HEIGHT AND CONTOUR PROCESSES OF CRUMP-MODE-JAGERS FORESTS (I): GENERAL DISTRIBUTION AND SCALING LIMITS IN THE CASE OF SHORT EDGES

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ABSTRACT. Crump–Mode–Jagers (CMJ) trees generalize Galton–Watson trees by allowing individuals to live for an arbitrary duration and give birth at arbitrary times during their life-time. In this paper, we are interested in the height and contour processes encoding a general CMJ tree.

We show that the one-dimensional distribution of the height process can be expressed in terms of a random transformation of the ladder height process associated with the underlying Lukasiewicz path. As an application of this result, when edges of the tree are "short" we show that, asymptotically, (1) the height process is obtained by stretching by a constant factor the height process of the associated genealogical Galton–Watson tree, (2) the contour process is obtained from the height process by a constant time change and (3) the CMJ trees converge in the sense of finite-dimensional distributions.

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1. Introduction and statement of main results

1.1. **Galton–Watson forests and their scaling limits.** A planar discrete rooted tree is a rooted tree where edges have unit length and which is endowed with an ordering on siblings, in such a way that it can be naturally embedded in the plane. Since the seminal work of Aldous, Neveu, Pitman and others [2, 3, 4, 17, 22, 23], it is well known that such a tree is conveniently encoded by its height and contour processes. To generate these processes, one can envision a particle starting from the root and traveling along the edges of the tree at unit speed, from left to right. The contour process is simply constructed by recording the distance of the particle from the root of the tree. To generate the height process, we start by labeling the vertices of the tree according to their order of visit by the exploration particle (i.e., from left to right): the height process evaluated at k is then given by the distance from the root of the kth vertex.

From a probabilistic standpoint, a particularly interesting case is the Galton–Watson case where each individual u in the tree begets a random number of offspring ξ_u, these random variables being i.i.d. with common distribution ξ. In the critical and subcritical cases – i.e., when $\mathbb{E}(\xi) \leq 1$ – the tree is almost surely finite. Considering an infinite sequence of such i.i.d. random rooted planar trees, we can generate a random (planar) forest with its corresponding contour and height processes – respectively denoted by $\mathscr{C}$ and $\mathscr{H}$ – obtained by pasting sequentially the height and contour processes of the trees composing the forest.

When $\mathbb{E}(\xi^2) < \infty$, the large time behavior of those processes properly normalized in time and space can be described in terms of a reflected Brownian motion. More precisely, if $\mathbb{E}(\xi) = 1$ and if $0 < \sigma = \mathrm{Var}(\xi^2) < \infty$ then we have

$$\left(\frac{1}{\sqrt{p}}\mathscr{H}([pt]), \frac{1}{\sqrt{p}}\mathscr{C}(pt)\right) \Longrightarrow \frac{2}{\sigma}\,(|w(t)|, |w(t/2)|)$$

with w a standard Brownian motion and the convergence holds weakly (in the functional sense), see Aldous [4], Bennies and Kersting [5] and Marckert and Mokkadem [19].

Le Gall and Le Jan [18] and then Duquesne and Le Gall [10] relaxed the finite variance assumption and proved, under suitable assumptions, the existence of a scaling sequence $(\varepsilon_p, p \in \mathbb{N})$ and a limiting continuous path $\mathscr{H}_\infty$ such that

$$\left(\varepsilon_p \mathscr{H}([pt]), \varepsilon_p \mathscr{C}(pt)\right) \Longrightarrow (\mathscr{H}_\infty(t), \mathscr{H}_\infty(t/2))$$

where $\mathscr{H}_\infty$ can be expressed as a functional of a spectrally positive Lévy process. In particular, we note that the height and contour processes are always asymptotically related by a simple deterministic and constant time change. The purpose of this paper is to extend these results to the more general class of Crump-Mode-Jagers forests.

1.2. **Crump-Mode-Jagers forests.** Chronological trees generalize discrete trees in the following way: each individual u is endowed with a pair $(V_u, \mathscr{P}_u)$ such that:

(1) $V_u \in (0, \infty)$ represents the life-length of u;
(2) $\mathscr{P}_u$ is a point measure which represents the age of u at childbearing. As individuals produce their offspring during their lifetime, we have $\mathscr{P}_u(V_u, \infty) = 0$.

Discrete trees are particular cases of chronological trees obtained with $V_u = 1$ and $\mathscr{P}_u = \xi_u \epsilon_1$, with $\xi_u \in \mathbb{N}$ the number of offspring and ϵ_1 the Dirac measure at 1.

As noted by Lambert in [14] (to which the reader is referred for background on chronological trees), a chronological tree can be regarded as a tree satisfying the rule "edges always grow to the right". This is illustrated in Figures 1 and 2 where we present a sequential construction of a planar chronological forest from a sequence of "sticks" $\omega = (\omega_n, n \geq 0)$, where $\omega_n = (V_n, \mathscr{P}_n)$.

1.2.1. *Sequential construction of a Crump-Mode-Jagers forest.* The reader is refered to Figures 1 and 2 for an illustration of this construction.

At time $n = 0$ we start with the empty forest and we add the stick ω_0 at time $n = 1$. In the case considered in Figure 2, $\mathscr{P}_0$ has two atoms which correspond to birth times of individuals, but these two atoms are not yet matched with the sticks corresponding to these individuals. These unmatched atoms are called *stubs*, and each time there is at least one stub we graft the next stick to the highest stub.

We iteratively apply this rule until there is no more stub, at which point we have built a complete chronological tree with a natural planar embedding. Figure 2 illustrates a particular case where at time 10 there is no more stub, and in each time this happens we start a new tree with the next stick.

Thus, starting at time $n = 0$ from the empty forest and iterating these two rules, we build in this way a forest $\mathbb{F}^\infty$, possibly consisting of infinitely many chronological trees. By definition, a CMJ forest is obtained when the initial sticks are i.i.d., and throughout the paper we will denote their common distribution by $(V^*, \mathscr{P}^*)$.

1.2.2. *Chronological height and contour processes of CMJ forests.* As for discrete trees, the contour process of a CMJ forest is obtained by recording the position of an exploration particle traveling at unit speed along the edges of the forest from left to right, moving, when a chronological tree is represented as in Figure 2, at infinite speed along dashed lines. This process will be referred to as the *chronological* contour process associated to the CMJ forest, and the chronological height of an individual is defined as its date of birth.

We define the genealogical contour and height processes as the contour and height processes associated to the discrete forest encoding the genealogy of $\mathbb{F}^\infty$, see Figures 3 and 4 for a pictorial representation. Throughout the paper, we use the following notation:

Genealogical processes: : $\mathscr{H}$ and $\mathscr{C}$ denote the *genealogical* height and contour processes, respectively;

Chronological processes: : $\mathbb{H}$ and $\mathbb{C}$ denote the *chronological* height and contour processes, respectively.

Contour processes of CMJ forests have been considered by Lambert in [14] in the particular setting where birth events are distributed in a Poissonian way along the sticks independently of the life-length – the so-called binary, homogeneous case. Under this assumption, the author showed that the (jumping) contour process is a spectrally positive Lévy process. See also [8, 9, 15, 16, 24, 25] for related works.

1.3. **Overview of main results.** Besides these results, little is known to our knowledge in the general case. One of the main result of the present paper is to describe in full generality the joint distribution of the chronological and genealogical height processes at a fixed time, see Theorem 1.1 and Lemma 2.12 below.

We believe that this description paves the way to a general study of Crump-Mode-Jagers forests. As an illustration, we treat here the so-called "short edge" case where edges of the chronological trees are short: in this case, the Crump-Mode-Jagers forest becomes asymptotically proportional to its genealogical forest. This loose statement is formalized in Theorems 1.3, 1.6 and and 1.9 below below.

Also, in current work in progress [32] we use these techniques to treat the case where the offspring distribution has finite variance: in this case, new scaling limits emerge, which are related to the Poisson snake [1, 6].

1.4. **First main result: joint distribution of the chronological and genealogical height processes at a fixed time.** Let $S = (S(n), n \in \mathbb{N})$ be the Lukasiewicz path: it is defined by

$S(0) = 0$ and, for $n \geq 1$,

$$S(n) = \sum_{k=0}^{n-1} (|\mathscr{P}_k| - 1)$$

(here and in the sequel, $|\nu|$ is mass of the measure ν). Let $T = (T(k), k \in \mathbb{N})$ be the sequence of weak ascending ladder height times, also referred to as record times: it is defined by $T(0) = 0$ and by

$$T(k+1) = \inf\{\ell > T(k) : S(\ell) \geq S(T(k))\}$$

for $k \geq 0$, with the convention $T(k+1) = \infty$ if $T(k) = \infty$. Let $\widetilde{T}^{-1}(n)$ for $n \in \mathbb{N}$ be the number of record times smaller than n, i.e., $\widetilde{T}^{-1}(n) = \max\{k \geq 0 : T(k) \leq n\}$. For $k \geq 1$ such that $T(k) < \infty$, define

$$\xi(k) \ = S(T(k-1)) - S(T(k)-1)$$

corresponding to the undershoot upon reaching the kth record time. For any measure $\mathscr{P}$ and any $k \leq |\mathscr{P}|$, denote by $A_k(\mathscr{P})$ the position of the kth largest atom of $\mathscr{P}$.

As explained above, all our objects are constructed from an initial sequence of sticks $\omega = (\omega_n, n \in \mathbb{N})$. For technical convenience, we actually assume that a sequence of sticks ω is indexed by $\mathbb{Z}$, i.e., $\omega = (\omega_n, n \in \mathbb{Z})$, and we denote by Ω the set of sequences of sticks. This makes it possible to consider, for each n, the dual (or time-reversal) operator $\vartheta^n : \Omega \to \Omega$ defined by $\vartheta^n(\omega) = (\omega_{n-k-1}, k \in \mathbb{Z})$. Recall that $\mathscr{H}$ is the height process of a classical Galton-Watson tree.

Theorem 1.1. *For $n \in \mathbb{N}$ let*

$$R(n) = \sum_{1 \leq k \leq n : T(k) < \infty} \mathbb{Y}(k) \ \textit{ where } \ \mathbb{Y}(k) = A_{\xi(k)}(\mathscr{P}_{T(k)-1}).$$

Then the genealogical and chronological height processes at time n are given by the following formula:

$$\big(\mathscr{H}(n), \mathbb{H}(n)\big) = \big(\widetilde{T}^{-1}(n) \ , \ R \circ \widetilde{T}^{-1}(n)\big) \circ \vartheta^n. \tag{1.1}$$

The functional $\mathbb{Y}(k) \circ \vartheta^n$ appearing in the above statement is depicted in Figures 5 and 8. Moreover, the functionals in the right-hand side of (1.1) are by definition computed with respect to the reversed sequence of sticks $(\omega_{n-k-1}, k \in \mathbb{Z})$, e.g., $\widetilde{T}^{-1}(n) \circ \vartheta^n$ is the nth record time associated to the sequence $(\omega_{n-1-k}, k \geq 0)$.

We note that the one-dimensional marginals of the *genealogical* height process $\mathscr{H}$ in terms of the ladder height time process is already known in the literature, see for instance Marckert and Mokkadem [19]. The previous result states that in order to describe the *chronological* height process, more structure of the ladder height process is needed: not only do we need to extract the record times (as in the Galton-Watson case), but also the corresponding undershoots.

We emphasize the fact that the previous result is *purely deterministic*. We now introduce the probabilistic set-up of Crump-Mode-Jagers forests and state our main results concerning the asymptotic behavior of the chronological height and contour processes.

1.5. **Main results: scaling limits.** We now present the main results of the paper concerning the asymptotic behavior of the chronological height and contour processes, see Theorems 1.3, 1.6 and 1.9 below.

1.5.1. *Probabilistic set-up.* A Crump-Mode-Jagers forest is obtained when the initial sequence of sticks is i.i.d.. We consider in this paper a triangular setting and consider for each $p \geq 1$ a stick-valued random variable $(V_p^*, \mathscr{P}_p^*)$ corresponding to a (sub)critical CMJ branching process, i.e., which satisfies

$$0 \leq \mathbb{E}(|\mathscr{P}_p^*|) \leq 1. \tag{1.2}$$

We assume moreover that the sequence $(\mathscr{P}_p^*)$ is near-critical in the sense that

$$\lim_{p\to\infty} \mathbb{E}(|\mathscr{P}_p^*|) = 1. \tag{1.3}$$

Let $\mathbb{P}_p$ be the probability distribution on Ω under which ω is an i.i.d. sequence of sticks with common distribution $(V_p^*, \mathscr{P}_p^*)$. We let $\Rightarrow$ denote weak convergence under $\mathbb{P}_p$ and $\overset{\text{fdd}}{\Rightarrow}$ denote convergence in the sense of finite-dimensional distributions under $\mathbb{P}_p$. For instance, $B_p \overset{\text{fdd}}{\Rightarrow} B_\infty$ if and only if $(B_p(t), t \in I)$ under $\mathbb{P}_p$ converges weakly to $(B_\infty(t), t \in I)$ for any finite set $I \subset [0,\infty)$.

1.5.2. *Preliminaries.* For a given sequence $(\varepsilon_p, p \in \mathbb{N})$, define the rescaled processes $\mathscr{H}_p$, $\mathbb{H}_p$, S_p, $\mathscr{C}_p$ and $\mathbb{C}_p$ as follows: for $t \in \mathbb{R}_+$:

$$\mathscr{H}_p(t) = \varepsilon_p \mathscr{H}([pt]),\ \mathbb{H}_p(t) = \varepsilon_p \mathbb{H}([pt]) \text{ and } S_p(t) = \frac{1}{p\varepsilon_p} S([pt]), \tag{1.4}$$

($[x] \in \mathbb{Z}$ denotes the integer part of $x \in \mathbb{R}$) and

$$\mathscr{C}_p(t) = \varepsilon_p \mathscr{C}(pt),\ \mathbb{C}_p(t) = \varepsilon_p \mathbb{C}(pt). \tag{1.5}$$

In the near-critical case, it is well-known since Duquesne and Le Gall [10] that if S_p converges, then under under additional mild assumptions the rescaled genealogical height and contour processes converge weakly toward a continuous process. This is summarized in the next theorem which involves the following condition.

Condition G. *The following three conditions are met:*

- (H1) $S_p \Rightarrow S_\infty$ *for some Lévy process* S_∞ *with infinite variation;*
- (H2) *the Laplace exponent* ψ *of* S_∞ *satisfies* $\int_1^\infty \mathrm{d}u/\psi(u) < \infty$*;*
- (H3) *if* $(Z_k^p, k \geq 0)$ *is a Galton-Watson process with offspring distribution* $|\mathscr{P}_p^*|$ *and started with* $[p\varepsilon_p]$ *individuals, then for every* $\delta > 0$*,*

$$\liminf_{p\to\infty} \mathbb{P}\left(Z^p_{[\delta/\varepsilon_p]} = 0\right) > 0.$$

When condition G holds, we can and will assume without loss of generality that as $p \to \infty$ we have $\varepsilon_p \to 0$ and $p\varepsilon_p \to \infty$. Moreover, since we are in triangular setting where the law of the jump size of S may depend on p, S_∞ is not necessarily a stable process.

