Axiomatizing the lexicographic products of modal logics with linear temporal logic

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Abstract

Given modal logics λ_1, λ_2, their lexicographic product $\lambda_1 \rhd \lambda_2$ is a new logic whose frames are the Cartesian products of a λ_1-frame and a λ_2-frame, but with the new accessibility relations reminiscent of a lexicographic ordering. This article considers the lexicographic products of several modal logics with linear temporal logic (LTL) based on "next" and "always in the future". We provide axiomatizations for logics of the form $\lambda \rhd \mathsf{LTL}$ and define *cover-simple* classes of frames; we then prove that, under fairly general conditions, our axiomatizations are sound and complete whenever the class of λ-frames is cover-simple. Finally, we prove completeness for several concrete logics of the form $\lambda \rhd \mathsf{LTL}$.

Keywords: Modal logics. Linear temporal logic. Lexicographic product. Axiomatization/completeness.

1 Introduction

There are a great many applications of modal logic to computer science and artificial intelligence that require the use of propositional languages mixing different sorts of modal connectives. By just considering the logical aspects of multi-agent systems, there are, for example, the combination of dynamic logic with epistemic logic [9] or the combination of temporal logic with epistemic logic [10]. There exist many ways to mix together given normal modal logics λ_1 and λ_2 defined over disjoint sets of modal connectives; the appropriate way to do so depends on the application at hand.

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If λ_1 and λ_2 are axiomatically presented, their fusion simply consists of putting together their axiomatical presentations. If λ_1 and λ_2 are semantically defined by means of the classes $\mathcal{C}_1$ and $\mathcal{C}_2$ of frames, their fusion simply consists of the modal logic determined by the class of all frames $(W, R_1, \ldots, R_m, S_1, \ldots, S_n)$ such that $(W, R_1, \ldots, R_m)$ is a $\mathcal{C}_1$-frame and $(W, S_1, \ldots, S_n)$ is a $\mathcal{C}_2$-frame. In both cases, the question arises whether the fusion operation preserves properties like decidability, interpolation, etc. [14,15,21].

However, in temporal epistemic logics with no learning and perfect recall, the fusion operation is not the most appropriate way of combining modal logics. In such a setting we may consider a different way to mix logics, given by the asynchronous product operation. Given modal logics λ_1 and λ_2, their asynchronous product is the logic of the products $(W_1 \times W_2, R'_1, \ldots, R'_m, S'_1, \ldots, S'_n)$ of $\mathcal{C}_1$-frames $(W_1, R_1, \ldots, R_m)$ and $\mathcal{C}_2$-frames $(W_2, S_1, \ldots, S_n)$ where $(x_1, x_2)R'_i(y_1, y_2)$ iff $x_1 R_i y_1$ and $x_2 = y_2$, and $(x_1, x_2)S'_j(y_1, y_2)$ iff $x_1 = y_1$ and $x_2 S_j y_2$. See [11,12,15].

More recently, first within the context of qualitative temporal reasoning and then within the context of ordinary modal logics, the first author [2,3] has introduced a third way of mixing modal logics: the lexicographic way. Given modal logics λ_1 and λ_2, their lexicographic product is the logic of the products $(W_1 \times W_2, R'_1, \ldots, R'_m, S'_1, \ldots, S'_n)$ of $\mathcal{C}_1$-frames $(W_1, R_1, \ldots, R_m)$ and $\mathcal{C}_2$-frames $(W_2, S_1, \ldots, S_n)$ where $(x_1, x_2)R'_i(y_1, y_2)$ iff $x_1 R_i y_1$ and $x_2 = y_2$ and $(x_1, x_2)S'_j(y_1, y_2)$ iff $x_2 S_j y_2$.

It has appeared later that the operation of lexicographic products has strong similarities with the operation of ordered sum considered, for example, by Beklemishev [7] within the context of the provability logic GLP. See also Babenyshev and Rybakov [1] and Shapirovsky [19]. The similarity between lexicographic products and ordered sums consists of the fact that, in many situations, the lexicographic product of two Kripke complete modal logics is equal to their ordered sum [6].

Layout of the article. This article considers the lexicographic products of modal logics with linear temporal logic based on "next" and "always in the future". It provides complete axiomatizations of the sets of valid formulas they give rise to. The section-by-section breakdown of the paper is as follows. In Sections 2–4, we present the syntax, the semantics and a minimal axiomatization of our lexicographic products. The aim of Section 5 is to define a requirement allowing us to assert a general completeness theorem. In Section 6, we provide more specific requirements making it possible to apply this general completeness theorem. Finally, Section 7 contains the proof that many familiar modal logics satisfy these more specific requirements. Easy proofs will be omitted.

2 Preliminaries

Let us review a few preliminary notions that we will use throughout the text. We assume the reader feels at home with tools and techniques in modal logic (generated subframes, bounded morphisms, etc.). For more on this, see [8].

2.1 Syntax

Let $\mathbb{P}$ be a countable set of propositional variables (with typical members denoted p, q, etc.) and $\mathfrak{S}$ be a countable set of modalities (with typical members denoted a, b, etc.). The set $\mathcal{L}_{\mathfrak{S}}$ of all formulas (with typical members denoted φ, ψ, etc.) is inductively defined as follows:

$$\varphi, \psi ::= p \mid \bot \mid \neg\varphi \mid (\varphi \vee \psi) \mid [a]\varphi.$$

The Boolean connectives $\top$, $\wedge$, $\rightarrow$ and $\leftrightarrow$ are defined by the usual abbreviations. As usual, $\langle a \rangle$ is the modal connective defined by $\langle a \rangle\varphi ::= \neg[a]\neg\varphi$.

Let $\mathfrak{S}^{\circ,G} = \mathfrak{S} \cup \{\circ, G\}$ and $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$ be the corresponding set of formulas. We will simply write $\circ\varphi$ ("at all next moments of time, φ") instead of $[\circ]\varphi$ and $\hat{\circ}\varphi$ ("at some next moment of time, φ") instead of $\langle\circ\rangle\varphi$. Similarly, we will simply write $G\varphi$ ("at all future moments of time, φ") instead of $[G]\varphi$ and $F\varphi$ ("at some future moment of time, φ") instead of $\langle G \rangle\varphi$. We adopt the standard rules for omission of parentheses.

For all formulas φ, let $SF(\varphi)$ be the set of all subformulas of φ and $SF^{\neg}(\varphi)$ be the closure of $SF(\varphi)$ under negations. In the sequel, we use $\varphi(p_1, \ldots, p_n)$ to denote a formula whose propositional variables form a subset of $\{p_1, \ldots, p_n\}$. For all sets Γ of $\mathcal{L}_{\mathfrak{S}}$-formulas, let $\neg\Gamma = \{\neg\varphi : \varphi \in \Gamma\}$. For all $I \subseteq \mathfrak{S}$, let Γ^I be the set of all $\mathcal{L}_{\mathfrak{S}}$-formulas in Γ of the form $[a]\varphi$ or of the form $\langle a \rangle\varphi$ for some $a \in I$; we will usually omit parentheses when elements of I are written extensionally, as in, e.g., $\Gamma^{\circ,G}$.

2.2 Semantics

A $\mathfrak{S}$-*frame* is a relational structure of the form $\mathcal{F} = (W, R)$ where W is a non-empty set of *states* (with typical members denoted s, t, etc.) and R is a function associating to each $a \in \mathfrak{S}$ a binary relation $R(a)$ on W. For all $a \in \mathfrak{S}$ and for all states s in W, let $R(a)(s) = \{t \in W : sR(a)t\}$. If $\mathfrak{S}'$ is a countable set of modalities containing $\mathfrak{S}$ and $\mathcal{F}' = (W', R')$ is a $\mathfrak{S}'$-frame then $\mathcal{F}'^{|\mathfrak{S}} = (W, R)$ is the $\mathfrak{S}$-frame defined as follows:

- $W = W'$,
- for all $a \in \mathfrak{S}$, $R(a) = R'(a)$.

A *model based on a* $\mathfrak{S}$-*frame* $\mathcal{F} = (W, R)$ is a relational structure — also called an $\mathfrak{S}$-*model* — of the form $\mathcal{M} = (W, R, V)$, where $V : \mathbb{P} \rightarrow 2^W$. The function V is called the *valuation of* $\mathcal{M}$. The relation "the $\mathcal{L}_{\mathfrak{S}}$-formula φ is true in the $\mathfrak{S}$-model $\mathcal{M}$ at state s" (in symbols $\mathcal{M}, s \models \varphi$) is inductively defined as follows:

- $\mathcal{M}, s \models p$ iff $s \in V(p)$,
- $\mathcal{M}, s \not\models \bot$,
- $\mathcal{M}s \models \neg\varphi$ iff $\mathcal{M}, s \not\models \varphi$,
- $\mathcal{M}, s \models \varphi \vee \psi$ iff either $\mathcal{M}, s \models \varphi$, or $\mathcal{M}, s \models \psi$,
- $\mathcal{M}, s \models [a]\varphi$ iff for all states t in $\mathcal{M}$, if $sR(a)t$ then $\mathcal{M}, t \models \varphi$.

We shall say that φ is globally true in $\mathcal{M}$ (in symbols $\mathcal{M} \models \varphi$) if $\mathcal{M}, s \models \varphi$ for every state s in $\mathcal{M}$.

Let $\mathcal{C}$ be a class of $\mathfrak{S}$-frames. We will denote by $\mathrm{Mod}(\mathcal{C})$ the class of all $\mathfrak{S}$-models based on some $\mathfrak{S}$-frame in $\mathcal{C}$. We shall say that a $\mathcal{L}_{\mathfrak{S}}$-formula φ is $\mathcal{C}$-satisfiable if there exists a $\mathfrak{S}$-frame $\mathcal{F} = (W, R)$ in $\mathcal{C}$, there exists a $\mathfrak{S}$-model $\mathcal{M} = (W, R, V)$ based on $\mathcal{F}$ and there exists $s \in W$ such that $\mathcal{M}, s \models \varphi$. Finally, we denote the class of elements of $\mathcal{C}$ with finite set of states by $\mathcal{C}^{\mathrm{fin}}$.

