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THE NONABELIAN TENSOR SQUARE OF A BIEBERBACH GROUP WITH SYMMETRIC POINT GROUP OF ORDER SIX

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Graphical abstract

$$B_1(4) = \left\langle a,b,l_1,l_2,l_3,l_4 \;\middle|\; \begin{array}{l} a^2 = l_3, b^3 = l_4, b^a = b^2 l_4^{-1}, \\ l_1^a = l_1 l_2^{-1}, l_2^a = l_2^{-1}, l_3^a = l_3, l_4^a = l_4^{-1}, \\ l_1^b = l_1^{-1} l_2, l_2^b = l_1^{-1}, l_3^b = l_3, l_4^b = l_4, \\ l_j^{l_i} = l_j, l_j^{l_i^{-1}} = l_j \text{ for } \quad j > i, \;\; 1 \leqslant i, j \leqslant 4 \end{array} \right\rangle$$

Abstract

Bieberbach groups are torsion free crystallographic groups. In this paper, our focus is given on the Bieberbach groups with symmetric point group of order six. The nonabelian tensor square of a group is a well known homological functor which can reveal the properties of a group. With the method developed for polycyclic groups, the nonabelian tensor square of one of the Bieberbach groups of dimension four with symmetric point group of order six is computed. The nonabelian tensor square of this group is found to be not abelian and its presentation is constructed.

Keywords: Bieberbach group, nonabelian tensor square, polycyclic group

Abstrak

Kumpulan Bieberbach merupakan kumpulan kristolografi yang bebas kilasan. Dalam makalah ini, fokus kami diberi kepada kumpulan Bieberbach dengan kumpulan titik simetrik berperingkat enam. Kuasa dua tensor tak abelan merupakan fungtor berhomologi terkenal yang dapat mendedahkan ciri-ciri sesuatu kumpulan. Dengan kaedah yang dibangunkan untuk kumpulan polikitaran, kuasa dua tensor tak abelan bagi salah satu kumpulan Bieberbach yang berdimensi empat dengan kumpulan titik simetrik berperingkat enam dikira. Hasil menunjukkan kuasa dua tensor tak abelan bagi kumpulan ini adalah tidak abelan dan persembahannya dibina.

Kata kunci: Kumpulan Bieberbach, kuasa dua tensor tak abelan, kumpulan polikitaran

1.0 INTRODUCTION

A Bieberbach group is a torsion free crystallographic group G with a point group *P* given by the short exact sequence $1 \to L \to G \to P \to 1$, where L is the lattice subgroup. The dimension of the Bieberbach group is the rank of the lattice subgroup.

Tan *et al* [1] found that there are 36 Bieberbach groups with symmetric point group of order six. Three of them are of dimension four, 10 of them are of dimension five and the rest of them are of dimension six. In this paper, our focus is on one of the groups with dimension four, denoted as $B_1(4)$, which has a consistent polycyclic presentation as in the following.

$$B_1(4)=\left\langle a,b,l_1,l_2,l_3,l_4 \,\middle|\, \begin{array}{l} a^2=l_3, b^3=l_4, b^a=b^2l_4^{-1}, \\ l_1^a=l_1l_2^{-1}, l_2^a=l_2^{-1}, l_3^a=l_3, l_4^a=l_4^{-1}, \\ l_1^b=l_1^{-1}l_2, l_2^b=l_1^{-1}, l_3^b=l_3, l_4^b=l_4, \\ l_j^{l_i}=l_j, l_j^{l_i^{-1}}=l_j \quad \text{for} \quad j>i, \quad 1\leqslant i,j\leqslant 4 \end{array} \right\rangle \quad (1)$$

The nonabelian tensor square is an essential homological functor which is defined as in the following.

Definition 1.1 [2]
The nonabelian tensor square of a group G, $G\otimes G$, is a group generated by the symbols $g\otimes h$ for all $g,h\in G$, subject to relations

$gh\otimes k=(g^h\otimes k^h)(h\otimes k)$ and $g\otimes hk=(g\otimes k)(g^k\otimes h^k)$

for all $g,h,k\in G$ where $g^h=h^{-1}gh$.

The study of the nonabelian tensor square was first started by Brown *et al.* [3]. Following that, many researchers started to study the properties of the nonabelian tensor square. The nonabelian tensor square of 2-generator Burnside group of exponent 4 [4] and of 2-Engel groups of order at most 16 [5], to name a few, were computed in the past. Since year 2009, some researchers started to invest on the properties of the Bieberbach groups with certain point groups. The nonabelian tensor squares of Bieberbach groups with cyclic point groups [6] and of Bieberbach groups with dihedral point group [7, 8] were determined. In continuing the efforts in this field, the nonabelian tensor square of $B_1(4)$, denoted as $B_1(4)\otimes B_1(4)$, is determined in this paper.

