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Full Paper

THE SCHUR MULTIPLIER OF PAIRS OF NONABELIAN GROUPS OF ORDER P^4

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Graphical abstract

$$M(G,N) \cong {}^{R \cap [S,F]}\!\big/\!{}_{[R,F]}$$

Abstract

Let (G,N) be a pair of groups where G is any group and N is a normal subgroup of G, then the Schur multiplier of pairs of groups is a functorial abelian group. The notion of the Schur multiplier of pairs of groups is an extension from the Schur multiplier of a group G. In this research, the Schur multiplier of pairs of finite nonabelian groups of order p^4, where p is an odd prime, is determined.

Keywords: Schur multiplier; pair of groups; normal subgroup; nonabelian group

Abstrak

Katalah (G,N) sepasang kumpulan yang mana G adalah suatu kumpulan dan N adalah subkumpulan normal bagi G. Maka pendarab Schur bagi sepasang kumpulan adalah suatu kumpulan abelan fungtorial. Idea pendarab Schur bagi sepasang kumpulan adalah lanjutan daripada pendarab Schur bagi suatu kumpulan G. Dalam kajian ini, pendarab Schur bagi sepasang kumpulan tak abelan peringkat p^4, dengan p adalah nombor perdana ganjil, ditentukan.

Kata kunci: Pendarab Schur; sepasang kumpulan; subkumpulan normal; kumpulan tak abelan

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1.0 INTRODUCTION

The Schur multiplier was introduced by Schur [1] in 1904. For a group G with a free presentation $1 \to R \to F \to G \to 1$, the Schur multiplier of G, $M(G)$, is isomorphic to ${}^{(R \cap [F,F])}\!\big/\!{}_{[R,F]}$. (Throughout this paper, we denote the trivial group as 1). The computation of the Schur multiplier for many different kinds of group have been done by Schur and can be found in [2]. Following Green [3], the order of the Schur multiplier of a finite p-group of order p^n is bounded by $p^{\frac{1}{2}n(n-1)}$. In [4], Ok computed the Schur multiplier of some nonabelian groups of order p^4 where p is an odd prime.

In 1998, Ellis [5] defined the notion of the Schur multiplier of a pair of groups. Let (G,N) be an arbitrary pair of finite groups where N is a normal subgroup of G, then the Schur multiplier of the pair, $M(G,N)$ is a functorial abelian group whose principal feature is a natural exact sequence

$$H_3(G)\xrightarrow{\eta}H_3\left(G/N\right)\to M(G,N)\to M(G)\xrightarrow{\mu}M\left(G/N\right)$$
$$\to N/[N,G]\to (G)^{ab}\xrightarrow{\alpha}\left(G/N\right)^{ab}\to 1 \qquad (1)$$

in which $H_3(-)$ denotes some finiteness-preserving functor from groups to abelian groups (to be precise, $H_3(-)$ is the third homology of a group with integer coefficients). The homomorphisms η, μ, α are those due to the functorial of $H_3(-)$, $M(-)$, and $(-)^{ab}$.

In [5], Ellis also showed that the Schur multiplier of (G,N) is bounded by $p^{\frac{1}{2}n(2m+n-1)}$ if G is a finite p-group with a normal subgroup N of order p^n and its quotient G/N of order p^m. In 2007, Moghaddam *et al.* [6] showed that $M(G,N)\cong R\cap[S,F]/[R,F]$ if S is a normal subgroup of F such that $N\cong S/R$.

In this research, the Schur multiplier of a pair of groups for some nonabelian groups of order p^4, where p is an odd prime, is determined.

2.0 PRELIMINARIES

This section includes some preliminary results that are used in proving our main theorem. Note that in this section and onwards, we denote p as an odd prime.

The classification of nonabelian groups of order p^4 is given by Burnside in [7]. The nonabelian groups of order p^4 which satisfy the conditions: $|Z(G)|=p^2$, $|G'|=p$, and $G'\subseteq Z(G)$ are listed in the following.

