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BI-IDEALS OF ORDERED SEMIGROUPS BASED ON THE INTERVAL-VALUED FUZZY POINT

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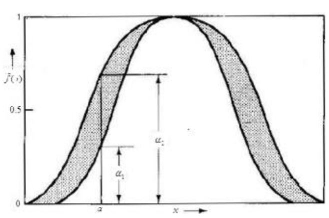
Graphical abstract



Abstract

Interval-valued fuzzy set theory (advanced generalization of Zadeh's fuzzy sets) is a more generalized theory that can deal with real world problems more precisely than ordinary fuzzy set theory. In this paper, we introduce the notion of generalized quasi-coincident with ($q_{\tilde{k}}$) relation of an interval-valued fuzzy point with an interval-valued fuzzy set. In fact, this new concept is a more generalized form of quasi-coincident with relation of an interval-valued fuzzy point with an interval-valued fuzzy set. Applying this newly defined idea, the notion of an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal is introduced. Moreover, some characterizations of interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideals are described. It is shown that an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal is an interval-valued fuzzy bi-ideal by imposing a condition on interval-valued fuzzy subset. Finally, the concept of implication-based interval-valued fuzzy bi-ideals, characterizations of an interval-valued fuzzy bi-ideal and an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal are considered.

Keywords: Interval-valued fuzzy bi-ideals, Interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal, Interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal, Interval-valued fuzzifying bi-ideal, $\tilde{t}$-implication-based interval-valued fuzzy bi-ideal

Abstrak

Teori set kabur bernilai-selang (pengitlakan lanjutan bagi set kabur Zadeh) adalah teori yang lebih menyeluruh yang dapat mengendalikan masalah-masalah dunia sebenar dengan lebih tepat berbanding teori set kabur biasa. Dalam kertas kerja ini, kami memperkenalkan idea kuasi-kebetulan teritlak ($q_{\tilde{k}}$) bagi suatu titik kabur bernilai-selang dengan suatu set kabur bernilai-selang. Malahan, konsep baru ini juga adalah bentuk yang lebih menyeluruh bagi kuasi-kebetulan suatu titik kabur bernilai-selang dengan suatu set kabur bernilai-selang. Dengan menggunakan idea yang baru didefinisikan ini, konsep bagi suatu dwi-unggulan kabur-$(\in, \in \vee q_{\tilde{k}})$ bernilai-selang telah diperkenalkan. Sebagai tambahan, beberapa pencirian bagi dwi-unggulan kabur-$(\in, \in \vee q_{\tilde{k}})$ bernilai-selang telah diterangkan. Telah ditunjukkan bahawa suatu dwi-unggulan kabur-$(\in, \in \vee q_{\tilde{k}})$ bernilai-selang adalah suatu dwi-unggulan kabur dengan diberikan satu syarat. Akhir sekali, konsep bagi dwi-unggulan kabur bernilai-selang yang berasaskan implikasi, pengitlakan bagi suatu dwi-unggulan kabur bernilai selang dan suatu dwi-unggulan kabur- $(\in,\in\vee q_{\tilde{k}})$ bernilai-selang telah dipertimbangkan.

Kata kunci: Dwi-unggulan kabur bernilai-selang, Dwi-unggulan kabur-$(\in,\in\vee q)$ bernilai-selang, Dwi-unggulan kabur-$(\in,\in\vee q_{\tilde{k}})$ bernilai-selang, Dwi-unggulan pengkaburan bernilai-selang, Dwi-unggulan kabur bernilai-selang berasaskan implikasi-$\tilde{t}$

1.0 INTRODUCTION

In mathematics, an ordered semigroup is a semigroup together with a partial order that is compatible with the semigroup operation. Ordered semigroups have many applications in the theory of sequential machines, formal languages, computer arithmetics, and error-correcting codes. The fundamental concept of a fuzzy set, introduced by L. A. Zadeh [1], provides a natural frame-work for generalizing several basic notions of algebra. Moreover, the study of fuzzy sets in semigroups was introduced by Kuroki [2-4]. Likewise, a systematic exposition of fuzzy semigroups was given by Mordeson *et al.* [5], where one can find theoretical results on fuzzy semigroups and their use in fuzzy coding, fuzzy finite state machines and fuzzy languages. In addition, the monograph by Mordeson and Malik [6] dealing with the application of fuzzy approach to the concepts of automata and formal languages. Moreover, Murali [7] proposed the definition of a fuzzy point belonging to a fuzzy subset under a natural equivalence on fuzzy subset. Besides, the idea of quasi-coincidence of a fuzzy point with a fuzzy set [8], played a vital role to generate some different types of fuzzy subgroups. Furthermore, Bhakat and Das [9-10] gave the concepts of (α,β)-fuzzy subgroups by using the "belongs to" relation $(\in)$ and "quasi-coincident with" relation (q) between a fuzzy point and a fuzzy subgroup, and introduced the concept of an $(\in,\in\vee q)$-fuzzy subgroup. Jun and Song [11] initiated the study of (α,β)-fuzzy interior ideals of a semigroup. In addition, Kazanci and Yamak [12] studied $(\in,\in\vee q)$-fuzzy bi-ideals of a semigroup and Shabir *et al.* [13] studied characterization of regular semigroups by (α,β)-fuzzy ideals. Moreover, Jun *et al.* [14] discussed generalization of an (α,β)-fuzzy ideals of a BCK/BCI-algebra. In addition, Shabir and Khan [15] characterized different classes of ordered semigroups by the properties of fuzzy quasi-ideals. Further, by applying fuzzy soft set theory the notions of fuzzy left (right, bi- and quasi-) ideals of type $(\in_{\gamma},\in_{\gamma}\vee q_{\delta})$ [16] are introduced in ordered semigroups. For further study on generalized fuzzy sets in ordered semigroups the reader is referred to [17-21]. The concept of a fuzzy bi-ideal in ordered semigroups was first introduced by Kehayopulu and Tsingelis in [22], where some basic properties of fuzzy bi-ideals were discussed. Furthermore, using the idea of a quasi-coincidence of a fuzzy point with a fuzzy set, Khan *et al.* [23] introduced the concept of interval-valued (α,β)-fuzzy bi-ideals in an ordered semigroup.

In this paper, we present a more general form of the idea presented in [23]. This new generalization is called an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal of an ordered semigroup. By constructing suitable examples, it is shown that there are interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideals which are not interval-valued $(\in,\in\vee q)$-fuzzy bi-ideals. In addition, ordered semigroups are characterized by the properties of this new concept. Further, a condition for an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal to be an interval-valued fuzzy bi-ideal is provided. It is important to note that several results of [23] are corollaries of our results obtained in this paper, which is the important achievement of this study.

2.0 PRELIMINARIES

By an *ordered semigroup* (or *po-semigroup*) we mean a structure $(S,\cdot,\leq)$ in which the following are satisfied for $x,a,b\in S$:

(OS1) $(S,\cdot)$ is a semigroup,

(OS2) $(S,\leq)$ is a poset,

(OS3) $a\leq b\Rightarrow a\cdot x\leq b\cdot x, x\cdot a\leq x\cdot b$.

Throughout this paper, $x\cdot y$ is simply denoted by xy for all $x,y\in S$.

A nonempty subset A of an ordered semigroup S is called a *subsemigroup* of S if $A^2\subseteq A$.

A non-empty subset A of an ordered semigroup S is called a *bi-ideal* of S if it satisfies the following three conditions:

(b1) for all $a\in S$ and $b\in A$, $a\leq b\Rightarrow a\in A$,

(b2) $A^2\subseteq A$,

(b3) $ASA\subseteq A$.

By an *interval number* $\tilde{a}$ we mean an interval $[a^-,a^+]$ where $0\leq a^-\leq a^+\leq 1$. The set of all interval numbers is denoted by $D[0,1]$. The interval $[a,a]$ can be simply identified by the number $a\in[0,1]$. We define the following for the interval numbers $\tilde{a}_i=[a_i^-,a_i^+]$, $\tilde{b}_i=[b_i^-,b_i^+]$ for all $i\in I$:

(i) $\mathrm{r}\max\{\tilde{a}_i,\tilde{b}_i\}=[\max(a^-_i,b^-_i),\max(a^+_i,b^+_i)]$,

(ii) $\mathrm{r}\min\{\tilde{a}_i,\tilde{b}_i\}=[\min(a^-_i,b^-_i),\min(a^+_i,b^+_i)]$,

(iii) $\mathrm{r}\inf\tilde{a}_i=[\wedge_{i\in I} a^-_i,\wedge_{i\in I} a^+]$, $\mathrm{r}\sup\tilde{a}_i=[\vee_{i\in I} a^-_i,\vee_{i\in I} a^+]$,

(iv) $\tilde{a}_1\le\tilde{a}_2 \Leftrightarrow a^-_1\le a^-_2$ and $a^+_1\le a^+_2$,

(v) $\tilde{a}_1=\tilde{a}_2 \Leftrightarrow a^-_1=a^-_2$ and $a^+_1=a^+_2$,

(vi) $\tilde{a}_1<\tilde{a}_2 \Leftrightarrow a^-_1\le a^-_2$ and $a^+_1\ne a^+_2$,

(vii) $k\tilde{a}_1=[ka^-_i,ka^+_i]$, whenever $0\le k\le 1$.