Theorem 1.2 (Corollary 2.5.1 in [10])**.** *Assume that condition* G *holds. Then* $(\mathscr{H}_p, \mathscr{C}_p) \Rightarrow (\mathscr{H}_\infty, \mathscr{H}_\infty(\cdot/2))$ *for some continuous process* $\mathscr{H}_\infty$ *satisfying* $\mathbb{P}(\mathscr{H}_\infty(t) > 0) = 1$ *for every* $t > 0$.

1.5.3. *Convergence of the chronological height process.* To explain our results we start with some notation. The strong Markov property implies that the random variables $\Upsilon(k)$ introduced in Theorem 1.1 are i.i.d. (under $\mathbb{P}_p$), and we denote by Υ_p^* a random variable with their common distribution. We will show in Lemma 2.12 that Υ_p^* is obtained by first size-biasing the random variable $|\mathscr{P}_p^*|$ and then recording the age of the individual when giving birth to a randomly chosen child. The mean of Υ_p^* has a simple expression, namely (see Lemma 2.12)

$$\mathbb{E}(\Upsilon_p^*) = \mathbb{E}\left(\int_0^\infty u \mathscr{P}_p^*(\mathrm{d}u)\right). \tag{1.6}$$

Nerman and Jagers [21] already noticed that Υ_p^* describes the age of an ancestor of a typical individual when giving birth to its next ancestor. For this reason, Υ_p^* and in particular the condition $\mathbb{E}(\Upsilon_p^*) < \infty$ – which is one way to formalize the "short edge" condition – plays a major role in previous works on CMJ processes, see for instance [26, 27, 28, 29, 30, 31]. In the present paper we prove that if $\mathbb{E}(\Upsilon_p^*) < \infty$, then in the near-critical regime the asymptotic behavior of the chronological height process is obtained by stretching the

genealogical height process by the deterministic factor $\mathbb{E}(\mathcal{Y}_p^*)$. The statement involves the following assumption which is alway satisfied (under (1.3)) in the non-triangular setting.

Condition Y. *The sequence of random variables* $(\mathcal{Y}_p^*)$ *is uniformly integrable and converges in law to a random variable* $\mathcal{Y}_\infty^*$ *with finite mean* α^*.

Theorem 1.3 (Short edges)**.** *Assume that conditions* G *and* Y *hold. Then*

$$\left(\mathscr{H}_p, \mathbb{H}_p\right) \overset{\text{fdd}}{\Longrightarrow} \left(\mathscr{H}_\infty, \alpha^* \mathscr{H}_\infty\right).$$

Remark 1.4. Theorem 1.3 is a consequence of a more general result: if, for a fixed t, $\mathscr{H}_p(t)$ is tight and a weaker condition than condition Y holds, then $\mathscr{H}_p(t) - \mathbb{H}_p(t) \Rightarrow 0$, see Theorem 3.1 below.

Remark 1.5. In [30], Sagitov investigated (in the non-triangular setting) the size of a CMJ process conditioned to survive at large time under the very short edge assumption introduced below, corresponding to $\mathbb{E}(V_1^*) < \infty$ and $\mathbb{E}(\mathcal{Y}_1^*) < \infty$ (see also Section 8 and Green [12]). The population size is described in the limit in terms of a continuous state branching process where space and time are scaled analogously as in Theorem 1.2. As a consequence, the previous result can be seen as a genealogical version of [30]. We also note that in [30], the results are obtained through an entirely different approach, namely analytic computations involving some non-trivial extension of the renewal theorem.

1.5.4. *Convergence of the chronological contour process.* The analysis of the contour process is significantly more delicate than that of the height process: compared to the Galton–Watson case, new difficulties are created by the chronological structure, see the discussion in Section 1.6.

For the chronological contour process, condition Y is not enough. Indeed, we note that $\mathbb{H}$ does not "see" what happens after an individual has given birth to its last child. In other words, two sequences of sticks $\omega = ((V_n, \mathscr{P}_n), n \in \mathbb{Z})$ and $\tilde{\omega} = ((\tilde{V}_n, \mathscr{P}_n), n \in \mathbb{Z})$ yield the same chronological height process. In contrast, the chronological contour process heavily depends on the life length of individuals and so an extra assumption on V_p^* is called upon.

Condition VP. *The sequence of random sticks* $(V_p^*, \mathscr{P}_p^*)$ *converges in law to a random stick* $(V_\infty^*, \mathscr{P}_\infty^*)$ *such that* V_∞^* *has mean* $\mathbb{E}(V_\infty^*) = \beta^* \in (0, \infty)$ *and* $\mathbb{E}(|\mathscr{P}_\infty^*|) = 1$. *Moreover, the sequence* (V_p^*) *is uniformly integrable.*

In light of the above discussion, condition VP is intuitively more stringent than condition Y and so we will refer to this case as to the case of "very short edges"[1]. Our main result shows that when conditions Y and VP hold, then the chronological contour process is obtained from the chronological height process by rescaling time by the deterministic factor $1/(2\beta^*)$. Hence, again provided that edges are short enough, this result provides a relation between the height and contour processes which is analogous to the discrete case. Moreover, the limits are proportional to the height process (up to multiplicative constant in time and space) of a continuous-state branching process as in the Galton-Watson case.

Theorem 1.6 (Very short edges case)**.** *Assume that conditions* G, Y *and* VP *hold, as well as the technical condition* V *in Section 4. Then*

$$\left(\mathscr{H}_p, \mathscr{C}_p, \mathbb{H}_p, \mathbb{C}_p\right) \overset{\text{fdd}}{\Longrightarrow} \left(\mathscr{H}_\infty, \mathscr{H}_\infty(\cdot/2), \alpha^* \mathscr{H}_\infty, \alpha^* \mathscr{H}_\infty \circ \varphi_\infty\right)$$

where $\varphi_\infty(t) = t/(2\beta^*)$.

[1] It follows from (1.6) that $\mathbb{E}(\mathcal{Y}^*) \le \mathbb{E}(V^* | \mathscr{P}^*|)$ and so if the life length is independent from the number of offspring, then we do obtain $\mathbb{E}(\mathcal{Y}^*) \le \mathbb{E}(V^*)$.

Remark 1.7. Condition V is here in order to have a generalized versions of the renewal theorem, see Section 5.2. Note that this condition is automatically satisfied in the non-triangular setting, in which case the assumptions of the previous result are simply condition G, $\mathbb{E}(V^*) < \infty$ and $\mathbb{E}(|\mathscr{P}^*|) = 1$.

Remark 1.8. Theorem 1.6 is a consequence of a more general result: if only condition VP holds (with no requirement on $\mathbb{Y}_p^*$), then the contour process can be obtained from the height process by a deterministic time change, see Theorem 4.1 below.

The previous result, and in particular the joint convergence

$$(\mathscr{C}_p, \mathbb{C}_p) \overset{\text{fdd}}{\Rightarrow} (\mathscr{H}_\infty(\cdot/2), \alpha^* \mathscr{H}_\infty \circ \varphi_\infty),$$

strongly suggests that the whole chronological forest can asymptotically be obtained from the genealogical one through a deterministic stretching of the edges. If instead of convergence of finite-dimensional marginals we had functional convergence in the previous display, then this would actually be exact. However, we exhibit counter-examples in Section 8 where the assumptions of Theorem 1.6 hold, but there cannot be functional convergence. Despite this negative result, the following result shows that the chronological and genealogical forests indeed become asymptotically proportional to one another in the sense of finite-dimensional distributions.

Theorem 1.9. *Assume that conditions* G, Y *and* VP *hold, as well as the technical condition* V *in Section 4. Then for every* $0 \le u \le v$ *we have*

$$\inf_{u \le t \le v} \mathbb{C}_p(t) - \alpha^* \inf_{u \le t \le v} \mathscr{C}_p(2\varphi_\infty(t)) \Rightarrow 0.$$

1.6. **Main ideas of the proof of Theorems 1.3 and 1.6 and technical challenges.** The proof of Theorem 1.3 is actually quite straightforward once Theorem 1.1 is established. Indeed, condition Y implies that the law of large numbers hold for R. It gives $R(n) \approx \alpha^* n$ for large n and as a consequence,

$$\mathbb{H}(n) = R \circ \widetilde{T}^{-1}(n) \circ \vartheta^n \approx \alpha^* \widetilde{T}^{-1}(n) \circ \vartheta^n = \alpha^* \mathscr{H}(n)$$

(note that since we are interested in convergence in distribution, the dual operator is actually irrelevant). Details are provided in Section 3.

In contrast, the proof of Theorem 1.6 is significantly more difficult. To explain this difficulty, it is useful to compare with the Galton–Watson case.

1.6.1. *The Galton-Watson case.* In the Galton-Watson case, the convergence of the contour process is obtained from the convergence of the height process by using the fact that the contour process somehow interpolates the height process (see details below). This observation leads to the inequality (see for instance [10, Equation (2.33)])

$$\sup_{0 \le s \le t} \left| \mathscr{C}_p(s) - \mathscr{H}_p(f_p(s)) \right| \le \varepsilon_p + \sup_{s \le t} \left| \mathscr{H}_p(s + 1/p) - \mathscr{H}_p(s) \right| \tag{1.7}$$

with

$$f_p(t) = \frac{1}{p} \inf\left\{ j \ge 0 : 2(j-1) - \mathscr{H}(j) \ge pt \right\}.$$

Because $\mathscr{H}(j) \ll j$, it is not hard to see that f_p converges (in a functional sense) to the linear function $t \mapsto t/2$. From (1.7), it is obvious that if $\mathscr{H}_p \Rightarrow \mathscr{H}_\infty$, again in a functional sense, with $\mathscr{H}_\infty$ continuous, then $\mathscr{H}_p$ and $\mathscr{C}_p$ converge jointly.

1.6.2. *The Crump-Mode-Jagers case.* Many of these ideas work in the present chronological setting, and we begin by explaining the interpolation alluded to above. Let in the sequel

$$\mathcal{V}(-1) = 0,\ \mathcal{V}(n) = V_0 + \cdots + V_n \ \text{ and } \ K_n = 2\mathcal{V}(n-1) - \mathbb{H}(n),\ n \geq 0.$$

Note that the sequence $(K_n, n \geq 0)$ is non-decreasing and that its terminal value is almost surely infinite (because of the subcritical assumption (1.2)). It can be checked from the definition of the chronological height and contour processes that:

- $\mathbb{C}(K_n) = \mathbb{H}(n)$ for every $n \in \mathbb{N}$;
- for $t \in [K_n, K_{n+1}]$, $\mathbb{C}$ first increases at rate $+1$ up to $\mathbb{H}(n) + V_n$ and then decreases at rate -1 to $\mathbb{H}(n+1)$.

Since $\mathbb{H}(n+1) \leq \mathbb{H}(n) + V_n$ and $V_n + (V_n + \mathbb{H}(n) - \mathbb{H}(n+1)) = K_{n+1} - K_n$, this interpolation is indeed well-defined. Moreover, it immediately entails the following bound (see for instance Figure 3, and note that it holds deterministically for any initial sequence of sticks):

$$\sup_{t \in [K_n, K_{n+1}]} |\mathbb{C}(t) - \mathbb{H}(n)| \leq |\mathbb{H}(n+1) - \mathbb{H}(n)| + V_n. \tag{1.8}$$

Let further φ be the left-continuous inverse of $(K_{\lfloor t \rfloor}, t \geq 0)$, defined by

$$\varphi(t) := \min\left\{ j \geq 0 : K_j \geq t \right\},\ t \geq 0. \tag{1.9}$$

Then defining

$$\varphi_p(t) := \frac{1}{p}\varphi(pt) = \frac{1}{p}\inf\left\{ j \geq 0 : 2\mathcal{V}(j-1) - \mathbb{H}(j) \geq pt \right\}, \tag{1.10}$$

the inequality (1.8) translates after scaling into

$$\left| \mathbb{C}_p(t) - \mathbb{H}_p(\varphi_p(t)) \right| \leq \varepsilon_p V_{\varphi(pt)} + \left| \mathbb{H}_p(\varphi_p(t) + 1/p) - \mathbb{H}_p(\varphi_p(t)) \right|,\ t \geq 0, \tag{1.11}$$

which is the chronological generalization of (1.7). Under condition VP, the law of large numbers applies to $\mathcal{V}$ and gives $\mathcal{V}(j-1) \approx \beta^* j$. As $\mathbb{H}(j) \ll j$, this gives similarly as in the Galton-Watson case $\varphi_p(t) \Rightarrow \varphi_\infty(t)$. However, the analogy with the Galton-Watson case stops here, and we now highlight the main differences with the Galton-Watson case, and the technical challenges to overcome in order to prove Theorem 1.6.

1.6.3. *Difference with the Galton-Watson case.* First of all, although in the Galton-Watson case the gap between convergence of finite-dimensional distributions and functional convergence of $\mathcal{H}_p$ is small (this is essentially condition (H2) above, and this can only happen in a triangular setting) this is not the case for the chronological height process. To illustrate this, we present in Section 8 simple *non-triangular* examples where $\mathbb{H}_p$ converges in the sense of finite-dimensional distributions but the limiting process is unbounded on any open interval. For this to happen in the Galton-Watson case, one has to consider very specific offspring distributions in a triangular setting (so that condition (H2) above does not hold), whereas here many simple examples, in a non-triangular setting, can be easily found. In other words, *assuming* functional convergence of $\mathbb{H}_p$ seems a strong hypothesis to make; and *finding* conditions under which $\mathbb{H}_p$ converges in a functional sense constitutes an interesting open problem which is not addressed here. More deranging, we also exhibit in Section 8 an example where $\mathbb{H}_p$ converges in a functional sense to a continuous process, but $\mathbb{C}_p$ fails to converge in a functional sense.

These various examples show that the usual techniques developed in the Galton-Watson case are insufficient, and new arguments are called upon.

1.6.4. *Technical challenges and new arguments.* Technically, one of the main difficulty comes from the fact that the random time $\varphi_p(t)$ appearing in (1.11) is not "nice": because the processes $\mathcal{V}$ and $\mathbb{H}$ appearing in its definition are dependent, we cannot readily rely on renewal-type arguments to control it, or to control other processes considered at this time. For instance, even the term $\varepsilon_p V_{\varphi(pt)}$ appearing in the right-hand side of (1.11), which seems innocuous as the rescaled length of a single individual (and which is just ε_p in the Galton-Watson case), is actually not straightforward to control and involved arguments are needed (see Section 6.3).

