

Keywords: Shooting method, predictor-corrector scheme, Runge-Kutta method, BVPs, ODEs.

Abstrak

Dalam artikel penyelidikan ini, penyelesaian berangka bagi masalah nilai sempadan dua titik tertib kedua tak linear (MNSDT) dengan bantuan kaedah penembakan tak linear (KPTL) dan kaedah terubahsuai penembakan tak linear (KTPTL) akan dibincangkan. Dalam KTPTL, kaedah Runge-Kutta tertib keempat untuk sistem telah digantikan dengan kaedah Adams Bashforth Moulton, iaitu skema peramal-pembetul. Keputusan yang diperoleh secara berangka melalui KPTL dan KTPTL daripada MNSDT turut dibincangkan dan dianalisis. Keputusan masalah yang diuji yang diperoleh secara berangka menunjukkan bahawa prestasi KTPTL adalah pantas dan memberikan kesan yang optimum kepada MNSDT. KTPTL juga memerlukan masa yang kurang untuk dilaksanakan berbanding KPTL untuk menyelesaikan MNSDT.

Kata kunci: Kaedah penembakan, skema peramal-pembetul, kaedah Runge-Kutta, masalah nilai sempadan (MNS), persamaan pembezaan biasa (PPB)

1.0 INTRODUCTION

For the two-point boundary value problems (TPBVPs) of ordinary differential equations (ODEs), some of the boundary conditions are stated at starting value of the independent variable, whereas the remaining boundary conditions are stated at end values of independent variable. Therefore, boundary conditions are divided between the initial points and terminal points of independent variable[1].

Numerous problems in engineering and in applied sciences are sculpted as TPBVPs like in fluid dynamics, heat energy distribution theory, reaction kinetics, space technology, optimization and control theory. The newest application of the TPBVPs has been described by [2] [3] [4] [5] [6] [7] [8] and many others. Since the TPBVPs has a large number of applications in science, therefore, more rapidly and precise solutions numerically of TPBVPs are much needed.

The tactic for the solution of a nonlinear second order TPBVP of the type

$$y'' = f(x, y, y'), x \in [a,b];\ a,b \in R$$

In association with boundary conditions

$$y(a) = \alpha \text{ and } y(b) = \beta .$$

here a and β are constants.

have been suggested by a different number of researchers like [9] [10] [11] [12] [13] [14]and[15].

It has been reported by [11] who considered multiple shooting methods (MSM) with Runge-Kutta method (RKM) to solve the nonlinear 2nd order TPBVPs using constant step size. In a research paper [14], discussed the multistep method regarding the backward difference formula and approaching solutions with NLSM. [10] discussed a numerical algorithm for the solution of TPBVPs directly by means of the divided-difference mode that comprises the differentiation and integration of coefficients in the code with MSM via adjustable order and step size.

In this paper, the NLSM is modified, which is named as a MNLSM. This method is applied to find the numerical solution of 2nd order nonlinear TPBVPs by substituting RKM for systems (which is a single step method) by Adam Bashforth Moulton method (ABMM) for systems (which is multi step method). Both methods are used to find solution of initial value problems (IVPs). The execution and convergence time of both these methods are also tested and discussed.

2.0 MATERIALS AND METHODS

In latest study of optimal control theory, engineering and mechanics, one frequently faces with a second order TPBVPs. Many techniques for solving TPBVPs are discussed and presented by many researchers. The common technique for solving TPBVPs is shooting method (SM). In SM, TPBVP is reduced to the solution of an IVP, with the supposition of initial values that would have been given if ODE is an IVP. The boundary value calculated is then matched with real boundary value. Using some scientific approach or trial and error, one wants to reach the boundary value as close as possible.

The SM workings by allowing for boundary conditions as multivariate functions of initial conditions (ICs) at specific points, reducing the TPBVP to finding ICs that gives a root. The SM takings advantage of adaptivity and speed of methods for IVPs. SM disadvantage is that it is not as strong as collocation or finite difference methods: some IVPs with increasing modes are inherently unstable even though the TPBVP itself may be somewhat well posed and stable.