Then, it is clear that $(D[0,1],\le,\vee,\wedge)$ forms a complete lattice with $\tilde{0}=[0,0]$ as its least element and $\tilde{1}=[1,1]$ as its greatest element.

The interval-valued fuzzy subsets provide a more adequate description of uncertainty than the traditional fuzzy subsets; it is therefore important to use interval-valued fuzzy subsets in applications. One of the main applications of fuzzy subsets is fuzzy control, and one of the most computationally intensive parts of fuzzy control is the defuzzification. Since a transition to interval-valued fuzzy subsets usually increase the amount of computations, it is vitally important to design faster algorithms for the corresponding defuzzification.

An interval-valued fuzzy subset $\tilde{f}:X\to D[0,1]$ of X is the set

$$\tilde{f}=\{x\in X\mid x,[f^-(x),f^+(x)]\in D[0,1]\}$$

where $f^-:X\to[0,1]$ and $f^+:X\to[0,1]$ are fuzzy subsets such that $0\le f^-(x)\le f^+(x)\le 1$ for all $x\in X$ and $[f^-(x),f^+(x)]$ is the interval degree of membership function of an element x to the set $\tilde{f}$.

Let $\tilde{f}$ be an interval-valued fuzzy subset of X. Then for every $\tilde{0}\le\tilde{t}\le\tilde{1}$, the crisp set $U(\tilde{f};\tilde{t})=\{x\in X\mid \tilde{f}(x)\ge\tilde{t}\}$ is called the *level set* of $\tilde{f}$.

The reader is referred to[23] for more details on operations on two interval-valued fuzzy sets of X.

Note that since every $a\in[0,1]$ is in correspondence with the interval $[a,a]\in D[0,1]$, hence a fuzzy set is a special case of the interval-valued fuzzy set.

For any $\tilde{f}=[f^-,f^+]$ and $\tilde{t}=[t^-,t^+]$, we define $\tilde{f}(x)+\tilde{t}=[f^-(x)+t^-,f^+(x)+t^+]$, for all $x\in X$. In particular, if $f^-(x)+t^->1$ and $f^+(x)+t^+>1$, then we write $\tilde{f}(x)+\tilde{t}>\tilde{1}$.

An interval-valued fuzzy subset $\tilde{f}$ of an ordered semigroup S is called an *interval-valued fuzzy bi-ideal*[23] of S if it satisfies the following for all $x,y,z\in S$:

(b4) $x\le y\Rightarrow \tilde{f}(x)\ge\tilde{f}(y)$,

(b5) $\tilde{f}(xy)\ge \mathrm{r}\min\{\tilde{f}(x),\tilde{f}(y)\}$,

(b6) $\tilde{f}(xyz)\ge \mathrm{r}\min\{\tilde{f}(x),\tilde{f}(z)\}$.

An interval-valued fuzzy subset $\tilde{f}$ of an ordered semigroup $(S,\cdot,\le)$ of the form

$$\tilde{f}(y):=\begin{cases}\tilde{t}\in D(0,1], & \text{if } y=x,\\ \tilde{0}, & \text{if } y\ne x,\end{cases}$$

is called an *interval-valued fuzzy point* with support x and *interval value* $\tilde{t}\in D[0,1]$ and is denoted by $x_{\tilde{t}}$. For an interval-valued fuzzy subset $\tilde{f}$ of S, we say that an interval-valued fuzzy point $x_{\tilde{t}}$ is:

(b7) *contained in* $\tilde{f}$, denoted by $x_{\tilde{t}}\in\tilde{f}$, if $\tilde{f}(x)\ge\tilde{t}$.

(b8) *quasi-coincident with* $\tilde{f}$, denoted by $x_{\tilde{t}}\mathrm{q}\tilde{f}$, if $\tilde{f}(x)+\tilde{t}>\tilde{1}$.

For an interval-valued fuzzy point $x_{\tilde{t}}$ and an interval-valued fuzzy subset $\tilde{f}$ of a set S, we say that:

(b9) $x_{\tilde{t}}\in\vee\mathrm{q}\tilde{f}$ if $x_{\tilde{t}}\in\tilde{f}$ or $x_{\tilde{t}}\mathrm{q}\tilde{f}$.

(b10) $x_{\tilde{t}}\overline{\alpha}\tilde{f}$ if $x_{\tilde{t}}\alpha\tilde{f}$ does not hold for $\alpha\in\{\in,\mathrm{q},\in\vee\mathrm{q}\}$.

3.0 GENERALIZATION OF INTERVAL-VALUED $(\in,\in\vee\mathrm{q})$-FUZZY BI-IDEALS

Throughout this paper, S is an ordered semigroup and let $\tilde{k}=[k^-,k^+]$ denote an arbitrary element of $D[0,1)$ unless otherwise specified. For an interval-valued fuzzy point $x_{\tilde{t}}$ and an interval-valued fuzzy subset $\tilde{f}$ of S, we say that

(c1) $x_{\tilde{t}}\mathrm{q}_{\tilde{k}}\tilde{f}$ if $\tilde{f}(x)+\tilde{t}+\tilde{k}>\tilde{1}$, where $f^-+t^-+k^->1$, $f^++t^++k^+>1$.

(c2) $x_{\tilde{t}}\in\vee\mathrm{q}_{\tilde{k}}\tilde{f}$ if $x_{\tilde{t}}\in\tilde{f}$ or $x_{\tilde{t}}\mathrm{q}_{\tilde{k}}\tilde{f}$.

(c3) $x_{\tilde{t}}\overline{\alpha}\tilde{f}$ if $x_{\tilde{t}}\alpha\tilde{f}$ does not hold for any α in $\{\in,\mathrm{q}_{\tilde{k}},\in\vee\mathrm{q}_{\tilde{k}}\}$.

We emphasize here that the interval-valued fuzzy subset $\tilde{f}(x)=[f^-(x),f^+(x)]$ must satisfy the following condition:

$$\left.\begin{array}{l}[f^-(x),f^+(x)]\le\left[\dfrac{1-k^+}{2},\dfrac{1-k^-}{2}\right]\\ \text{or}\\ \left[\dfrac{1-k^+}{2},\dfrac{1-k^-}{2}\right]<[f^-(x),f^+(x)]\quad \text{for all } x\in X\end{array}\right\}\qquad (E)$$

In what follows, we emphasize that all the interval-valued fuzzy subsets of X must satisfy the condition (E) and any two elements of $D[0,1]$ are comparable unless otherwise specified.

3.1 Theorem

Let $\tilde{f}$ be an interval-valued fuzzy subset of S. Then the following are equivalent:

(1) $\left(\forall \tilde{t} \in D\left(\frac{1-k}{2},1\right]\right)$ $U(\tilde{f};\tilde{t}) \neq \phi \Rightarrow U(\tilde{f};\tilde{t})$ is a bi-ideal of S.

(2) $\tilde{f}$ satisfies the following assertions for all $x, y, z \in S$:

(2.1) $x \leq y \Rightarrow \tilde{f}(y) \leq \mathrm{r\,max}\left\{\tilde{f}(x),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$,

(2.2) $\mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} \leq \mathrm{r\,max}\left\{\tilde{f}(xy),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$,

(2.3) $\mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(z)\right\} \leq \mathrm{r\,max}\left\{\tilde{f}(xyz),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$.

Proof. Assume that $U(\tilde{f};\tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D\left(\frac{1-k}{2},1\right]$ with $U(\tilde{f};\tilde{t}) \neq \phi$. If there exist $a,b \in S$ with $a \leq b$ such that the condition (2.1) is not valid, then $\tilde{f}(b) > \mathrm{r\,max}\left\{\tilde{f}(a),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$. In which it follows that $\tilde{f}(b) \in D\left(\frac{1-k}{2},1\right]$ and $b \in U(\tilde{f},\tilde{f}(b))$. However, $\tilde{f}(a) < \tilde{f}(b)$ which implies that $a \notin U(\tilde{f},\tilde{f}(b))$, a contradiction. Hence (2.1) is valid.

Suppose that (2.2) is false, that is

$$\tilde{s} = \mathrm{r\,min}\left\{\tilde{f}(a),\tilde{f}(c)\right\} > \mathrm{r\,max}\left\{\tilde{f}(ac),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\},$$

for some $a,c \in S$. Then $\tilde{s} \in D\left(\frac{1-k}{2},1\right]$ and $a,c \in U(\tilde{f},\tilde{s})$ but $ac \notin U(\tilde{f},\tilde{s})$, a contradiction, and so (2.2) holds for all $x,y \in S$.