2.3 Relative covers

It will be convenient to work with relative cover modalities as well as global covers; these are a variation of the cover modalities $\nabla_i\Gamma$ [18]. Let $a \in \mathfrak{S}$ and Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$. We define the formula $\binom{\Sigma}{\Phi}_a$ as follows:

- $\binom{\Sigma}{\Phi}_a = \bigwedge_{\varphi \in \Phi} \langle a \rangle \varphi \wedge \bigwedge_{\varphi \in \Sigma \setminus \Phi} [a] \neg \varphi$.

This expression states that every formula in Φ holds at some $R(a)$-successor, and if $\sigma \in \Sigma \setminus \Phi$, then σ does not hold at any $R(a)$-successor. To be precise:

Lemma 2.1 *Let $\mathcal{M} = (W, R, V)$ be an $\mathfrak{S}$-model, and Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$. Then, for all $a \in \mathfrak{S}$ and all $s \in W$, the two following conditions are equivalent:*

(i) $\mathcal{M}, s \models \binom{\Sigma}{\Phi}_a$,

(ii) *for all $\varphi \in \Phi$, there exists $t \in R(a)(s)$ such that $\mathcal{M}, t \models \varphi$, and for all $\varphi \in \Sigma \setminus \Phi$ and $t \in R(a)(s)$, $\mathcal{M}, t \not\models \varphi$.*

Given a $\mathfrak{S}$-model $\mathcal{M} = (W, R, V)$ and sets Φ, Σ of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$, we say Φ *covers* $\mathcal{M}$ *relative to* Σ if for all $\varphi \in \Phi$, there exists $s \in W$ such that $\mathcal{M}, s \models \varphi$ and for all $\varphi \in \Sigma \setminus \Phi$ and for all $s \in W$, $\mathcal{M}, s \not\models \varphi$; in other words, Φ is precisely the set of formulas from Σ that are satisfied on $\mathcal{M}$. If Φ covers $\mathcal{M}$ relative to Σ, we will write $\mathcal{M} \models \binom{\Sigma}{\Phi}_\forall$. We warn the reader that $\binom{\Sigma}{\Phi}_\forall$ is not actually a formula of $\mathcal{L}_{\mathfrak{S}}$. In fact, when $\mathfrak{S}$ includes a universal modality, we will denote it by '$[U]$' rather than '$[\forall]$'.

Finally, we may also consider relative covers based on definable modalities. Broadly construed, a definable modality is any formula $\psi(p)$, where p is a propositional variable. We may then define $\widehat{\psi}(p) = \neg\psi(\neg p)$, and as above set

- $\binom{\Sigma}{\Phi}_\psi = \bigwedge_{\varphi \in \Phi} \widehat{\psi}(\varphi) \wedge \bigwedge_{\varphi \in \Sigma \setminus \Phi} \psi(\neg\varphi)$.

2.4 Generated subframes

If $\mathcal{F}$ is an $\mathfrak{S}$-frame and s is a state of $\mathcal{F}$, we denote by $\mathcal{F}_s$ the subframe of $\mathcal{F}$ generated by s. Let $\leq_{\mathcal{F}}$ be the binary relation on W defined by $s \leq_{\mathcal{F}} t$ iff t is a state in $\mathcal{F}_s$. If $\mathcal{M}$ is an $\mathfrak{S}$-model, we define $\mathcal{M}_s$ analogously.

We want to give a syntactic characterization of validity in a generated submodel. To do this, for all sets Γ of $\mathcal{L}_{\mathfrak{S}}$-formulas and $I \subseteq \mathfrak{S}$, let $[I]^*\Gamma$ be the set of all $\mathcal{L}_{\mathfrak{S}}$-formulas of the form $[a_1] \ldots [a_n]\varphi$ such that $a_1, \ldots, a_n \in I$ and $\varphi \in \Gamma$ (where we allow $n = 0$). Then, the following is straighforward and we omit the proof:

Lemma 2.2 *Let $\mathcal{M} = (W, R, V)$ be a $\mathfrak{S}$-model and Γ be a set of $\mathcal{L}_{\mathfrak{S}}$-formulas. For all $s \in W$, the three following conditions are equivalent:*

(i) $\mathcal{M}_s \models \Gamma$,

(ii) $\mathcal{M}_s, s \models [\mathfrak{S}]^*\Gamma$,

(iii) $\mathcal{M}, s \models [\mathfrak{S}]^*\Gamma$.

With these preliminary notions in mind, we are ready to define lexicographic products of modal logics.

3 Lexicographic products

In this paper, we will be interested in $\mathfrak{S}^{\circ,G}$-frames, but of a specific kind: lexicographic products of $\mathfrak{S}$-frames with $(\mathbb{N}, +1, <)$. If (W, R) is a $\mathfrak{S}^{\circ,G}$-frame, we will often write S instead of $R(\circ)$, and $<$ instead of $R(G)$. The *lexicographic product of a $\mathfrak{S}$-frame $\mathcal{F} = (W, R)$ with* $(\mathbb{N}, +1, <)$ is the relational structure $\mathcal{F}' = (W', R', S', <')$ defined as follows:

- $W' = W \times \mathbb{N}$,
- R' is the function associating to each $a \in \mathfrak{S}$ the binary relation $R'(a)$ on W' defined by $(s, i)R'(a)(t, j)$ iff $sR(a)t$ and $i = j$,
- S' is the binary relation on W' defined by $(s, i)S'(t, j)$ iff $i + 1 = j$,
- $<'$ is the binary relation on W' defined by $(s, i) <' (t, j)$ iff $i < j$.

Lemma 3.1 *Let $\mathcal{F}' = (W', R', S', <')$ be the lexicographic product of a $\mathfrak{S}$-frame with $(\mathbb{N}, +1, <)$. Then, $<' = S'^{+}$ and*

(i) *S' is serial,*

(ii) $S'^{-1} \circ S' \circ S' \subseteq S'$,

(iii) $<' \circ <' \subseteq <'$,

(iv) $S' \subseteq <'$,

(v) *for all $a \in \mathfrak{S}$, $S' \circ R'(a) \subseteq S'$,*

(vi) *for all $a \in \mathfrak{S}$, $R'(a) \circ S' \subseteq S'$,*

(vii) *for all $a \in \mathfrak{S}$, $R'(a)^{-1} \circ S' \subseteq S'$,*

(viii) *for all $a \in \mathfrak{S}$, $<' \circ R'(a) \subseteq <'$,*

(ix) *for all $a \in \mathfrak{S}$, $R'(a) \circ <' \subseteq <'$,*

(x) *for all $a \in \mathfrak{S}$, $R'(a)^{-1} \circ <' \subseteq <'$.*

These conditions may be easily verified by the reader. Obviously, lexicographic products of $\mathfrak{S}$-frames with $(\mathbb{N}, +1, <)$ can be considered as $\mathfrak{S}^{\circ,G}$-frames. We shall say that a $\mathfrak{S}^{\circ,G}$-frame is *concrete* if it is isomorphic to the lexicographic product of a $\mathfrak{S}$-frame with $(\mathbb{N}, +1, <)$. For all classes $\mathcal{C}$ of $\mathfrak{S}$-frames, let $\mathcal{C}^{\triangleright}$ be the class of concrete $\mathfrak{S}^{\circ,G}$-frames it corresponds to. In the next sections, for multifarious classes $\mathcal{C}$ of $\mathfrak{S}$-frames, we will consider the axiomatization/completeness of the sets of valid $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas given rise to by $\mathcal{C}^{\triangleright}$.

4 The basic logic

A *logic in the signature* $\mathfrak{S}$ is a set λ of $\mathcal{L}_{\mathfrak{S}}$-formulas containing all propositional tautologies and closed under modus ponens and substitution. The logic λ is said to be *normal* if it also contains $[a](p \to q) \to ([a]p \to [a]q)$ for each $a \in \mathfrak{S}$ and is closed under *necessitation*, $\frac{\varphi}{[a]\varphi}$. We write $\lambda \vdash \varphi$ instead of $\varphi \in \lambda$. A $\mathfrak{S}$-frame $\mathcal{F}$ is said to be a λ-*frame* when for all $\mathcal{L}_{\mathfrak{S}}$-formulas φ, if $\lambda \vdash \varphi$ then $\mathcal{F} \models \varphi$. We shall say that a $\mathcal{L}_{\mathfrak{S}}$-formula φ is λ-consistent if $\lambda \not\vdash \neg\varphi$. A set Σ of $\mathcal{L}_{\mathfrak{S}}$-formulas is said to be λ-consistent if for all finite subsets Γ of Σ, the $\mathcal{L}_{\mathfrak{S}}$-formula $\bigwedge \Gamma$ is λ-consistent.

Now, let us define the minimal lexicographic logic:

Definition 4.1 Given a logic λ in the signature $\mathfrak{S}$, let $(\lambda \rhd \mathsf{LTL})_0$ be the least set of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas containing λ, closed under modus ponens, necessitation for all modalities, and all substitution instances of the induction rule $\frac{p \to \circ p}{p \to Gp}$ and of the following axioms:

(i) $\circ(p \to q) \to (\circ p \to \circ q)$,

(ii) $G(p \to q) \to (Gp \to Gq)$,

(iii) $\hat{\circ}\top$,

(iv) $\hat{\circ}\hat{\circ}p \to \circ\hat{\circ}p$,

(v) $Gp \to GGp$,

(vi) $Gp \to \circ p$,

(vii) $\circ p \to \circ[a]p$,

(viii) $\circ p \to [a]\circ p$,

(ix) $\langle a\rangle \circ p \to \circ p$,

(x) $Gp \to G[a]p$,

(xi) $Gp \to [a]Gp$,

(xii) $\langle a\rangle Gp \to Gp$.

The next result follows from the well-known completeness of LTL, but can also be verified directly. We omit the proofs.

Lemma 4.2 *Given any normal logic λ in the signature $\mathfrak{S}$, the following formulas are derivable in $(\lambda \rhd \mathsf{LTL})_0$:*

(i) $\circ p \to \hat{\circ}p$,

(ii) $\hat{\circ}\circ p \to \circ\circ p$,

(iii) $\circ Gp \leftrightarrow G\circ p$,

(iv) $Gp \leftrightarrow \circ p \wedge \circ Gp$.

Given a logic λ in the signature $\mathfrak{S}$, a *lexicographic λ-logic* is any logic Λ in the signature $\mathfrak{S}^{\circ,G}$ containing $(\lambda \rhd \mathsf{LTL})_0$. Below, we make use of the notations Γ^a and $[I]^*\Gamma$ introduced in Sections 2.1 and 2.4, respectively.