To circumvent this problem, the main idea is to approximate φ by a "nicer" random time $\bar{\varphi}$: since, as mentioned above, $\mathcal{V}(j) \gg \mathbb{H}(j)$, a natural approximation of φ is given by

$$\bar{\varphi}(t) = \inf\big\{j \geq 0 : 2\mathcal{V}(j) \geq t\big\}.$$

It turns out that $\bar{\varphi}$ indeed exhibits many useful properties and that the other processes are much easier to control when considered at $\bar{\varphi}$ than at φ. For instance, $2V_{\bar{\varphi}(pt)}$ is the jump of the renewal process $2\mathcal{V}$ straddling pt, and can thus be controlled by the renewal theorem. As another illustration, we will show in Lemma 6.7 that $\mathbb{H}$ shifted at $\bar{\varphi}$ has a simple and useful probabilistic description (which is not the case for $\mathbb{H}$ shifted at φ).

Thus, the global idea of the proof is to:

- show that $\bar{\varphi}$ and φ are close;
- use this to transfer problems on φ to problems on $\bar{\varphi}$;
- leverage the nicer structure of $\bar{\varphi}$ to solve problems on $\bar{\varphi}$.

In addition, one of the main ingredient to fulfill this program is a refined decomposition of the spine of an individual. This decomposition relies on the spine process, which generalizes the exploration process of Le Gall and Le Jan in [18] to the present chronological setting. This process lies at the heart of the proof of Theorem 1.1 and of many other results: it is presented in the next section.

1.7. **Notation.** Before going on we collect some general notation used throughout the paper.

1.7.1. *General notation.* Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of non-negative integers. For $x \in \mathbb{R}$ let $[x] = \max\{n \in \mathbb{Z} : n \leq x\}$ and $x^+ = \max(x, 0)$ be its integer and positive parts, respectively. If $A \subset \mathbb{R}$ is a finite set we denote by $|A|$ its cardinality. Throughout we adopt the convention $\max\emptyset = \sup\emptyset = -\infty$, $\min\emptyset = \inf\emptyset = +\infty$ and $\sum_{k=a}^{b} u_k = 0$ if $b < a$, with (u_k) any real-valued sequence.

1.7.2. *Measures.* Let $\mathscr{M}$ be the set of finite point measures on $(0,\infty)$ endowed with the weak topology, $\epsilon_x \in \mathscr{M}$ for $x > 0$ be the Dirac measure at x and $\mathbf{z}$ be the zero measure, the only measure with mass 0. For a measure $\nu \in \mathscr{M}$ we denote its mass by $|\nu| = \nu(0,\infty)$ and the supremum of its support by $\pi(\nu) = \inf\{x > 0 : \pi(x,\infty) = 0\}$ with the convention $\pi(\mathbf{z}) = 0$. For $k \in \mathbb{N}$ we define $\Upsilon_k(\nu) \in \mathscr{M}$ as the measure obtained by removing the k largest atoms of ν, i.e., $\Upsilon_k(\nu) = \mathbf{z}$ for $k \geq |\nu|$ and, writing $\nu = \sum_{i=1}^{|\nu|} \epsilon_{a(i)}$ with $0 < a(|\nu|) \leq \cdots \leq a(1)$, $\Upsilon_k(\nu) = \sum_{i=k+1}^{|\nu|} \epsilon_{a(i)}$ for $k = 0, \ldots, |\nu| - 1$.

1.7.3. *Finite sequences of measures.* We let $\mathscr{M}^* = \cup_{n\in\mathbb{N}}(\mathscr{M} \setminus \{\mathbf{z}\})^n$ be the set of finite sequences of non-zero measures in $\mathscr{M}$. For $Y \in \mathscr{M}^*$ we denote by $\mathrm{Len}(Y)$ the only integer $n \in \mathbb{N}$ such that $Y \in (\mathscr{M} \setminus \{\mathbf{z}\})^n$, which we call the length of Y, and identify $\mathbf{z}$ with the only sequence of length 0. For two sequences $Y_1 = (Y_1(1), \ldots, Y_1(H_1))$ and $Y_2 = (Y_2(1), \ldots, Y_2(H_2))$ in $\mathscr{M}^*$ with lengths $H_1, H_2 \geq 1$, we define $[Y_1, Y_2] \in \mathscr{M}^*$ as their concatenation:

$$[Y_1, Y_2] = \big(Y_1(1), \ldots, Y_1(H_1), Y_2(1), \ldots, Y_2(H_2)\big).$$

Further, by convention we set $[\mathbf{z}, Y] = [Y, \mathbf{z}] = Y$ for any $Y \in \mathscr{M}^*$ and we then define inductively

$$[Y_1, \dots, Y_N] = \big[[Y_1, \dots, Y_{N-1}], Y_N\big],\ N \geq 2.$$

Note that, with these definitions, we have $\mathrm{Len}\,([Y_1, \dots, Y_N]) = \mathrm{Len}\,(Y_1) + \cdots + \mathrm{Len}\,(Y_N)$ for any $N \geq 1$ and $Y_1, \dots, Y_N \in \mathscr{M}^*$.

Identifying a measure $\nu \in \mathscr{M} \setminus \{\mathbf{z}\}$ with the sequence of length one $(\nu) \in \mathscr{M}^*$, the above definitions give sense to, say, $[Y, \nu]$ with $Y \in \mathscr{M}^*$ and $\nu \in \mathscr{M} \setminus \{\mathbf{z}\}$. The operator π defined on $\mathscr{M}$ is extended to $\mathscr{M}^*$ through the relation

$$\pi(Y) = \sum_{k=1}^{\mathrm{Len}(Y)} \pi(Y(k)),\ Y = (Y(1), \dots, Y(\mathrm{Len}\,(Y)) \in \mathscr{M}^*.$$

Recalling the convention $\sum_{k=1}^{0} = 0$, we see that $\pi(\mathbf{z}) = 0$ and further, it follows directly from the above relation that $\pi([Y_1, \dots, Y_N]) = \pi(Y_1) + \cdots + \pi(Y_N)$.

1.7.4. *Measurable space.* We define $\mathbb{L} = \{(v, \nu) \in (0, \infty) \times \mathscr{M} : v \geq \pi(\nu)\}$ and call an element $s \in \mathbb{L}$ either a *stick* or a *life descriptor*. We work on the measurable space $(\Omega, \mathscr{F})$ with $\Omega = \mathbb{L}^{\mathbb{Z}}$ the space of doubly infinite sequences of sticks and $\mathscr{F}$ the σ-algebra generated by the coordinate mappings. An elementary event $\omega \in \Omega$ is written as $\omega = (\omega_n, n \in \mathbb{Z})$ and $\omega_n = (V_n, \mathscr{P}_n)$. For $n \in \mathbb{Z}$ we consider the three operators $\theta_n, \vartheta^n, \mathscr{G} : \Omega \to \Omega$ defined as follows:

- θ_n is the shift operator, defined by $\theta_n(\omega) = (\omega_{n+k}, \in \mathbb{Z})$;
- ϑ^n is the dual (or time-reversal) operator, defined by $\vartheta^n(\omega) = (\omega_{n-k-1}, k \in \mathbb{Z})$;
- $\mathscr{G}$ is the genealogical operator, mapping the sequence $((V_n, \mathscr{P}_n), n \in \mathbb{Z})$ to the sequence $((1, |\mathscr{P}_n|\epsilon_1), n \in \mathbb{Z})$.

Note that the genealogical and chronological height and contour processes are related by the relations $\mathscr{H} = \mathbb{H} \circ \mathscr{G}$ and $\mathscr{C} = \mathbb{C} \circ \mathscr{G}$. We say that a mapping $\Gamma : \Omega \to \mathfrak{X}$ (valued in an arbitrary space $\mathfrak{X}$) is a genealogical mapping if it is invariant by the genealogical operator, i.e., if $\Gamma \circ \mathscr{G} = \Gamma$. The shift and dual operators are related by the following relations:

$$\vartheta^m \circ \vartheta^n = \theta_{n-m} \text{ and } \vartheta^n \circ \theta_m = \vartheta^{n+m},\ m, n \in \mathbb{Z}, \tag{1.12}$$

and for any random time $\Gamma : \Omega \to \mathbb{Z}$ we have

$$\mathscr{P}_\Gamma \circ \vartheta^n = \mathscr{P}_{n-1-\Gamma \circ \vartheta^n}. \tag{1.13}$$

2. Spine process and Lukasiewicz path

In this section, we introduce the spine process and relate it to the well-known Lukasiewicz path. The spine process is introduced in Sections 2.1 and 2.2 and the Lukasiewicz in Section 2.4. We prove in Section 2.5 a crucial formula for the spine process (see Proposition 2.4) from which Theorem 1.1 is readily derived. More precisely, the spine process is expressed in terms of a random functional of the weak ascending ladder height process associated to the dual Lukasiewicz path. Sections 2.6 and 2.7 continue the study of the spine process and give a description of the.

2.1. **Overview of the spine process.** The idea underlying the definition of the spine process relies on the decomposition of the "spine" – or "ancestral line" – lying below the point of the tree corresponding to the birth of the nth individual. In the nth step of the sequential construction presented on Figure 2, this corresponds to the path in the forest starting from the root and reaching up to n (which also corresponds to the right-most path in the planar forest constructed at step n). As can be seen from the figure, this path is naturally decomposed into finitely many segments that correspond to each ancestor's contribution to the spine: these segments are highlighted in bold on Figure 6.

The spine process at n is then defined as a sequence of measures that encodes this decomposition. More precisely, we start by labeling ancestors from highest to lowest. Then, the kth element of the spine process (evaluated at time n) is simply the measure

that records the location of the stubs on the kth segment – crosses on Figure 6 – and the age of the kth ancestor upon giving birth to the $(k-1)$st ancestor – circles on Figure 6.

2.2. **Spine process.** Consider the operator $\Phi : \mathscr{M}^* \times \mathscr{M} \to \mathscr{M}^*$ defined for $v \in \mathscr{M}$ and $Y = (Y(1), \dots, Y(\mathrm{Len}(Y)) \in \mathscr{M}^*$ by

$$\Phi(Y,v) = \begin{cases} [Y,v] & \text{if } v \neq \mathbf{z}, \\ \big(Y(1),\dots,Y(H-1),\Upsilon_1(Y(H))\big) & \text{if } v = \mathbf{z} \text{ and } H \geq 1, \\ \mathbf{z} & \text{else,} \end{cases} \tag{2.1}$$

where $H = \max\{k \geq 1 : |Y(k)| \geq 2\}$. Note that by definition, we have $\Phi(Y,v) \in \mathscr{M}^*$ for $Y \in \mathscr{M}^*$ and $v \in \mathscr{M}$ and that further, if $v \neq \mathbf{z}$ then $\Phi(Y,v) \neq \mathbf{z}$.

The spine process $\mathbb{S}_0 = (\mathbb{S}_0^n, n \geq 0)$ (the subscript 0 will be justified below, see (2.9)) is the $\mathscr{M}^*$-valued sequence defined recursively by

$$\mathbb{S}_0^0 = \mathbf{z} \text{ and } \mathbb{S}_0^{n+1} = \Phi(\mathbb{S}_0^n, \mathscr{P}_n),\ n \geq 0. \tag{2.2}$$

This dynamic is illustrated on Figure 6. As already discussed in the introduction, the kth element of $\mathbb{S}_0^n$ (ordered from top to bottom) records (1) the location of the stubs on the kth segment in the spine decomposition illustrated in Figure 6, and (2) the age of the kth ancestor (of n) when begetting the $(k-1)$st ancestor (identifying, for $k = 1$, the individual with its 0th ancestor). In words, the recursive relation (2.2) encodes the fact that the birth event corresponding to the $(n+1)$st individual coincides with the next available stub after grafting the nth stick on top of $\mathbb{S}_0^n$. In particular, if no stub is available, a new spine is started from scratch (third relation).

We note that when $\mathbb{S}_0^n \neq \mathbf{z}$, any element of the sequence $\mathbb{S}_0^n$ contains at least one atom: the one corresponding the birth of an ancestor, which is not counted as a *stub*. In particular, the condition $H = \max\{k \geq 1 : |Y(k)| \geq 2\}$ in (2.1) reads "look for the first available segment with a stub".

2.3. **Link between the spine process and the height and exploration processes.** As discussed above, the spine process encodes the spine of an individual by breaking it into the different sticks of its ancestors as in Figure 6. In particular, the birth time of the individual is recovered by summing up the lengths of the sticks appearing in the spine process: this means precisely that the spine process and the chronological height process are related as follows:

$$\mathbb{H}(n) = \pi(\mathbb{S}_0^n),\ n \geq 0.$$

The spine process can be seen as a chronological generalization of the exploration process of Le Gall and Le Jan [18], and for this reason we will define $\rho_0^n = \mathbb{S}_0^n \circ \mathscr{G}$ as the *exploration process*. This process is not exactly the one of Le Gall and Le Jan. Therein, the authors only consider the stubs attached to the spine. However, in the chronological case, not only do we need to keep track of the number of available stubs, but one needs to also record the length of the segments carrying those stubs (in the discrete case, this is always equal to 1). This is done by adding the additional atom corresponding to the birth of the "previous" ancestor (when ancestors are labelled from top to bottom), and whose location coincides with the length of the corresponding segment.

2.4. **Lukasiewicz path.** We define the Lukasiewicz path $S = (S(n), n \in \mathbb{Z})$ by $S(0) = 0$ and, for $n \geq 1$,

$$S(n) = \sum_{k=0}^{n-1} (|\mathscr{P}_k| - 1) \text{ and } S(-n) = -\sum_{k=-n}^{-1} (|\mathscr{P}_k| - 1).$$

Note that if Γ is a random time, the dual operator acts as follows:

$$S(\Gamma) \circ \vartheta^n = S(n) - S(n - \Gamma \circ \vartheta^n),\ n \in \mathbb{Z}. \tag{2.3}$$

We consider the following functionals associated to S, which will be used repeatedly in the rest of the paper:

- the sequence of weak ascending ladder height times: $T(0) = 0$ and for $k \geq 0$,
$$T(k+1) = \inf\{\ell > T(k) : S(\ell) \geq S(T(k))\} = T(1) \circ \theta_{T(k)} + T(k);$$
- the hitting times upward and downward:
$$\tau_\ell = \inf\{k > 0 : S(k) \geq \ell\} \text{ and } \tau_\ell^- = \inf\{k \geq 0 : S(k) = -\ell\},\ \ell \geq 0,$$
so that in particular $\tau_0 = T(1)$;
- for $\ell \in \mathbb{N}$ with $\tau_\ell < \infty$,
$$\zeta_\ell = \ell - S(\tau_\ell - 1) \text{ and } \mu_\ell = \Upsilon_{\zeta_\ell}(\mathscr{P}_{\tau_\ell - 1}),$$
so that ζ_ℓ is the undershoot upon reaching level ℓ;
- and the backward maximum
$$L(m) = \max_{k=0,\dots,m} S(-k),\ m \geq 0.$$

Note that, since $S(\tau_0) \geq 0$, $\zeta_0 = -S(\tau_0 - 1) \leq S(\tau_0) - S(\tau_0 - 1) = |\mathscr{P}_{\tau_0 - 1}| - 1$, so that $\mu_0 \neq \mathbf{z}$. We will pay special attention to the following functionals of the ladder height process:

- for $k \geq 1$ with $T(k) < \infty$,
$$\mathscr{Q}(k) = \mu_0 \circ \theta_{T(k-1)} \text{ and } \Upsilon(k) = \pi \circ \mathscr{Q}(k) = A_{\xi(k)}\left(\mathscr{P}_{T(k)-1}\right);$$
where $A_k(\nu)$ is the position of the kth largest atom of ν;
- the following two inverses associated to the sequence $(T(k), k \geq 0)$:
$$T^{-1}(n) = \min\{k \geq 0 : T(k) \geq n\} \text{ and } \widetilde{T}^{-1}(n) = \max\{k \geq 0 : T(k) \leq n\},\ n \geq 0.$$

The fact that $\mu_0 \neq \mathbf{z}$ implies that $\mathscr{Q}(k) \neq \mathbf{z}$ whenever it is well-defined, a simple fact that will be used later on. If n is a weak ascending ladder height time, then $\widetilde{T}^{-1}(n) = T^{-1}(n)$ with $T(\widetilde{T}^{-1}(n)) = n = T(T^{-1}(n))$, while if n is not a weak ascending ladder height time, then $\widetilde{T}^{-1}(n) + 1 = T^{-1}(n)$ with $T(\widetilde{T}^{-1}(n)) < n < T(T^{-1}(n))$. Define
$$\mathscr{A}(n) = \{n - T(k) : k \geq 0\} \circ \vartheta^n,\ n \geq 0.$$
It is well-known that $\mathscr{A}(n) \cap \mathbb{R}_+$ is the set of n's ancestors, see for instance Duquesne and Le Gall [10]. This property relates the height process and the weak ascending ladder height times T through the following identity:
$$\mathscr{H}(n) = \widetilde{T}^{-1}(n) \circ \vartheta^n,\ n \geq 0. \tag{2.4}$$
The genealogical height is also given by the length of $\mathbb{S}_0^n$ as we show now.