2.0 PRELIMINARIES

In this section, some basic results that are used throughout this paper are included. We start with the definition of the group $\nu(G)$.

Definition 2.1 [9]
Let G be a group with presentation $\langle G|R\rangle$ and let G^{φ} be an isomorphic copy of G via the mapping $\varphi: g\to g^{\varphi}$ for all $g\in G$. Then the group $\nu(G)$ is defined to be

$\nu(G)=\left\langle G,G^{\varphi} \,\middle|\, R,R^{\varphi}, {}^{x}[g,h^{\varphi}]=[{}^{x}g,({}^{x}h)^{\varphi}]={}^{x\varphi}[g,h^{\varphi}], \forall x,g,h\in G\right\rangle.$

Theorem 2.1 [11]
Let G be any group whose abelianization is finitely generated by the independent set $x_iG', i=1,...,n$. Let K be the kernel of the epimorphism $\nabla(G)\to\nabla(G/G')$, and let $E(G)$ be the subgroup of $\nu(G)$ defined by $E(G)=\langle [x_i,x_j^{\varphi}] \,|\, 1\le i<j\le n\rangle\,[G,(G')^{\varphi}]$. Then

$\nabla(G)\cap E(G)=K$ and $\nabla(G)E(G)=[G,G^{\varphi}]$.

Theorem 2.2 shows that the nonabelian tensor square of group G is isomorphic to the normal subgroup of $\nu(G)$. Therefore, some of the computations can be done using the commutator calculus as listed in Proposition 2.1, Lemma 2.1, Lemma 2.2 and Corollary 2.1.

Theorem 2.2 [4]
Let G be a group. The map $\sigma: G\otimes G\to[G,G^{\varphi}]\triangleleft\nu(G)$ defined by $\sigma(g\otimes h)=[g,h^{\varphi}]$ for all g,h in G is an isomorphism.

Theorem 2.3 [3]
There is a homomorphism of a group $\kappa: G\otimes G\to G'$ such that $\kappa(g\otimes h)=g^{-1}g^h=[g,h]$.

Proposition 2.1 [9]
Let x, y and z be elements in a group G. Then for right conjugation, $y^x=x^{-1}yx$, $[x,y]=x^{-1}y^{-1}xy$ and the following applies

(i) $[xy,z]=[x,z]^y\cdot[y,z]=[x,z]\cdot[[x,z],y]\cdot[y,z]$;
(ii) $[x,yz]=[x,z]\cdot[x,y]^z=[x,z]\cdot[x,y]\cdot[[x,y],z]$;
(iii) $[x^{-1},y]=[x,y]^{-x^{-1}}=[x^{-1},[x,y]].[x,y]^{-1}$;
(iv) $[x,y^{-1}]=[x,y]^{-y^{-1}}=[y^{-1},[x,y]].[x,y]^{-1}$.

Lemma 2.1 [9, 10]
Let x and y be elements of G such that $[x,y]=1$. Then in $\nu(G)$,

(i) $[x^n,y^{\varphi}]=[x,y^{\varphi}]^n=[x,(y^{\varphi})^n]$ for all integer n;
(ii) $[x,y^{\varphi}]$ is central in $\nu(G)$.

Lemma 2.2 [9, 10]
Let G be a group. The following relations hold in $\nu(G)$:

(i) $[g_1,g_2{}^{\varphi}]^{[g_3,g_4^{\varphi}]}=[g_1,g_2{}^{\varphi}]^{[g_3,g_4]}$ for all g_1,g_2,g_3,g_4 in G;
(ii) $[g,g^{\varphi}]$ is central in $\nu(G)$ for all g in G;
(iii) $[g_1,g_2^{\varphi}][g_2,g_1^{\varphi}]$ is central in $\nu(G)$ for all g_1,g_2 in G;
(iv) $[g,g^{\varphi}]=1$ for all g in G';
(v) $[[g_1,g_2],[g_3,g_4]^{\varphi}]=[[g_1,g_2{}^{\varphi}],[g_3,g_4{}^{\varphi}]]$.

Corollary 2.1 [11]
Let G be a group. Then the following hold.

(i) If $g_1\in G'$ or $g_2\in G'$, then $[g_1,g_2^{\varphi}]^{-1}=[g_2,g_1^{\varphi}]$;
(ii) $[G',Z(G)^{\varphi}]=1$.

Lemma 2.3 [6]
Let G and H be groups and let $g\in G$. Suppose ϕ is a homomorphism from G to H. If $\phi(g)$ has finite order then $|\phi(g)|$ divides $|g|$. Otherwise the order of g equals the order of $\phi(g)$.

In [1], the polycyclic presentation of $B_1(4)$ as in (1) has been shown to be consistent. The following lemmas and theorem, which are also obtained in [1], are some basic results related to $B_1(4)$.