Assume that there exist $a,b,c \in S$ such that

$$\tilde{r} = \mathrm{r\,min}\left\{\tilde{f}(a),\tilde{f}(c)\right\} > \mathrm{r\,max}\left\{\tilde{f}(abc),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}.$$

This implies $\tilde{r} \in D\left(\frac{1-k}{2},1\right]$ and $a,c \in U(\tilde{f},\tilde{r})$ but $abc \notin U(\tilde{f},\tilde{r})$, this is impossible and therefore

$$\mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(z)\right\} \leq \mathrm{r\,max}\left\{\tilde{f}(xyz),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\},$$

for all $x,y,z \in S$.

Conversely, assume that $\tilde{f}$ satisfies all the three conditions (2.1), (2.2) and (2.3). Suppose that $U(\tilde{f};\tilde{t}) \neq \phi$ for all $\tilde{t} \in D\left(\frac{1-k}{2},1\right]$. Let $x,y \in S$ be such that $x \leq y$ and $y \in U(\tilde{f};\tilde{t})$. Then $\tilde{f}(y) \geq \tilde{t}$, and so by (2.1);

$$\mathrm{r\,max}\left\{\tilde{f}(x),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\} \geq \tilde{f}(y) \geq \tilde{t} > [\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}].$$

Hence $\tilde{f}(x) \geq \tilde{t}$, that is $x \in U(\tilde{f};\tilde{t})$.

If $x,y \in U(\tilde{f};\tilde{t})$, it follows from (2.2) that

$$\mathrm{r\,max}\left\{\tilde{f}(xy),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\} \geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} \geq \tilde{t} > [\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}],$$

in which it follows that $\tilde{f}(xy) \geq \tilde{t}$ i.e., $xy \in U(\tilde{f};\tilde{t})$.

Take $x,y \in U(\tilde{f};\tilde{t})$, it follows from (2.3) that

$$\mathrm{r\,max}\left\{\tilde{f}(xyz),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\} \geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(z)\right\} \geq \tilde{t} > [\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}].$$

This implies $\tilde{f}(xyz) \geq \tilde{t}$ i.e., $xyz \in U(\tilde{f};\tilde{t})$. Therefore $U(\tilde{f};\tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D\left(\frac{1-k}{2},1\right]$ with $U(\tilde{f};\tilde{t}) \neq \phi$.

Taking $\tilde{k} = [0,0]$ in Theorem 3.1 the following corollary arises.

3.2 Corollary

The following are equivalent for every interval-valued fuzzy subset $\tilde{f}$ of S:

(1) The level subset $U(\tilde{f};\tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D(0.5,1]$, whenever $U(\tilde{f};\tilde{t}) \neq \phi$.

(2) The interval-valued fuzzy subset $\tilde{f}$ satisfies the following assertions for all $x,y,z \in S$:

(2.1). $\tilde{f}(y) \leq \mathrm{r\,max}\left\{\tilde{f}(x),[0.5,0.5]\right\}$ for $x \leq y$,

(2.2). $\mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} \leq \mathrm{r\,max}\left\{\tilde{f}(xy),[0.5,0.5]\right\}$,

(2.3). $\mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(z)\right\} \leq \mathrm{r\,max}\left\{\tilde{f}(xyz),[0.5,0.5]\right\}$.

3.3 Definition

An interval-valued fuzzy subset $\tilde{f}$ of S is called an interval-valued $(\in,\in\vee q_{\tilde{k}})$-*fuzzy bi-ideal* of S if it satisfies the following conditions for all $x,y,z \in S$ and for all $\tilde{t},\tilde{t}_1,\tilde{t}_2 \in D(0,1]$:

(i) $(\forall x \leq y)\ (y_{\tilde{t}} \in \tilde{f} \rightarrow x_{\tilde{t}} \in \vee q_{\tilde{k}} \tilde{f})$,

(ii) $x_{\tilde{t}_1} \in \tilde{f}, y_{\tilde{t}_2} \in \tilde{f} \rightarrow (xy)_{\mathrm{r\,min}\{\tilde{t}_1,\tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{f}$,

(iii) $x_{\tilde{t}_1} \in \tilde{f}, z_{\tilde{t}_2} \in \tilde{f} \rightarrow (xyz)_{\mathrm{r\,min}\{\tilde{t}_1,\tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{f}$.

The following example is constructed to support the newly defined notion of interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideals in ordered semigroups.

3.4 Example

Consider the ordered semigroup $S = \{a,b,c,d,e\}$ with the following order relation "$\leq$" and multiplication Table 3.1:

$$\leq := \left\{\begin{matrix}(a,a),(a,c),(a,d),(a,e),(b,b),(b,d), \\ (b,e),(c,c),(c,e),(d,d),(d,e),(e,e)\end{matrix}\right\}.$$

Table 3.1 Multiplication table of S

$\cdot$	a	b	c	d	e
a	a	d	a	d	d
b	a	b	a	d	d
c	a	d	c	d	e
d	a	d	a	d	d
e	a	d	c	d	e

Define an interval-valued fuzzy subset $\tilde{f}: S \to D[0,1]$ by

$$\tilde{f}(x)=\begin{cases}[0.40,0.60], & \text{if } x=a,\\ [0.35,0.45], & \text{if } x\in\{b,c\},\\ [0.25,0.35], & \text{if } x=d,\\ [0.15,0.30], & \text{if } x=e.\end{cases}$$

Then using Definition 3.3, $\tilde{f}$ is an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S for all $\tilde{k}\geq[0.3,0.5]$.

The necessary and sufficient conditions for an interval-valued fuzzy subset to be an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal are given in the following result.

3.5 Theorem

An interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S if and only if:

(1) $\tilde{f}(x)\geq \mathrm{r}\min\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$ for $x\leq y$.

(2) $\tilde{f}(xy)\geq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$.

(3) $\tilde{f}(xyz)\geq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$.

Proof. Suppose that $\tilde{f}$ is an interval-valued $(\in,\in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S. Let $x,y\in S$ such that $x\leq y$. If $\tilde{f}(x)<\tilde{f}(y)$, then $\tilde{f}(x)<\tilde{t}\leq\tilde{f}(y)$ for some $\tilde{t}\in D\left(\frac{1-k}{2},1\right]$. It follows that $y_{\tilde{t}}\in\tilde{f}$, but $x_{\tilde{t}}\,\overline{\in}\,\tilde{f}$ and $x_{\tilde{t}}\overline{q}_{\tilde{k}}\tilde{f}$. Therefore, $x_{\tilde{t}}\,\overline{\in\vee q_{\tilde{k}}}\,\tilde{f}$, a contradiction and hence $\tilde{f}(x)\geq\tilde{f}(y)$. Now if $\tilde{f}(y)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $y_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\tilde{f}$ and so $x_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\vee q_{\tilde{k}}\tilde{f}$ that is $\tilde{f}(x)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$ or $\tilde{f}(x)+[\frac{1-k^+}{2},\frac{1-k^-}{2}]>\tilde{1}-\tilde{k}$. Hence, $\tilde{f}(x)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, otherwise,

$$\tilde{f}(x)+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]<[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]=\tilde{1}-\tilde{k},$$

again a contradiction. Consequently, $\tilde{f}(x)\geq \mathrm{r}\min\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$ for all $x,y\in S$ with $x\leq y$.

Let $x,y\in S$ be such that $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$. We claim that $\tilde{f}(xy)\geq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}$. If not, then $\tilde{f}(xy)<\tilde{t}\leq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}$ for some $\tilde{t}\in D\left(\frac{1-k}{2},1\right]$. It follows that $x_{\tilde{t}}\in\tilde{f}$ and $y_{\tilde{t}}\in\tilde{f}$, but $(xy)_{\tilde{t}}\,\overline{\in}\,\tilde{f}$ and $(xy)_{\tilde{t}}\overline{q}_{\tilde{k}}\tilde{f}$, a contradiction. Thus $\tilde{f}(xy)\geq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}$ for all $x,y\in S$ with $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$.

If $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y)\right\}\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $x_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\tilde{f}$ and $y_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\tilde{f}$. By using Definition 3.3 (ii), it implies that

$$(xy)_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}=(xy)_{\mathrm{r}\min\left\{[\frac{1-k^+}{2},\frac{1-k^-}{2}],[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}}\in\vee q_{\tilde{k}}\tilde{f}.$$

Hence $\tilde{f}(xy)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$ or $\tilde{f}(xy)+[\frac{1-k^+}{2},\frac{1-k^-}{2}]>\tilde{1}-\tilde{k}$. Now if $\tilde{f}(xy)<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then

$$\tilde{f}(xy)+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]<[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]=\tilde{1}-\tilde{k},$$

a contradiction and hence $\tilde{f}(xy)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$. Consequently, $\tilde{f}(xy)\geq \mathrm{r}\min\{\tilde{f}(x),\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\}$ for all $x,y\in S$.