For solving these TPBVPs, a couple of other methods such as nonlinear SM (NLSM) and its variation, and multiple shooting methods (MSM) are present in the literature. In this study a new scheme is proposed and designed from favorable aspects of both NLSM and MSM. The modified nonlinear SM (MNLSM) covers discrepancies of both previously mentioned methods to give up a faster and superior method for solving nonlinear TPBVPs. The convergence of MNLSM is proved under mild conditions on second order nonlinear TPBVP. A comparison for a problem by MNLSM and MSM is made where both methods converge.

MNLSM is the modified version of existing shooting techniques using predictor-corrector method (PCM) which proceeds in two steps. Firstly, prediction step computes a rough approximation of essential quantity. Secondly, the corrector step improves initial approximation using another means. The idea behind PCM is to use a suitable combination of an implicit and an explicit technique to find a method with better convergence characteristics.

The fourth order classical RKM for systems is a single step method, has been used in NLSM to approximate the solution of the nonlinear TPBVPs. In MNLSM, ABMM for systems, which is a multistep method, is used in the replacement of the Classical fourth order RKM. The execution time of algorithms for both NLSM and MNLSM were also checked.

Considered a nonlinear 2nd order TPBVP

$$y'' = f(x, y, y'),\ y(a) = \alpha,\ y(b) = \beta \qquad (1)$$

Here a and β are constants and $x \in [a,b]$.

For solutions of IVPs in the form of a sequence of

$$y'' = f(x, y, y'),\ y(a) = \alpha,\ y'(a) = t \qquad (2)$$

including t a parameter, and $x \in [a,b]$, is applied to estimate solution of BVP (1).

Express this through selecting $t = t_k$ as a parameters in a manner that make assure that

$$\lim_{k\to\infty} y(b,t_k) = y(b) = \beta \qquad (3)$$

Here $y(x,t_k)$ is a solution of IVP (2) with $t = t_k$ and y(x) is a solution to the BVP (1).
This procedure is known as a NLSM.

Initiated with parameter t_0 that set up out initial elevation by which object is fired from point (a,α) and close to curve termed by solution for IVP.

$$y'' = f(x,y,y'),\ y(a) = \alpha,\ y'(a) = t_0 \qquad (4)$$

If $y(b,t_0)$ is not satisfactorily nearby to β, tried to accurate approximation by selecting a new elevation t_1 and so on, up to $y(b,t_k)$ is appropriately near to strike β.

Decide that in what way the parameter t_k might be selected, assume a TPBVP (3) has single solution. If y(x, t) is solution to IVP (2), then there is requirement to conclude t so

$$y(b,t) - \beta = 0 \qquad (5)$$

Since (5) is a nonlinear, Newton's method $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$ is applied to solve this problem.

The early approximation t_0 is chosen and then produces sequence by

$$t_k = t_{k-1} - \frac{y(b,t_{k-1}) - \beta}{\dfrac{dy}{dt}(b,t_{k-1})} \qquad (6)$$

This needs the information of $\dfrac{dy}{dt}(b,t_{k-1})$, which offered a trouble, meanwhile an explicit drawing for y(b, t) was not known; and only acknowledged of values $y(b,t_0),\ y(b,t_1),\ \dots\dots\dots,\ y(b,t_{k-1})$.

Hence reformed IVP (2), give emphasis that solution depending on together x and t.

$$y''(x,t) = f(x,y,y');\ a \le x \le b,\ y(a,t) = \alpha,\ y'(a,t) = t \qquad (7)$$

recalling prime notation to specify differentiation w.r.t x.

Then to determined $\dfrac{dy}{dt}(b,t)$, when $t = t_{k-1}$, take partial derivative of (7) w.r.t t.