Lemma 2.4 [1]
The derived subgroup of $B_1(4)$ is given by $B_1(4)'=\langle b^2,l_2,l_4^2,l_2^{-1}l_1^2,l_1l_2\rangle$ while the abelianization of $B_1(4)$ is $B_1(4)^{ab}\cong\langle aB_1(4)',bB_1(4)'\rangle\cong C_0\times C_2$.

Lemma 2.5 [1]
Let $B_1(4)$ be a group which has a polycyclic presentation as in (1). Then, $[a,a^{\varphi}]$ has infinite order, $[a,b^{\varphi}][b,a^{\varphi}]$ has order 2 and $[b,b^{\varphi}]$ has order 4.

Theorem 2.4 [1]
The subgroup $\nabla(B_1(4))$ is given as

$$\nabla(B_1(4)) = \langle [a,a^{\varphi}],[b,b^{\varphi}],[a,b^{\varphi}][b,a^{\varphi}] \rangle \cong C_0 \times C_4 \times C_2.$$

3.0 MAIN RESULTS

In this section, the presentation of the nonabelian tensor square of $B_1(4)$, denoted as $B_1(4)\otimes B_1(4)$ is constructed.

Theorem 3.1
The nonabelian tensor square of $B_1(4)$ is nonabelian and its presentation is given as follows:

$$B_1(4)\otimes B_1(4) = \left\langle g_1,g_2\ldots g_8 \;\middle|\; \begin{array}{l} g_2^4 = g_3^2 = g_4^2 = [g_6,g_7]=[g_7,g_8]=1, \\ [g_5,g_6]=g_6^{-1}g_8^{-1}g_7^{2}g_4^{-1},[g_5,g_7]=g_6^{-2}g_8^{-2}g_7^{4}, \\ [g_5,g_8]=g_6^{-1}g_7^{2}g_4^{-1},[g_6,g_8]=g_4^{-1},[g_i,g_j]=1 \end{array} \right\rangle$$

for $1\le i\le 4, 1\le j\le 8$ where

$$g_1 = a\otimes a, g_2 = b\otimes b, g_3 = (a\otimes b)(b\otimes a), g_4 = l_1\otimes l_2,$$
$$g_5 = a\otimes b, g_6 = a\otimes l_1, g_7 = a\otimes l_2 \text{ and } g_8 = b\otimes l_2.$$

Proof. By Theorem 2.2, the isomorphism σ maps $g\otimes h$ onto $[g,h^{\varphi}]$ for all $g,h\in G$. Hence, the computation is done by using the commutators. By Lemma 2.4, $B_1(4)' = \langle b^2, l_2, l_4^2, l_2^{-1}l_1^2, l_1l_2\rangle$ and $B_1(4)^{ab} \cong \langle aB_1(4)', bB_1(4)'\rangle$. By Theorem 2.1, $E(B_1(4)) = \langle [a,b^{\varphi}]\rangle [B_1(4),B_1(4)'^{\varphi}]$ where $[B_1(4),B_1(4)'^{\varphi}]$

$$= \left\langle \begin{array}{l} [a,b^{2\varphi}],[b,b^{2\varphi}],[l_1,b^{2\varphi}],[l_2,b^{2\varphi}],[l_3,b^{2\varphi}],[l_4,b^{2\varphi}], \\ [a,l_2^{\varphi}],[b,l_2^{\varphi}],[l_1,l_2^{\varphi}],[l_2,l_2^{\varphi}],[l_3,l_2^{\varphi}],[l_4,l_2^{\varphi}], \\ [a,l_2^{2\varphi}],[b,l_2^{2\varphi}],[l_1,l_2^{2\varphi}],[l_2,l_2^{2\varphi}],[l_3,l_2^{2\varphi}],[l_4,l_2^{2\varphi}], \\ [a,l_4^{2\varphi}],[b,l_4^{2\varphi}],[l_1,l_4^{2\varphi}],[l_2,l_4^{2\varphi}],[l_3,l_4^{2\varphi}],[l_4,l_4^{2\varphi}], \\ [a,(l_2^{-1}l_1^2)^{\varphi}],[b,(l_2^{-1}l_1^2)^{\varphi}],[l_1,(l_2^{-1}l_1^2)^{\varphi}],[l_2,(l_2^{-1}l_1^2)^{\varphi}], \\ [l_3,(l_2^{-1}l_1^2)^{\varphi}],[l_4,(l_2^{-1}l_1^2)^{\varphi}],[a,(l_1l_2)^{\varphi}],[b,(l_1l_2)^{\varphi}], \\ [l_1,(l_1l_2)^{\varphi}],[l_2,(l_1l_2)^{\varphi}],[l_3,(l_1l_2)^{\varphi}],[l_4,(l_1l_2)^{\varphi}] \end{array} \right\rangle. \quad (2)$$