Lemma 2.1. *For any $n \geq 0$ we have* $\mathrm{Len}\left(\mathbb{S}_0^n\right) = \mathscr{H}(n)$.

Proof. As highlighted in Section 2.3, the exploration process $\rho_0^n = \mathbb{S}_0^n \circ \mathscr{G}$ slightly differs from the classical definition of the exploration process in Le Gall and Le Jan [18]: however, this slight difference does not alter the length of the sequence, which remains unchanged between the two definitions.

Since the length of the sequence in the classical exploration process coincides with the height process, this implies that $\mathrm{Len}\left(\rho_0^n\right) = \mathscr{H}(n)$. Thus, $\mathrm{Len}\left(\mathbb{S}_0^n\right) = \mathrm{Len}\left(\mathbb{S}_0^n\right) \circ \mathscr{G} = \mathrm{Len}\left(\mathbb{S}_0^n \circ \mathscr{G}\right)$, which proves the desired result. □

Define
$$m \wedge n = \max\left(\mathscr{A}(m) \cap \mathscr{A}(n)\right),\ m, n \geq 0.$$
Then $m \wedge n \in \mathbb{Z}$ and m and n have an ancestor in common (i.e., belong to the same tree) if and only if $m \wedge n \geq 0$ in which case $m \wedge n$ is the lexicographic index of their most recent common ancestor – see for instance [10]. We end this section by listing the following identities, which are proved in the Appendix A. The second identity involves the condition $L(n-m) \circ \vartheta^m > 0$: it is readily checked that
$$L(n-m) \circ \vartheta^m = S(m) - \min_{\{m,\dots,n\}} S,\ 0 \leq m \leq n, \tag{2.5}$$

Lemma 2.2. *For any $0 \le m \le n$, m is an ancestor of n, i.e., $m \wedge n = m$, if and only if $L(n-m) \circ \vartheta^m = 0$.*

Proof. The exploration of the subtree rooted at the mth individual starts at time m and ends at time $\inf\{n \ge m : S(n) = S(m) - 1\}$. Thus, $S(m) \ge \min_{\{m,\dots,n\}} S$ is a necessary and sufficient condition for n to be in the subtree rooted at m, i.e., for m to be an ancestor of n, and so this gives the result by (2.5). □

Lemma 2.3. *For any $n \ge m \ge 0$ with $L(n-m) \circ \vartheta^m > 0$, we have*

$$m \wedge n = n - T(T^{-1}(n-m)) \circ \vartheta^n = m - \tau_{L(n-m)} \circ \vartheta^m \tag{2.6}$$

and

$$\mathcal{Q}(T^{-1}(n-m)) \circ \vartheta^n = \mu_{L(n-m)} \circ \vartheta^m. \tag{2.7}$$

2.5. **Fundamental formula for $\mathbb{S}_0^n$ and proof of Theorem 1.1.** The goal of this section is to prove the following fundamental formula for $\mathbb{S}_0^n$. As we show right after, Theorem 1.1 is easily derived from it.

Proposition 2.4. *We have*

$$\mathbb{S}_0^n = \left(\mathcal{Q}(\widetilde{T}^{-1}(n)), \dots, \mathcal{Q}(1)\right) \circ \vartheta^n, \ n \ge 0. \tag{2.8}$$

Proof of Theorem 1.1 based on Proposition 2.4. As explained in Section 2.3 we have $\mathbb{H}(n) = \pi(\mathbb{S}_0^n)$, and so by definition of π this gives

$$\mathbb{H}(n) = \sum_{k=1}^{\mathrm{Len}\left(\mathbb{S}_0^n\right)} \pi\left(\mathbb{S}_0^n(k)\right).$$

Therefore, we obtain by plugging in (2.8)

$$\mathbb{H}(n) = \sum_{k=1}^{\widetilde{T}^{-1}(n) \circ \vartheta^n} \pi\left(\mathcal{Q}(k) \circ \vartheta^n\right) = \left(\sum_{k=1}^{\widetilde{T}^{-1}(n)} \pi \circ \mathcal{Q}(k)\right) \circ \vartheta^n.$$

Since $\mathbb{Y}(k) = \pi \circ \mathcal{Q}(k)$ this gives $\mathbb{H}(n) = (R \circ \widetilde{T}^{-1}(n)) \circ \vartheta^n$, and as $\mathscr{H}(n) = \widetilde{T}^{-1}(n) \circ \vartheta^n$ the proof of Theorem 1.1 is complete. □

The rest of this section is devoted to proving Proposition 2.4. We prove it through several lemmas, several of which will be used in the sequel. To prove these results, for $m \ge 0$ and $k \in \{0, \dots, |\mathscr{P}_m|\}$ we introduce

$$\chi(m,k) = \tau_k^- \circ \theta_{m+1} + m + 1 = \inf\{n \ge m+1 : S(n) = S(m+1) - k\}$$

and define $\chi(m) = \chi(m, |\mathscr{P}_m|)$ so that

$$\chi(m) = \inf\{n \ge m+1 : S(n) = S(m+1) - |\mathscr{P}_m|\} = \inf\{n \ge m+1 : S(n) = S(m) - 1\}$$

which is also equal to $\tau_1^- \circ \theta_m + m$. Intuitively, for $k \in \{0, \dots, |\mathscr{P}_m| - 1\}$, $\chi(m,k)$ corresponds to the index of $(k+1)$st child of the mth individual (with the convention that children are ranked from youngest to oldest); whereas $\chi(m)$ is the index of the highest stub on $\mathbb{S}_0^m$ (i.e., right before attaching the mth individual). In particular, any individual $n \in \{m+1, \dots, \chi(m) - 1\}$ belongs to a subtree attached to m. In view of this interpretation, the two following lemmas seem quite natural. For the proof of Lemma 2.8 we will need the following identity, whose proof is deferred to Appendix B.

Lemma 2.5. *Let $n \ge 0$, $m = n - \tau_0 \circ \vartheta^n$ and $i = \zeta_0 \circ \vartheta^n$. If $m \ge 0$, then it holds that $i \in \{0, \dots, |\mathscr{P}_m| - 1\}$ and $\chi(m,i) = n$.*

Lemma 2.6. *For any $m \ge 0$ such that $|\mathscr{P}_m| > 0$, $n \in \{m+1, \dots, \chi(m)-1\}$ and $\ell \in \{1, \dots, \mathscr{H}(m)\}$ we have*

$$\mathscr{H}(n) > \mathscr{H}(m) \ \text{ and } \ \mathbb{S}_0^n(\ell) = \mathbb{S}_0^m(\ell).$$

Proof. Let ℓ, m and n be as in the statement: we first prove that $\mathscr{H}(n) > \mathscr{H}(m)$. Since S only makes negative jumps of size -1, we have by definition of $\chi(m)$

$$\min_{\{m+1,\dots,\chi(m)-1\}} S \geq S(m).$$

This inequality implies that, since $n \in \{m+1,\dots,\chi(m)-1\}$, there is at least one more ladder height time for the dual Lukasiewicz process seen from n as compared to the dual Lukasiewicz process seen from m. In view of the relation (2.4) which expresses $\mathscr{H}(n) = \widetilde{T}^{-1}(n) \circ \vartheta^n$ as the number of weak ascending ladder height times of the dual Lukasiewicz process, this means precisely that $\mathscr{H}(n) > \mathscr{H}(m)$.

We now prove that $\mathbb{S}_0^n(\ell) = \mathbb{S}_0^m(\ell)$. Since $n \in \{m+1,\dots,\chi(m)-1\}$, in order to prove this it is enough to prove that $\chi' \geq \chi(m)$ where we define

$$\chi' = \inf\left\{k \geq m+1 : \mathbb{S}_0^k(\ell) \neq \mathbb{S}_0^m(\ell)\right\}.$$

In view of the definition (2.1) of Φ and the dynamic (2.2), we see that the ℓth element of the spine between m and n is modified only if the length of the spine goes below ℓ between m and n. Since the length of the spine coincides with $\mathscr{H}$, this implies $\mathscr{H}(\chi') = \ell \leq \mathscr{H}(m)$. Finally, since $\mathscr{H}(m) < \min_{\{m+1,\dots,\chi(m)-1\}} \mathscr{H}$, this implies that $\chi' \geq \chi(m)$ and concludes the proof. □

Lemma 2.7. *For $m \geq 0$ such that $|\mathscr{P}_m| > 0$ and $k \in \{0,\dots,|\mathscr{P}_m|-1\}$ we have*

$$\mathscr{H}(\chi(m,k)) = \mathscr{H}(m)+1 \text{ and } \mathbb{S}_0^{\chi(m,k)}(\mathscr{H}(m)+1) = \Upsilon_k(\mathscr{P}_m).$$

Proof. By definition of $\chi(m,k)$ and the fact that S only makes jumps of negative size -1, we have

$$S(\chi(m,k)) = \min_{m+1,\dots,\chi(m,k)} S \geq S(m).$$

A similar argument as in the proof of the previous lemma then leads to the conclusion $\mathscr{H}(\chi(m,k)) = \mathscr{H}(m)+1$ (i.e., by showing that there is exactly one extra ladder height time for the dual walk seen from $\chi(m,k)$).

We now prove that $\mathbb{S}_0^{\chi(m,k)}(\mathscr{H}(m)+1) = \Upsilon_k(\mathscr{P}_m)$. For $k=0$ this is seen to be true by looking at the dynamic (2.2). We now prove that this is true by induction: so assume this is true for $k \in \{0,\dots,|\mathscr{P}_m|-2\}$ and let us prove that this continues to hold for $k+1$. In order to do so, it is sufficient to combine the induction hypothesis with the following claim:

$$\mathbb{S}_0^{\chi(m,k+1)}(\mathscr{H}(m)+1) = \Upsilon_1\left(\mathbb{S}_0^{\chi(m,k)}(\mathscr{H}(m)+1)\right).$$

In order to prove this identity, we first note that (again, this is seen by comparing the number of ladder height times of the dual processes seen from the two times)

$$\mathscr{H}(n) > \mathscr{H}(\chi(m,k)) = \mathscr{H}(m)+1 \text{ for } n = \chi(m,k)+1,\dots,\chi(m,k+1)-1.$$

Finally, we already know that $\mathscr{H}(\chi(m,k+1)) = \mathscr{H}(m)+1$. From the dynamic (2.2), this implies that the $(\mathscr{H}(m)+1)$st element of $\mathbb{S}_0^n$ remains unchanged for $n = \chi(m,k)+1,\dots,\chi(m,k+1)-1$, but that one stub is removed at time $\chi(m,k+1)$, i.e.,

$$\mathbb{S}_0^{\chi(m,k+1)}(\mathscr{H}(m)+1) = \Upsilon_1\left(\mathbb{S}_0^{\chi(m,k)}(\mathscr{H}(m)+1)\right).$$

This proves the claim made earlier and ends the proof of Lemma 2.7. □

The purpose of the next lemma is to decompose the spine at time n before and after the kth ancestor: that n has $\geq k$ ancestors if and only if $T(k) \circ \vartheta^n \leq n$ explains the condition in the following statement.

Lemma 2.8. *For any $n, k \geq 0$ with $T(k) \circ \vartheta^n \leq n$ we have*

$$\mathbb{S}_0^n = \left[\mathbb{S}_0^{n-T(k)\circ\vartheta^n}, \mathscr{Q}(k) \circ \vartheta^n, \dots, \mathscr{Q}(1) \circ \vartheta^n\right].$$

Recall the convention $[\mathbf{z}, Y] = Y$ for any $Y \in \mathscr{M}^*$: in particular,

$$\left[\mathbb{S}_0^{n-T(k)\circ\vartheta^n}, \mathscr{Q}(k)\circ\vartheta^n, \dots, \mathscr{Q}(1)\circ\vartheta^n\right] = \left[\mathscr{Q}(k)\circ\vartheta^n, \dots, \mathscr{Q}(1)\circ\vartheta^n\right]$$

when $\mathbb{S}_0^{n-T(k)\circ\vartheta^n} = \mathbf{z}$.

Proof of Lemma 2.8. Let us first prove the result for $k = 1$, so we consider $n \geq 0$ with $\tau_0 \circ \vartheta^n \leq n$ and we prove that $\mathbb{S}_0^n = [\mathbb{S}_0^{n-T(1)\circ\vartheta^n}, \mathscr{Q}(1)\circ\vartheta^n]$. Combining the two previous lemmas, we see that

$$\mathbb{S}_0^{\chi(m,i)} = \left[\mathbb{S}_0^m, \Upsilon_i(\mathscr{P}_m)\right]$$

for any $m \geq 0$ and any $i \in \{0, \dots, |\mathscr{P}_m| - 1\}$. In particular, Lemma 2.5 shows that we can apply this to $m = n - \tau_0 \circ \vartheta^n$ and $i = \zeta_0 \circ \vartheta^n$, which gives

$$\mathbb{S}_0^{\chi(m,i)} = \left[\mathbb{S}_0^{n-\tau_0\circ\vartheta^n}, \Upsilon_{\zeta_0\circ\vartheta^n}(\mathscr{P}_{n-\tau_0\circ\vartheta^n})\right].$$

On the one hand, we have $\chi(m,i) = n$ (again by Lemma 2.5) and so in particular $\mathbb{S}_0^{\chi(m,i)} = \mathbb{S}_0^n$, while on the other hand, we have

$$\Upsilon_{\zeta_0\circ\vartheta^n}(\mathscr{P}_{n-\tau_0\circ\vartheta^n}) = \Upsilon_{\zeta_0}(\mathscr{P}_{\tau_0-1})\circ\vartheta^n = \mathscr{Q}(1)\circ\vartheta^n.$$

Combining the above arguments concludes the proof for $k = 1$. The general case follows by induction left to the reader. □

We can now prove Proposition 2.4.