Assume that $x,y,z\in S$ be such that $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z)\right\}<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$ and claim that $\tilde{f}(xyz)<\tilde{s}\leq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z)\right\}$ for some $\tilde{s}\in D\left(\frac{1-k}{2},1\right]$. It follows that $x_{\tilde{s}}\in\tilde{f}$ and $z_{\tilde{s}}\in\tilde{f}$, but $(xyz)_{\tilde{s}}\,\overline{\in}\,\tilde{f}$ and $\tilde{f}(xyz)+\tilde{s}<2\tilde{s}<\tilde{1}-\tilde{k}$, i.e., $(xyz)_{\tilde{s}}\overline{q}_{\tilde{k}}\tilde{f}$, a contradiction and hence $\tilde{f}(xyz)\geq \mathrm{r}\min\{\tilde{f}(x),\tilde{f}(z)\}$ for all $x,y,z\in S$ with $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z)\right\}<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$.

If $\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z)\right\}\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $x_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\tilde{f}$ and $z_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}\in\tilde{f}$ and by Definition 3.3 (iii),

$$(xyz)_{[\frac{1-k^+}{2},\frac{1-k^-}{2}]}=(xyz)_{\mathrm{r}\min\left\{[\frac{1-k^+}{2},\frac{1-k^-}{2}],[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}}\in\vee q_{\tilde{k}}\tilde{f}.$$

In which it follows that $\tilde{f}(xyz)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$ or $\tilde{f}(xyz)+[\frac{1-k^+}{2},\frac{1-k^-}{2}]>\tilde{1}-\tilde{k}$. If $\tilde{f}(xyz)<[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then

$$\tilde{f}(xyz)+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]<[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]+[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]=\tilde{1}-\tilde{k},$$

a contradiction and hence $\tilde{f}(xyz)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$. Consequently,

$$\tilde{f}(xyz)\geq \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\},$$

for all $x,y,z\in S$.

Conversely, let $\tilde{f}$ be an interval-valued fuzzy subset of S that satisfies the Conditions (1), (2) and (3). Let $x,y\in S$ and $\tilde{t}\in D(0,1]$ be such that $x\leq y$ and $y_{\tilde{t}}\in\tilde{f}$. Then $\tilde{f}(y)\geq\tilde{t}$, and so

$$\begin{aligned}\tilde{f}(x)&\geq \mathrm{r}\min\left\{\tilde{f}(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &\geq \mathrm{r}\min\left\{\tilde{t},[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &=\begin{cases}\tilde{t}, & \text{if } \tilde{t}\leq[\frac{1-k^+}{2},\frac{1-k^-}{2}],\\ [\frac{1-k^+}{2},\frac{1-k^-}{2}], & \text{if } \tilde{t}>[\frac{1-k^+}{2},\frac{1-k^-}{2}].\end{cases}\end{aligned}$$

It follows that $x_{\tilde{t}}\in\tilde{f}$ or $x_{\tilde{t}}q_{\tilde{k}}\tilde{f}$ i.e., $x_{\tilde{t}}\in\vee q_{\tilde{k}}\tilde{f}$.

Let $x,y\in S$ and $\tilde{t}_1,\tilde{t}_2\in D(0,1]$ be such that $x_{\tilde{t}_1}\in\tilde{f}$ and $y_{\tilde{t}_2}\in\tilde{f}$. Then $\tilde{f}(x)\geq\tilde{t}_1$ and $\tilde{f}(y)\geq\tilde{t}_2$. It follows from (2) that

$$\tilde{f}(xy) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$
$$\geq \text{r}\min\left\{\tilde{t}_1, \tilde{t}_2, [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$
$$= \begin{cases} \text{r}\min\{\tilde{t}_1, \tilde{t}_2\}, & \text{if} \quad \text{r}\min\{\tilde{t}_1, \tilde{t}_2\} \leq [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], \\ [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], & \text{if} \quad \text{r}\min\{\tilde{t}_1, \tilde{t}_2\} > [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], \end{cases}$$

This implies $(xy)_{\text{r}\min\{\tilde{t}_1, \tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{f}$.

If $x_{\tilde{t}_1} \in \tilde{f}$ and $z_{\tilde{t}_2} \in \tilde{f}$ for some $x, y, z \in S$ and $\tilde{t}_1, \tilde{t}_2 \in D(0,1]$, then $\tilde{f}(x) \geq \tilde{t}_1$ and $\tilde{f}(z) \geq \tilde{t}_2$. It follows from (3) that

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$
$$\geq \text{r}\min\{\tilde{t}_1, \tilde{t}_2, [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}$$
$$= \begin{cases} \text{r}\min\{\tilde{t}_1, \tilde{t}_2\}, & \text{if} \quad \text{r}\min\{\tilde{t}_1, \tilde{t}_2\} \leq [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], \\ [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], & \text{if} \quad \text{r}\min\{\tilde{t}_1, \tilde{t}_2\} > [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], \end{cases}$$

This implies, $(xyz)_{\text{r}\min\{\tilde{t}_1, \tilde{t}_2\}} \in \tilde{f}$ or $(xyz)_{\text{r}\min\{\tilde{t}_1, \tilde{t}_2\}} q_{\tilde{k}} \tilde{f}$ and consequently, $(xyz)_{\text{r}\min\{\tilde{t}_1, \tilde{t}_2\}} \in \vee q_{\tilde{k}} \tilde{f}$.

The following corollary arises by taking $\tilde{k} = [0,0]$ in Theorem 3.5.

3.6 Corollary

An interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued $(\in, \in\vee q)$-fuzzy bi-ideal of S if and only if for all $x, y, z \in S$ the following assertions hold:

(1) $(\forall x \leq y)\ (\tilde{f}(x) \geq \text{r}\min\{\tilde{f}(y), [0.5, 0.5]\})$.

(2) $\tilde{f}(xy) \geq \text{r}\min\{\tilde{f}(x), \tilde{f}(y), [0.5, 0.5]\}$.

(3) $\tilde{f}(xyz) \geq \text{r}\min\{\tilde{f}(x), \tilde{f}(z), [0.5, 0.5]\}$.

Clearly every interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal is an interval-valued fuzzy generalized bi-ideal of type $(\in, \in\vee q_{\tilde{k}})$. The following example illustrates that its converse is not true in general.

3.7 Example

Consider the ordered semigroup $S = \{a, b, c, d\}$ with the order relation "$\leq$" and multiplication given in Table 3.2:

$$\leq := \{(a,a), (b,b), (c,c), (d,d), (a,b)\}$$

Table 3.2 Multiplication table of S

.	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	b	a
d	a	a	b	b

Define an interval-valued fuzzy subset $\tilde{f} : S \rightarrow D[0,1]$ by

$$\tilde{f}(x) = \begin{cases} [0.8, 0.9], & \text{if} \quad x = a, \\ [0,0], & \text{if} \quad x = b, \\ [0.7, 0.8], & \text{if} \quad x = c, \\ [0,0], & \text{if} \quad x = d. \end{cases}$$

Then $\tilde{f}$ is interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy generalized bi-ideal. However, $cc = b$ but $\tilde{f}(cc) = \tilde{f}(b) = [0,0] < \text{r}\min\left\{\tilde{f}(c) = [0.7, 0.8], [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$ for all $\tilde{k} \in D[0,1)$ and hence Theorem 3.5 (2) is not satisfied. Therefore, $\tilde{f}$ is not an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal.

Since, every interval-valued fuzzy bi-ideal of S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S for some $\tilde{k} \in D[0,1)$. Therefore, the following example is constructed to show that there exists $\tilde{k} \in D[0,1)$ such that, the interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S but not an interval-valued fuzzy bi-ideal of S.

3.8 Example

The interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal $\tilde{f}$ of S in Example 3.4 is not an interval-valued fuzzy bi-ideal of S. Since $\tilde{f}(b) = [0.35, 0.45] = \tilde{f}(c)$ and $[0.25, 0.35] = \tilde{f}(d) = \tilde{f}(cb)$ but on the other hand;

$$\text{r}\min\left\{\tilde{f}(c), \tilde{f}(b)\right\} = \text{r}\min\{[0.35, 0.45], [0.35, 0.45]\} = [0.35, 0.45].$$

It follows that that $\tilde{f}(cb) < \text{r}\min\left\{\tilde{f}(c), \tilde{f}(b)\right\}$.

In the next result the condition under which an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal is an interval-valued fuzzy bi-ideal is provided.

3.9 Theorem

If for all $x \in S$ the value of the interval-valued fuzzy subset $\tilde{f}$ of S is less than the interval $[\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]$. Then every interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal $\tilde{f}$ of S is an interval-valued fuzzy bi-ideal of S.

Proof. Consider $\tilde{f}$ be an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S and $\tilde{f}(x) < [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]$ for all $x \in S$. Let $x, y \in S$ be such that $x \leq y$, then by Theorem 3.5 (1)

$$\tilde{f}(x) \geq \text{r}\min\left\{\tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$
$$= \tilde{f}(y) \left(\text{since } \tilde{f}(y) < [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right),$$

it follows that $\tilde{f}(x) \geq \tilde{f}(y)$.