However, some of the generators in (2) are identities. Since $l_3 \in Z(B_1(4))$ and $b^2, l_2, l_4^2, l_2^{-1}l_1^2, l_1l_2 \in B_1(4)'$, then by Corollary 2.1 (ii), $[l_3,b^{2\varphi}]=[l_3,l_2^{\varphi}]=[l_3,l_2^{2\varphi}]=[l_3,l_4^{2\varphi}]=[l_3,(l_2^{-1}l_1^2)^{\varphi}]=[l_3,(l_1l_2)^{\varphi}]=1$. Also, by Lemma 2.2 (iv), $[l_2,l_2^{\varphi}]=1$ since $l_2\in B_1(4)'$. Then by Lemma 2.1 (i), $[l_2,l_2^{2\varphi}]=[l_2,l_2^{\varphi}]^2=1$. By Lemma 2.2 (iv), $[l_1,l_1^{\varphi}]=1$ as $(l_2)^2(l_2^{-1}l_1^2)(l_1l_2)^{-1}=l_1\in B_1(4)'$. Moreover, the generator $[l_4,(l_1l_2)^{\varphi}]$ is an identity as shown below.

$$\begin{aligned} &[l_4,(l_1l_2)^{\varphi}] \\ &=[l_4,l_2^{\varphi}][l_4,l_1^{\varphi}]^{l_2} && \text{by Proposition 2.1 (ii)} \\ &=[l_4,l_2^{\varphi}]^{b}[l_4,l_1^{\varphi}] && \text{by Lemma 2.1 (ii)} \\ &=[l_4,l_1^{-\varphi}][l_4,l_1^{\varphi}] && \text{by (1)} \\ &=[l_4,l_1^{\varphi}]^{-1}[l_4,l_1^{\varphi}] && \text{by Corollary 2.1 (i)} \\ &=1. \end{aligned}$$

The generator $[l_4,(l_2^{-1}l_1^2)^{\varphi}]$ can also be shown to be an identity by using similar arguments.

In addition to that, some of the generators in (2) can be written as the products of other generators. By Lemma 2.1 (i), $[b,b^{2\varphi}]=[b,b^{\varphi}]^2$, $[l_1,l_2^{2\varphi}]=[l_1,l_2^{\varphi}]^2$, $[l_4,l_2^{2\varphi}]=[l_4,l_2^{\varphi}]^2$, $[l_1,l_4^{2\varphi}]=[l_1,l_4^{\varphi}]^2$ and $[l_2,l_4^{2\varphi}]=[l_2,l_4^{\varphi}]^2$. Also, by (1) and Lemma 2.1 (i), we proved that $[l_4,b^{2\varphi}]=[b^3,b^{2\varphi}]=[b,b^{\varphi}]^6$, $[b,l_4^{2\varphi}]=[b,(b^3)^{2\varphi}]=[b,b^{\varphi}]^6$ and $[l_4,l_4^{2\varphi}]=[b^3,(b^3)^{2\varphi}]=[b,b^{\varphi}]^{18}$. Besides, we also have

$$\begin{aligned} &[a,b^{2\varphi}] \\ &=[a,b^{\varphi}][a,b^{\varphi}][[a,b],b^{\varphi}] && \text{by Proposition 2.1 (ii)} \\ &=[a,b^{\varphi}]^2[b^2,b^{\varphi}] \\ &=[a,b^{\varphi}]^2[b,b^{\varphi}]^2. && \text{by Lemma 2.1 (i)} \end{aligned}$$

Using similar method, we have $[a,l_4^{2\varphi}]=[a,b^{\varphi}]^6[b,b^{\varphi}]^{30}$, $[a,l_2^{2\varphi}]=[a,l_2^{\varphi}]^2$ and $[b,l_2^{2\varphi}]=[b,l_2^{\varphi}]^2[l_1,l_2^{\varphi}]$.

Again by Proposition 2.1 (i), we have

$$\begin{aligned} &[a,(l_1l_2)^{\varphi}] \\ &=[a,l_2^{\varphi}][a,l_1^{\varphi}][[a,l_1],l_2^{\varphi}] && \text{by Proposition 2.1 (ii)} \\ &=[a,l_2^{\varphi}][a,l_1^{\varphi}][l_2,l_2^{\varphi}] \\ &=[a,l_2^{\varphi}][a,l_1^{\varphi}]. \end{aligned}$$

With the similar arguments, we show that $[b,(l_1l_2)^{\varphi}]=[b,l_2^{\varphi}][b,l_1^{\varphi}][l_1,l_2^{\varphi}]^2$, $[a,(l_2^{-1}l_1^2)^{\varphi}]=[a,l_1^{\varphi}]^2[a,l_2^{\varphi}]^{-1}[l_1,l_2^{\varphi}]^3$ and $[b,(l_2^{-1}l_1^2)^{\varphi}]=[b,l_1^{\varphi}]^2[b,l_2^{\varphi}]^{-1}[l_1,l_2^{\varphi}]^4$.