Lemma 4.3 *Let Λ be a lexicographic λ-logic. Let Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas such that $\Phi \subseteq \Sigma$. If λ is normal, then*

(i) *if Φ is Λ-consistent then $\Phi \cup [\Phi^{\circ,G}]^{\mathfrak{S}}$ is Λ-consistent,*

(ii) *if $\binom{\Sigma}{\Phi}_\circ$ is Λ-consistent then for every $\varphi \in \Phi$, $\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)$ is Λ-consistent.*

Proof. (i) Suppose Φ is Λ-consistent. Using Axioms (viii), (ix), (xi) and (xii), the reader may easily obtain that $\Lambda \vdash \circ\varphi \to [a_1]\dots[a_n]\circ\varphi$, $\Lambda \vdash \hat{\circ}\varphi \to [a_1]\dots[a_n]\hat{\circ}\varphi$, $\Lambda \vdash G\varphi \to [a_1]\dots[a_n]G\varphi$ and $\Lambda \vdash F\varphi \to [a_1]\dots[a_n]F\varphi$ for any $a_1, \dots, a_n \in \mathfrak{S}$. Hence, for all $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas $\psi \in [\mathfrak{S}]^*\Phi^\circ \cup \Phi^G$, $\Lambda \vdash \bigwedge \Phi \to \psi$. Since Φ is Λ-consistent, therefore $\Phi \cup [\mathfrak{S}]^*\Phi^\circ \cup \Phi^G$ is Λ-consistent.

(ii) Suppose $\binom{\Sigma}{\Phi}_\circ$ is Λ-consistent. Let $\varphi \in \Phi$. Suppose $\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)$ is Λ-inconsistent. Hence, let Γ be a finite subset of $[\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)$ such that $\Lambda \vdash \neg(\varphi \wedge \bigwedge \Gamma)$. Using Axiom (vii), the reader may easily obtain that for all $\psi \in \Gamma$, $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \circ\psi$. Thus, $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \circ \bigwedge \Gamma$. Since $\varphi \in \Phi$, therefore $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \hat{\circ}\varphi$. Since $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \circ \bigwedge \Gamma$, therefore $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \hat{\circ}(\varphi \wedge \bigwedge \Gamma)$. Since $\Lambda \vdash \neg(\varphi \wedge \bigwedge \Gamma)$, by $\circ$-necessitation we see that $\binom{\Sigma}{\Phi}_\circ$ is Λ-inconsistent: a contradiction. □

Next we will show that, under fairly general conditions, logics extending $(\lambda \rhd \mathsf{LTL})_0$ are complete for their class of lexicographic products.

5 A general completeness theorem

To state our general completeness results, we will need a few preliminary notions. Let $\mathcal{C}$ be a class of $\mathfrak{S}$-frames. A *universal frame for* $\mathcal{C}$ is a $\mathfrak{S}$-frame $\mathcal{F} \in \mathcal{C}$ such that for all $\mathcal{L}_{\mathfrak{S}}$-formulas φ, if φ is satisfiable in $\mathcal{C}$ then φ is satisfiable in $\mathcal{F}$. We shall say that $\mathcal{C}$ is *simple* if $\mathcal{C}$ possesses a universal frame. A *cover-universal frame for* $\mathcal{C}$ is a $\mathfrak{S}$-frame $\mathcal{F} \in \mathcal{C}$ such that for all finite sets Φ, Σ of $\mathcal{L}_{\mathfrak{S}}$-formulas with $\Phi \subseteq \Sigma$, if there is a model $\mathcal{M} \in \mathrm{Mod}(\mathcal{C})$ such that $\mathcal{M} \models \binom{\Sigma}{\Phi}_\forall$, then there is a model $\mathcal{M}$ based on $\mathcal{F}$ such that $\mathcal{M} \models \binom{\Sigma}{\Phi}_\forall$. We shall say that $\mathcal{C}$ is *cover-simple* if $\mathcal{C}$ possesses a cover-universal frame. Observe that if $\mathcal{C}$ contains a single frame, then it is trivially cover-simple, but larger classes of frames may also be cover-simple.

Lemma 5.1 *Let $\mathcal{C}$ be a class of $\mathfrak{S}$-frames. If $\mathcal{C}$ is cover-simple then $\mathcal{C}$ is simple.*

Proof. Given a $\mathcal{L}_{\mathfrak{S}}$-formula φ, simply take $\Phi = \{\varphi\}$ and $\Sigma = \{\varphi\}$. □

In this article, we will consider logics λ in the signature $\mathfrak{S}$ such that the class of all $\mathfrak{S}$-frames for λ is cover-simple. As we will see, a large number of familiar logics have this property. Now, our goal is to prove that under certain general conditions, a given λ-logic Λ in the signature $\mathfrak{S}^{\circ,G}$ is complete with respect to a class of concrete $\mathfrak{S}^{\circ,G}$-frames. We will first focus on constructing the temporal part of a concrete $\mathfrak{S}^{\circ,G}$-frame and then building a lexicographic product on top of it; a similar strategy is used in [11] for establishing completeness of other products of modal logics.

Let x, y be sets of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas. We shall say that the couple (x, y) is *temporally adequate* if for all $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas φ, the following conditions hold:

- if $\hat{\circ}\varphi \in x$ then $\varphi \in y$,
- if $\circ\varphi \in x$ then $\neg\varphi \notin y$,
- if $G\varphi \in x$ then $\neg\varphi \notin y$ and $G\varphi \in y$,
- if $\neg G\varphi \in x$ then $\neg\varphi \in y$ or $\neg G\varphi \in y$.

Thus, temporally adequate pairs are similar to bricks in mosaics [5]. Given a finite set Σ of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas closed under subformulas and single negations, let $\mathcal{T}_\Sigma^\Lambda = (T_\Sigma^\Lambda, S_\Sigma^\Lambda, <_\Sigma^\Lambda)$ be the relational structure defined as follows:

- T_Σ^Λ is the set of all $x \subseteq \Sigma$ such that $\binom{\Sigma}{x}_\circ$ is Λ-consistent,

- S^Λ_Σ is the binary relation on T^Λ_Σ defined by $xS^\Lambda_\Sigma y$ if and only if (x, y) is temporally adequate,
- $<^\Lambda_\Sigma$ is the transitive closure of S^Λ_Σ.

The next lemma lists the basic properties of $\mathcal{T}^\Lambda_\Sigma$. Below, we follow the convention that $\bigvee \varnothing = \bot$.

Lemma 5.2 *Let Λ be any logic in the signature $\mathfrak{S}^{\circ,G}$ and $\Sigma \subseteq \mathcal{L}^{\circ,G}_{\mathfrak{S}}$ be finite and closed under subformulas and single negations. Then:*

(i) $\Lambda \vdash \bigvee \left\{ \binom{\Sigma}{x}_\circ : x \in T^\Lambda_\Sigma \right\}$.

(ii) *For all $\Phi \subseteq \Sigma$,*

$$\Lambda \vdash \bigwedge \Phi \to \bigvee \left\{ \binom{\Sigma}{y}_\circ : (\Phi, y) \textit{ is temporally adequate} \right\}.$$

In particular, the latter set is non-empty whenever Φ is consistent.

(iii) *For all $x \in T^\Lambda_\Sigma$, $\Lambda \vdash \binom{\Sigma}{x}_\circ \to \bigvee \left\{ \circ\binom{\Sigma}{y}_\circ : xS^\Lambda_\Sigma y \right\}$. The latter set is always non-empty.*

(iv) *For all $\neg G\varphi \in x$, there exists $y \in T^\Lambda_\Sigma$ such that $x <^\Lambda_\Sigma y$ and $\neg\varphi \in y$.*

Proof. (i) First note that $\bigvee \left\{ \binom{\Sigma}{x}_\circ : x \subseteq \Sigma \right\}$ is a tautology. Hence, $\Lambda \vdash \bigvee \left\{ \binom{\Sigma}{x}_\circ : x \subseteq \Sigma \right\}$. Let $x \subseteq \Sigma$. If $x \notin T^\Lambda_\Sigma$ then $\Lambda \vdash \neg\binom{\Sigma}{x}_\circ$. Thus, $\Lambda \vdash \bigvee \left\{ \binom{\Sigma}{x}_\circ : x \in T^\Lambda_\Sigma \right\}$.

(ii) Suppose $\bigwedge \Phi$ is Λ-consistent. By item (i), $\Lambda \vdash \bigvee \left\{ \binom{\Sigma}{y}_\circ : y \in T^\Lambda_\Sigma \right\}$. Let Q be the set of all $y \in T^\Lambda_\Sigma$ such that $\bigwedge \Phi \wedge \binom{\Sigma}{y}_\circ$ is Λ-consistent. Since $\Lambda \vdash \bigvee \left\{ \binom{\Sigma}{y}_\circ : y \in T^\Lambda_\Sigma \right\}$, therefore $\Lambda \vdash \bigwedge \Phi \to \bigvee \left\{ \binom{\Sigma}{y}_\circ : y \in Q \right\}$. The reader can then check that if $y \in Q$, then (Φ, y) is temporally adequate; the proof is very similar to that of item (iii) below. It follows that if $\bigwedge \Phi$ is Λ-consistent, then $Q \neq \varnothing$.