If $x, y \in S$, then $\tilde{f}(x) < [\frac{1-k^+}{2}, \frac{1-k^-}{2}]$ and $\tilde{f}(y) < [\frac{1-k^+}{2}, \frac{1-k^-}{2}]$ this implies $\mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\} < [\frac{1-k^+}{2}, \frac{1-k^-}{2}]$ and Theorem 3.5 (2) implies

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &= \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\} \\ &\quad \left(\text{since } \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\} < [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right),\end{aligned}$$

that is $\tilde{f}(xy) \geq \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\}$. Finally, for $x, y, z \in S$ using Theorem 3.5 (3)

$$\begin{aligned}\tilde{f}(xyz) &\geq \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &= \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z)\} \left(\text{since } \tilde{f}(x) < [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\ \forall x \in S\right),\end{aligned}$$

in which it follows that $\tilde{f}(xyz) \geq \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z)\}$ for all $x, y, z \in S$. The above discussion shows that $\tilde{f}$ is an interval-valued fuzzy bi-ideal of S.

By taking $\tilde{k} = [0,0]$ in the Theorem 3.9, reduced to the following corollary.

3.10 Corollary

An interval-valued $(\in, \in\vee \mathrm{q})$-fuzzy bi-ideal $\tilde{f}$ of S is ordinary interval-valued fuzzy bi-ideal of S, if $\tilde{f}(x) < [0.5, 0.5]$ for all $x \in S$.

The following result shows that intersection of any finite collection of interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal of an ordered semigroup S is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal.

3.11 Theorem

If $\{\tilde{f}_i\}_{i\in I} \neq \phi$ is a collection of interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal of S, then $\bigcap_{i\in I} \tilde{f}_i$ is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal of S.

Proof. Let $\tilde{f}_i$ be an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal of S for all $i \in I$ and $a, b \in S$ with $a \leq b$. Consider

$$\begin{aligned}\bigcap_{i\in I} \tilde{f}_i(a) &= \bigwedge_{i\in I} \tilde{f}_i(a) \\ &\geq \bigwedge_{i\in I} \{\mathrm{r\,min}\{\tilde{f}_i(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}\} \\ &= \mathrm{r\,min}\{\bigcap_{i\in I} \tilde{f}_i(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}.\end{aligned}$$

Next we take $a, b \in S$ and consider

$$\begin{aligned}\bigcap_{i\in I} \tilde{f}_i(ab) &= \bigwedge_{i\in I} \tilde{f}_i(ab) \\ &\geq \bigwedge_{i\in I} \{\mathrm{r\,min}\{\tilde{f}_i(a), \tilde{f}_i(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}\} \\ &= \mathrm{r\,min}\{\bigcap_{i\in I} \tilde{f}_i(a), \bigcap_{i\in I} \tilde{f}_i(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}.\end{aligned}$$

Finally, if $a, b, c \in S$, then

$$\begin{aligned}\bigcap_{i\in I} \tilde{f}_i(abc) &= \bigwedge_{i\in I} \tilde{f}_i(abc) \\ &\geq \bigwedge_{i\in I} \{\mathrm{r\,min}\{\tilde{f}_i(a), \tilde{f}_i(c), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}\} \\ &= \mathrm{r\,min}\{\bigcap_{i\in I} \tilde{f}_i(a), \bigcap_{i\in I} \tilde{f}_i(c), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\}.\end{aligned}$$

Consequently by Theorem 3.5 $\bigcap_{i\in I} \tilde{f}_i$ is an interval-valued $(\in, \in\vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal of S.

Based on the level subsets, the following three theorems establish links between interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideals and ordinary interval-valued fuzzy bi-ideals.

3.12 Theorem

For an interval-valued fuzzy subset $\tilde{f}$ of S, the following assertions are equivalent:

(1) An interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S.

(2) The level subset $U(\tilde{f}; \tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D\left(0, \frac{1-k}{2}\right]$ whenever, $U(\tilde{f}; \tilde{t}) \neq \phi$.

Proof. Assume that $U(\tilde{f}; \tilde{t}) \neq \phi$ for all $\tilde{t} \in D(\frac{1-k}{2}, 1]$ and $\tilde{f}$ be an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S. If $x, y \in S$ with $x \leq y$ such that $y \in U(\tilde{f}; \tilde{t})$, then by Theorem 3.5 (1),

$$\begin{aligned}\tilde{f}(x) &\geq \mathrm{r\,min}\{\tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &\geq \mathrm{r\,min}\{\tilde{t}, [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &= \tilde{t},\end{aligned}$$

in which it follows that $x \in U(\tilde{f}; \tilde{t})$.

If $x, y \in U(\tilde{f}; \tilde{t})$, then $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(y) \geq \tilde{t}$ and Theorem 3.5 (2) induces that

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &\geq \mathrm{r\,min}\{\tilde{t}, \tilde{t}, [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\} \\ &= \tilde{t}.\end{aligned}$$

Thus $xy \in U(\tilde{f}; \tilde{t})$.

Now let $x, z \in U(\tilde{f}; \tilde{t})$, then $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(z) \geq \tilde{t}$. Hence, Theorem 3.5 (3) implies that

$$\begin{aligned}\tilde{f}(xyz) &\geq \mathrm{r\,min}\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}\\ &\geq \mathrm{r\,min}\left\{\tilde{t}, \tilde{t}, [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}\\ &= \tilde{t}.\end{aligned}$$

In which it follows that $xyz \in U(\tilde{f};\tilde{t})$. Consequently, $U(\tilde{f};\tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$.

Conversely, let $\tilde{f}$ be an interval-valued fuzzy subset of S such $U(\tilde{f};\tilde{s}) \neq \phi$ is a bi-ideal for all $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$. If there exist $a, b \in S$ with $a \leq b$ and such that

$$\tilde{f}(a) < \mathrm{r\,min}\left\{\tilde{f}(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

then $\tilde{f}(a) < \tilde{t} \leq \mathrm{r\,min}\left\{\tilde{f}(b), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$ for some $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$. This implies $b \in U(\tilde{f};\tilde{t})$ but $a \,\overline{\in}\, U(\tilde{f};\tilde{t})$, a contradiction. Therefore, $\tilde{f}(x) \geq \mathrm{r\,min}\left\{\tilde{f}(y), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$ for all $x, y \in S$ with $x \leq y$. Let there exist $a, b \in S$ such that

$$\tilde{f}(ab) < \mathrm{r\,min}\left\{\tilde{f}(a), \tilde{f}(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

then $\tilde{f}(ab) < \tilde{s} \leq \mathrm{r\,min}\left\{\tilde{f}(a), \tilde{f}(b), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$ for some $\tilde{s} \in D\left(\frac{1-k}{2}, 1\right]$. It follows that $a, b \in U(\tilde{f};\tilde{s})$ but $ab \notin U(\tilde{f};\tilde{s})$. This is impossible because $U(\tilde{f};\tilde{s})$ is a bi-ideal and thus $\tilde{f}(xy) \geq \mathrm{r\,min}\left\{\tilde{f}(x), \tilde{f}(y), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$ for all $x, y \in S$.
Let there exist $a, b, c \in S$ such that $\tilde{f}(abc) < \mathrm{r\,min}\left\{\tilde{f}(a), \tilde{f}(c), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$. Then $\tilde{f}(abc) < \tilde{r} \leq \mathrm{r\,min}\left\{\tilde{f}(a), \tilde{f}(c), [\frac{1-k^+}{2}, \frac{1-k^-}{2}]\right\}$ for some $\tilde{r} \in (0, \frac{1-k}{2}]$. It follows that $a, c \in U(\tilde{f};\tilde{r})$ but $abc \notin U(\tilde{f};\tilde{r})$, a contradiction and hence

$$\tilde{f}(xyz) \geq \mathrm{r\,min}\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} \text{ for all } x, y, z \in S.$$

Using Theorem 3.1 it is concluded that $\tilde{f}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S.
Theorem 3.12 induces the following corollary by taking $\tilde{k} = [0,0]$.

3.13 Corollary

If $\tilde{f}$ is an interval-valued fuzzy subset of S, then the following assertions are equivalent:
(1) $\tilde{f}$ is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.
(2) For all $\tilde{t} \in (0, 0.5]$ the non-empty level subset $U(\tilde{f};\tilde{t})$ is a bi-ideal of S.

3.14 Theorem

If $\tilde{f}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S, then the non-empty level subset $Q^{\tilde{k}}(\tilde{f};\tilde{t})$ is a bi-ideal of S for all $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$.

Proof. Assume that $\tilde{f}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S. Let $\tilde{t} \in D\left(\frac{1-k}{2}, 1\right]$ be such that $Q^{\tilde{k}}(\tilde{f};\tilde{t}) \neq \phi$. If $y \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$ and $x \in S$ be such that $x \leq y$, then $\tilde{f}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and by Theorem 3.5 (1),

$$\begin{aligned}\tilde{f}(x) &\geq \mathrm{r\,min}\left\{\tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}[\frac{1-k^+}{2}, \frac{1-k^-}{2}], & \text{if } \tilde{f}(y) \geq [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\\ \tilde{f}(y), & \text{if } \tilde{f}(y) < [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\end{cases}\\ &> \tilde{1} - \tilde{t} - \tilde{k},\end{aligned}$$

in which it follows that $x \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$.