Proof of Proposition 2.4. By definition, $T(\widetilde{T}^{-1}(n)) \leq n$ and so Lemma 2.8 with $k = \widetilde{T}^{-1}(n)$ yields

$$\mathbb{S}_0^n = \left[\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n))\circ\vartheta^n}, \mathscr{Q}(\widetilde{T}^{-1}(n))\circ\vartheta^n, \cdots, \mathscr{Q}(1)\circ\vartheta^n\right].$$

Since $\mathscr{Q}(k) \neq \mathbf{z}$ whenever it is well-defined, in particular for $k \in \{1, \dots, \widetilde{T}^{-1}(n)\}$, it follows that

$$\mathrm{Len}\left(\mathbb{S}_0^n\right) = \widetilde{T}^{-1}(n)\circ\vartheta^n + \mathrm{Len}\left(\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n))\circ\vartheta^n}\right).$$

However, $\mathrm{Len}\left(\mathbb{S}_0^n\right) = \widetilde{T}^{-1}(n)\circ\vartheta^n$ by (2.4), and thus $\mathrm{Len}\left(\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n))\circ\vartheta^n}\right) = 0$ which means $\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n))\circ\vartheta^n} = \mathbf{z}$. This achieves the proof of Proposition 2.4. □

2.6. **Right decomposition of the spine.** In the case of i.i.d. life descriptors, the spine process is by construction (2.2) a Markov process and the present section can be seen as a description of its transition probabilities: we show in Proposition 2.9 that for $m \leq n$, the spine at n is deduced from the spine at m by truncating $\mathbb{S}_0^m$ and then by concatenating a spine that is independent of the past up to m, a construction reminiscent of the snake property – see Duquesne and Le Gall [10]. As we shall now see, the independent "increment" will be given by

$$\mathbb{S}_m^n := \mathbb{S}_0^{n-m}\circ\theta_m,\ 0 \leq m \leq n, \tag{2.9}$$

which, when life descriptors are i.i.d., is distributed as the original spine at time $n - m$. In particular, since $\pi(\mathbb{S}_m^n) = \pi(\mathbb{S}_0^{n-m}\circ\theta_m) = \pi(\mathbb{S}_0^{n-m})\circ\theta_m = \mathbb{H}(n-m)\circ\theta_m$, we note that an immediate consequence of (1.1) and (1.12) is that

$$\pi(\mathbb{S}_m^n) = \left(\sum_{k=1}^{\widetilde{T}^{-1}(n-m)} \mathbb{Y}(k)\right)\circ\vartheta^n,\ 0 \leq m \leq n. \tag{2.10}$$

Proposition 2.9. *Let $n \geq m \geq 0$. If $m \wedge n \geq 0$, then $\mathbb{S}_0^n = [\mathbb{S}_0^{m\wedge n}, \mathbb{S}_{m\wedge n}^n]$ and*

$$\mathbb{S}_{m\wedge n}^n = \begin{cases} \left[\mu_{L(n-m)}\circ\vartheta^m, \mathbb{S}_m^n\right] & \text{if } L(n-m)\circ\vartheta^m > 0, \\ \mathbb{S}_m^n & \text{else.} \end{cases} \tag{2.11}$$

In order to prove Proposition 2.9, we will need the following lemma.

Lemma 2.10. *For any $n \geq m \geq 0$ we have*

$$\mathbb{S}_m^n = \left(\mathcal{Q}(\widetilde{T}^{-1}(n-m)), \ldots, \mathcal{Q}(1)\right) \circ \vartheta^n. \tag{2.12}$$

If in addition $m \wedge n \geq 0$, then

$$\mathbb{S}_{m\wedge n}^n = \left(\mathcal{Q} \circ T^{-1}(n-m), \ldots, \mathcal{Q}(1)\right) \circ \vartheta^n. \tag{2.13}$$

Proof. By definition we have $\mathbb{S}_m^n = \mathbb{S}_0^{n-m} \circ \theta_m$ and so Proposition 2.4 implies that

$$\mathbb{S}_m^n = \left(\mathcal{Q}(\widetilde{T}^{-1}(n-m)), \ldots, \mathcal{Q}(1)\right) \circ \vartheta^{n-m} \circ \theta_m. \tag{2.14}$$

The first relation (2.12) thus follows from the identity $\vartheta^{n-m} \circ \theta_m = \vartheta^n$ of (1.12). To prove the other relation (2.13), we use (2.12) with m random, which in this case reads as follows: for any random time Γ, the relation

$$\mathbb{S}_{\Gamma\circ\vartheta^n}^n = \left(\mathcal{Q}(\widetilde{T}^{-1}(n-\Gamma)), \ldots, \mathcal{Q}(1)\right) \circ \vartheta^n \tag{2.15}$$

holds in the event $0 \leq \Gamma \circ \vartheta^n \leq n$. Apply now this relation to $\Gamma = n - T(T^{-1}(n-m))$, so that $m \wedge n = \Gamma \circ \vartheta^n$ by (2.6). Then we always have $\Gamma \leq n$ and so under the assumption $m \wedge n \geq 0$, we obtain

$$\mathbb{S}_{m\wedge n}^n = \left(\mathcal{Q}(\widetilde{T}^{-1}(T(\Gamma'))), \ldots, \mathcal{Q}(1)\right) \circ \vartheta^n$$

with $\Gamma' = T^{-1}(n-m)$. Since $\widetilde{T}^{-1}(T(k)) = k$ for any $k \geq 0$, we obtain the result. □

Remark 2.11. Let us comment on (2.15) as similar identities will be used in the sequel. To see how it follows from (2.14), write (2.14) in the form $\mathbb{S}_m^n = (U \circ \vartheta^n)(m)$ for some mapping U with domain Ω and values in the space of $\mathscr{M}^*$-valued sequence, so that $(U \circ \vartheta^n)(m)$ is the mth element of the dual sequence. With this notation, we can directly plug in a random time, i.e., if $m = \Gamma$ is random then we have $\mathbb{S}_\Gamma^n = (U \circ \vartheta^n)(\Gamma)$ and in particular, $\mathbb{S}_{\Gamma\circ\vartheta^n}^n = (U \circ \vartheta^n)(\Gamma \circ \vartheta^n) = U(\Gamma) \circ \vartheta^n$.

Proof of Proposition 2.9. By (2.6), $m \wedge n \geq 0$ implies that $T(T^{-1}(n-m)) \circ \vartheta^n \leq n$ and so Lemma 2.8 with $k = T^{-1}(n-m)$ gives

$$\mathbb{S}_0^n = \left[\mathbb{S}_0^{n-T(T^{-1}(n-m))\circ\vartheta^n}, \mathcal{Q}(T^{-1}(n-m)) \circ \vartheta^n, \ldots, \mathcal{Q}(1) \circ \vartheta^n\right].$$

Combining (2.6), which shows that $\mathbb{S}_0^{n-T(T^{-1}(n-m))\circ\vartheta^n} = \mathbb{S}_0^{m\wedge n}$, and the expression for $\mathbb{S}_{m\wedge n}^n$ given in (2.13) under the assumption $m \wedge n \geq 0$ gives the first part of the result, namely that $\mathbb{S}_0^n = \left[\mathbb{S}_0^{m\wedge n}, \mathbb{S}_{m\wedge n}^n\right]$. In order to show (2.11) and thus complete the proof, we distinguish between the two cases $L(n-m) \circ \vartheta^m = 0$ and $L(n-m) \circ \vartheta^m > 0$.

If $L(n-m) \circ \vartheta^n = 0$, then $m \wedge n = m$ according to Lemma 2.2 which proves (2.11).

Assume now that $L(n-m) \circ \vartheta^n > 0$: in view of (2.5), this means that $n-m$ is not a weak ascending ladder height time of $S \circ \vartheta^n$ and so $T^{-1}(n-m) \circ \vartheta^n = \widetilde{T}^{-1}(n-m) \circ \vartheta^n + 1$. We then obtain by Lemma 2.10 the relation $\mathbb{S}_{m\wedge n}^n = [\mathcal{Q}(T^{-1}(n-m)) \circ \vartheta^n, \mathbb{S}_m^n]$ and since $\mathcal{Q}(T^{-1}(n-m)) \circ \vartheta^n = \mu_{L(n-m)} \circ \vartheta^m$ in this case by (2.7), we obtain the result. □

2.7. **Probabilistic description of the spine.** Let $G = \inf\{k \geq 0 : T(k) = \infty\}$: Proposition 2.4 shows that the spine process at time n is a measurable function of the random variables

$$\left(\left(T(k) - T(k-1), \mathcal{Q}(k)\right), k = 1, \ldots, G-1\right) \circ \vartheta^n.$$

Therefore, the next lemma implicitly characterizes the law of $\mathbb{S}_0^n$. Recall that $\tau_\ell^- = \inf\{k \geq 0 : S(k) = -\ell\}$ for $\ell \geq 0$.

Lemma 2.12. *Let $\mathbb{P}$ be a probability distribution on Ω such that ω under $\mathbb{P}$ is i.i.d. with common distribution $(V^*, \mathscr{P}^*)$ where $\mathbb{E}(|\mathscr{P}^*|) \leq 1$. Then under $\mathbb{P}$, the sequence*

$$\left(\left(T(k) - T(k-1), \mathcal{Q}(k)\right), k = 1, \ldots, G-1\right)$$

is equal in distribution to $((T^*(k),\mathscr{Q}^*(k)),k=1,\dots,G^*-1)$*, where the random variables* $((T^*(k),\mathscr{Q}^*(k)),k\geq 1)$ *are i.i.d. with common distribution* $(T^*,\mathscr{Q}^*)$ *satisfying*

$$\mathbb{E}\left[f(\mathscr{Q}^*)g(T^*)\right]=\frac{1}{\mathbb{E}(|\mathscr{P}^*|)}\sum_{t\geq 1}\sum_{x\geq 0}\mathbb{E}\left[f\circ\Upsilon_x(\mathscr{P}^*);|\mathscr{P}^*|\geq x+1\right]g(t)\,\mathbb{P}\left(\tau_x^-=t-1\right) \tag{2.16}$$

for every bounded and measurable functions $f:\mathscr{M}\to\mathbb{R}_+$ *and* $g:\mathbb{Z}_+\to\mathbb{R}_+$*, and* G^* *is an independent geometric random variable with parameter* $1-\mathbb{E}(|\mathscr{P}^*|)$.

We note that the random variable $\mathbb{Y}^*=\pi(\mathscr{Q}^*(1))$ admits a natural interpretation. Indeed, the previous result implies that

$$\mathbb{E}\left[f(\mathbb{Y}^*)\right]=\sum_{k=1}^{\infty}\sum_{r=0}^{k-1}\frac{1}{k}\mathbb{E}\left[f\circ\pi\circ\Upsilon_r(\mathscr{P}^*)\,|\,|\mathscr{P}^*|=k\right]\times\frac{k\mathbb{P}(|\mathscr{P}^*|=k)}{\mathbb{E}(|\mathscr{P}^*|)}. \tag{2.17}$$

Identifying $(k\mathbb{P}(|\mathscr{P}^*|=k)/\mathbb{E}(|\mathscr{P}^*|),k\geq 0)$ as the size-biased distribution of $|\mathscr{P}^*|$, we see that if we bias the life descriptor $\mathscr{P}^*$ by its number of children, then $\mathbb{Y}^*$ is the age of the individual when its begets a randomly chosen child. As mentioned in the introduction, in the critical case $\mathbb{E}(\mathbb{Y}^*)=1$, the random variable $\mathbb{Y}^*$ and its genealogical interpretation can already be found in Nerman and Jagers [21].

Proof of Lemma 2.12. The strong Markov property implies that G is a geometric random variable with parameter $\mathbb{P}(\tau_0=T(1)=\infty)$ and that conditionally on G, the random variables $((T(k)-T(k-1),\mathscr{Q}(k)),k=1,\dots,G-1)$ are i.i.d. with common distribution $(\tau_0,\mathscr{Q}(1))$ conditioned on $\{\tau_0<\infty\}$. Thus in order to prove Lemma 2.12, we only have to show that $(\tau_0,\mathscr{Q}(1))$ under $\mathbb{P}(\cdot\,|\,\tau_0<\infty)$ is equal in distribution to $(T^*,\mathscr{Q}^*)$. Recalling that $\mathscr{Q}(1)=\Upsilon_{\zeta_0}(\mathscr{P}_{\tau_0-1})$, we will actually show a more complete result and characterize the joint distribution of $(\mathscr{P}_{\tau_0-1},\tau_0,\zeta_0)$ under $\mathbb{P}(\cdot\,|\,\tau_0<\infty)$.

Fix in the rest of the proof $x,t\in\mathbb{N}$ with $t\geq 1$ and $h:\mathscr{M}\to[0,\infty)$ measurable: we will prove that

$$\mathbb{E}\left[h\left(\mathscr{P}_{\tau_0-1}\right)\mathbb{1}_{\{\zeta_0=x\}}\mathbb{1}_{\{\tau_0=t\}}\right]=\mathbb{E}\left[h(\mathscr{P}^*);|\mathscr{P}^*|\geq x+1\right]\mathbb{P}\left(\tau_x^-=t-1\right). \tag{2.18}$$

By standard arguments, this characterizes the law of $(\mathscr{P}_{\tau_0-1},\tau_0,\zeta_0)$ and implies for instance that for any bounded measurable function $F:\mathscr{M}\times\mathbb{N}\times\mathbb{N}\to[0,\infty)$, we have

$$\begin{aligned}\mathbb{E}\left[F\left(\mathscr{P}_{\tau_0-1},\zeta_0,\,\tau_0\right)\,|\,\tau_0<\infty\right]&\\ =\frac{1}{\mathbb{P}(\tau_0<\infty)}\sum_{t\geq 1}\sum_{x\geq 0}\mathbb{E}\left[F(\mathscr{P}^*,x,t);|\mathscr{P}^*|\geq x+1\right]\mathbb{P}\left(\tau_x^-=t-1\right).&\end{aligned}$$

Since τ_x^- is $\mathbb{P}$-almost surely finite, the above relation for $F(v,x,t)=1$ entails the relation $\mathbb{P}(\tau_0<\infty)=\mathbb{E}(|\mathscr{P}^*|)$ which implies in turn the desired result by taking $F(v,x,t)=f(\Upsilon_x(v))g(t)$. Thus we only have to prove (2.18), which we do now. First of all, note that if

$$B=\left\{S(t-1)=-x\ \text{ and }\ S(k)<0\ \text{ for }\ k=1,\dots,t-1\right\},$$

then the two events $\{\zeta_0=x,\tau_0=t\}$ and $B\cap\{|\mathscr{P}_{t-1}|\geq x+1\}$ are equal. It follows from this observation that

$$\mathbb{E}\left[h(\mathscr{P}_{\tau_0-1})\mathbb{1}_{\{\zeta_0=x\}}\mathbb{1}_{\{\tau_0=t\}}\right]=\mathbb{E}\left[h(\mathscr{P}_{t-1})\mathbb{1}_{\{|\mathscr{P}_{t-1}|\geq x+1\}};B\right]$$

and since $\mathscr{P}_{t-1}$ and the indicator function of the event B are independent and $\mathscr{P}_{t-1}$ under $\mathbb{P}$ is equal in distribution to $\mathscr{P}^*$, we obtain

$$\mathbb{E}\left[h(\mathscr{P}_{\tau_0-1})\mathbb{1}_{\{\zeta_0=x\}}\mathbb{1}_{\{\tau_0=t\}}\right]=\mathbb{E}\left[h(\mathscr{P}^*);|\mathscr{P}^*|\geq x+1\right]\mathbb{P}(B).$$

Since $\mathbb{P}(B)=\mathbb{P}(\tau_x^-=t-1)$ by duality, this proves Lemma 2.12. □

3. Convergence of the height process and proof of Theorem 1.3

In this section we prove the following result from which Theorem 1.3 is immediately derived.