(iii) Let $x \in T^\Lambda_\Sigma$. Using Lemma 5.2, necessitation, axiom (iv) of $(\lambda \triangleright \mathsf{LTL})_0$ and formulas (i) and (ii) of Lemma 4.2, we see that $\Lambda \vdash \bigvee \left\{ \circ\binom{\Sigma}{y}_\circ : y \in T^\Lambda_\Sigma \right\}$. Let $y \in T^\Lambda_\Sigma$ be such that not $xS^\Lambda_\Sigma y$. Hence, we have to consider the following four cases:

- **Case "there is $\hat{\circ}\varphi \in x$ such that $\varphi \notin y$":** First, observe that $\Lambda \vdash \binom{\Sigma}{x}_\circ \to \hat{\circ}\hat{\circ}\varphi$. Hence, by Axiom (iv), we also have $\Lambda \vdash \binom{\Sigma}{x}_\circ \to \circ\hat{\circ}\varphi$. Since $\varphi \notin y$, therefore $\Lambda \vdash \binom{\Sigma}{y}_\circ \to \circ\neg\varphi$. Thus, using necessitation and formula (i) of Lemma 4.2, $\Lambda \vdash \circ\binom{\Sigma}{y}_\circ \to \hat{\circ}\circ\neg\varphi$. Since $\Lambda \vdash \binom{\Sigma}{x}_\circ \to \circ\hat{\circ}\varphi$, therefore $\binom{\Sigma}{x}_\circ \wedge \circ\binom{\Sigma}{y}_\circ$ is Λ-inconsistent.
- **Case "there is $\circ\varphi \in x$ such that $\neg\varphi \in y$":** First, observe that $\Lambda \vdash$

$\binom{\Sigma}{x}_{\circ} \to \hat{\circ}\circ\varphi$. Since $\neg\varphi \in y$, therefore $\Lambda \vdash \binom{\Sigma}{y}_{\circ} \to \hat{\circ}\neg\varphi$. Thus, using necessitation, $\Lambda \vdash \circ\binom{\Sigma}{y}_{\circ} \to \circ\hat{\circ}\neg\varphi$. Since $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\circ\varphi$, therefore $\binom{\Sigma}{x}_{\circ} \wedge \circ\binom{\Sigma}{y}_{\circ}$ is Λ-inconsistent.

- **Case "there is $G\varphi \in x$ such that $\neg\varphi \in y$ or $G\varphi \notin y$":** Suppose $\neg\varphi \in y$. Since $G\varphi \in x$, therefore using Axiom (vi), $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\circ\varphi$. Since $\neg\varphi \in y$, therefore we can proceed as in the second case in order to conclude that $\binom{\Sigma}{x}_{\circ} \wedge \circ\binom{\Sigma}{y}_{\circ}$ is Λ-inconsistent. Suppose $G\varphi \notin y$. Since $G\varphi \in x$, therefore using Axioms (v), (vi), and formula (i) of Lemma 4.2, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \circ\hat{\circ}G\varphi$. Since $G\varphi \notin y$, therefore $\Lambda \vdash \binom{\Sigma}{y}_{\circ} \to \circ\neg G\varphi$. Hence, we can proceed as in the first case in order to conclude that $\binom{\Sigma}{x}_{\circ} \wedge \circ\binom{\Sigma}{y}_{\circ}$ is Λ-inconsistent.
- **Case "there is $\neg G\varphi \in x$ such that $\neg\varphi \notin y$ and $\neg G\varphi \notin y$":** Hence, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\neg G\varphi$. Since $\neg\varphi \notin y$ and $\neg G\varphi \notin y$, therefore $\Lambda \vdash \binom{\Sigma}{y}_{\circ} \to \circ\varphi \wedge \circ G\varphi$. Thus, by formula (iv) in Lemma 4.2, $\Lambda \vdash \binom{\Sigma}{y}_{\circ} \to G\varphi$. Thus, using necessitation, $\Lambda \vdash \circ\binom{\Sigma}{y}_{\circ} \to \circ G\varphi$. Since $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\neg G\varphi$, therefore $\binom{\Sigma}{x}_{\circ} \wedge \circ\binom{\Sigma}{y}_{\circ}$ is Λ-inconsistent.

Consequently, for all $y \in T^{\Lambda}_{\Sigma}$, if not $xS^{\Lambda}_{\Sigma}y$ then $\binom{\Sigma}{x}_{\circ} \wedge \circ\binom{\Sigma}{y}_{\circ}$ is Λ-inconsistent. Since $\Lambda \vdash \bigvee\left\{\circ\binom{\Sigma}{y}_{\circ} :\ y \in T^{\Lambda}_{\Sigma}\right\}$, therefore $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \bigvee\left\{\circ\binom{\Sigma}{y}_{\circ} :\ xS^{\Lambda}_{\Sigma}y\right\}$.

(iv) Let $\neg G\varphi \in x$. Thus, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\neg G\varphi$. By (iii), $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \bigvee_{xS^{\Lambda}_{\Sigma}y} \circ\binom{\Sigma}{y}_{\circ}$ and $\Lambda \vdash \bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ} \to \bigvee_{x<^{\Lambda}_{\Sigma}y} \bigvee_{yS^{\Lambda}_{\Sigma}z} \circ\binom{\Sigma}{z}_{\circ}$. Hence, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \circ\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ}$ and $\Lambda \vdash \bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ} \to \circ\bigvee_{x<^{\Lambda}_{\Sigma}z} \binom{\Sigma}{z}_{\circ}$ Consequently, using induction, $\Lambda \vdash \bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ} \to G\bigvee_{x<^{\Lambda}_{\Sigma}z} \binom{\Sigma}{z}_{\circ}$. Since $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \circ\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ}$, therefore $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \circ G\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ}$ and, by formula (iv) in Lemma 4.2, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to G\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ}$. Suppose there is no $y \in T^{\Lambda}_{\Sigma}$ such that $x <^{\Lambda}_{\Sigma} y$ and $\neg\varphi \in y$. Thus, $\Lambda \vdash \bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ} \to \circ\varphi$. Consequently, using necessitation, $\Lambda \vdash G\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ} \to G\circ\varphi$. Since $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to G\bigvee_{x<^{\Lambda}_{\Sigma}y} \binom{\Sigma}{y}_{\circ}$, therefore $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to G\circ\varphi$. Hence, by formula (iii) in Lemma 4.2, $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \circ G\varphi$. Since $\Lambda \vdash \binom{\Sigma}{x}_{\circ} \to \hat{\circ}\neg G\varphi$, therefore $\binom{\Sigma}{x}_{\circ}$ is Λ-inconsistent: a contradiction. □

Now, we are ready to build the temporal part of our models. A *good Σ-path* is an infinite sequence $(x_i)_{i\in\mathbb{N}}$ of subsets of Σ such that for all $i \in \mathbb{N}$,

- (x_i, x_{i+1}) is temporally adequate,
- for all $\neg G\varphi \in x_i$, there exists $j \in \mathbb{N}$, $i < j$, such that $\neg\varphi \in x_j$.

We can then use Lemma 5.2 to construct good paths in $\mathcal{T}^{\Lambda}_{\Sigma}$. Let us denote the set of positive integers by $\mathbb{N}^{\star}$.

Lemma 5.3 *Let $\Phi \subseteq \Sigma$. If Φ is Λ-consistent then there exists a good path $(x_i)_{i\in\mathbb{N}^{\star}}$ such that for all $i \in \mathbb{N}^{\star}$, $x_i \in T^{\Lambda}_{\Sigma}$ and (Φ, x_1) is temporally adequate.*

Proof. For each $i \in \mathbb{N}^\star$, we build a sequence $x_1, \ldots, x_{n_i}$ in T^Λ_Φ recursively on i. First, for $i = 1$, we merely take $n_1 = 1$ and x_1 to be any element of T^Λ_Φ such that $(\Phi, , x_1)$ is temporally adequate, which exists by Lemma 5.2(ii). Then, consider $i \in \mathbb{N}^\star$ and let $\psi_1, \ldots, \psi_k$ be a list of all formulas ψ such that $\neg G\psi \in x_{n_i}$. In the case that $k = 0$, just let $n_{i+1} = n_i + 1$ and choose $x_{n_{i+1}}$ arbitrary so that $x_{n_i} S^\Lambda_\Sigma x_{n_{i+1}}$. This will ensure that the construction does not get stuck at x_{n_i}. If the case that $k \geq 1$, we construct a path $x_{n_i+1}, \ldots, x_{m_j}$ for all $j = 1 \ldots k$ as follows. By 5.2(iv), let $y \in T^\Lambda_\Sigma$ be such that $x_{n_i} <^\Lambda_\Sigma y$ and $\neg\psi_1 \in y$. Since $<^\Lambda_\Phi$ is the transitive closure of S^Λ_Φ, therefore we have a path $x_{n_i}, \ldots, x_{n_i+m_1} = y$. Now, suppose we have constructed $x_{n_i}, \ldots, x_{n_i+m_j}$ and consider two cases. If $\neg\psi_{j+1} \in x_{n_i+r}$ for some r, then $m_{j+1} = 0$ and we do nothing. Otherwise, it follows that $\neg G\psi_{j+1} \in x_{n_i+m_j}$, and thus we can construct $x_{n_i+m_{j+1}}$ with $\neg\psi_{j+1} \in x_{n_i+m_{j+1}}$ by once again using 5.2(iv). It is straightforward to check that the path thus constructed is a good path. □

Before stating our general completeness result, we need a last technical assumption. In order to state it, it will be convenient to treat temporal formulas as if they were propositional variables. For each formula of $\mathcal{L}^{\circ,G}_{\mathfrak{S}}$ of the form $\psi = \circ\varphi$ or $\psi = G\varphi$, let $\underline{\psi}$ denote a fresh propositional variable. If Σ is a set of formulas, let $\mathbb{P}_\Sigma$ denote the set of propositional variables which include all variables $\underline{\psi}$ for $\psi \in \Sigma$. We extend the notation $\underline{\psi}$ to an arbitrary $\mathcal{L}^{\circ,G}_{\mathfrak{S}}$-formula ψ by replacing the outermost occurrences of its $\circ$-subformulas and G-subformulas belonging to Σ by the corresponding variables, and if Φ is a set of formulas set $\underline{\Phi} = \{\underline{\psi} : \psi \in \Phi\}$. A Σ*-valuation* on an $\mathfrak{S}$-frame $\mathcal{F} = (W, R)$ is a function $V : \mathbb{P}_\Sigma \mapsto 2^W$ such that for all $\varphi \in \Sigma$ of the form $\circ\psi$ or $G\psi$, $V(\underline{\varphi}) = W$ or $V(\underline{\varphi}) = \varnothing$.

If $\mathcal{C}$ is a class of $\mathfrak{S}$-frames, a $(\mathcal{C}, \Sigma)$*-moment* is a pair $\mathfrak{m} = (\mathcal{F}, V)$ where $\mathcal{F} \in \mathcal{C}$ and V is a Σ-valuation on $\mathcal{F}$. Given a $(\mathcal{C}, \Sigma)$-moment $\mathfrak{m} = (W, R, V)$, we define $\chi^\Sigma(\mathfrak{m})$ to be the set of all $\varphi \in \Sigma$ such that $V(\underline{\varphi}) \neq \varnothing$. A $(\mathcal{C}, \Sigma)$-*quasimodel* is a sequence $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ of $(\mathcal{C}, \Sigma)$-moments such that $(\chi^\Sigma(\mathfrak{m}_i))_{i\in\mathbb{N}}$ is a good path. As one would expect, quasimodels are useful because they may be used to construct models.