Next, we have $[l_1,(l_1l_2)^{\varphi}] = [l_1,l_2^{\varphi}][l_1,l_1^{\varphi}]^{l_2} = [l_1,l_2^{\varphi}]$. We also have $[l_1,(l_2^{-1}l_1^2)^{\varphi}]=[l_1,l_2^{\varphi}]^{-1}$, $[l_2,(l_2^{-1}l_1^2)^{\varphi}]=[l_1,l_2^{\varphi}]^{-2}$ and $[l_2,(l_1l_2)^{\varphi}]=[l_1,l_2^{\varphi}]^{-1}$. Moreover,

$$\begin{aligned} &[l_1,b^{2\varphi}] \\ &=[b^2,l_1^{\varphi}]^{-1} && \text{by Corollary 2.1 (i)} \\ &=([b,l_1^{\varphi}][[b,l_1],b^{\varphi}][b,l_1^{\varphi}])^{-1} && \text{by Proposition 2.1 (i)} \\ &=([b,l_1^{\varphi}][b,[b,l_1]^{\varphi}]^{-1}[b,l_1^{\varphi}])^{-1} && \text{by Corollary 2.1 (i)} \\ &=([b,l_1^{\varphi}][b,(l_2^{-1}l_1^2)^{\varphi}]^{-1}[b,l_1^{\varphi}])^{-1} \\ &=[b,l_1^{\varphi}]^{-1}[b,(l_2^{-1}l_1^2)^{\varphi}][b,l_1^{\varphi}]^{-1} \\ &=[b,l_1^{\varphi}]^{-1}[b,l_1^{\varphi}]^2[b,l_2^{\varphi}]^{-1}[b,l_1^{\varphi}]^{-1} \\ &=[b,l_1^{\varphi}][b,l_2^{\varphi}]^{-1}[b,l_1^{\varphi}]^{-1} \end{aligned}$$

By using the same method, we show that $[l_2,b^{2\varphi}]=[b,l_1^{\varphi}][b,l_2^{\varphi}]^{-1}$ and $[l_4,l_2^{\varphi}]=[l_1,l_2^{\varphi}]^{-6}$.

From the above computations, we can conclude that the subgroup $[B_1(4),B_1(4)'^{\varphi}]$ is generated by seven generators: $[b,b^{\varphi}]$, $[a,b^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_2^{\varphi}]$, $[b,l_1^{\varphi}]$, $[b,l_2^{\varphi}]$ and $[l_1,l_2^{\varphi}]$. However, $[b,l_1^{\varphi}]=[a,l_2^{\varphi}]^{-2}[b,l_2^{\varphi}]^2[a,l_1^{\varphi}][l_1,l_2^{\varphi}]$. Thus, $[B_1(4),B_1(4)'^{\varphi}]=\langle [b,b^{\varphi}],[a,b^{\varphi}],[a,l_1^{\varphi}],[a,l_2^{\varphi}],[b,l_2^{\varphi}],[l_1,l_2^{\varphi}]\rangle$.

By Theorem 2.1,

$$[B_1(4),B_1(4)^{\varphi}] = \langle [a,a^{\varphi}],[b,b^{\varphi}],[a,b^{\varphi}][b,a^{\varphi}],[l_1,l_2^{\varphi}],[a,b^{\varphi}],[a,l_1^{\varphi}],[a,l_2^{\varphi}],[b,l_2^{\varphi}]\rangle.$$

Hence, by Theorem 2.2,

$$B_1(4)\otimes B_1(4) = \langle a\otimes a, b\otimes b, (a\otimes b)(b\otimes a), l_1\otimes l_2, a\otimes b, a\otimes l_1, a\otimes l_2, b\otimes l_2\rangle.$$

Next, the order of each generator in $[B_1(4), B_1(4)^{\varphi}]$ is determined. By Lemma 2.5, $[a,a^{\varphi}]$ has infinite order, $[b,b^{\varphi}]$ has order 4 and $[a,b^{\varphi}][b,a^{\varphi}]$ has order 2. The order of the other five generators are determined as follows.