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial t}(x, y(x,t), y'(x,t)) = \frac{\partial f}{\partial x}(x, y(x,t), y'(x,t))\frac{\partial x}{\partial t}$$

$$+\frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t)$$

Since x and t are independent, $\dfrac{\partial x}{\partial t} = 0$, so

$$\frac{\partial y''}{\partial t}(x,t) = \frac{\partial f}{\partial y}(x, y(x,t), y'(x,t))\frac{\partial y}{\partial t}(x,t) + \frac{\partial f}{\partial y'}(x, y(x,t), y'(x,t))\frac{\partial y'}{\partial t}(x,t) \qquad (8)$$

for $a \le x \le b$. The initial conditions give

$$\frac{\partial y}{\partial t}(a,t) = 0, \text{ and } \frac{\partial y'}{\partial t}(a,t) = 1.$$

Making simpler the representation by using $z(x,t)$ to indicate $\dfrac{\partial y}{\partial t}(x,t)$ and consider that order of the differentiation of x and t can be reversed, Eq. (8) with initial conditions become IVP

$$z''(x,t) = \frac{\partial f}{\partial y}(x,y,y')z(x,t) + \frac{\partial f}{\partial y'}(x,y,y')z'(x,t),$$

$$a \le x \le b;\ z(a,t) = 0 \text{ and } z'(a,t) = 1 \qquad (9)$$

So, one requires that two IVPs (2) and (9) be solved for every single iteration.
Then from Eq. (6),

$$t_k = t_{k-1} - \frac{y(b,\ t_{k-1}) - \beta}{Z(b,\ t_{k-1})} \qquad (10)$$

In exercise, no one of these IVPs are solved accurately; as an alternative, the numerical solutions are found through one of IVP solvers.

Hence, in SM for 2nd order nonlinear TPBVPs, classical fourth order RKM is used to find together the solutions essential by Newton's method.

2.2 Adams-Bashforth-Moulton Method

The PCMs also named multistep methods, are not self-starting. They need four starting points $(x_0,y_0).(x_1,y_1),(x_2,y_2),\ (x_3,y_3)$, in order to create the point (x_4,y_4).

Let the two first order IVPs are

$$m'_{i+1} = f(x_{i+1},\ m_{i+1},\ n_{i+1}),\ m(x_0) = m_0$$

$$n'_{i+1} = g(x_{i+1},\ m_{i+1},\ n_{i+1}),\ \ n(x_0) = n_0 \text{ for } a \le x \le b.$$

By using four step Adams Bashforth methods as predictor formula, is

$$m_{i+1} = m_i + \frac{h}{24}\left(55f_i' - 59f_{i-1}' + 37f_{i-2}' - 9f_{i-3}'\right)$$

$$n_{i+1} = n_i + \frac{h}{24}\left(55g_i' - 59g_{i-1}' + 37g_{i-2}' - 9g_{i-3}'\right)$$

The above predictor formulas are used one time in iteration, by using three step Adams Moulton methods as corrector formula, is

$$m_{i+1} = m_i + \frac{h}{24}\left(9f_{i+1}'^{p} + 19f_i' - 5f_{i-1}' + f_{i-2}'\right)$$

$$n_{i+1} = n_i + \frac{h}{24}\left(9g'^{p}_{i+1} + 19g'_i - 5g'_{i-1} + g'_{i-2}\right)$$

Here p is the predicted value. The above correctors formulas are used as several times as required to obtain the essential level of accuracy.

3.0 RESULTS AND DISCUSSION

Here, we discussed two examples to display the working of both NLSM and proposed MNLSM algorithm.

For simulation, MATLAB codes are written and that codes are implemented on Core I3 Windows 7 system.

3.1 Example-1

Suppose a 2nd order nonlinear TPBVP of the form

$y'' = \frac{1}{8}\left(32 + 2x^3 - yy'\right),\ 1 \le x \le 3,$ with boundary conditions $y(1) = 17,\ y(3) = \frac{43}{3}$ and actual solution is $y(x) = x^2 + \frac{16}{x}$.

Take h=0.2 and the error bound 10^{-5}.

Table 1 indicates that when value of the variable x increased from $x = 1\ to\ x = 3$, the numerical results of the MNLSM are further accurate than numerical results of NLSM, while compared to the exact solution, but results attained by NLSM and MNLSM are suitable as related to the exact solution and for the results reported by researcher [11] , [13]and [15].