Lemma 5.4 *Let $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ be a $(\mathcal{C}, \Sigma)$-quasimodel. If $\mathcal{C}$ is cover-simple then there exists a frame $\mathcal{F} = (W, R)$ in $\mathcal{C}$ and a valuation V' on the lexicographic product $\mathcal{F}' = (W', R', S', <')$ of $\mathcal{F}$ with $(\mathbb{N}, +1, <)$ such that for all $\varphi \in \Sigma$ and for all $i \in \mathbb{N}$, $\varphi \in \chi^\Sigma(\mathfrak{m}_i)$ iff there exists $w \in W$ such that $(\mathcal{F}', V'), (w, i) \models \varphi$.*

Proof. Suppose $\mathcal{C}$ is cover-simple. Let $\mathcal{F} = (W, R)$ be a cover-universal frame for $\mathcal{C}$. Since $(\mathfrak{m}_i)_{i\in\mathbb{N}}$ is a $(\mathcal{C}, \Sigma)$-quasimodel, therefore for all $i \in \mathbb{N}$, $\mathfrak{m}_i$ is a $(\mathcal{C}, \Sigma)$-moment. Hence, for all $i \in \mathbb{N}$, $\mathfrak{m}_i$ is a pair $(\mathcal{F}_i, V_i)$ where $\mathcal{F}_i \in \mathcal{C}$ and V_i is a Σ-valuation on $\mathcal{F}_i$. Let $\Phi = \chi^\Sigma(\mathfrak{m}_i)$; note that by definition, for all $i \in \mathbb{N}$, $\mathfrak{m}_i \models \binom{\underline{\Sigma}}{\underline{\Phi}}_\forall$. Since $\mathcal{F}$ is a cover-universal frame for $\mathcal{C}$, therefore for all $i \in \mathbb{N}$, let V'_i be a valuation on $\mathcal{F}$ such that $(\mathcal{F}, V'_i) \models \binom{\underline{\Sigma}}{\underline{\Phi}}_\forall$. Let V' be the valuation on the lexicographic product $\mathcal{F}' = (W', R', S', <')$ of $\mathcal{F}$ with $(\mathbb{N}, +1, <)$ such that for all $p \in \mathbb{P}$, for all $w \in W$ and for all $i \in \mathbb{N}$, $(w, i) \in V'(p)$ iff $w \in V'_i(p)$. A routine induction would show that for any $\varphi \in \Sigma$, for all $w \in W$ and for all $i \in \mathbb{N}$, $(\mathcal{F}', V'), (w, i) \models \psi$ if and only if $(\mathcal{F}, V_i'), w \models \underline{\psi}$. Thus, for all $\varphi \in \Sigma$ and for all $i \in \mathbb{N}$, $\varphi \in \chi^{\Sigma}(\mathfrak{m}_i)$ iff there exists $w \in W$ such that $(\mathcal{F}', V'), (w, i) \models \varphi$.□

As a result, we may turn our attention to building quasimodels rather than concrete models. A lexicographic λ-logic Λ is said to be *moment-complete for* $\mathcal{C}$ if for all finite sets Σ of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas closed under subformulas and single negations and for all $\Phi \subseteq \Sigma$,

(C_1) if Φ is Λ-consistent then there is a $(\mathcal{C}, \Sigma)$-moment $\mathfrak{m}$ satisfying $\bigwedge \underline{\Phi}$,

(C_2) if $\binom{\Sigma}{\Phi}_\circ$ is Λ-consistent then there is $(\mathcal{C}, \Sigma)$-moment $\mathfrak{m}$ such that $\mathfrak{m} \models \binom{\Sigma}{\underline{\Phi}}_\forall$.

With this, we are ready to state our general completeness theorem from which completeness for many specific logics will follow later.

Theorem 5.5 *Let $\mathcal{C}$ be a cover-simple class of $\mathfrak{S}$-frames and Λ be a λ-logic in the signature $\mathfrak{S}^{\circ,G}$. If Λ is moment-complete for $\mathcal{C}$ then for all $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas φ, if φ is Λ-consistent then φ is satisfiable in $\mathcal{C}^{\triangleright}$.*

Proof. Suppose Λ is moment-complete for $\mathcal{C}$. Let φ be a Λ-consistent $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formula and $\Sigma = SF^{\neg}(\varphi)$. Since $\mathcal{C}$ is a cover-simple class of $\mathfrak{S}$-frames, let $\mathcal{F} = (W, R)$ be a cover-universal frame for $\mathcal{C}$. Since φ is Λ-consistent, let Φ_0 be a maximal Λ-consistent subset of Σ containing φ. By Lemma 5.3, let $(x_i)_{i \in \mathbb{N}^\star}$ be a good path such that for all $i \in \mathbb{N}^\star$, $x_i \in T_\Sigma^\Lambda$ and $(\Phi_0, , x_1)$ is temporally adequate.

Since Λ is moment-complete for $\mathcal{C}$, by (C_1), let $\mathfrak{m}_0$ be a $(\mathcal{C}, \Sigma)$-moment satisfying $\bigwedge \underline{\Phi}_0$. Note that, for all, $i \geq 1$, $\binom{\Sigma}{x_i}_\circ$ is Λ-consistent. Hence, for all $i \geq 1$, by (C_2), there is a $(\mathcal{C}, \Sigma)$-moment $\mathfrak{m}_i$ satisfying $\binom{\Sigma}{\underline{x_i}}_\forall$. Obviously, the sequence $(\mathfrak{m}_i)_{i \in \mathbb{N}}$ of $(\mathcal{C}, \Sigma)$-moments constitute a $(\mathcal{C}, \Sigma)$-quasimodel. Consequently, by Lemma 5.4, there exists a valuation V' on the lexicographic product $\mathcal{F}' = (W', R', S', <')$ of $\mathcal{F}$ with $(\mathbb{N}, +1, <)$ such that for all $\psi \in \Sigma$ and for all $i \in \mathbb{N}$, $\psi \in \chi^{\Sigma}(\mathfrak{m}_i)$ iff there exists $w \in W$ such that $(\mathcal{F}', V'), (w, i) \models \psi$. Since $\varphi \in \chi^{\Sigma}(\mathfrak{m}_0)$, therefore there exists $w \in W$ such that $(\mathcal{F}', V'), (w, 0) \models \varphi$. □

6 Completeness for special classes of frames

In this section, we consider classes of frames satisfying specific conditions which will allow us to apply Theorem 5.5. Many of the well-known modal logics satisfy at least one of the conditions we will give.

6.1 Local classes

A class $\mathcal{C}$ of $\mathfrak{S}$-frames is *local* if it is closed under generated subframes and disjoint unions. The idea behind local classes of $\mathfrak{S}$-frames is that we can build models by looking only at what individual states see. Below, recall that a logic λ is *strongly complete* for $\mathcal{C}$ if whenever Φ is a (possibly infinite) set of formulas that is λ-consistent, then there is a model $\mathcal{M} \in \mathcal{C}$ with a world w such that $\mathcal{M}, w \models \varphi$ for every $\varphi \in \Phi$.

Lemma 6.1 *Let λ be a logic in the signature $\mathfrak{S}$, $\mathcal{C}$ be a cover-simple local class of $\mathfrak{S}$-frames and Λ be a lexicographic λ-logic. If λ is strongly complete with respect to $\mathcal{C}$ then Λ is moment-complete.*

Proof. Suppose λ is strongly complete with respect to $\mathcal{C}$. Since $\mathcal{C}$ is a cover-simple class of $\mathfrak{S}$-frames, let $\mathcal{F} = (W, R)$ be a cover-universal frame for $\mathcal{C}$. Let Σ be a finite set of $\mathcal{L}^{\circ,G}_{\mathfrak{S}}$-formulas closed under subformulas and single negations and let $\Phi_0 \subseteq \Sigma$.

(i) Suppose Φ_0 is Λ-consistent. Let Φ be a maximal Λ-consistent subset of Σ containing Φ_0. Hence, by Lemma 4.3, $\Phi \cup [\mathfrak{S}]^*\Phi^\circ \cup \Phi^G$ is Λ-consistent and $\underline{\Phi \cup [\mathfrak{S}]^*\Phi^\circ \cup \Phi^G}$ is λ-consistent. Since λ is strongly complete with respect to $\mathcal{C}$, let $\mathcal{M}$ be a $\mathcal{C}$-model and s be a state in $\mathcal{M}$ such that $\mathcal{M}, s \models \underline{\Phi \cup [\mathfrak{S}]^*\Phi^\circ \cup \Phi^G}$. Thus, by Lemma 2.2, $\mathcal{M}_s, s \models \underline{\Phi^\circ \cup \Phi^G}$. Consequently, $\mathcal{M}_s$ is a $(\mathcal{C}, V)$-moment.

(ii) Suppose $\binom{\Sigma}{\Phi}_{\circ}$ is Λ-consistent and let $\varphi \in \Phi$. Hence, by Lemma 4.3, $\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)$ is Λ-consistent and $\underline{\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)}$ is λ-consistent. Since λ is strongly complete with respect to $\mathcal{C}$, let $\mathcal{M}^\varphi$ be a $\mathcal{C}$-model and s_φ be a state in $\mathcal{M}^\varphi$ be such that $\mathcal{M}^\varphi, s_\varphi \models \underline{\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)}$. Thus, $\mathcal{M}^\varphi_{s_\varphi}, s_\varphi \models \underline{\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)}$. Since $\mathcal{C}$ is a local class of $\mathfrak{S}$-frames, therefore $\mathcal{M}^\varphi_{s_\varphi}$ belongs to $Mod(\mathcal{C})$. Let $\mathcal{M}_\Phi$ be the disjoint union of $\{\mathcal{M}^\varphi_{s_\varphi} : \varphi \in \Phi\}$. Since $\mathcal{C}$ is a local class of $\mathfrak{S}$-frames, therefore $\mathcal{M}_\Phi$ belongs to $Mod(\mathcal{C})$. Since for all $\varphi \in \Phi$, $\mathcal{M}^\varphi_{s_\varphi}, s_\varphi \models \underline{\{\varphi\} \cup [\mathfrak{S}]^*\neg(\Sigma \setminus \Phi)}$, therefore $\mathcal{M}_\Phi \models \binom{\Sigma}{\underline{\Phi}}_{\forall}$. Since $\mathcal{F}$ is a cover-universal frame for $\mathcal{C}$, there exists a valuation V on $\mathcal{F}$ such that $(\mathcal{F}, V) \models \binom{\Sigma}{\underline{\Phi}}_{\forall}$.