$$\begin{aligned}
&[l_1,l_2^{\varphi}]^2 \\
&=[l_1,l_2^{\varphi}]^{a}[l_1,l_2^{\varphi}] && \text{by Lemma 2.2 (iii)} \\
&=[l_1^{a},(l_2^{a})^{\varphi}][l_1^{-1},l_2^{-\varphi}] && \text{by Lemma 2.1 (i)} \\
&=[l_1l_2^{-1},l_2^{-\varphi}][l_1^{-1},l_2^{-\varphi}] && \text{by (1)} \\
&=[l_1l_2^{-1},l_2^{-\varphi}]^{l_1^{-1}}[l_1^{-1},l_2^{-\varphi}] && \text{by Lemma 2.2 (iii)} \\
&=[l_1l_2^{-1}l_1^{-1},l_2^{-\varphi}] && \text{by Proposition 2.1 (i)} \\
&=[l_2,l_2^{\varphi}] && \text{by Lemma 2.1 (i)} \\
&=1.
\end{aligned}$$

This means that the order of $[l_1,l_2^{\varphi}]$ divides 2. Since $[l_1,l_2^{\varphi}]$ cannot be trivial, then $[l_1,l_2^{\varphi}]$ has order 2.

By Theorem 2.3, we have $\kappa : B_1(4)\otimes B_1(4) \to B_1(4)'$ such that

$$\kappa([a,b^{\varphi}])=[a,b]=b^2,$$
$$\kappa([a,l_1^{\varphi}])=[a,l_1]=l_2,$$
$$\kappa([a,l_2^{\varphi}])=[a,l_2]=l_2^2 \text{ and}$$
$$\kappa([b,l_2^{\varphi}])=[b,l_2]=l_1l_2.$$

Since $[a,b],[a,l_1],[a,l_2]$ and $[b,l_2]$ are all in $B_1(4)'$ and all of them have infinite order, by Lemma 2.3, $[a,b^{\varphi}]$, $[a,l_1^{\varphi}]$, $[a,l_2^{\varphi}]$ and $[b,l_2^{\varphi}]$ have infinite order.

The following example shows that not all generators in $B_1(4)\otimes B_1(4)$ commute to each other.

$$\begin{aligned}
&\left[[a,l_1^{\varphi}],[b,l_2^{\varphi}]\right] \\
&=\left[[a,l_1],[b,l_2]^{\varphi}\right] && \text{by Lemma 2.2 (v)} \\
&=[l_2,(l_1l_2)^{\varphi}] \\
&=[l_1,l_2^{\varphi}]^{-1} \\
&\neq 1.
\end{aligned}$$

Thus, we can conclude that $B_1(4)\otimes B_1(4)$ is nonabelian.

Lastly, the presentation of $B_1(4)\otimes B_1(4)$ is constructed. Let

$$g_1=a\otimes a,\ g_2=b\otimes b,\ g_3=(a\otimes b)(b\otimes a),\ g_4=l_1\otimes l_2,$$
$$g_5=a\otimes b,\ g_6=a\otimes l_1, g_7=a\otimes l_2 \text{ and } g_8=b\otimes l_2.$$

Since g_2 has order 4 and both g_3 and g_4 have order 2, then $g_2^4=g_3^2=g_4^2=1$. Moreover, by Lemma 2.2 (ii) and (iii), g_1,g_2,g_3 and g_4 are central in $\nu(B_1(4))$. Hence, $[g_i,g_j]=1$ for $1\le i\le 4, 1\le j\le 8$. Then, the rest of the commutators that are not in the central of $\nu(B_1(4))$ are $[g_5,g_6]$, $[g_5,g_7]$, $[g_5,g_8]$, $[g_6,g_7]$, $[g_6,g_8]$ and $[g_7,g_8]$. These commutators are computed as in the following.

$$\begin{aligned}
&[g_6,g_7] \\
&=[[a,l_1^{\varphi}],[a,l_2^{\varphi}]] && \text{by Lemma 2.2 (v)} \\
&=[[a,l_1],[a,l_2]^{\varphi}] \\
&=[l_2,l_2^{2\varphi}] && \text{by Lemma 2.1 (i)} \\
&=[l_2,l_2^{\varphi}]^2 \\
&=1,
\end{aligned}$$

$$\begin{aligned}
&[g_6,g_8] \\
&=[[a,l_1^{\varphi}],[b,l_2^{\varphi}]] && \text{by Lemma 2.2 (v)} \\
&=[[a,l_1],[b,l_2]^{\varphi}] \\
&=[l_2,(l_1l_2)^{\varphi}] \\
&=[l_1,l_2^{\varphi}]^{-1} \\
&=g_4^{-1},
\end{aligned}$$

$$\begin{aligned}
&[g_7,g_8] \\
&=[[a,l_2^{\varphi}],[b,l_2^{\varphi}]] && \text{by Lemma 2.2 (v)} \\
&=[[a,l_2],[b,l_2]^{\varphi}] \\
&=[l_2^2,(l_1l_2)^{\varphi}] && \text{by Lemma 2.1 (i)} \\
&=[l_2,(l_1l_2)^{\varphi}]^2 \\
&=([l_1,l_2^{\varphi}]^{-1})^2 \\
&=[l_1,l_2^{\varphi}]^{-2} \\
&=1,
\end{aligned}$$