Table 1 Showing Numerical Results and Exact Solution

I	X(I)	Exact Solution	Results by NLSM	Results by MNLSM
0	1.00000000	17.00000000	17.00000000	17.00000000
1	1.10000000	15.75545455	15.75549614	15.75531210
2	1.20000000	14.77333333	14.77339116	14.77305380
3	1.30000000	13.99769231	13.99775428	13.99728621
4	1.40000000	13.38857143	13.38863177	13.38745291
5	1.50000000	12.91666667	12.91672269	12.92664767
6	1.60000000	12.56000000	12.56005059	12.55470098
7	1.70000000	12.30176471	12.30180955	12.30418467
8	1.80000000	12.12888889	12.12892807	12.12538183
9	1.90000000	12.03105263	12.03108645	12.03201527
10	2.00000000	12.00000000	12.00002885	11.99844632
11	2.00000000	12.02904762	12.02907192	12.02944466
12	2.20000000	12.11272727	12.11274744	12.11205785
13	2.30000000	12.24652174	12.24653819	12.24664626
14	2.40000000	12.42666667	12.42667979	12.42637097
15	2.50000000	12.65000000	12.65001016	12.65001916
16	2.60000000	12.91384615	12.91385369	12.91370993
17	2.70000000	13.21592593	13.21593115	13.21591762
18	2.80000000	13.55428571	13.55428891	13.55422821
19	2.90000000	13.92724138	13.92724281	13.92724524
20	3.00000000	14.33333333	14.33333324	14.33333336

Results in Table 2 of example-1 showed that when value of the variable x increased from $x = 1\ to\ x = 3$, absolute error for MNLSM decreased when compared with the absolute error of NLSM, and with results reported by the researcher [11] , [13]and [15].

Numerical results in Table 3 of example-1 indicates that NLSM with tk = -1.4000192e+001 converges in 7 iterations and its execution time is 2.459359 seconds, whereas MNLSM with tk = -1.4002225e+001 converges in 14 iterations and its execution time is 1.598757 seconds, which is also less than the execution time of NLSM, and from execution time described by [15]. The numerical results acquired by MNLSM are also suitable, as compared with exact solution.

Table 2 Showing Absolute Error and Exact Solution

I	X(I)	Exact Solution	Absolute Error by NLSM	Absolute Error by MNLSM
0	1.00000000	17.00000000	0.00000000	0.00000000
1	1.10000000	15.75545455	0.00004159	0.00014245
2	1.20000000	14.77333333	0.00005783	0.00027953
3	1.30000000	13.99769231	0.00006189	0.00040610
4	1.40000000	13.38857143	0.00006034	0.00111852
5	1.50000000	12.91666667	0.00005602	0.00998100
6	1.60000000	12.56000000	0.00005059	0.00529902
7	1.70000000	12.30176471	0.00004484	0.00241996
8	1.80000000	12.12888889	0.00003918	0.00350706
9	1.90000000	12.03105263	0.00003382	0.00096254
10	2.00000000	12.00000000	0.00002885	000155368
11	2.00000000	12.02904762	0.00002430	0.00039684
12	2.20000000	12.11272727	0.00002017	0.00066942
13	2.30000000	12.24652174	0.00001645	0.00012452
14	2.40000000	12.42666667	0.00001312	0.00029570
15	2.50000000	12.65000000	0.00001016	0.00001916
16	2.60000000	12.91384615	0.00000754	0.00013622
17	2.70000000	13.21592593	0.00000522	0.00000831
18	2.80000000	13.55428571	0.00000320	0.00005750
19	2.90000000	13.92724138	0.00000143	0.00000386
20	3.00000000	14.33333333	0.00000009	0.00000003

Table 3 Showing Execution Time and Convergence

	NLSM	MNLSM
Convergence in	7 iterations with tk = -1.4000192e+001	14 iterations with tk = -1.4002225e+001
Execution Time is	2.459359 seconds	1.598757 seconds.

3.2 Example-2

Considered another 2nd order nonlinear TPBVP of the form $y'' = 2y^3,\ 1 \le x \le 2,$ with the boundary conditions $y(1) = \frac{1}{4},\ y(2) = \frac{1}{5}$ and exact solution of the problem is $y(x) = (x+3)^{-1}$. Take h=0.1 and error bound 10^{-5}.

Numerical results in Table 4 of example-2 indicates that when value of the variable x increased from $x = 1\ to\ x = 2$, the results of NLSM are further accurate as results of MNLSM, when compared with exact solution, but results obtained with both methods are suitable when compared with exact solution and with results reported by researcher [11], [13]and [15].