Let $x, y \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$, then $\tilde{f}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{f}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. It follows from Theorem 3.5 (2) that

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\left\{\tilde{f}(x), \tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}\mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\}, & \text{if } \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\} < [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\\ [\frac{1-k^+}{2}, \frac{1-k^-}{2}], & \text{if } \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(y)\} \geq [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\end{cases}\\ &> \tilde{1} - \tilde{t} - \tilde{k},\end{aligned}$$

this shows that $xy \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$.

If $x, z \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$, then $\tilde{f}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{f}(z) + \tilde{t} > \tilde{1} - \tilde{k}$. It follows from Theorem 3.5 (3) that

$$\begin{aligned}\tilde{f}(xyz) &\geq \mathrm{r\,min}\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}\mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z)\}, & \text{if } \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z)\} < [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\\ [\frac{1-k^+}{2}, \frac{1-k^-}{2}], & \text{if } \mathrm{r\,min}\{\tilde{f}(x), \tilde{f}(z)\} \geq [\frac{1-k^+}{2}, \frac{1-k^-}{2}],\end{cases}\\ &> \tilde{1} - \tilde{t} - \tilde{k},\end{aligned}$$

hence $xyz \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$. From the above discussion it is concluded that the level subset $Q^{\tilde{k}}(\tilde{f};\tilde{t})$ is a bi-ideal of S.

The following corollary is derived from Theorem 3.14 by taking $\tilde{k} = [0,0]$.

3.15 Corollary

If $\tilde{f}$ is an interval-valued $(\in, \in \vee \mathrm{q})$-fuzzy bi-ideal of S and $Q(\tilde{f};\tilde{t}) \neq \phi$ for all $\tilde{t} \in (0.5, 1]$, then $Q(\tilde{f};\tilde{t})$ is a bi-ideal of S.

3.16 Theorem

An interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued fuzzy $(\in, \in \vee \mathrm{q}_{\tilde{k}})$-fuzzy bi-ideal if and only if $[\tilde{f}]_{\tilde{t}}^{\tilde{k}}$ is a bi-ideal of S for all $\tilde{t} \in D(0,1]$.

Proof. Assume that $\tilde{f}$ is an interval-valued $(\in, \in\vee q_{\tilde{k}})$-fuzzy bi-ideal of S and $[\tilde{f}]_{\tilde{t}}^{\tilde{k}} \neq \phi$ for all $\tilde{t} \in D(0,1]$. Let $y \in [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$ and $x \in S$ be such that $x \leq y$. Then $y \in U(\tilde{f};\tilde{t})$ or $y \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$ i.e., $\tilde{f}(x) \geq \tilde{t}$ or $\tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$ and by Theorem 3.5 (1), we have $\tilde{f}(x) \geq \mathrm{r\,min}\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\}$.
The following two cases are considered here:
(i). $\tilde{f}(y) \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}]$,
(ii). $\tilde{f}(y) > [\frac{1-k^+}{2},\frac{1-k^-}{2}]$.

If $\tilde{f}(y) \geq \tilde{t}$, then the first case implies that $\tilde{f}(x) \geq \tilde{f}(y) \geq \tilde{t}$ and therefore $x \in U(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$ and if $\tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$, then $\tilde{f}(x)+\tilde{t} > \tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$, it follows that $x \in Q^{\tilde{k}}(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$. Combining the second case with Theorem 3.5 (1) induces $\tilde{f}(x) \geq [\frac{1-k^+}{2},\frac{1-k^-}{2}]$. If $\tilde{t} \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $\tilde{f}(x) \geq \tilde{t}$ and hence $x \in U(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$. Now, if $\tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $\tilde{f}(x)+\tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}]+[\frac{1-k^+}{2},\frac{1-k^-}{2}] = \tilde{1}-\tilde{k}$, it follows that $x \in Q^{\tilde{k}}(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$.

If $x, y \in [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$, then $\tilde{f}(x) \geq \tilde{t}$ or $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$, $\tilde{f}(y) \geq \tilde{t}$ or $\tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$. The following four cases are considered here:
(i) If $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(y) \geq \tilde{t}$.
(ii) If $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$.
(iii) If $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$ and $\tilde{f}(y) \geq \tilde{t}$.
(iv) If $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$ and $\tilde{f}(y)+\tilde{t} > \tilde{1}-\tilde{k}$.

Using Case (i) in Theorem 3.5 (2) leads to

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &\geq \mathrm{r\,min}\left\{\tilde{t},\tilde{t},[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}[\frac{1-k^+}{2},\frac{1-k^-}{2}], & \text{if } \tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}],\\ \tilde{t}, & \text{if } \tilde{t} \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}],\end{cases}\end{aligned}$$

it follows that $xy \in U(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$ or $xy \in Q^{\tilde{k}}(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$ and hence $xy \in [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$.

For the second if $\tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $1-\tilde{t}-\tilde{k} < [\frac{1-k^+}{2},\frac{1-k^-}{2}]$ and Theorem 3.5 (2) it implies that

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}\mathrm{r\,min}\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} > \tilde{1}-\tilde{t}-\tilde{k},\\ \text{if r min}\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \leq \tilde{f}(x),\\ \tilde{f}(x) \geq \tilde{t},\\ \text{if r min}\left\{\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} > \tilde{f}(x),\end{cases}\end{aligned}$$

in which it follows that $xy \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$. On the other hand if $\tilde{t} \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}\mathrm{r\,min}\left\{\tilde{f}(x),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \geq \tilde{t},\\ \text{if r min}\left\{\tilde{f}(x),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \leq \tilde{f}(y),\\ \tilde{f}(y) > \tilde{1}-\tilde{t}-\tilde{k},\\ \text{if r min}\left\{\tilde{f}(x),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} > \tilde{f}(y),\end{cases}\end{aligned}$$

it follows that $xy \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$. The similar result can be obtained for Case (iii).

For the final case, if $\tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $1-\tilde{t}-\tilde{k} < [\frac{1-k^+}{2},\frac{1-k^-}{2}]$. Hence $\tilde{f}(xy) \geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} > 1-\tilde{t}-\tilde{k}$, thus $xy \in Q^{\tilde{k}}(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$. If $\tilde{t} \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then

$$\begin{aligned}\tilde{f}(xy) &\geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}[\frac{1-k^+}{2},\frac{1-k^-}{2}] \geq \tilde{t},\\ \text{if r min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} \geq [\frac{1-k^+}{2},\frac{1-k^-}{2}],\\ \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} > \tilde{1}-\tilde{t}-\tilde{k},\\ \text{if r min}\left\{\tilde{f}(x),\tilde{f}(y)\right\} < [\frac{1-k^+}{2},\frac{1-k^-}{2}],\end{cases}\end{aligned}$$

this implies $xy \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$.

Let $x, z \in [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$, then $x, z \in U(\tilde{f};\tilde{t})$ or $x, z \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$ that is $\tilde{f}(x) \geq \tilde{t}$ or $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$ and $\tilde{f}(z) \geq \tilde{t}$ or $\tilde{f}(z)+\tilde{t} > \tilde{1}-\tilde{k}$. In this regard the following four cases are considered:
(i) If $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(z) \geq \tilde{t}$.
(ii) If $\tilde{f}(x) \geq \tilde{t}$ and $\tilde{f}(z)+\tilde{t} > \tilde{1}-\tilde{k}$.
(iii) If $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$ and $\tilde{f}(z) \geq \tilde{t}$.
(iv) If $\tilde{f}(x)+\tilde{t} > \tilde{1}-\tilde{k}$ and $\tilde{f}(z)+\tilde{t} > \tilde{1}-\tilde{k}$.

For the case (i), Theorem 3.5(3) implies that

$$\begin{aligned}\tilde{f}(xyz) &\geq \mathrm{r\,min}\left\{\tilde{f}(x),\tilde{f}(z),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &\geq \mathrm{r\,min}\left\{\tilde{t},\tilde{t},[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\\ &= \begin{cases}[\frac{1-k^+}{2},\frac{1-k^-}{2}], & \text{if } \tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}],\\ \tilde{t}, & \text{if } \tilde{t} \leq [\frac{1-k^+}{2},\frac{1-k^-}{2}],\end{cases}\end{aligned}$$

in which it follows that $xyz \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]_{\tilde{t}}^{\tilde{k}}$.