□

From this, we may use Theorem 5.5 to immediately obtain the following:

Corollary 6.2 *Let $\mathcal{C}$ be a cover-simple local class of $\mathfrak{S}$-frames and Λ be a lexicographic λ-logic. If λ is strongly complete with respect to $\mathcal{C}$ then Λ is complete for $\mathcal{C}^\triangleright$.*

We remark that even if λ is strongly complete, we cannot expect Λ to be; this is because the set of formulas $\{\neg Gp\} \cup \{\circ^n p : n \in \mathbb{N}\}$ is consistent, but unsatisfiable.

6.2 Linear classes

Recall that the notation $\leq_{\mathcal{F}}$ was introduced in Section 2.4. A cover-simple class $\mathcal{C}$ of $\mathfrak{S}$-frames is *linear* if there exists a $\mathcal{L}_{\mathfrak{S}}$-formula $\theta_{\mathcal{C}}(p)$ and a cover-universal frame $\mathcal{F}_{\mathcal{C}}$ for $\mathcal{C}$ such that $\leq_{\mathcal{F}_{\mathcal{C}}}$ is a total order and for all valuations V on $\mathcal{F}_{\mathcal{C}}$ and for all states $s \in \mathcal{F}_{\mathcal{C}}$, $(\mathcal{F}_{\mathcal{C}}, V), s \models \theta_{\mathcal{C}}(p)$ iff for all states $t \in \mathcal{F}_{\mathcal{C}}$, if $s \leq_{\mathcal{F}_{\mathcal{C}}} t$ then $(\mathcal{F}_{\mathcal{C}}, V), t \models p$; in other words, $\theta_{\mathcal{C}}(p)$ expresses that p is true in the generated submodel of s.

Lemma 6.3 *Suppose $\mathcal{C}$ is a linear class of $\mathfrak{S}$-frames and the $\mathfrak{S}$-logic λ is complete for $\mathcal{C}$. Then,*

(i) $\lambda \vdash \theta_{\mathcal{C}}(p) \to p$,

(ii) $\lambda \vdash \theta_{\mathcal{C}}(p) \to \theta_{\mathcal{C}}(\theta_{\mathcal{C}}(p))$,

(iii) $\lambda \vdash \widehat{\theta_{\mathcal{C}}}(p) \wedge \widehat{\theta_{\mathcal{C}}}(q) \to \widehat{\theta_{\mathcal{C}}}(p \wedge \widehat{\theta_{\mathcal{C}}}(q)) \vee \widehat{\theta_{\mathcal{C}}}(q \wedge \widehat{\theta_{\mathcal{C}}}(p))$.

Proof. It is a well-known fact that these formulas are valid on any total order $(T, \leq)$ when one interprets $\theta_{\mathcal{C}}(\cdot)$ in T as a modal connective with $\leq$ playing the role of its accessibility relation. Since $\leq_{\mathcal{F}_{\mathcal{C}}}$ is a total order, therefore these formulas are valid in $\mathcal{F}_{\mathcal{C}}$. Since $\mathcal{F}_{\mathcal{C}}$ is a cover-universal frame for $\mathcal{C}$ and λ is complete for $\mathcal{C}$, therefore these formulas are in λ. $\square$

We will need a few extra axioms to axiomatize lexicographic logics over linear classes of frames.

Definition 6.4 Let $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$ be the least lexicographic λ-logic containing the following formulas:

(LC$_1$) $\hat{\circ} p \wedge \hat{\circ} q \to \hat{\circ}(\widehat{\theta_{\mathcal{C}}}(p) \wedge \widehat{\theta_{\mathcal{C}}}(q))$

(LC$_2$) $\circ p \to \circ\theta_{\mathcal{C}}(p)$

(LC$_3$) $\circ p \to \theta_{\mathcal{C}}(\circ p)$

(LC$_3$) $\widehat{\theta_{\mathcal{C}}}(\circ p) \to \circ p$

(LC$_3$) $Gp \to \theta_{\mathcal{C}}(Gp)$

(LC$_3$) $\widehat{\theta_{\mathcal{C}}}(Gp) \to Gp$

Our goal for the remainder of the section is to show that, in many cases where $\mathcal{C}$ is a linear class of $\mathfrak{S}$-frames and the $\mathfrak{S}$-logic λ is complete for $\mathcal{C}$, $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$ is complete with respect to $\mathcal{C}^{\triangleright}$. We will obtain this using Theorem 5.5, and in the following lemmas, we establish that $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$ has the required conditions to apply said theorem.

Lemma 6.5 *Let $\Phi \subseteq \Sigma$ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas. If $\mathcal{C}$ is closed under generated subframes and $\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is λ-consistent then there exists a valuation V on $\mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{F}_{\mathcal{C}}, V) \models \binom{\Sigma}{\Phi}_{\forall}$.*

Proof. Suppose $\mathcal{C}$ is closed under generated subframes and $\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is λ-consistent. Since $\mathcal{F}_{\mathcal{C}}$ is a cover-universal frame for $\mathcal{C}$ and λ is complete for $\mathcal{C}$, there exists a valuation $V_{\mathcal{C}}$ on $\mathcal{F}_{\mathcal{C}}$ and there exists a state $s \in \mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{F}_{\mathcal{C}}, V_{\mathcal{C}}), s \models \binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$. Let $\mathcal{F}_s$ be the subframe of $\mathcal{F}_{\mathcal{C}}$ generated from s. Obviously, $(\mathcal{F}_s, V_{\mathcal{C}}) \models \binom{\Sigma}{\Phi}_{\forall}$. Moreover, since $\mathcal{C}$ is closed under generated subframes, therefore $\mathcal{F}_s$ is in $\mathcal{C}$. Since $\mathcal{F}_{\mathcal{C}}$ is a cover-universal frame for $\mathcal{C}$ and $(\mathcal{F}_s, V_{\mathcal{C}}) \models \binom{\Sigma}{\Phi}_{\forall}$, therefore there exists a valuation V on $\mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{F}_{\mathcal{C}}, V) \models \binom{\Sigma}{\Phi}_{\forall}$. $\square$

Lemma 6.6 *Let Λ be a lexicographic λ-logic containing $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$. Let $\Phi \subseteq \Sigma$ be finite sets of $\mathcal{L}_{\mathfrak{S}}^{\circ,G}$-formulas. Then:*

(i) *If Φ is consistent then so is $\Phi \cup \left\{\theta_{\mathcal{C}}(\psi) : \psi \in \Phi^{\circ} \cup \Phi^{G}\right\}$.*

(ii) *If $\binom{\Sigma}{\Phi}_{\circ}$ is Λ-consistent then $\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is Λ-consistent.*

Proof. (i) Immediate using axioms (LC$_3$)-(LC$_6$).
(ii) Suppose $\binom{\Sigma}{\Phi}_{\circ}$ is Λ-consistent. Using Formulas (LC$_1$) and (LC$_2$), the reader may easily verify that $\Lambda \vdash \bigwedge_{\varphi\in\Phi} \hat{\circ}\varphi \to \hat{\circ} \bigwedge_{\varphi\in\Phi} \widehat{\theta_{\mathcal{C}}}(\varphi)$ and $\Lambda \vdash \bigwedge_{\varphi\in\Sigma\setminus\Phi} \circ\neg\varphi \to$

$\circ \bigwedge_{\varphi\in\Sigma\setminus\Phi} \theta_{\mathcal{C}}(\neg\varphi)$. Hence, $\Lambda \vdash \binom{\Sigma}{\Phi}_\circ \to \hat{\circ}\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$. Since $\binom{\Sigma}{\Phi}_\circ$ is Λ-consistent, therefore $\hat{\circ}\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is Λ-consistent. Thus, $\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is Λ-consistent. □

Lemma 6.7 *If $\mathcal{C}$ is closed under generated subframes, λ is complete for $\mathcal{C}$, and Λ is a lexicographic λ-logic containing $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$ then Λ is moment-complete.*

Proof. Suppose $\mathcal{C}$ is closed under generated subframes and Λ is a lexicographic λ-logic containing $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$. Let Σ be a finite set of $\mathcal{L}^{\circ;G}_{\mathfrak{S}}$-formulas closed under subformulas and single negations and let $\Phi_0 \subseteq \Sigma$.

(i) Suppose $\Phi_0 \subseteq \Sigma$ is Λ-consistent; without loss of generality, we may assume it is maximal consistent. Then, by Lemma 6.6(i), $\Gamma = \Phi_0 \cup \{\theta_{\mathcal{C}}(\psi) : \psi \in \Phi_0^\circ \cup \Phi_0^G\}$ is consistent, so that $\underline{\Gamma}$ is also λ-consistent. Thus there is a model $\mathcal{M}$ based on $\mathcal{F}_{\mathcal{C}}$ and a state s of $\mathcal{M}$ such that $\mathcal{M}, s \models \bigwedge \underline{\Gamma}$, so that $\mathcal{M}_s, s \models \bigwedge \underline{\Gamma}$. But since $\mathcal{M}_s$ is generated by s, if $x\psi \in \Phi_0$ with $x \in \{\circ, G\}$, from $\mathcal{M}_s, s \models \theta_{\mathcal{C}}(\underline{x\psi})$ we obtain $\mathcal{M}_s \models \underline{x\psi}$, and from $\mathcal{M}_s, s \models \theta_{\mathcal{C}}(\neg\underline{x\psi})$ we obtain $\mathcal{M}_s \models \neg\underline{x\psi}$; since Φ_0 was maximal consistent one of these two occurs for each such $x\psi$ so $\mathcal{M}$ is a Σ-moment satisfying $\bigwedge \Phi_0$, as needed.