Theorem 2.1 [7] Let G be a nonabelian group of order p^4 such that $|Z(G)|=p^2$, $|G'|=p$, and $G'\subseteq Z(G)$. Then G is isomorphic to exactly one of the following groups:

(i) $G_1\cong\left\langle x,y \mid x^{p^3}=y^p=1, x^y=x^{1+p^2}\right\rangle$

(ii) $G_2\cong\left\langle x,y,z \mid x^p=y^p=z^{p^2}=1,[x,z]=[y,z]=1,[x,y]=z^p\right\rangle$

(iii) $G_3\cong\left\langle x,y \mid x^{p^2}=y^{p^2}=1, x^y=x^{1+p}\right\rangle$

(iv) $G_4\cong M_p\times\langle w\rangle$, where
$M_p\cong\left\langle x,y,z \mid x^p=y^p=z^p=1,[x,z]=[y,z]=1,[x,y]=z\right\rangle$,
$\langle w\rangle\cong\left\langle w \mid w^p=1\right\rangle$

(v) $G_5\cong\left\langle x,y,z \mid x^{p^2}=y^p=z^p=1,[x,z]=[y,z]=1,x^y=x^{1+p}\right\rangle$

(vi) $G_6\cong\left\langle x,y,z \mid x^{p^2}=y^p=z^p=1,[x,y]=z,[x,z]=[y,z]=1\right\rangle$

where $[x,y]=xyx^{-1}y^{-1}$.

Note that:
$G_1\cong\mathbb{Z}_{p^3}\rtimes\mathbb{Z}_p$,
$G_2\cong\left(\mathbb{Z}_{p^2}\times\mathbb{Z}_p\right)\rtimes\mathbb{Z}_p$,
$G_3\cong\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_{p^2}$,
$G_4\cong\mathbb{Z}_p\times\left(\left(\mathbb{Z}_p\times\mathbb{Z}_p\right)\rtimes\mathbb{Z}_p\right)$,
$G_5\cong\mathbb{Z}_p\times\left(\mathbb{Z}_{p^2}\rtimes\mathbb{Z}_p\right)$ and
$G_6\cong\left(\mathbb{Z}_p\times\mathbb{Z}_p\right)\rtimes\mathbb{Z}_{p^2}$.

Next, some results that are essential to compute the Schur multiplier and the nonabelian tensor product are stated below.

Theorem 2.2 [8] Let $G\cong\mathbb{Z}_m$ and $H\cong\mathbb{Z}_n$ be cyclic groups that act trivially on each other. Then $G\otimes H\cong\mathbb{Z}_{(m,n)}$.

Theorem 2.3 [9] Let A, B, C be groups, with given actions of A on B and C, and of B and C on A. Suppose that the latter actions

(i) commute: ${}^{bc}a={}^{cb}a$, so that $B\times C$ acts on A,

(ii) induce the trivial action of B on $A\otimes C$; ${}^{b}(a\otimes c)=a\otimes c$, and

(iii) induce the trivial action of C on $A\otimes B$; ${}^{c}(a\otimes b)=a\otimes b$,

for all $a\in A, b\in B, c\in C$. Then $A\otimes(B\times C)\cong(A\otimes B)\times(A\otimes C)$.

Theorem 2.4 [2] Let G be a finite group. Then

(i) $M(G)$ is a finite group whose elements have order dividing the order of G.

(ii) $M(G)=1$ if G is cyclic.

The following theorem gives the Schur multiplier of nonabelian groups of order p^3.

Theorem 2.5 [2] Let G be an extra-special p-group of order p^{2n+1}. Suppose that $|G| = p^3$. Then

$$M(G)=\begin{cases}\mathbb{Z}_p\times\mathbb{Z}_p; & \text{if } G \text{ is of exponent } p\\ 1; & \text{if } G \text{ is of exponent } p^2.\end{cases}$$

The Schur multiplier of nonabelian groups of order p^4 which satisfy the conditions: $|Z(G)|=p^2$, $|G'|=p$ and $G'\subseteq Z(G)$, obtained by Ok in [4] is stated in the following theorems.