Table 4 Showing Numerical Results and Exact Solution

I	X(I)	Exact Solution	Results by NLSM	Results by MNLSM
0	1.00000000	0.25000000	0.25000000	0.25000000
1	1.10000000	0.24390244	0.24390244	0.24390218
2	1.20000000	0.23809524	0.23809524	0.23809472
3	1.30000000	0.23258514	0.23255815	0.23255736
4	1.40000000	0.22727273	0.22727274	0.22727167
5	1.50000000	0.22222222	0.22222224	0.22222855
6	1.60000000	0.21739130	0.21739132	0.21739040
7	1.70000000	0.21276596	0.21276598	0.21276612
8	1.80000000	0.20833333	0.20833336	0.20833302
9	1.90000000	0.20408163	0.20408166	0.20408167
10	2.00000000	0.20000000	0.20000003	0.20000004

Results in Table 5 of example-2 indicates that as the value of variable x increased from $x = 1 \, to \, x = 2$, the absolute error for MNLSM is higher when compared with absolute error of NLSM, and with exact solution, and with results reported by the researcher [11], [13]and [15], but absolute errors of both methods are acceptable.

Table 5 Showing Absolute Error and Exact Solution

I	X(I)	Exact Solution	Absolute Error by NLSM	Absolute Error by MNLSM
0	1.00000000	0.25000000	0.00000000	0.00000000
1	1.10000000	0.24390244	0.00000000	0.00000026
2	1.20000000	0.23809524	0.00000000	0.00000052
3	1.30000000	0.23258514	0.00000001	0.00002778
4	1.40000000	0.22727273	0.00000001	0.00000106
5	1.50000000	0.22222222	0.00000002	0.00000633
6	1.60000000	0.21739130	0.00000002	0.00000090
7	1.70000000	0.21276596	0.00000002	0.00000016
8	1.80000000	0.20833333	0.00000003	0.00000031
9	1.90000000	0.20408163	0.00000003	0.00000004
10	2.00000000	0.20000000	0.00000003	0.00000004

Numerical results in Table 6 of example-2 indicates that NLSM with tk = -6.2499975e-002 converges in 3 iterations and its execution time is 1.483343 seconds, whereas MNLSM with tk = -6.2502598e-002 converges in 10 iterations and its execution time is 1.029948 seconds, which is much less than the execution time of NLSM and from execution time observed by [15]. The results obtained by MNLSM are also suitable, as related with exact solution.

Table 6 Showing Execution Time and Convergence

	NLSM	MNLSM
Convergence in	3 iterations with tk = -6.2499975e-002	10 iterations with tk = -6.2502598e-002
Execution Time is	1.483343 seconds.	1.029948 seconds.

Results found numerically of both the tested problems clearly indicated that MNLSM in which ABMM for systems is used, will always require less execution time however perhaps with some loss in the accuracy. The fact is: ABMM which used in the MNLSM needs two function evaluations inspite of fourth order classical RKM used in NLSM which needs four function evaluations, make it more efficient [11] has applied NLSM on nonlinear 2nd order TPBVPs and attained the desired results, while in this paper, NLSM and MNLSM are applied on same TPBVPs, which presented further accurate results than [11], when compared with exact solution. Also, results in this paper are much better than results reported by [11],[13]and[15], obtained by using NLSM.

The reason is that the PCM which we used in MNLSM needs two function evaluations as a substitute of fourth order classical RKM used in NLSM which needs four function evaluations, make it more efficient.

4.0 CONCLUSION

Numerical simulations of tested problems pointed out that MNLSM all the time needs a smaller amount of time to execute, though possibly with certain loss in accuracy. Numerical results achieved by MNLSM are also acceptable, when compared with NLSM and with the exact solutions of the 2nd order nonlinear TPBVPs. For future research, higher order TPBVPs will be solved by using parallel computing techniques [16-20].

References

[1] Burden, R., J. Faires, and A. Burden. 2015. *Numerical Analysis*. Nelson Education.