(ii) Suppose $\binom{\Sigma}{\Phi}_\circ$ is Λ-consistent. Hence, by Lemma 6.6(ii), $\binom{\Sigma}{\Phi}_{\theta_{\mathcal{C}}}$ is Λ-consistent. By Lemma 6.5, since $\mathcal{C}$ is closed under generated subframes, therefore there exists a valuation V on $\mathcal{F}_{\mathcal{C}}$ such that $(\mathcal{F}_{\mathcal{C}}, V) \models \underline{\binom{\Sigma}{\Phi}}_\forall$. □

With this we obtain a general completeness result for linear classes of frames.

Corollary 6.8 *Let $\mathcal{C}$ be a linear class of $\mathfrak{S}$-frames and Λ be a lexicographic λ-logic containing $(\lambda \triangleright \mathsf{LTL})_{\theta_{\mathcal{C}}}$. If $\mathcal{C}$ is closed under generated subframes and λ is complete for $\mathcal{C}$ then Λ is complete for $\mathcal{C}^{\triangleright}$.*

Proof. By Lemma 6.7 and Theorem 5.5. □

6.3 Global classes

A class $\mathcal{C}$ of $\mathfrak{S}$-frames is *global* if there exists a $\mathcal{L}_{\mathfrak{S}}$-formula $\mu_{\mathcal{C}}(p)$ such that for all models $\mathcal{M}$ in $\mathrm{Mod}(\mathcal{C})$ and for all states s in $\mathcal{M}$, $\mathcal{M}, s \models \mu_{\mathcal{C}}(p)$ iff for all states t in $\mathcal{M}$, $\mathcal{M}, t \models p$.

Our work on linear classes readily applies to global classes.

Lemma 6.9 *If $\mathcal{C}$ is cover-simple and global then $\mathcal{C}$ is linear.*

Proof. Suppose $\mathcal{C}$ is cover-simple and global. Let $\mathcal{F}$ be a cover-universal frame for $\mathcal{C}$ and $\mu_{\mathcal{C}}(p)$ be a $\mathcal{L}_{\mathfrak{S}}$-formula such that for all models $\mathcal{M}$ in $\mathrm{Mod}(\mathcal{C})$ and for all states s in $\mathcal{M}$, $\mathcal{M}, s \models \mu_{\mathcal{C}}(p)$ iff for all states t in $\mathcal{M}$, $\mathcal{M}, t \models p$. Obviously, the only generated subframe of $\mathcal{F}$ is $\mathcal{F}$ itself. Taking $\theta_{\mathcal{C}}(p) = \mu_{\mathcal{C}}(p)$, the reader may easily verify that $\mathcal{C}$ is linear. □

The main reason that global classes are particularly useful is that they allow us to define global covers directly within our language.

Lemma 6.10 *Let Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$. If $\mathcal{C}$ is global then $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\forall}$ if and only if $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\mu_{\mathcal{C}}}$ for every model $\mathcal{M}$ based on any frame from $\mathcal{C}$.*

In view of this, the property of being simple becomes equivalent to that of being cover-simple over any global class of frames.

Lemma 6.11 *If $\mathcal{C}$ is simple and global then $\mathcal{C}$ is cover-simple.*

Proof. Suppose $\mathcal{C}$ is simple and global. Let $\mathcal{F}$ be a universal frame for $\mathcal{C}$. Let Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$. Let $\mathcal{M}$ be a model in $\mathrm{Mod}(\mathcal{C})$ such that $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\forall}$. Hence, by Lemma 6.10, $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\mu_{\mathcal{C}}}$. Since $\mathcal{F}$ is a universal frame for $\mathcal{C}$, therefore let V be a valuation on $\mathcal{F}$ such that $(\mathcal{F}, V) \models \binom{\Sigma}{\Phi}_{\mu_{\mathcal{C}}}$. Thus, by Lemma 6.10, $(\mathcal{F}, V) \models \binom{\Sigma}{\Phi}_{\forall}$. □

In fact, the above result applies to classes of frames that are *not* global, provided they can be made global by appropriately extending the signature.

Lemma 6.12 *Let $\mathfrak{S}'$ be a countable set of modalities containing $\mathfrak{S}$. Let $\mathcal{C}'$ be a simple and global class of $\mathfrak{S}'$-frames. If $\mathcal{C}$ is the class of all $\mathcal{F}'^{|\mathfrak{S}}$ where $\mathcal{F}'$ is in $\mathcal{C}'$ then $\mathcal{C}$ is cover-simple.*

Proof. Suppose $\mathcal{C}'$ is simple and global. Let $\mathcal{F}'$ be a universal frame for $\mathcal{C}'$ and $\mathcal{F} = \mathcal{F}'^{|\mathfrak{S}}$. Let Φ, Σ be finite sets of $\mathcal{L}_{\mathfrak{S}}$-formulas such that $\Phi \subseteq \Sigma$. Let $\mathcal{M}$ be a model in $\mathrm{Mod}(\mathcal{C})$ such that $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\forall}$. Let $\mathcal{M}'$ be the corresponding model in $\mathrm{Mod}(\mathcal{C}')$. Since $\mathcal{M} \models \binom{\Sigma}{\Phi}_{\forall}$, therefore $\mathcal{M}' \models \binom{\Sigma}{\Phi}_{\forall}$. Since $\mathcal{C}'$ is global, therefore by Lemma 6.10, $\mathcal{M}' \models \binom{\Sigma}{\Phi}_{\mu_{\mathcal{C}}}$. Since $\mathcal{F}'$ is a universal frame for $\mathcal{C}'$, therefore let V' be a valuation on $\mathcal{F}'$ such that $(\mathcal{F}', V') \models \binom{\Sigma}{\Phi}_{\mu_{\mathcal{C}}}$. Thus, by Lemma 6.10, $(\mathcal{F}', V') \models \binom{\Sigma}{\Phi}_{\forall}$. Remark that V' is also a valuation on $\mathcal{F}$. Moreover, $(\mathcal{F}, V') \models \binom{\Sigma}{\Phi}_{\forall}$. It follows that $\mathcal{F}$ is a cover-universal frame for $\mathcal{C}$. Consequently, $\mathcal{C}$ is cover-simple. □

This will often mean that logics that are known to have a simple extension with a universal modality will be cover-simple. We will use this technique in the following section.

7 Completeness for specific logics

Now we proceed to show that many familiar classes of frames fall into one of the above frameworks, and use this to prove that several lexicographic logics are complete for their class of concrete frames.

7.1 Logics with tree-like models

We begin with unimodal logics containing the D axiom. Recall that KD is the least normal logic in a unimodal signature containing the formula $\Box p \to \Diamond p$. Then, T is the least extension of KD with the formula $\Box p \to p$, $\mathsf{KD4}$ is the least extension of KD with the formula $\Box p \to \Box\Box p$, $\mathsf{S4}$ is the least extension of KD with the formulas $\Box p \to p$ and $\Box p \to \Box\Box p$ and $\mathsf{S5}$ is the least extension of KD with the formulas $\Box p \to \Box\Box p$ and $p \to \Box\Diamond p$. Let $\mathcal{C}_{\mathsf{KD}}$ be the class of all serial frames, $\mathcal{C}_{\mathsf{T}}$ be the class of all reflexive frames, $\mathcal{C}_{\mathsf{KD4}}$ be the class of all serial, transitive frames, $\mathcal{C}_{\mathsf{S4}}$ be the class of all reflexive, transitive frames and $\mathcal{C}_{\mathsf{S5}}$ be the class of all serial, transitive, symmetric frames.

If our unimodal signature is extended by the universal modality U then for $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$, we let λU be the least normal logic in the extended signature containing the formulas $[U]p \to \Box p$, $[U]p \to p$, $[U]p \to [U][U]p$ and $p \to [U]\langle U\rangle p$. For $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$, let $\mathcal{C}_{\lambda U}$ be the class of all frames in $\mathcal{C}_\lambda$ extended by the universal accessibility relation. Obviously, $\mathcal{C}_{\lambda U}$ is global.

The following is well-known (see, e.g., [8]):

Theorem 7.1 *Let $\eta \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$ and λ be either η or ηU. Then, the following are equivalent:*

(i) $\lambda \vdash \varphi$,

(ii) $\mathcal{C}_\lambda \models \varphi$,

(iii) $\mathcal{C}_\lambda^{\mathrm{fin}} \models \varphi$.

Let us show that all of these classes of frames are cover-simple. For this, define the *ω-forest* to be the set $\mathbb{N}^{[0,\omega)}$ of all sequences $\boldsymbol{a} = (a_0, \ldots, a_n)$ where $n \geq 0$ and each $a_i \in \mathbb{N}$ (note that the empty sequence is *not* allowed). Let R_{KD} be the binary relation on $\mathbb{N}^{[0,\omega)}$ such that $\boldsymbol{a} R_{\mathsf{KD}} \boldsymbol{b}$ iff there exists $n \geq 0$ such that $\boldsymbol{a} = (a_0, \ldots, a_n)$ and $\boldsymbol{b} = (a_0, \ldots, a_n, b)$ for some $a_0, \ldots, a_n, b \in \mathbb{N}$. Let R_{T} be the reflexive closure of R_{KD}, R_{KD4} be its transitive closure, R_{S4} be its reflexive, transitive closure and R_{S5} be its transitive, symmetric closure. For $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$, we let $\mathcal{F}_\lambda = (\mathbb{N}^{[0,\omega)}, R_\lambda)$ and $\mathcal{F}_{\lambda U}$ be the extension of $\mathcal{F}_\lambda$ by the universal accessibility relation.