Theorem 2.6 [4] Let G be a nonabelian group of order p^4 such that $Z(G)$ is cyclic group of order p^2. And let $M(G)$ be a Schur multiplier of G. Then exactly one of the following holds:

(i) $G = G_1$ and $M(G) \cong 1$.

(ii) $G = G_2$ and $M(G) \cong (\mathbb{Z}_p)^2$.

Theorem 2.7 [4] Let G be a nonabelian group of order p^4 such that $Z(G)$ is elementary abelian group of order p^2. And let $M(G)$ be a Schur multiplier of G. Then exactly one of the following holds:

(i) $G = G_3$ and $M(G) \cong \mathbb{Z}_p$.

(ii) $G = G_4$ and $M(G) \cong (\mathbb{Z}_p)^4$.

(iii) $G = G_5$ and $M(G) \cong (\mathbb{Z}_p)^2$.

(iv) $G = G_6$ and $M(G) \cong (\mathbb{Z}_p)^2$.

The following theorems are some of the basic results of the Schur multiplier of a pair deduced by Ellis [5].

Theorem 2.8 [5] Let $N = 1$, then $M(G,N) = 1$.

Theorem 2.9 [5] Let $N = G$, then $M(G,G) = M(G)$.

The structure for the Schur multiplier of a direct product of finite groups given by Schur in [2] is shown as follows:

Theorem 2.10 [2] If G_1 and G_2 are finite groups, then $M(G_1 \times G_2) = M(G_1) \times M(G_2) \times (G_1^{ab} \otimes G_2^{ab})$.

As a consequence of the above fact, Mohammadzadeh *et al.* [10] gave the following result.

Theorem 2.11 [10] Let (G,N) be a pair of groups and K be the complement of N in G. Then $|M(G,N)| = |M(N)||N^{ab} \otimes K^{ab}|$.

3.0 RESULTS AND DISCUSSION

In the following two theorems, the Schur multiplier of pairs of nonabelian groups of order p^4 is stated and proven.

Theorem 3.1

Let G be a nonabelian group of order p^4 such that $|G'| = p$ and $G' \subseteq Z(G)$ where $Z(G)$ is cyclic group order p^2. If $N \lhd G$, then the Schur multiplier of pairs of G, is given in the following:

$$M(G,N) = \begin{cases} 1 & ; \text{if } (G = G_1) \text{ or } (G = G_2 \text{ when } N = 1), \\ (\mathbb{Z}_p)^2 & ; \text{if } \begin{pmatrix} G = G_2 \text{ when } N = G, \mathbb{Z}_p, \mathbb{Z}_{p^2}, (\mathbb{Z}_p)^2, \\ \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p \text{ or } (\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p \end{pmatrix}. \end{cases}$$

Proof:

Let G be a nonabelian group of order p^4 such that $|G'| = p$ and $G' \subseteq Z(G)$ where $Z(G)$ is cyclic group order p^2. Suppose $N \lhd G$, then the Schur multiplier of pairs of G is computed below. Note that if $N = 1$, then by Theorem 2.8, $M(G,N) = 1$.

Case 1: Let $G = G_1$. Then by Theorem 2.6, $M(G) = 1$. Since $M(G,N) \le M(G)$ (see [6]), for all normal subgroups N of G, $M(G,N) \le M(G) = 1$.

Case 2: If $G = G_2$, then by Theorem 2.6, $M(G) \cong (\mathbb{Z}_p)^2$.

(i) If $N = G$ then by Theorem 2.9, $M(G,N) = M(G,G) = M(G) = (\mathbb{Z}_p)^2$.

(ii) If $N = G' = \mathbb{Z}_p$ then $|G/N| = p^3$. Assume that $G/N \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, nonabelian group of order p^3 of exponent p^2. By Theorem 2.5, $M(G/N) \cong 1$. Thus the exact sequence $M(G,N) \to M(G) \to M(G/N) = 1$ shows that $M(G,N)/\kappa \cong 1$ where κ is the kernel of homomorphism $M(G,N)$ to $M(G)$ Thus, $M(G,N) \cong (\mathbb{Z}_p)^2$.