$$\begin{aligned}
&[g_5,g_7] \\
&=[[a,b^{\varphi}],[a,l_2^{\varphi}]] \\
&=[[a,b],[a,l_2]^{\varphi}] && \text{by Lemma 2.2 (v)} \\
&=[b^2,l_2^{2\varphi}] \\
&=[b^2,l_2^{\varphi}][b^2,l_2^{\varphi}][[b^2,l_2],l_2^{\varphi}] && \text{by Proposition 2.1 (ii)} \\
&=[b^2,l_2^{\varphi}][b^2,l_2^{\varphi}][(l_2^{-1}l_1^2)^{-1},l_2^{\varphi}] \\
&=[b^2,l_2^{\varphi}][b^2,l_2^{\varphi}][l_2^{-1}l_1^2,l_2^{\varphi}]^{-1} && \text{by Lemma 2.1 (i)} \\
&=[l_2,b^{2\varphi}]^{-1}[l_2,b^{2\varphi}]^{-1}[l_2,(l_2^{-1}l_1^2)^{\varphi}] && \text{by Corollary 2.1 (i)} \\
&=([b,l_1^{\varphi}][b,l_2^{\varphi}]^{-1})^{-1}([b,l_1^{\varphi}][b,l_2^{\varphi}]^{-1})^{-1}[l_1,l_2^{\varphi}]^{-2} \\
&=[b,l_2^{\varphi}][b,l_1^{\varphi}]^{-1}[b,l_2^{\varphi}][b,l_1^{\varphi}]^{-1} \\
&=[b,l_2^{\varphi}]([a,l_2^{\varphi}]^{-2}[b,l_2^{\varphi}]^2[a,l_1^{\varphi}][l_1,l_2^{\varphi}])^{-1}[b,l_2^{\varphi}] \\
&\quad([a,l_2^{\varphi}]^{-2}[b,l_2^{\varphi}]^2[a,l_1^{\varphi}][l_1,l_2^{\varphi}])^{-1} \\
&=[b,l_2^{\varphi}][l_1,l_2^{\varphi}]^{-1}[a,l_1^{\varphi}]^{-1}[b,l_2^{\varphi}]^{-2}[a,l_2^{\varphi}]^2[b,l_2^{\varphi}] \\
&\quad[l_1,l_2^{\varphi}]^{-1}[a,l_1^{\varphi}]^{-1}[b,l_2^{\varphi}]^{-2}[a,l_2^{\varphi}]^2 \\
&=([a,l_1^{\varphi}]^{[b,l_2^{\varphi}]})^{-1}([a,l_2^{\varphi}]^{[b,l_2^{\varphi}]})^2[a,l_1^{\varphi}]^{-1}[b,l_2^{\varphi}]^{-2} \\
&\quad[a,l_2^{\varphi}]^2[l_1,l_2^{\varphi}]^{-2} \\
&=[a,l_1^{\varphi}]^{-1}[a,l_2^{\varphi}]^2[a,l_1^{\varphi}]^{-1}[b,l_2^{\varphi}]^{-2}[a,l_2^{\varphi}]^2 \\
&=[a,l_1^{\varphi}]^{-2}[b,l_2^{\varphi}]^{-2}[a,l_2^{\varphi}]^4 \\
&=g_6^{-2}g_8^{-2}g_7^4,
\end{aligned}$$