Theorem 3.1. *Fix some $t > 0$. If Condition* Y *holds and the sequence $(\mathscr{H}_p(t), p \geq 1)$ is tight, then $\mathbb{H}_p(t) - \mathbb{E}(\mathbb{Y}_p^*)\mathscr{H}_p(t) \Rightarrow 0$.*

As will be clear from the proof, condition Y is here to ensure that the overall supremum of the random walk with step distribution $\mathbb{Y}_p^* - \mathbb{E}(\mathbb{Y}_p^*) - \eta$ (for some $\eta > 0$) converges in distribution to the overall supremum of the random walk with step distribution $\mathbb{Y}_\infty^* - \mathbb{E}(\mathbb{Y}_\infty^*) - \eta$. For this, we invoke Theorem 22 in Borovkov [7], which actually holds under the following weaker condition (i.e., condition Y implies condition Y').

Condition Y'. *For every $p \geq 1$, $\mathbb{E}(\mathbb{Y}_p^*) < \infty$. Moreover, there exists an integrable random variable $\bar{\mathbb{Y}}$ with $\mathbb{E}\bar{\mathbb{Y}} = 0$ such that $\mathbb{Y}_p^* - \mathbb{E}(\mathbb{Y}_p^*) \Rightarrow \bar{\mathbb{Y}}$ and $\mathbb{E}[(\mathbb{Y}_p^* - \mathbb{E}(\mathbb{Y}_p^*))^+] \to \mathbb{E}(\bar{\mathbb{Y}}^+)$.*

Thus, Theorem 3.1, and therefore Theorem 1.3, hold if we assume condition Y' instead of condition Y.

Proof of Theorem 3.1. First of all, note that $\mathscr{H}([pt]) \Rightarrow \infty$ since $\mathscr{H}(n)$ and $\widetilde{T}^{-1}(n)$ are equal in distribution by duality (we are working under $\mathbb{P}_p$). Further, the fundamental formula (1.1) gives

$$\mathbb{H}_p(t) - \mathbb{E}(\mathbb{Y}_p^*)\mathscr{H}_p(t) = \mathscr{H}_p(t) \times \left(\frac{1}{\widetilde{T}^{-1}([pt])} \sum_{k=1}^{\widetilde{T}^{-1}([pt])} \left(\mathbb{Y}(k) - \mathbb{E}(\mathbb{Y}_p^*) \right) \right) \circ \vartheta^{[pt]}.$$

Let in the sequel $W_p(n) = \bar{\mathbb{Y}}_p(1) + \cdots + \bar{\mathbb{Y}}_p(n)$ and $W(n) = \bar{\mathbb{Y}}(1) + \cdots + \bar{\mathbb{Y}}(n)$, where the two sequences $(\bar{\mathbb{Y}}_p(k), k \geq 1)$ and $(\bar{\mathbb{Y}}(k), k \geq 1)$ are i.i.d. with common distribution $\mathbb{Y}_p^* - \mathbb{E}(\mathbb{Y}_p^*)$ and $\bar{\mathbb{Y}}$ introduced in Condition G, respectively. Fix $\eta > 0$ and $M, N \geq 1$: by duality, it follows from Lemma 2.12 and standard manipulations that

$$\mathbb{P}_p\left(\left| \mathbb{H}_p(t) - \mathbb{E}(\mathbb{Y}_p^*)\mathscr{H}_p(t) \right| \geq \eta \right) \leq \mathbb{P}_p\left(\mathscr{H}_p(t) \geq M \right) + \mathbb{P}_p\left(\mathscr{H}([pt]) \leq N \right) + \mathbb{P}\left(\sup_{n \geq N} \frac{1}{n} \left| W_p(n) \right| \geq \eta / M \right).$$

Letting first $p \to \infty$, then $N \to \infty$ and finally $M \to \infty$ makes the two first terms of the above upper bound vanish: the first one because the sequence $(\mathscr{H}_p(t), n \geq 1)$ is tight and the second one because $\mathscr{H}([pt]) \Rightarrow \infty$, and so we end up with

$$\limsup_{p \to \infty} \mathbb{P}_p\left(\left| \mathbb{H}_p(t) - \mathbb{E}(\mathbb{Y}_p^*)\mathscr{H}_p(t) \right| \geq \eta \right) \leq \limsup_{N \to \infty} \limsup_{p \to \infty} \mathbb{P}\left(\sup_{n \geq N} \frac{1}{n} \left| W_p(n) \right| \geq 2\eta' \right) \tag{3.1}$$

with $\eta' = \eta/(2M)$. We omit the $\limsup_{M \to \infty}$ because, as we now show, the previous limit is equal to 0 for each fixed $M > 0$. In the non-triangular case where the law of $\mathbb{Y}_p^*$ (and thus W_p) does not depend on p, this follows from the strong law of large numbers, and we now extend this to the triangular setting under Condition Y. Writing

$$\sup_{n \geq N} \frac{1}{n} \left| W_p(n) \right| \leq \frac{1}{N} \left| W_p(N) \right| + \sup_{n \geq N} \frac{1}{n} \left| W_p(n) - W_p(N) \right|$$

and using that $(W_p(n) - W_p(N), n \geq N)$ is equal in distribution to W_p, we get

$$\mathbb{P}\left(\sup_{n \geq N} \frac{1}{n} \left| W_p(n) \right| \geq 2\eta' \right) \leq \mathbb{P}\left(\frac{1}{N} |W_p(N)| \geq \eta' \right) + \mathbb{P}\left(\sup_{n \geq 0} \frac{1}{n+N} \left| W_p(n) \right| \geq \eta' \right).$$

By the Portmanteau Theorem, we have

$$\limsup_{p \to \infty} \mathbb{P}\left(\frac{1}{N} |W_p(N)| \geq \eta' \right) \leq \mathbb{P}\left(\frac{1}{N} |W(N)| \geq \eta' \right),$$

which entails

$$\limsup_{p\to\infty} \mathbb{P}\left(\frac{1}{N}|W_p(N)| \geq \eta'\right) \underset{N\to\infty}{\longrightarrow} 0.$$

As for the second term, if we define $W_p^{\pm}(n) = W_p(n) \pm \eta' n$ and $W^{\pm}(n) = W(n) \pm \eta' n$, then simple manipulations lead to

$$\mathbb{P}\left(\sup_{n\geq 0} \frac{1}{n+N}\left|W_p(n)\right| \geq \eta'\right) \leq \mathbb{P}\left(\sup_{n\geq 0} W_p^- \geq \eta' N\right) + \mathbb{P}\left(\inf_{n\geq 0} W_p^+ \leq -\eta N\right).$$

Under Condition Y, we have $\sup W_p^- \Rightarrow \sup W^-$ and $\inf W_p^+ \Rightarrow \inf W^+$, see for instance Theorem 22 in Borovkov [7]. The result thus follows from the fact that, since W^+ (resp. W^-) is a random walk drifting to $+\infty$ (resp. $-\infty$), its infimum (resp. supremum) is finite. □

Remark 3.2. By the exact same argument, we leave the reader convince himself that if t_p is a deterministic sequence such that $t_p/p \to 0$, then $\varepsilon_p \mathbb{H}(t_p) \Rightarrow 0$. This fact will be used later in proving the convergence of the contour process.

4. Convergence of the contour process and proof of Theorem 1.6

In this section we state a stronger result than Theorem 1.6. Indeed, Theorem 4.1 below proves that $\mathbb{H}_p$ and $\mathbb{C}_p$ may converge jointly even if condition Y does not hold, i.e., $\mathbb{E}(\mathbb{Y}_p^*) = \infty$. In this case, the law of large numbers suggests, in view of Theorem 1.1, that the chronological and genealogical processes obey to different scalings, i.e., $\mathbb{H}(n) \gg \mathscr{H}(n)$ for large n. Considering different scalings may seem peculiar at first sight and we begin by justifying this. We then state Theorem 4.1 which is proved in Section 6 after preliminary results have been established in Section 5.

4.1. **Generalization of the scalings.** Consider the classical binary-homogeneous case (considered for instance in [14]) in a non-triangular setting: $(V_p^*, \mathscr{P}_p^*) = (V^*, \mathscr{P}^*)$ where $\mathscr{P}^*$ is a Poisson process stopped at V^*. Assume moreover that $|\mathscr{P}^*|$ has finite variance and that V^* satisfies $\mathbb{P}(V^* \geq x) \propto x^{-\gamma}$ for some $\gamma \in (1,2)$: in particular, V^* has finite first moment but infinite variance.

The law of large numbers suggests that $|\mathscr{P}^*|$ and V^* are proportional to one another when they get large and it can be shown that they indeed have the same tail behavior. In particular, it is well known that condition G is satisfied with $\varepsilon_p = 1/p^{1-1/\gamma}$, which implies in particular that macroscopic jumps of S are of the order of $p\varepsilon_p = p^{1/\gamma}$.

On the other hand, at the chronological level, macroscopic jumps of S correspond to macroscopic edges: in particular, $\mathbb{H}(p)$ and $\mathbb{C}(p)$ are typically of the order $p^{1/\gamma}$, whereas as argued above $\mathscr{H}(p)$ and $\mathscr{C}(p)$ are of the order of $1/\varepsilon_p = p^{1-1/\gamma}$. Thus in this simple case, we see that the chronological and genealogical processes scale in different ways.

To allow for this, we consider in this section another sequence $\bar{\varepsilon}_p$ and, instead of scaling $\mathbb{H}$ and $\mathbb{C}$ as in (1.4) and (1.5), we consider the following scalings: the scaling at the chronological level remains unchanged, i.e., $\mathbb{H}_p$ and $\mathbb{C}_p$ are still given by

$$\mathbb{H}_p(t) = \varepsilon_p \mathbb{H}([pt]) \text{ and } \mathbb{C}_p(t) = \varepsilon_p \mathbb{C}(pt).$$

whereas for the genealogical processes $\mathscr{H}_p$, $\mathscr{C}_p$ and S_p we use $\bar{\varepsilon}_p$ instead of ε_p and consider

$$\mathscr{H}_p(t) = \bar{\varepsilon}_p \mathscr{H}([pt]),\ \mathscr{C}_p(t) = \bar{\varepsilon}_p \mathscr{C}(pt) \text{ and } S_p(t) = \frac{1}{p\bar{\varepsilon}_p} S([pt]).$$

Thus as mentioned earlier, when condition G holds we can and will assume without loss of generality that $\bar{\varepsilon}_p \to 0$ and $p\bar{\varepsilon}_p \to \infty$. Furthermore, we assume in the sequel that (ε_p) obeys a similar behavior, i.e., $\varepsilon_p \to 0$ and $p\varepsilon_p \to \infty$.

4.2. **Main result concerning convergence of the chronological contour process.** In the triangular setting, condition V below is needed to ensure the convergence of the contour process. This condition entails a generalized version of the renewal theorem in a triangular setting.

Let $V > 0$ be some random variable and G be the additive subgroup generated by the support of its distribution. In the sequel we say that V is *non-arithmetic* if G is dense in $\mathbb{R}$; otherwise, we say that V is *arithmetic* and in this case, there exists a unique $h > 0$, called the *span* of V, such that $G = h\mathbb{Z}$.

Condition V. *If $V_p^* \Rightarrow V_\infty$ for some random variable V_∞^*, then either all the random variables $(V_p^*, p \in \mathbb{N} \cup \{\infty\})$ are arithmetic, or all the random variables $(V_p^*, p \in \mathbb{N} \cup \{\infty\})$ are non-arithmetic.*

Note that this condition is clearly satisfied in the non-triangular setting.

Theorem 4.1. *Assume that conditions* G, VP *and* V *hold and that moreover $\mathbb{H}_p \overset{\text{fdd}}{\Longrightarrow} \mathbb{H}_\infty$ for some process $\mathbb{H}_\infty$ which is (almost surely) continuous at* 0 *and satisfies the condition $\mathbb{P}(\mathbb{H}_\infty(t) > 0) = 1$ for every $t > 0$. Then*

$$\big(\mathbb{H}_p, \mathbb{C}_p\big) \overset{\text{fdd}}{\Longrightarrow} \big(\mathbb{H}_\infty, \mathbb{H}_\infty \circ \varphi_\infty\big). \tag{4.1}$$

4.3. **Proof of Theorem 1.6 based on Theorem 4.1.** As mentioned earlier, Theorem 1.6 directly follows from Theorem 4.1: the idea is to make repeated use of the following simple lemma.