(iii) If $N = Z(G) = \mathbb{Z}_{p^2}$ then the complement of N, $K \cong \mathbb{Z}_p \times \mathbb{Z}_p$. Thus we have

$$\begin{aligned} |M(G,\mathbb{Z}_{p^2})| &= |M(\mathbb{Z}_{p^2})||(\mathbb{Z}_{p^2})^{ab} \otimes (\mathbb{Z}_p \times \mathbb{Z}_p)^{ab}| && \text{by Theorem 2.11} \\ &= |(\mathbb{Z}_{p^2}) \otimes (\mathbb{Z}_p \times \mathbb{Z}_p)| && \text{by Theorem 2.4} \\ &= |(\mathbb{Z}_{p^2} \otimes \mathbb{Z}_p) \times (\mathbb{Z}_{p^2} \otimes \mathbb{Z}_p)| && \text{by Theorem 2.3} \\ &= |(\mathbb{Z}_{(p^2,p)}) \times (\mathbb{Z}_{(p^2,p)})| && \text{by Theorem 2.2} \\ &= |\mathbb{Z}_p \times \mathbb{Z}_p|. \end{aligned}$$

Therefore, $M(G,N) \cong (\mathbb{Z}_p)^2$.

(iv) If $N = \mathbb{Z}_p \times \mathbb{Z}_p$ then $|G/N| = p^2$. Assume that $G/N = \mathbb{Z}_{p^2}$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong (\mathbb{Z}_p)^2$.

(v) If $N = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ or $(\mathbb{Z}_p \times \mathbb{Z}_p) \rtimes \mathbb{Z}_p$ then $|G/N| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong (\mathbb{Z}_p)^2$.

Theorem 3.2

Let G be a nonabelian group of order p^4 such that $|G'| = p$ and $G' \subseteq Z(G)$ where $Z(G)$ is elementary abelian group order p^2. If $N \lhd G$, then the Schur multiplier of pairs of G, is given in the following:

$$M(G,N) = \begin{cases} 1 & ; \text{if } G = G_3, G_4, G_5, G_6 \text{ when } N = 1, \\ \mathbb{Z}_p & ; \text{if } G = G_3 \text{ when } N = G, \mathbb{Z}_p, (\mathbb{Z}_p)^2, \mathbb{Z}_{p^2} \text{ or } \mathbb{Z}_{p^2} \times \mathbb{Z}_p, \\ (\mathbb{Z}_p)^2 & ; \text{if } \begin{pmatrix} G = G_5, \text{when } N = G, \mathbb{Z}_p, (\mathbb{Z}_p)^2, \mathbb{Z}_{p^2}, \mathbb{Z}_{p^2} \times \mathbb{Z}_p \\ \text{or } (\mathbb{Z}_p)^3 \end{pmatrix} \\ & \text{or } \begin{pmatrix} G = G_6 \text{ when } N = G, \mathbb{Z}_p, (\mathbb{Z}_p)^2, \mathbb{Z}_{p^2} \times \mathbb{Z}_p \\ \text{or } (\mathbb{Z}_p)^3 \end{pmatrix}, \\ (\mathbb{Z}_p)^3 & ; \text{if } G = G_4 \text{ when } N = \mathbb{Z}_p, \\ (\mathbb{Z}_p)^4 & ; \text{if } G = G_4 \text{ when } N = G, (\mathbb{Z}_p)^2, (\mathbb{Z}_p)^3 \text{ or } \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p. \end{cases}$$

Proof:

Let G be a nonabelian group of order p^4 such that $|G'| = p$ and $G' \subseteq Z(G)$ where $Z(G)$ is elementary group order p^2. Suppose $N \lhd G$, then the Schur multiplier of pairs of G is computed below. Note that if $N = 1$, then by Theorem 2.8, $M(G,N) = 1$.

Case 1: If $G = G_3$, then by Theorem 2.7, $M(G) = \mathbb{Z}_p$.

(i) If $N = G$ then by Theorem 2.9, $M(G,N) = M(G,G) = M(G) = \mathbb{Z}_p$.