$$
\begin{aligned}
&[g_5, g_8] \\
&= [[a, b^{\varphi}], [b, l_2^{\varphi}]] \\
&= [[a, b], [b, l_2]^{\varphi}] && \text{by Lemma 2.2 (v)} \\
&= [b^2, (l_1 l_2)^{\varphi}] \\
&= [b, (l_1 l_2)^{\varphi}][[b, l_1 l_2], b^{\varphi}][b, (l_1 l_2)^{\varphi}] && \text{by Proposition 2.1 (ii)} \\
&= [b, (l_1 l_2)^{\varphi}][l_1^3, b^{\varphi}][b, (l_1 l_2)^{\varphi}] \\
&= [b, (l_1 l_2)^{\varphi}][b, l_1^{3\varphi}]^{-1}[b, (l_1 l_2)^{\varphi}] && \text{by Corollary 2.1 (i)} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}][l_1, l_2^{\varphi}]^2([b, l_1^{\varphi}]^3[l_1, l_2^{\varphi}]^3)^{-1}[b, l_2^{\varphi}] \\
&\quad [b, l_1^{\varphi}][l_1, l_2^{\varphi}]^2 \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-2}[b, l_2^{\varphi}][b, l_1^{\varphi}][l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}([b, l_2^{\varphi}]^{[b, l_1^{\varphi}]})[l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}([b, l_2^{\varphi}]^{[b, l_1]})[l_1, l_2^{\varphi}]^{-1} && \text{by Lemma 2.2 (i)} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}([b, l_2^{\varphi}]^{l_2^{-1} l_1^2})[l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}[l_1^{-3} b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}[l_1^{-3}, l_2^{\varphi}][[l_1^{-3}, l_2], b^{\varphi}] \\
&\quad [b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}[l_1^{-3}, l_2^{\varphi}][b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1} && \text{by Proposition 2.1 (i)} \\
&= [b, l_2^{\varphi}][b, l_1^{\varphi}]^{-1}[l_1, l_2^{\varphi}]^{-3}[b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1} \\
&= [b, l_2^{\varphi}]([a, l_2^{\varphi}]^{-2}[b, l_2^{\varphi}]^2[a, l_1^{\varphi}][l_1, l_2^{\varphi}])^{-1} && \text{by Lemma 2.1 (i)} \\
&\quad [b, l_2^{\varphi}] \\
&= [b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1}[a, l_1^{\varphi}]^{-1}[b, l_2^{\varphi}]^{-2}[a, l_2^{\varphi}]^2[b, l_2^{\varphi}] \\
&= [b, l_2^{\varphi}][a, l_1^{\varphi}]^{-1}[b, l_2^{\varphi}]^{-1}[b, l_2^{\varphi}]^{-1}[a, l_2^{\varphi}]^2[b, l_2^{\varphi}] \\
&\quad [l_1, l_2^{\varphi}]^{-1} \\
&= ([a, l_1^{\varphi}]^{[b, l_2^{\varphi}]})^{-1}([a, l_2^{\varphi}]^{[b, l_2^{\varphi}]})^2[l_1, l_2^{\varphi}]^{-1} \\
&= [a, l_1^{\varphi}]^{-1}[a, l_2^{\varphi}]^2[l_1, l_2^{\varphi}]^{-1} \\
&= g_6^{-1} g_7^2 g_4^{-1},
\end{aligned}
$$

$$
\begin{aligned}
&[g_5, g_6] \\
&= [[a, b^{\varphi}], [a, l_1^{\varphi}]] \\
&= [[a, b], [a, l_1]^{\varphi}] && \text{by Lemma 2.2 (v)} \\
&= [b^2, l_2^{\varphi}] \\
&= [l_2, b^{2\varphi}]^{-1} && \text{by Corollary 2.1 (i)} \\
&= ([b, l_1^{\varphi}][b, l_2^{\varphi}]^{-1})^{-1} \\
&= ([a, l_2^{\varphi}]^{-2}[b, l_2^{\varphi}]^2[a, l_1^{\varphi}][l_1, l_2^{\varphi}][b, l_2^{\varphi}]^{-1})^{-1} \\
&= [b, l_2^{\varphi}][l_1, l_2^{\varphi}]^{-1}[a, l_1^{\varphi}]^{-1}[b, l_2^{\varphi}]^{-2}[a, l_2^{\varphi}]^2 \\
&= ([a, l_1^{\varphi}]^{[b, l_2^{\varphi}]})^{-1}[b, l_2^{\varphi}]^{-1}[a, l_2^{\varphi}]^2[l_1, l_2^{\varphi}]^{-1} \\
&= [a, l_1^{\varphi}]^{-1}[b, l_2^{\varphi}]^{-1}[a, l_2^{\varphi}]^2[l_1, l_2^{\varphi}]^{-1} \\
&= g_6^{-1} g_8^{-1} g_7^2 g_4^{-1}.
\end{aligned}
$$

Hence, the presentation of $B_1(4) \otimes B_1(4)$ is shown as follows:

$$
B_1(4) \otimes B_1(4) = \left\langle g_1, g_2 \ldots g_8 \;\middle|\; \begin{array}{l} g_2^4 = g_3^2 = g_4^2 = [g_6, g_7] = [g_7, g_8] = 1, \\ [g_5, g_6] = g_6^{-1} g_8^{-1} g_7^2 g_4^{-1}, [g_5, g_7] = g_6^{-2} g_8^{-2} g_7^4, \\ [g_5, g_8] = g_6^{-1} g_7^2 g_4^{-1}, [g_6, g_8] = g_4^{-1}, [g_i, g_j] = 1 \end{array} \right\rangle.
$$

for $1 \le i \le 4, 1 \le j \le 8$. □

4.0 CONCLUSION

In this paper, the nonabelian tensor square of a Bieberbach group of dimension four with symmetry point group of order six is computed and is shown to be nonabelian. This computation leads us to the construction of the presentation of the nonabelian tensor square of this group. This finding can be further applied to compute other homological functors such as the exterior square and the Schur multiplier of the group.

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