Lemma 7.2 *Let λ be in $\{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$. Then,*

(i) *$\mathcal{F}_\lambda$ is a λ-frame,*

(ii) *$\mathcal{F}_{\lambda U}$ is a λU-frame.*

To show that $\mathcal{F}_\lambda$ is cover-universal, we will use well-known results on bounded morphisms, along with the following:

Lemma 7.3 *Let $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$ and $\mathcal{F}$ be any λU-frame. If $\mathcal{F}$ is finite then $\mathcal{F}$ is a bounded morphic image of $\mathcal{F}_{\lambda U}$.*

Proof. Suppose $\mathcal{F}$ is finite. Our aim is to construct a surjective bounded morphism $f\colon \mathbb{N}^{[0,\omega)} \to W$. We will define $f(\boldsymbol{a})$ by induction on the length of $\boldsymbol{a}$. For the base case, let $g\colon \mathbb{N} \to W$ be an arbitrary surjection. We define $f((a_0)) = g(a_0)$ for any $(a_0) \in \mathbb{N}^{[0,\omega)}$. Now, suppose that $f((a_0, \ldots, a_n)) = w$ is defined and let $h\colon \mathbb{N} \to R(w)$ be an arbitrary surjection. Then, for any $b \in \mathbb{N}$, set $f((a_0, \ldots, a_n, b)) = h(b)$. One can then check that for each of the listed λ, the map f thus defined is a bounded morphism. Surjectivity comes from the way we chose g. $\Box$

Lemma 7.4 *If $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$ then both $\mathcal{C}_\lambda$ and $\mathcal{C}_{\lambda U}$ are cover-simple.*

Proof. Let λ be in $\{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$. It is well-known that λU has the finite model property over $\mathrm{Mod}(\mathcal{C}_{\lambda U})$. Hence, by Lemma 7.3 and the fact that bounded morphic images preserve validity, we obtain that $\mathcal{F}_{\lambda U}$ is a universal frame for $\mathcal{C}_{\lambda U}$. Thus, $\mathcal{C}_{\lambda U}$ is simple. Since $\mathcal{C}_{\lambda U}$ is global, therefore by Lemmas 6.12 and 6.11, both $\mathcal{C}_{\lambda}$ and $\mathcal{C}_{\lambda U}$ are cover-simple. $\square$

Corollary 7.5 *If* $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$ *then* $(\lambda \rhd \mathsf{LTL})_0$ *is complete for* $\mathcal{C}^{\rhd}_{\lambda}$.

Proof. Let λ be in $\{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$. By Lemma 7.4, $\mathcal{C}_{\lambda}$ is cover-simple. Since $\mathcal{C}_{\lambda}$ is closed under generated subframes and disjoint unions, therefore $\mathcal{C}_{\lambda}$ is local. In other respect, it is well-known that λ is strongly complete with respect to $\mathcal{C}_{\lambda}$. Since $\mathcal{C}_{\lambda}$ is cover-simple, therefore by Corollary 6.2, $(\lambda \rhd \mathsf{LTL})_0$ is complete for $\mathcal{C}^{\rhd}_{\lambda}$. $\square$

7.2 Examples of linear logics

Next we turn to giving examples of logics with linear classes of frames. We begin by those that contain a universal modality. Let λ be in $\{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$. Obviously, $\leq_{\mathcal{F}_{\mathcal{C}_{\lambda U}}}$ is the universal relation on $\mathbb{N}^{[0,\omega)}$. Hence, $\leq_{\mathcal{F}_{\mathcal{C}_{\lambda U}}}$ is a total order on $\mathbb{N}^{[0,\omega)}$. Let $\theta_{\mathcal{C}_{\lambda U}}(p)$ be the formula $[U]p$.

Corollary 7.6 *If* $\lambda \in \{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$ *then* $(\lambda \rhd \mathsf{LTL})_{\theta_{\mathcal{C}_{\lambda U}}}$ *is complete for* $\mathcal{C}^{\rhd}_{\lambda U}$.

Proof. Let λ be in $\{\mathsf{KD}, \mathsf{T}, \mathsf{KD4}, \mathsf{S4}, \mathsf{S5}\}$. By Lemma 7.4, $\mathcal{C}_{\lambda U}$ is cover-simple. Since $\mathcal{C}_{\lambda U}$ is global, therefore by Lemma 6.9, $\mathcal{C}_{\lambda U}$ is linear. Since $\mathcal{C}_{\lambda U}$ is closed under generated subframes and λU is complete for $\mathcal{C}_{\lambda U}$, therefore by Corollary 6.8, $(\lambda \rhd \mathsf{LTL})_{\theta_{\mathcal{C}_{\lambda U}}}$ is complete for $\mathcal{C}^{\rhd}_{\lambda U}$. $\square$

Finally, we conclude with examples of linear logics which are not global.

Lemma 7.7 *Let* $\mathcal{C}$ *be either:*

(i) *the class of all linear orders, with signature* $\{\leq\}$*,*

(ii) *the class of all linear orders without a maximal element, with signature* $\{<\}$*,*

(iii) $\{\mathbb{N}\}$*, with signature* $\{S, <\}$ *or* $\{S, \leq\}$*.*

Then, $\mathcal{C}$ *is linear.*

Proof. Note that in signatures with $<$, we may define $[\leq]p = p \wedge [<]p$. Now, in the first two cases, $[0, \infty) \cap \mathbb{Q}$ is readily checked to be a cover-universal frame, while in the third $\mathbb{N}$ is already the only allowed frame, hence it is cover-universal. Moreover, the class of all linear orders is closed under generated subframes, while $\mathbb{N}$ is isomorphic to its own generated subframes. $\square$

Let $\mathsf{S4.3}$ be the ordinary normal modal logic determined by the class of all linear orders (reflexive, transitive and weakly-connected frames) and $\mathsf{KD4.3}$ be the ordinary normal modal logic determined by the class of all strict linear orders without a maximal element (serial, transitive and weakly-connected frames). See [13, Chapter 3].

Corollary 7.8 *Let* $\lambda \in \{\mathsf{S4.3}, \mathsf{KD4.3}, \mathsf{LTL}\}$ *and* $\mathcal{C}_\lambda$ *be the class of all linear orders, the class of all strict linear orders, or* $\{\mathbb{N}\}$*, respectively. Then,* $(\lambda \rhd \mathsf{LTL})_\leq$ *is complete for* $\mathcal{C}_\lambda \rhd \{\mathbb{N}\}$*.*

Proof. Immediate from Lemma 7.7 and Corollary 6.8. □

8 Conclusion

In this article, we have considered the lexicographic products of modal logics with linear temporal logic based on "next" and "always in the future". We have provided axiomatizations of the sets of valid formulas they give rise to. The proof of their completeness uses tools and techniques like universal frames, cover-universal frames, etc. Much remains to be done.

There is the issue of the axiomatization of the lexicographic products of modal logics with a linear temporal logic based on "until". We believe the tools and techniques that we have developed can be applied as well. Can they still be applied if one considers the lexicographic product of a linear temporal logic based on "until" with a linear temporal logic based on "until"? Considering a linear temporal logic based either on "always in the future" or on "until", this time interpreted over the class of all dense linear orders without endpoints, how to axiomatize its lexicographic products with modal logics? Is it possible to obtain in our lexicographic setting complete axiomatizations by following the line of reasoning suggested by [4]?

There is also the question of the complexity of the temporal logic characterized by the lexicographic products of modal logics with linear temporal logic. Is it possible to obtain in our lexicographic setting complexity results by following either of the lines of reasoning suggested by [5] or [20]?

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References

[1] Babenyshev, S., Rybakov, V. *Logics of Kripke meta-models.* Logic Journal of the IGPL **18** (2010) 823–836.

[2] Balbiani, P. Time representation and temporal reasoning from the perspective of non-standard analysis. In Brewka, G., Lang, J. (editors): Eleventh International Conference on Principles of Knowledge Representation and Reasoning. AAAI (2008) 695–704.

[3] Balbiani, P. Axiomatization and completeness of lexicographic products of modal logics. In Ghilardi, S., Sebastiani, R. (editors): Frontiers of Combining Systems. Springer (2009) 165–180.

[4] Balbiani, P. Axiomatizing the temporal logic defined over the class of all lexicographic products of dense linear orders without endpoints In Markey, N., Wijsen, J. (editors): Temporal Representation and Reasoning. IEEE (2010) 19–26.

[5] Balbiani, P., Mikulás, S. Decidability and complexity via mosaics of the temporal logic of the lexicographic products of unbounded dense linear orders. In Fontaine, P., Ringeissen, C., Schmidt, R. (editors): Frontiers of Combining Systems. Springer (2013) 151–164.

[6] Balbiani, P., Shapirovsky, I., Shehtman, V. Complete axiomatizations of lexicographic sums and products of modal logics. To appear.

[7] Beklemishev, L. *Kripke semantics for provability logic GLP*. Annals of Pure and Applied Logic **161** (2010) 756–774.

[8] Blackburn, P., de Rijke, M., Venema, Y. Modal Logic. Cambridge University Press (2001).

[9] Van Ditmarsch, H., van der Hoek, W., Kooi, B. Dynamic Epistemic Logic. Springer (2007).

[10] Fagin, R., Halpern, J., Moses, Y., Vardi, M. Reasoning About Knowledge. MIT Press (1995).

[11] Gabbay, D., Kurucz, A., Wolter, F., Zakharyaschev, M. Many-Dimensional Modal Logics: Theory and Applications. Elsevier (2003).

[12] Gabbay, D., Shehtman, V. Products of modal logics, part 1. Logic Journal of the IGPL **6** (1998) 73–146.

[13] Goldblatt, R. *Logics of Time and Computation.* CSLI (1992).

[14] Kracht, M., Wolter, F. Properties of independently axiomatizable bimodal logics. Journal of Symbolic Logic **56** (1991) 1469–1485.

[15] Kurucz, A. Combining modal logics. In Blackburn, P., van Benthem, J., Wolter, F. (editors): Handbook of Modal Logic. Elsevier (2007) 869–924.

[16] Litak, T., Wolter, F. All finitely axiomatizable tense logics of linear time flows are CoNP-complete. Studia Logica **81** (2005) 153–165.

[17] Marx, M., Mikulás, S., Reynolds, M. The mosaic method for temporal logics. In Dyckhoff, R. (editor): Automated Reasoning with Analytic Tableaux and Related Methods. Springer (2000) 324–340.

[18] Moss, L. S. Coalgebraic logic. Annals of Pure and Applied Logic **96** (1999) 277–317.

[19] Shapirovsky, I. *PSPACE-decidability of Japaridze's polymodal logic.* In Areces, C., Goldblatt, R. (editors): Advances in Modal Logic, Volume 7. College Publications (2008) 289–304.

[20] Sistla, A., Clarke, E.: *The complexity of propositional linear temporal logics.* Journal of the Association for Computing Machinery **32** (1985) 733–749.

[21] Wolter, F. Fusions of modal logics revisited. In Kracht, M., de Rijke, M., Wansing, H., Zakharyaschev, M. (editors): Advances in Modal Logic. CSLI Publications (1998) 361–379.