(ii) If $N = G' = \mathbb{Z}_p$ then $|G/N| = p^3$. Assume that $G/N \cong \mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$, nonabelian group of order p^3 of exponent p^2. By Theorem 2.5, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong \mathbb{Z}_p$.

(iii) If $N = Z(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ then $|G/N| = p^2$. Assume that $G/N = \mathbb{Z}_{p^2}$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong \mathbb{Z}_p$.

(iv) If $N = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ then $|G/N| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong \mathbb{Z}_p$.

Case 2: If $G = G_4$, then by Theorem 2.7, $M(G) = (\mathbb{Z}_p)^4$.

(i) If $N = G$ then by Theorem 2.9, $M(G,N) = M(G,G) = M(G) = (\mathbb{Z}_p)^4$.

(ii) If $N = G' = \mathbb{Z}_p$ then the complement of N, $K \cong \mathbb{Z}_p \times \mathbb{Z}_p \times \mathbb{Z}_p$. Thus by similar way as in Case 2 (iii), $M(G,N) \cong (\mathbb{Z}_p)^3$.

(iii) If $N = Z(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ then $|G/N| = p^2$. Assume that $G/N = \mathbb{Z}_{p^2}$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong (\mathbb{Z}_p)^4$.

(iv) If $N = (\mathbb{Z}_p)^3$ or $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ then $|G/N| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) \cong (\mathbb{Z}_p)^4$.

Case 3: If $G = G_5$, then by Theorem 2.7, $M(G) = (\mathbb{Z}_p)^2$.

(i) If $N = G$ then by Theorem 2.9, $M(G,N) = M(G,G) = M(G) = (\mathbb{Z}_p)^2$.

(ii) If $N = G' = \mathbb{Z}_p$ then $|G/N| = p^3$. Assume that $G/N \cong \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, nonabelian group of order p^3 of exponent p^2. By Theorem 2.5, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) = (\mathbb{Z}_p)^2$.

(iii) If $N = Z(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ or $\mathbb{Z}_{p^2}$ then $|G/N| = p^2$. Assume that $G/N = \mathbb{Z}_{p^2}$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) = (\mathbb{Z}_p)^2$.

(iv) If $N = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$, $\mathbb{Z}_{p^2} \rtimes \mathbb{Z}_p$ or $(\mathbb{Z}_p)^3$ then $|G/N| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) = (\mathbb{Z}_p)^2$.

Case 4: If $G = G_6$, then by Theorem 2.7, $M(G) = (\mathbb{Z}_p)^2$.

(i) If $N = G$ then by Theorem 2.9, $M(G,N) = M(G,G) = M(G) = (\mathbb{Z}_p)^2$.

(ii) If $N = G' = \mathbb{Z}_p$ then the complement of N, $K = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ Thus by similar way as in Case 2 (iii), $M(G,N) = (\mathbb{Z}_p)^2$.

(iii) If $N = Z(G) = \mathbb{Z}_p \times \mathbb{Z}_p$ then $|G/N| = p^2$. Assume that $G/N = \mathbb{Z}_{p^2}$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) = (\mathbb{Z}_p)^2$.

(iv) If $N = \mathbb{Z}_{p^2} \times \mathbb{Z}_p$ or $(\mathbb{Z}_p)^3$ then $|G/N| = p$ which implies $G/N \cong \mathbb{Z}_p$. By Theorem 2.4, $M(G/N) \cong 1$. Thus by similar way as in Case 2 (ii), $M(G,N) = (\mathbb{Z}_p)^2$.

4.0 CONCLUSION

There are six nonisomorphic nonabelian groups of order p^4 where p is an odd prime which satisfy the conditions: $|Z(G)| = p^2$, $|G'| = p$ and $G' \subseteq Z(G)$. In this paper, we determined the Schur multiplier of pairs of this six nonabelian groups of order p^4 and our proofs show that they are equal to 1, $\mathbb{Z}_p$, $(\mathbb{Z}_p)^2$, $(\mathbb{Z}_p)^3$, or $(\mathbb{Z}_p)^4$ depending on their normal subgroups.

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