For the second case assume that $\tilde{t} > [\frac{1-k^+}{2},\frac{1-k^-}{2}]$, then $1-\tilde{t}-\tilde{k} < [\frac{1-k^+}{2},\frac{1-k^-}{2}]$. From Theorem 3.5 (3) it can be seen that

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$

$$= \begin{cases} \text{r}\min\left\{\tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} > \tilde{1} - \tilde{t} - \tilde{k}, \\ \text{if r}\min\left\{\tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} \leq \tilde{f}(x), \\ \tilde{f}(x) \geq \tilde{t}, \\ \text{if r}\min\left\{\tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} > \tilde{f}(x), \end{cases}$$

it follows that $xyz \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$. Now if $\tilde{t} \leq [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]$, then Theorem 3.5 (3) implies that

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$

$$= \begin{cases} \text{r}\min\left\{\tilde{f}(x), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} \geq \tilde{t}, \\ \text{if r}\min\left\{\tilde{f}(x), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} \leq \tilde{f}(z), \\ \tilde{f}(x) > \tilde{1} - \tilde{t} - \tilde{k}, \\ \text{if r}\min\left\{\tilde{f}(x), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} > \tilde{f}(z), \end{cases}$$

hence $xyz \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$.
Similarly, the result can be obtained for Case (iii).
For the final case, if $\tilde{t} > [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]$, then by Theorem 3.5 (3)

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} > \tilde{1} - \tilde{t} - \tilde{k},$$

in which it follows that $xyz \in Q^{\tilde{k}}(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$. On the other hand if $\tilde{t} \leq [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]$, then

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$$

$$= \begin{cases} [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}] \geq \tilde{t}, \\ \text{if r}\min\left\{\tilde{f}(x), \tilde{f}(z)\right\} \geq [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}], \\ \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z)\right\} > \tilde{1} - \tilde{t} - \tilde{k}, \\ \text{if r}\min\left\{\tilde{f}(x), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\} < \tilde{f}(z), \end{cases}$$

it follows that $xyz \in U(\tilde{f};\tilde{t}) \cup Q^{\tilde{k}}(\tilde{f};\tilde{t}) = [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$. Consequently, $[\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ is a bi-ideal of S for all $\tilde{t} \in D(0,1]$.

Conversely, suppose that $[\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ is a bi-ideal of S for all $\tilde{t} \in D(0,1]$. If there exist $a,b \in S$ with $a \leq b$ such that

$$\tilde{f}(a) < \text{r}\min\left\{\tilde{f}(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

then

$$\tilde{f}(a) < \tilde{t}_a \leq \text{r}\min\left\{\tilde{f}(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

for some $\tilde{t}_a \in D\left(0, \dfrac{1-k}{2}\right]$. It follows that $b \in U(\tilde{f};\tilde{t}_a) \subseteq [\tilde{f}]^{\tilde{k}}_{\tilde{t}_a}$ but $a \,\overline{\in}\, U(\tilde{f};\tilde{t}_a)$ and $a \,\overline{\in}\, Q^{\tilde{k}}(\tilde{f};\tilde{t}_a)$ it follows that, $a \,\overline{\in}\, [\tilde{f}]^{\tilde{k}}_{\tilde{t}_a}$, a contradiction and hence $\tilde{f}(x) \geq \text{r}\min\left\{\tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$ for all $x, y \in S$ with $x \leq y$.
If there exist $a,b \in S$ such that

$$\tilde{f}(ab) < \tilde{t} \leq \text{r}\min\left\{\tilde{f}(a), \tilde{f}(b), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

for some $\tilde{t} \in D\left(0, \dfrac{1-k}{2}\right]$, then $a,b \in U(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ and since $[\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ is a bi-ideal therefore $ab \in [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$. In which it follows that $\tilde{f}(ab) \geq \tilde{t}$ or $\tilde{f}(ab) + \tilde{t} > \tilde{1} - \tilde{k}$, a contradiction. Therefore, $\tilde{f}(xy) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(y), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$ for all $x, y \in S$.
Assume that $\tilde{f}(abc) < \tilde{t} \leq \text{r}\min\left\{\tilde{f}(a), \tilde{f}(c), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\}$ for some $a,b,c \in S$ and $\tilde{t} \in D\left(0, \dfrac{1-k}{2}\right]$. It follows that, $a, c \in U(\tilde{f};\tilde{t}) \subseteq [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$. This implies $abc \in [\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ (since $[\tilde{f}]^{\tilde{k}}_{\tilde{t}}$ is a bi-ideal) again a contradiction. Therefore,

$$\tilde{f}(xyz) \geq \text{r}\min\left\{\tilde{f}(x), \tilde{f}(z), [\tfrac{1-k^+}{2}, \tfrac{1-k^-}{2}]\right\},$$

for all $x, y, z \in S$. By Theorem 3.5 it is concluded that $\tilde{f}$ is an $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S.

Taking $\tilde{k} = [0,0]$ in Theorem 3.16, we get the following corollary.

3.17 Corollary

For any interval-valued fuzzy subset $\tilde{f}$ of S, the following are equivalent:
(1) The interval-valued fuzzy subset $\tilde{f}$ of S is an interval-valued $(\in, \in \vee q)$-fuzzy bi-ideal of S.
(2) The non-empty level subset $[\tilde{f}]_{\tilde{t}}$ is a bi-ideal of S for all $\tilde{t} \in D(0,1]$.

3.16 Theorem

For $0 \leq \tilde{k} < \tilde{r} < \tilde{1}$ the interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal $\tilde{f}$ of S, the $q_{\tilde{k}}$-level subset $Q^{\tilde{r}}(\tilde{f};\tilde{t})$ ($Q^{\tilde{r}}(\tilde{f};\tilde{t}) \neq \phi$) is a bi-ideal of S for all $\tilde{t} \in D(\tfrac{1-r}{2}, 1]$.

Proof. Suppose that $Q^{\tilde{k}}(\tilde{f};\tilde{t}) \neq \phi$ be such that $0 \leq \tilde{k} < \tilde{r} < \tilde{1}$ and $\tilde{f}$ be an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S. Then by Theorem 3.14 $Q^{\tilde{k}}(\tilde{f};\tilde{t}) \neq \phi$ is a bi-ideal of S for all $\tilde{t} \in D(\tfrac{1-k}{2}, 1]$. It follows that, if $x, y \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$ for some $\tilde{t} \in D(\tfrac{1-k}{2}, 1]$, then $\tilde{f}(x) + \tilde{t} > \tilde{1} - \tilde{k}$ and $\tilde{f}(y) + \tilde{t} > \tilde{1} - \tilde{k}$. By hypothesis $\tilde{k} < \tilde{r}$, therefore, $1 - \tilde{k} > 1 - \tilde{r}$ and hence the above inequalities imply, $\tilde{f}(x) + \tilde{t} > \tilde{1} - \tilde{r}$ and $\tilde{f}(y) + \tilde{t} > \tilde{1} - \tilde{r}$. In which it follows that $x, y \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$ for $\tilde{t} \in D(\tfrac{1-r}{2}, 1]$

(since $\tilde{t} > [\frac{1-k^+}{2}, \frac{1-k^-}{2}] > [\frac{1-r^+}{2}, \frac{1-r^-}{2}]$). Again, since $Q^{\tilde{k}}(\tilde{f};\tilde{t})$ is a bi-ideal therefore, $xy \in Q^{\tilde{k}}(\tilde{f};\tilde{t})$ and hence $\tilde{f}(xy)+\tilde{t} > \tilde{1}-\tilde{k} > \tilde{1}-\tilde{r}$. This implies $xy \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$ for all $\tilde{t} \in D(\frac{1-r}{2},1]$. Similarly, we can show that $xyz \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$ for $x,y,z \in S$ such that $x,z \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$. Likewise, $x \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$ whenever $x,y \in S$ and that $x \le y \in Q^{\tilde{r}}(\tilde{f};\tilde{t})$. This completes the proof.

4.0 IMPLICATION-BASED INTERVAL-VALUED FUZZY BI-IDEALS

Fuzzy logic is based on fuzzy set theory while the classical one uses classical set theory. Thus it is clear that fuzzy logic is an extension of classical logic. In fuzzy logic the truth values are linguistic variables or terms of the linguistic variable truth. In fuzzy logic, the truth value of fuzzy proposition Φ is denoted by $[\Phi]$.

For a universe U of discourse, the well-known fuzzy logical and corresponding set-theoretical notations used in this paper are displayed in the following lines:

$$\begin{aligned}&[x \in \tilde{f}] = \tilde{f}(x),\\ &[\Phi \wedge \Psi] = \min\{[\Phi],[\Psi]\},\\ &[\Phi \to \Psi] = \min\{1, 1-[\Phi]+[\Psi]\},\\ &[\forall \Phi(x)] = \inf_{x\in U}[\Phi(x)].\end{aligned} \qquad (4.1)$$

$\models \Phi$ if and only if $[\Phi]=1$ for all valuations (4.2)

The truth valuation rules given in (4.1) are those in the Łukasiewicz system of continuous-valued logic.

Of course, various implication operators have been defined. We show only a selection of them in the following.

(a) Gaines-Rescher implication operator (I_{GR}):

$$I_{GR}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Gődel implication operator (I_G):

$$I_G(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ b & \text{otherwise.} \end{cases}$$

(c) The contraposition of Gődel implication operator (I_{cG}):

$$I_{cG}(a,b) = \begin{cases} 1 & \text{if } a \le b, \\ 1-a & \text{otherwise.} \end{cases}$$

Ying [24] introduced an important concept of fuzzifying topology. This new concept of fuzzifying topology can be extended to other algebraic structures. In this connection, the notion of interval-valued fuzzifying bi-ideal of ordered semigroup is defined as follows.