[2] Abbasbandy, S., B. Azarnavid, and M.S. Alhuthali. 2015. A Shooting Reproducing Kernel Hilbert Space Method For Multiple Solutions Of Nonlinear Boundary Value Problems. *Journal of Computational and Applied Mathematics*. 279: 293-305.

[3] Jamshidi, N. and M. Rostami., 2008. Gait Optimization Of Biped Robot During Double Support Phase By Pure

Dynamic Synthesis. *American Journal of Applied Sciences.* 5(9): 1175-1181.

[4] Filipov, S .M., I. D. Gospodinov, and J. Angelova. 2015. Solving Two-Point Boundary Value Problems for Integro-Differential Equations Using the Simple Shooting-Projection Method, Numerical Methods and Applications. *Springer.* 169-177.

[5] Manaf, A., M. Habib, and M. Ahmad. 2015. Review of Numerical Schemes for Two Point Second Order Non-Linear Boundary Value Problems. *Proceedings of Pakistan Academy of Sciences*. 2(52): 151-158.

[6] Adegbie, K. and A. Alao., 2007. On Thermal Explosion Of Sensitised Reaction With Variable Heat Loss. *American Journal of Applied Sciences.* 4(1): 53-55.

[7] Xu, X. and F. Zhou., 2015. Numerical Solutions for the Eighth-Order Initial and Boundary Value Problems Using the Second Kind Chebyshev Wavelets. *Advances in Mathematical Physics.*

[8] Arqub, O. A. and M. Al-Smadi. 2014. *Numerical Algorithm For Solving Two-Point, Second-Order Periodic Boundary Value Problems For Mixed Integro-Differential Equations. Applied Mathematics and Computation*. 243: 911-922.

[9] Roberts, S. M. and J. S. Shipman. 1972. *Two-Point Boundary Value Problems.* American Elsevier Pub. Co.

[10] Malathi, V. 1999. *Solving Boundary Value Problems For Ordinary Differential Equations Using Direct Integration And Shooting Techniques.* Ph. D diss., Universiti Putra Malaysia.

[11] Ha, S. N., 2001. *A Nonlinear Shooting Method For Two-Point Boundary Value Problems.* Computers & Mathematics with Applications. 42(10): 1411-1420.

[12] Auzinger, W., et al. 2003. A Collocation Code For Singular Boundary Value Problems In Ordinary Differential Equations*. Numerical Algorithms.* 33(1-4): 27-39.

[13] Attili, B. S. and M. I. Syam. 2008. Efficient Shooting Method For Solving Two Point Boundary Value Problems. *Chaos, Solitons & Fractals*. 35(5): 895-903.

[14] Jafri, M. D., et al., 2009. Solving Directly Two Point Boundary Value Problems Using Direct Multistep Method. *Sains Malaysiana.* 38(5): 723-728.

[15] Abbas, Y. A. and A. J. Al-Sawoor. 2006. Speeding-Up The Performance Of The Nonlinear Shooting Method For Solving Nonlinear BVPs. *Rafidain Sci. J.* 2(17): 59- 71.

[16] Alias, N. and Islam, M. 2010. A Review Of The Parallel Algorithms For Solving Multidimensional PDE Problems. *Journal of Applied Sciences.* 2187-2197.

[17] Alias, N., Hafizah, F. S. S. and Asnida, C. A. G. 2012. Numerical Method For Solving Multipoints Elliptic-Parabolic Equation For Dehydration Process. *Proceedings of 2nd Regional Conference On Applied And Engineering Mathematics (RCAEM-II) 2012*.

[18] Alias, N. et al. 2014. High Performance Large Sparse PDEs With Parabolic And Elliptic Types Using AGE Method On DPCS. *Advanced Science Letters.* 1956-1960.

[19] Alias, N., et al. 2008. 3D Parallel Algorithm Parabolic Equation For Simulation Of The Laser Glass Cutting Using Parallel Computing Platform. *The Pacific Rim Applications and Grid Middleware Assembly (PRAGMA15),* Penang, Malaysia.

[20] Alias, Norma, et al. 2014. Parallel Performance Comparison Of Alternating Group Explicit Method Between Parallel Virtual Machine And Matlab Distributed Computing For Solving Large Sparse Partial Differential Equations. *Advanced Science Letters* 20(2): 477-482.