Lemma 4.2. *If X_p, Y_p, Z_p are random variables such that $(X_p, Y_p) \Rightarrow (X_\infty, f(X_\infty))$ and $(X_p, Z_p) \Rightarrow (X_\infty, g(X_\infty))$ for some measurable functions f and g, then $(X_p, Y_p, Z_p) \Rightarrow (X_\infty, f(X_\infty), g(X_\infty))$.*

Proof. As (X_p, Z_p) and (X_p, Z_p) converge weakly, the sequence (X_p, Y_p, Z_p) is tight. Let (X, Y, Z) be any accumulation point and assume without loss of generality (working along subsequences) that $(X_p, Y_p, Z_p) \Rightarrow (X, Y, Z)$. Then the continuous mapping implies $(X_p, Y_p) \Rightarrow (X, Y)$ and so $Y = f(X)$, and similarly $Z = g(X)$. Since X is necessarily equal in distribution to X_∞ this uniquely characterizes the law of (X, Y, Z) and thus proves the result. □

We now prove Theorem 1.6 assuming that Theorem 4.1 holds. Assume therefore that conditions G, Y, VP and V hold. Then $(\mathscr{H}_p, \mathscr{C}_p) \overset{\text{fdd}}{\Rightarrow} (\mathscr{H}_\infty, \mathscr{C}_\infty)$ by Theorem 1.2, $(\mathscr{H}_p, \mathbb{H}_p) \overset{\text{fdd}}{\Rightarrow} (\mathscr{H}_\infty, \alpha^* \mathscr{H}_\infty)$ by Theorem 1.3 and thus $(\mathbb{H}_p, \mathbb{C}_p) \overset{\text{fdd}}{\Rightarrow} (\alpha^* \mathscr{H}_\infty, \alpha^* \mathscr{H}_\infty \circ \varphi_\infty)$ by Theorem 4.1 (with $\mathbb{H}_\infty = \alpha^* \mathscr{H}_\infty$ since $\mathbb{H}_p \overset{\text{fdd}}{\Rightarrow} \alpha^* \mathscr{H}_\infty$). Repeated use of the previous lemma then entails the joint convergence

$$\big(\mathscr{H}_p, \mathscr{C}_p, \mathbb{H}_p, \mathbb{C}_p\big) \overset{\text{fdd}}{\Longrightarrow} \big(\mathscr{H}_\infty, \mathscr{H}_\infty(\cdot/2), \alpha^* \mathscr{H}_\infty, \alpha^* \mathscr{H}_\infty \circ \varphi_\infty\big)$$

which is the content of Theorem 1.6.

The next three sections are devoted to proving Theorems 4.1 and 1.9. Some preliminary results are established in the next section, Theorem 4.1 is proved in Section 6 and Theorem 1.9 in Section 7.

5. Preliminary results

5.1. **Right decomposition of the spine continued.** In this section we continue the study of the spine process initiated in Section 2.6 and prove some further useful identities.

Lemma 5.1. *For any $n \geq m \geq 0$ with $0 \leq m \wedge n < m$, we have*

$$\mathbb{S}_0^m = \big[\mathbb{S}_0^{m \wedge n}, \mathscr{Q} \circ \widetilde{T}^{-1}(\tau_{L(n-m)}) \circ \vartheta^m, \ldots, \mathscr{Q}(1) \circ \vartheta^m\big].$$

Proof. By Lemma 2.8, for every k such that $T(k)\circ\vartheta^m \le m$ we have

$$\mathbb{S}_0^m = \left[\mathbb{S}_0^{m-T(k)\circ\vartheta^m}, \mathcal{Q}(k)\circ\vartheta^m, \ldots, \mathcal{Q}(1)\circ\vartheta^m\right]. \tag{5.1}$$

Let $k = \widetilde{T}^{-1}(\tau_{L(n-m)})$: then $T(k) = \tau_{L(n-m)}$ (as $T(\widetilde{T}^{-1}(i)) = i$ for every $i \ge 0$) and so $T(k)\circ\vartheta^m = m - m\wedge n$ by (2.6). Since by assumption $m\wedge n \ge 0$, we have $T(k)\circ\vartheta^m \le m$ and (5.1) gives the result as $m - T(k)\circ\vartheta^m = m\wedge n$. □

In the sequel, we consider the measurable function $D_\ell : \mathscr{M}^* \to \mathbb{R}_+$ that satisfies $D_0 \equiv 0$ and for $\ell \in \mathbb{N}\setminus\{0\}$:

$$D_\ell(\mathbb{S}_0^n) = \left(\sum_{i:0<T(i)\le \min(\tau_\ell, n)} \mathbb{Y}(i) - \mathbb{1}(\tau_\ell \le n)\pi(\mu_\ell)\right)\circ\vartheta^n, n \in \mathbb{N}. \tag{5.2}$$

The fact that the right hand side is measurable with respect to $\mathbb{S}_0^n$ (and thus can be written as a function of $\mathbb{S}_0^n$) is a consequence of Proposition 2.4 and the fact that the random variables appearing in the formula are related to the dual Lukasiewicz path $S\circ\vartheta^n$.

Moreover, we leave the reader check that for any $Y \in \mathscr{M}^*$ the sequence $(D_\ell(Y), \ell \in \mathbb{N})$ is increasing. Actually, this comes from a more general fact, namely that $D_\ell(Y)$ for $Y \in \mathscr{M}^*$ gives the distance between $\pi(Y)$ and the ℓ-th stub of Y as is illustrated on Figure 9.

The following result relates the two shifts which play a key role in this paper : on the one hand, the canonical shift θ which acts on the initial sequence of sticks $((V_n, \mathscr{P}_n), n \in \mathbb{Z})$ through the term $\pi(\mathbb{S}_m^n) = \pi(\mathbb{S}_0^{n-m})\circ\theta_m$, and on the other hand, the shift in time through the term $\mathbb{H}(n) - \mathbb{H}(m)$.

Proposition 5.2. *For every $0 \le m \le n$ we have*

$$\mathbb{H}(n) - \mathbb{H}(m) = \pi(\mathbb{S}_m^n) - D_{L(n-m)\circ\vartheta^m}(\mathbb{S}_0^m). \tag{5.3}$$

Proof. Applying 5.2 to the random $\ell = L(n-m)\circ\vartheta^m$, we obtain (see Remark 2.11)

$$D_{L(n-m)\circ\vartheta^m}(\mathbb{S}_0^m) = \left(\sum_{i=1}^{\widetilde{T}^{-1}(\min(\tau_{L(n-m)}, m))} \mathbb{Y}(i) - \mathbb{1}(\tau_{L(n-m)} \le m)\pi(\mu_{L(n-m)})\right)\circ\vartheta^m. \tag{5.4}$$

To prove (5.3) we distinguish the two cases $m\wedge n < 0$ and $m\wedge n \ge 0$.
Case 1: $m\wedge n < 0$. By (2.6) this condition is equivalent to $\tau_{L(n-m)}\circ\vartheta^m > m$: in view of (5.4), we thus need to show that

$$\mathbb{H}(n) - \mathbb{H}(m) = \pi(\mathbb{S}_m^n) - \left(\sum_{i=1}^{\widetilde{T}^{-1}(m)} \mathbb{Y}(i)\right)\circ\vartheta^m.$$

Using the expression for $\mathbb{H}(n)$, $\mathbb{H}(m)$ and $\pi(\mathbb{S}_m^n)$ provided by Proposition 1.1 and (2.10), we see that in order to show the above relation we only have to show that $\widetilde{T}^{-1}(n-m)\circ\vartheta^n = \widetilde{T}^{-1}(n)\circ\vartheta^n$. This in turn follows from the fact that the condition $m\wedge n < 0$ implies that $T(T^{-1}(n-m))\circ\vartheta^n > n$ (again by (2.6)), which is equivalent to saying that the sets $\{T(i) : i \in \mathbb{N}\}\circ\vartheta^n$ and $\{n-m, \ldots, n\}$ do not intersect and gives $\widetilde{T}^{-1}(n-m)\circ\vartheta^n = \widetilde{T}^{-1}(n)\circ\vartheta^n$. The proof in this case is thus complete.

Case 2: $m\wedge n \ge 0$. The result is obvious in the case $m\wedge n = m$, while in the other case we can invoke Proposition 2.9 and Lemma 5.1 that give

$$\mathbb{H}(n) = \pi(\mathbb{S}_0^{m\wedge n}) + \pi(\mu_{L(n-m)})\circ\vartheta^m + \pi(\mathbb{S}_m^n) \text{ and } \mathbb{H}(m) = \pi(\mathbb{S}_0^{m\wedge n}) + \left(\sum_{i=1}^{\widetilde{T}^{-1}(\tau_{L(n-m)})} \mathbb{Y}(i)\right)\circ\vartheta^m.$$

Taking the difference between these two expressions yield the result in view of (5.4) (recall that $m\wedge n \ge 0$ is equivalent to $\tau_{L(n-m)}\circ\vartheta^m \le m$). □

The following lemma relates the shifted spine to the Skorohod reflection.

Lemma 5.3. *For any $0 \le m \le n$, we have $\pi(\mathbb{S}_m^n) = \mathbb{H}(n) - \min_{k=m,\dots,n} \mathbb{H}(k)$.*

Proof. It follows from (2.10) that

$$\pi(\mathbb{S}_m^n) = \left(\sum_{i=1}^{\widetilde{T}^{-1}(n)} \mathbb{Y}(i) \right) \circ \vartheta^n - \left(\sum_{i=\widetilde{T}^{-1}(n-m)+1}^{\widetilde{T}^{-1}(n)} \mathbb{Y}(i) \right) \circ \vartheta^n = \mathbb{H}(n) - \left(\sum_{i=\widetilde{T}^{-1}(n-m)+1}^{\widetilde{T}^{-1}(n)} \mathbb{Y}(i) \right) \circ \vartheta^n.$$

Next, we have from Proposition 2.4 that

$$\mathbb{S}_0^n = \big(\mathcal{Q}(\widetilde{T}^{-1}(n)), \mathcal{Q}(\widetilde{T}^{-1}(n)-1), \dots, \mathcal{Q}(1) \big) \circ \vartheta^n$$

while Lemma 2.8 with $k = \widetilde{T}^{-1}(n-m)$ gives

$$\mathbb{S}_0^n = \left[\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n-m))\circ\vartheta^n}, \mathcal{Q}(\widetilde{T}^{-1}(n-m)) \circ \vartheta^n, \dots, \mathcal{Q}(1) \circ \vartheta^n \right].$$

Comparing the two expressions for $\mathbb{S}_0^n$,we see that

$$\mathbb{S}_0^{n-T(\widetilde{T}^{-1}(n-m))\circ\vartheta^n} = \big(\mathcal{Q}(\widetilde{T}^{-1}(n)), \dots, \mathcal{Q}(\widetilde{T}^{-1}(n-m)+1) \big) \circ \vartheta^n$$

and in particular,

$$\left(\sum_{i=\widetilde{T}^{-1}(n-m)+1}^{\widetilde{T}^{-1}(n)} \mathbb{Y}(i) \right) \circ \vartheta^n = \mathbb{H}\big(n - T(\widetilde{T}^{-1}(n-m)) \circ \vartheta^n \big).$$

We let the reader convince himself that $\mathbb{H}\big(n - T(\widetilde{T}^{-1}(n-m)) \circ \vartheta^n\big) = \min_{\{m,\dots,n\}} \mathbb{H}$ (again by comparing the number of ladder height times at $n - T(\widetilde{T}^{-1}(n-m)) \circ \vartheta^n$ and $k \in \{m,\dots,n\}$), so that gathering the previous relations we finally obtain the desired result. □

Corollary 5.4. *For any $0 \le m \le n$,*

$$\min_{K_m \le t \le K_n} \mathbb{C}(t) = \mathbb{H}(m) - D_{L(n-m)\circ\vartheta^m}(\mathbb{S}_0^m). \tag{5.5}$$

Proof. Let $I_m^n = \min_{[K_m,K_n]} \mathbb{C}$. Since $\mathbb{H}(n) - \mathbb{H}(m) = \pi(\mathbb{S}_m^n) - D_{L(n-m)\circ\vartheta^m}(\mathbb{S}_0^m)$ by Proposition 5.2, in order to prove (5.5) it is enough to prove that

$$\pi\left(\mathbb{S}_m^n\right) = \mathbb{H}(n) - I_m^n.$$

Local minima of $\mathbb{C}$ are by construction attained on the set $\{K_n : n \in \mathbb{N}\}$ and since $\mathbb{H}(k) = \mathbb{C}(K_k)$ for any $k \in \mathbb{N}$, this implies $I_m^n = \min_{k=m,\dots,n} \mathbb{H}(k)$. The result then follows from Lemma 5.3. □

5.2. **Triangular renewal theorem on a macroscopic horizon.** Let us now introduce

$$\bar{\varphi}_p(t) = \frac{1}{p} \bar{\varphi}(pt) \text{ with } \bar{\varphi}(t) = \inf\big\{ j \ge 0 : 2\mathcal{V}(j) \ge t \big\},$$

the first passage time of the renewal process $2\mathcal{V}$ above level t. In Appendix C, we shall prove the following results on renewal processes that can be seen as an extension of results of Miller [20]. Proposition 5.5 is an extension of the renewal theorem, where steps of the renewal process are marked and which holds in a triangular setting. And Proposition 5.6 is also an extension of the renewal theorem, allowing to consider a growing number of terms before the jump straddling t and also in a triangular setting. The only purpose of condition VP is for these two results to hold.

Proposition 5.5. *Assume that conditions Y, VP and V hold. Then for every $t > 0$ we have $\big(V_{\bar{\varphi}(pt)}, \mathscr{P}_{\bar{\varphi}(pt)}\big) \Rightarrow \big(\hat{V}_\infty^*, \hat{\mathscr{P}}_\infty^*\big)$ where $(\hat{V}_\infty^*, \hat{\mathscr{P}}_\infty^*)$ has the following distribution: for every measurable function $f : \mathbb{R}_+ \times \mathscr{M} \to \mathbb{R}_+$,*

- *if V_∞^* is non-arithmetic,*

$$\mathbb{E}\big[f(\hat{V}_\infty^*, \hat{\mathscr{P}}_\infty^*) \big] = \frac{1}{\mathbb{E}(V_\infty^*)} \int_0^\infty \mathbb{E}\big[f(v, \mathscr{P}_\infty^*) \mid V_\infty^* = v \big] \mathbb{P}(V_\infty^* > v) \mathrm{d}v;$$

- *if V^*_∞ is arithmetic with span h,*

$$\mathbb{E}\left[f(\hat{V}^*_\infty, \hat{\mathscr{P}}^*_\infty)\right] = \frac{1}{\mathbb{E}(V^*_\infty)} \sum_{i\geq 1} \mathbb{E}\left[f(ih, \mathscr{P}^*_\infty) \mid V^*_\infty = ih\right] \mathbb{P}(V^*_\infty > ih).$$

In the following statement we use the notation $\mathscr{P}^\ell_k = (\mathscr{P}_k, \dots, \mathscr{P}_{\ell-1})$.

Proposition 5.6. *Assume that conditions Y, VP and V hold and for each $p \geq 1$ let $\Xi_p : \mathscr{M}^p \to \mathbb{R}$ be a measurable mapping such that $\Xi_p(\mathscr{P}^p_0) \Rightarrow \Xi_\infty$ for some random variable Ξ_∞. Then $\Xi_{\lfloor \delta p \rfloor}(\mathscr{P}^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}) \Rightarrow \Xi_\infty$ for any $0 < \delta < t/(2\beta^*)$.*

Recall the exploration process $\rho^n_0 = \mathbb{S}^n_0 \circ \mathscr{G}$, which similarly as (2.9) is extended by setting $\rho^n_m = \mathbb{S}^n_m \circ \mathscr{G} = \rho^{n-m}_0 \circ \theta_m$. The following corollary to Propositions 5.5 and 5.6 gathers the results needed in the sequel.