4.1 Definition

An interval-valued fuzzy subset $\tilde{f}$ of S is called an interval-valued fuzzifying bi-ideal of S if it satisfies the following conditions for all $x,y,z \in S$:

(i) $\models [y \in \tilde{f}] \to [x \in \tilde{f}]$ for all $x \le y$,

(ii) $\models \mathrm{r}\min\{[x \in \tilde{f}],[y \in \tilde{f}]\} \to [xy \in \tilde{f}]$,

(iii) $\models \mathrm{r}\min\{[x \in \tilde{f}],[z \in \tilde{f}]\} \to [xyz \in \tilde{f}]$.

4.2 Definition

An interval-valued fuzzy subset $\tilde{f}$ of S and $\tilde{t} \in D(0,1]$ is called a $\tilde{t}$-implication-based bi-ideal of S if it satisfies the following assertions for all $x,y,z \in S$:

(i) $\models_{\tilde{t}} [y \in \tilde{f}] \to [x \in \tilde{f}]$ for all $x \le y$,

(ii) $\models_{\tilde{t}} \mathrm{r}\min\{[x \in \tilde{f}],[y \in \tilde{f}]\} \to [xy \in \tilde{f}]$,

(iii) $\models_{\tilde{t}} \mathrm{r}\min\{[x \in \tilde{f}],[z \in \tilde{f}]\} \to [xyz \in \tilde{f}]$.

Let I be an implication operator. Clearly, $\tilde{f}$ is a $\tilde{t}$-implication-based interval-valued fuzzy bi-ideal of S if and only if the following assertions are satisfied by all $x,y,z \in S$:

(i) $x \le y \Rightarrow I\left(\tilde{f}(x),\tilde{f}(y)\right) \ge \tilde{t}$,

(ii) $I\left(\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y),\tilde{f}(xy)\right\}\right) \ge \tilde{t}$,

(iii) $I\left(\mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z),\tilde{f}(xyz)\right\}\right) \ge \tilde{t}$.

In the next theorem ordered semigroups are characterized by the properties of ordered $[\frac{1-k^+}{2},\frac{1-k^-}{2}]$-implication-based interval-valued fuzzy bi-ideal.

4.3 Theorem

For any interval-valued fuzzy subset $\tilde{f}$ of S, we have the following two results:

(1) If $I = I_G$ and $\tilde{f}$ is a $[\frac{1-k^+}{2},\frac{1-k^-}{2}]$-implication-based interval-valued fuzzy bi-ideal of S, then $\tilde{f}$ is an interval-valued $(\in, \in \vee q_{\tilde{k}})$-fuzzy bi-ideal of S.

(2) If $\tilde{f}$ is a $[\frac{1-k^+}{2},\frac{1-k^-}{2}]$-implication-based interval-valued fuzzy bi-ideal of S for $I = I_{CG}$, then the following conditions hold for all $x,y,z \in S$:

(2.1) $x \le y \Rightarrow \mathrm{r}\max\left\{\tilde{f}(x),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \ge \mathrm{r}\min\left\{\tilde{f}(y),\tilde{1}\right\}$,

(2.2) $\mathrm{r}\max\left\{\tilde{f}(xy),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \ge \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(y),\tilde{1}\right\}$,

(2.3) $\mathrm{r}\max\left\{\tilde{f}(xyz),[\frac{1-k^+}{2},\frac{1-k^-}{2}]\right\} \ge \mathrm{r}\min\left\{\tilde{f}(x),\tilde{f}(z),\tilde{1}\right\}$.

Proof. (1). If $\tilde{f}$ is an $[\frac{1-k^-}{2},\frac{1-k^+}{2}]$-implication-based interval-valued fuzzy bi-ideal of S, then the following assertions hold:

(i) $x \le y \Rightarrow I_G\left(\tilde{f}(x),\tilde{f}(y)\right) \ge [\frac{1-k^+}{2},\frac{1-k^-}{2}]$,

(ii) $I_G\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\},\tilde f(xy)\right)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$,

(iii) $I_G\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\},\tilde f(xyz)\right)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$.

for all $x,y,z\in S$.

Let $x,y\in S$ be such that $x\leq y$. From (i) we have $\tilde f(y)\geq\tilde f(x)$ or $\tilde f(x)>\tilde f(y)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$ and hence

$$\tilde f(x)\geq \mathrm{r}\min\left\{\tilde f(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}.$$

Condition (ii) implies, $\tilde f(xy)\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\}$ or $\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\}>\tilde f(xy)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$. It follows that

$$\tilde f(xy)\geq\mathrm{r}\min\{\tilde f(x),\tilde f(y),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\}.$$

Using Condition (iii), we get $\tilde f(xyz)\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\}$ or $\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\}>\tilde f(xyz)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, in which it follows that

$$\tilde f(xyz)\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(z),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}.$$

From the above discussion, it is concluded in the light of Theorem 3.5 that $\tilde f$ is an interval-valued $(\in,\in\vee q_{\tilde k})$-fuzzy bi-ideal of S.

(2) Assume that $\tilde f$ is an $[\frac{1-k^+}{2},\frac{1-k^-}{2}]$-implication-based interval-valued fuzzy bi-ideal of S, then the following hold for all $x,y,z\in S$:

(iv) $x\leq y\Rightarrow I_{cG}\left(\tilde f(x),\tilde f(y)\right)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$,

(v) $I_{cG}\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\},\tilde f(xy)\right)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$,

(vi) $I_{cG}\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\},\tilde f(xyz)\right)\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$.

If $x,y\in S$ such that $x\leq y$, then by (iv) we have, $I_{cG}(\tilde f(x),\tilde f(y))=\tilde 1$ or $\tilde 1-\tilde f(y))\geq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$, it follows that $\tilde f(y)\geq\tilde f(x)$ or $\tilde f(y)\leq[\frac{1-k^+}{2},\frac{1-k^-}{2}]$. Therefore,

$$\mathrm{r}\max\left\{\tilde f(x),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\geq\tilde f(y)=\mathrm{r}\min\left\{\tilde f(y),\tilde 1\right\}.$$

From (v), we have;

$$I_{cG}(\mathrm{r}\min\{\tilde f(x),\tilde f(y)\},\tilde f(xy))=\tilde 1,$$

or

$$I_{cG}\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\},\tilde f(xy)\right)=\tilde 1-\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\},$$

that is,

$$\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\}\leq\tilde f(xy)$$

or

$$\tilde 1-\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\}\geq[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}].$$

Hence

$$\mathrm{r}\max\left\{\tilde f(xy),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\}=\mathrm{r}\min\left\{\tilde f(x),\tilde f(y),\tilde 1\right\}$$

for all $x,y\in S$.

Finally, from (vi), we have;

$$I_{cG}\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\},\tilde f(xyz)\right)=\tilde 1$$

or

$$I_{cG}\left(\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\},\tilde f(xyz)\right)=\tilde 1-\mathrm{r}\min\left\{\tilde f(x),\tilde f(y)\right\},$$

that is,

$$\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\}\leq\tilde f(xyz)$$

or

$$\tilde 1-\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\}\geq[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}].$$

Hence,

$$\mathrm{r}\max\left\{\tilde f(xyz),[\tfrac{1-k^+}{2},\tfrac{1-k^-}{2}]\right\}\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(z)\right\}=\mathrm{r}\min\left\{\tilde f(x),\tilde f(z),\tilde 1\right\}$$

for all $x,y\in S$. This completes the proof.

Taking $\tilde k=[0,0]$ in Theorem 4.3 leads to the following corollaries.

4.4 Corollary

If $I=I_G$, then any interval-valued fuzzy subset $\tilde f$ of S is a $[0.5,0.5]$-implication-based interval-valued fuzzy bi-ideal of S is an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal of S.

4.5 Corollary

If $I=I_{CG}$ and $\tilde f$ is a $[0.5,0.5]$-implication-based interval-valued fuzzy bi-ideal of S, then the following conditions hold for all $x,y,z\in S$:

(1) $x\leq y\Rightarrow\mathrm{r}\max\left\{\tilde f(x),[0.5,0.5]\right\}\geq\mathrm{r}\min\left\{\tilde f(y),\tilde 1\right\}$,

(2) $\mathrm{r}\max\left\{\tilde f(xy),[0.5,0.5]\right\}\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(y),\tilde 1\right\}$,

(3) $\mathrm{r}\max\left\{\tilde f(xyz),[0.5,0.5]\right\}\geq\mathrm{r}\min\left\{\tilde f(x),\tilde f(z),\tilde 1\right\}$.

5.0 CONCLUSION

The notion of an interval-valued $(\in,\in\vee q_{\tilde k})$-fuzzy bi-ideal in ordered semigroups is introduced as a more general form of an interval-valued $(\in,\in\vee q)$-fuzzy bi-ideal. Further, ordered semigroups are characterised by the properties of this new type of interval-valued fuzzy bi-ideal and several properties are investigated. In addition, characterizations of an interval-valued fuzzy bi-ideal and an interval-valued $(\in,\in\vee q_{\tilde k})$-fuzzy bi-ideal are considered by using implication operators and the notion of implication-based an interval-valued fuzzy bi-ideal.

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