Corollary 5.7. *For $t \geq 0$, the three sequences $\varepsilon_p V_{\bar\varphi(pt)}$, $\varepsilon_p \pi(\mathscr{P}_{\bar\varphi(pt)})$ and $\varepsilon_p |\mathscr{P}_{\bar\varphi(pt)}|$ converge weakly to 0 as $p \to \infty$. If in addition $0 < \delta < t/(2\beta^*)$, then*

$$\varepsilon_p \pi\left(\mathbb{S}^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right) \Rightarrow \mathbb{H}_\infty(\delta),\ \bar\varepsilon_p \pi\left(\rho^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right) \Rightarrow \mathscr{H}_\infty(\delta)$$

and

$$\sup_{0\leq u\leq \delta} S_p(u) \circ \vartheta^{\bar\varphi(pt)} \Rightarrow \sup_{0\leq u\leq \delta} S_\infty(u).$$

Proof. The convergence of the three sequences $\varepsilon_p V_{\bar\varphi(pt)}$, $\varepsilon_p \pi(\mathscr{P}_{\bar\varphi(pt)})$ and $\varepsilon_p |\mathscr{P}_{\bar\varphi(pt)}|$ is a direct consequence of Proposition 5.5 (note that, for point processes, the functionals π and $|\cdot|$ are continuous for the weak topology).

Let us now discuss the remaining convergence of $\varepsilon_p \pi\left(\mathbb{S}^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right)$, $\bar\varepsilon_p \pi\left(\rho^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right)$ and $\sup_{[0,\delta]} S_p \circ \vartheta^{\bar\varphi(pt)}$. From their definition, each of these random variables can be expressed in the form $\Xi_{\lfloor \delta p\rfloor}\left(\mathscr{P}^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right)$ for some measurable mappings $\Xi_p : \mathscr{M}^p \to [0,\infty)$. Proposition 5.6 implies that $\Xi_{\lfloor \delta p\rfloor}\left(\mathscr{P}^{\bar\varphi(pt)}_{\bar\varphi(pt)-\lfloor p\delta\rfloor}\right)$ converges if $\Xi\left(\mathscr{P}^{\lfloor p\delta\rfloor}_0\right)$ does, in which case they have the same limit. This means that we are brought back to the convergence of $\mathbb{H}_p(\delta)$, $\mathscr{H}_p(\delta)$ and $\sup_{[0,\delta]} S_p$ and since each of these three terms convergences under condition G the result follows. □

6. Proof of Theorem 4.1

We assume in this section that the assumptions of Theorem 4.1 hold, i.e., conditions G, VP and V hold and $\mathbb{H}_p \overset{\text{fdd}}{\Longrightarrow} \mathbb{H}_\infty$ for some process $\mathbb{H}_\infty$ which is (almost surely) continuous at 0 and satisfies the condition $\mathbb{P}(\mathbb{H}_\infty(t) > 0) = 1$ for every $t > 0$. Let also $\bar\varphi$ and $\bar\varphi_p$ be as in Section 5.2. We first reduce the proof of Theorem 4.1 to the next three propositions, and then prove them.

Proposition 6.1. *For any $t > 0$ and any $\eta > 1/(2\beta^*)$,*

$$\lim_{p\to\infty} \mathbb{P}_p\left(\varphi(pt) - \bar\varphi(pt) > \eta \mathbb{H}(\bar\varphi(pt))\right) = 0.$$

Proposition 6.2. *For any $t > 0$ we have*

$$\mathbb{H}_p(\bar\varphi_p(t)) - \mathbb{H}_p(\varphi_\infty(t)) \Rightarrow 0. \tag{6.1}$$

Proposition 6.3. *For any $t > 0$ we have*

$$\mathbb{H}_p(\varphi_p(t) + 1/p) - \mathbb{H}_p(\bar\varphi_p(t)) \Rightarrow 0 \ \text{ and } \ \mathbb{H}_p(\varphi_p(t)) - \mathbb{H}_p(\bar\varphi_p(t)) \Rightarrow 0. \tag{6.2}$$

6.1. **Proof of Theorem 4.1 based on Propositions 6.1, 6.2 and 6.3.** Pursuing from (1.11), we obtain

$$\left|\mathbb{C}_p(t)-\mathbb{H}_p(\varphi_\infty(t))\right| \le \varepsilon_p V_{\varphi(pt)} + \left|\mathbb{H}_p(\varphi_p(t)+1/p)-\mathbb{H}_p(\varphi_\infty(t))\right| + 2\left|\mathbb{H}_p(\varphi_p(t))-\mathbb{H}_p(\varphi_\infty(t))\right|. \tag{6.3}$$

Propositions 6.2 and 6.3 imply that the two last terms in the right-hand side vanish, and so we are left with showing that $\varepsilon_p V_{\varphi(pt)} \Rightarrow 0$. Since $\bar\varphi(pt) \le \varphi(pt)$, for any $M,\eta>0$ we have

$$\mathbb{P}_p\left(\varepsilon_p V_{\varphi(pt)} \ge \eta\right) \le \mathbb{P}_p\left(\varphi(pt)-\bar\varphi(pt) > M/\varepsilon_p\right) + \mathbb{P}_p\left(\varepsilon_p \max\left\{V_k : k=\bar\varphi(pt),\dots,\bar\varphi(pt)+[M/\varepsilon_p]\right\} \ge \eta\right)$$

which gives

$$\mathbb{P}_p\left(\varepsilon_p V_{\varphi(pt)} \ge \eta\right) \le \mathbb{P}_p\left(\varphi_p(t)-\bar\varphi_p(t) > \mathbb{H}(\bar\varphi(pt))/\beta^*\right) + \mathbb{P}_p\left(\mathbb{H}_p(\bar\varphi_p(t)) > \beta^* M\right) + \mathbb{P}_p\left(\varepsilon_p V_{\bar\varphi(pt)} \ge \eta\right) + \mathbb{P}_p\left(\varepsilon_p \max\left\{V_{\bar\varphi(pt)+k} : k=1,\dots,[M/\varepsilon_p]\right\} \ge \eta\right).$$

As $p\to\infty$, the first term vanishes by Proposition 6.1 and the third one by Corollary 5.7. The second one vanishes when $p\to\infty$ and then $M\to\infty$ because $\mathbb{H}_p(t)$ is tight by assumption. As for the last term, we use the union bound (using the fact that the random variables $(V_{\bar\varphi(pt)+k}, k\ge 1)$ under $\mathbb{P}_p$ are i.i.d. with common distribution V_p^*) and then Markov inequality to get

$$\mathbb{P}_p\left(\varepsilon_p \max\left\{V_{\bar\varphi(pt)+k} : k=1,\dots,[M/\varepsilon_p]\right\} \ge \eta\right) \le \frac{M}{\varepsilon_p}\mathbb{P}\left(\varepsilon_p V_p^* \ge \eta\right) \le \frac{M}{\eta}\mathbb{E}\left(V_p^*; V_p^* \ge \frac{\eta}{\varepsilon_p}\right).$$

Since the (V_p^*) are uniformly integrable, this last bound vanishes as $p\to\infty$, which completes the proof.

6.2. **Proof of Proposition 6.2.** We start with three lemmas. The following lemma, which states a triangular weak law of large numbers, is a direct consequence of the uniform integrability of the (V_p^*), see for instance [11, §22].

Lemma 6.4. *For any sequence $u_p\to\infty$ we have $\mathcal{V}([u_p])/u_p \Rightarrow \beta^*$. In particular, $\mathcal{V}([pt])/p \Rightarrow \beta^* t$ for any $t\ge 0$.*

Do we need this result earlier in the paper?

Lemma 6.5. *For every $t\ge 0$ we have $\varphi_p(t) \Rightarrow \varphi_\infty(t)$.*

Proof. Consider any $t' < \varphi_\infty(t)$: using the definition of φ_p, the fact that $\mathbb{H}(j)\ge 0$ and that $\mathcal{V}$ is increasing, one obtains that

$$\mathbb{P}_p\left(\varphi_p(t) < t'\right) \le \mathbb{P}_p\left(2\mathcal{V}\left(pt'\right) < pt\right).$$

Since $\mathcal{V}(ps)/p \Rightarrow \beta^* s$ for any $s\ge 0$ by Lemma 6.4, we obtain $\mathbb{P}_p\left(\varphi_p(t) < t'\right)\to 0$ for $t' < \varphi_\infty(t)$. Let now $t' > \varphi_\infty(t)$, and write

$$\mathbb{P}_p\left(\varphi_p(t) > t'\right) \le \mathbb{P}_p\left(2\mathcal{V}(pt') - \mathbb{H}(pt') \le pt\right).$$

Since the sequence $(\varepsilon_p \mathbb{H}([pt']), p\ge 1)$ is tight and $p\varepsilon_p\to\infty$, we obtain $\mathbb{H}(pt')/p \Rightarrow 0$ and so $(2\mathcal{V}(pt') - \mathbb{H}(pt'))/p \Rightarrow 2\beta^* t'$. We thus obtain $\mathbb{P}_p\left(2\mathcal{V}(pt') - \mathbb{H}(pt') \le pt\right)\to 0$ which concludes the proof. □

Lemma 6.6. *For any $1\le m\le n$,*

$$0 \le \pi(\mathbb{S}^n_{m-1}) - \pi(\mathbb{S}^n_m) \le \pi(\mathscr{P}_{m-1}). \tag{6.4}$$

Proof. Relation (2.10) gives

$$\pi(\mathbb{S}^n_{m-1}) = \left(\sum_{k=1}^{\tilde T^{-1}(n-m+1)} \mathbb{Y}(k)\right)\circ\vartheta^n.$$

If $\widetilde{T}^{-1}(n-m+1)\circ\vartheta^n = \widetilde{T}^{-1}(n-m)\circ\vartheta^n$, then we obtain $\pi(\mathbb{S}^n_{m-1}) = \pi(\mathbb{S}^n_m)$ and so the result holds in this case. Otherwise, we have $\widetilde{T}^{-1}(n-m+1)\circ\vartheta^n = \widetilde{T}^{-1}(n-m)\circ\vartheta^n + 1$ and so isolating the last term, we obtain

$$\pi(\mathbb{S}^n_{m-1}) = \pi(\mathbb{S}^n_m) + \mathbb{Y}\big(\widetilde{T}^{-1}(n-m+1)\big)\circ\vartheta^n.$$

Further, for any $k\in\mathbb{N}$ we have

$$\mathbb{Y}\left(\widetilde{T}^{-1}(k)\right) = \pi\circ\mathcal{Q}\left(\widetilde{T}^{-1}(k)\right) = \pi\circ\Upsilon_{\zeta_0}(\mathscr{P}_{\tau_0-1})\circ\theta_{T(\widetilde{T}^{-1}(k))-1} \le \pi(\mathscr{P}_{\tau_0-1})\circ\theta_{T(\widetilde{T}^{-1}(k))-1}.$$

As $\tau_0\circ\theta_{T(\widetilde{T}^{-1}(k))-1} = 1$, this gives $\mathbb{Y}(\widetilde{T}^{-1}(k)) \le \pi\left(\mathscr{P}_{T(\widetilde{T}^{-1}(k))-1}\right)$ and consequently,

$$\mathbb{Y}\left(\widetilde{T}^{-1}(n-m+1)\right)\circ\vartheta^n \le \pi\left(\mathscr{P}_{T(\widetilde{T}^{-1}(n-m+1))-1}\right)\circ\vartheta^n = \pi\left(\mathscr{P}_{n-T(\widetilde{T}^{-1}(n-m+1))\circ\vartheta^n}\right).$$

The condition $\widetilde{T}^{-1}(n-m+1)\circ\vartheta^n = \widetilde{T}^{-1}(n-m)\circ\vartheta^n + 1$ means that $n-m+1$ is a weak ascending ladder height time (for the dual process $S\circ\vartheta^n$) and thus implies the relation $T(\widetilde{T}^{-1}(n-m+1))\circ\vartheta^n = n-m+1$. Plugging in this relation in the previous display achieves the proof. □

Let for simplicity $m_p = \bar{\varphi}(pt)\wedge[\varphi_\infty(pt)]$. Since we have $\mathbb{H}(m_p)\le\mathbb{H}(\bar{\varphi}(pt))$ as well as $\mathbb{H}(m_p)\le\mathbb{H}([\varphi_\infty(pt)])$, the triangular inequality reads

$$\left|\mathbb{H}_p(\bar{\varphi}_p(t)) - \mathbb{H}_p(\varphi_\infty(t))\right| \le \varepsilon_p\left(\mathbb{H}(\bar{\varphi}(pt)) - \mathbb{H}(m_p)\right) + \varepsilon_p\left(\mathbb{H}([\varphi_\infty(pt)]) - \mathbb{H}(m_p)\right)$$

and since $m_p\le\min(\varphi(pt),[\varphi_\infty(pt)])$, (5.3) gives by neglecting the terms $D\ge 0$

$$\left|\mathbb{H}_p(\bar{\varphi}_p(t)) - \mathbb{H}_p(\varphi_\infty(t))\right| \le \varepsilon_p\pi\left(\mathbb{S}^{\bar{\varphi}(pt)}_{m_p}\right) + \varepsilon_p\pi\left(\mathbb{S}^{[\varphi_\infty(pt)]}_{m_p}\right).$$

In particular, we only need to show that $\varepsilon_p\pi(\mathbb{S}^{\phi_p}_{m_p})\Rightarrow 0$ for $\phi_p = \bar{\varphi}(pt)$ or $[\varphi_\infty(pt)]$. Using the monotonicity of $\pi(\mathbb{S}^n_m)$ in m given by Lemma 6.6, we obtain for any $0<\delta<\varphi_\infty(t)$

$$\mathbb{P}_p\left(\varepsilon_p\pi\left(\mathbb{S}^{\phi_p}_{m_p}\right)\ge\eta\right) \le \mathbb{P}_p\left(m_p\le\phi_p-[p\delta]\right) + \mathbb{P}_p\left(\varepsilon_p\pi\left(\mathbb{S}^{\phi_p}_{\phi_p-[p\delta]}\right)\ge\eta\right).$$

The second term converges to $\mathbb{P}_p\left(\mathbb{H}_\infty(\delta)\ge\eta\right)$: for $\phi_p = [\varphi_\infty(pt)]$ this is a consequence of Theorem 1.2, and for $\phi_p = \bar{\varphi}(pt)$ this was proved in Corollary 5.7 for δ small enough. Since this inequality holds for every δ small enough and since $\mathbb{H}_\infty$ is almost surely continuous at 0 by assumption, in order to conclude the proof it remains to show that $\mathbb{P}_p(m_p\le\phi_p-[p\delta])\to 0$ as $p\to\infty$ for each fixed $0<\delta<\varphi_\infty(t)$, which we do now.

The genealogical contour process $\mathscr{C}_p$ converges weakly to a continuous process $\mathscr{C}_\infty$ by Theorem 1.2. Since $\phi_p/p\Rightarrow\varphi_\infty(t)$, this implies that $\mathscr{C}_p(t_p) - \inf_{I_p}\mathscr{C}_p\Rightarrow 0$ with $t_p = \phi_p/p$ or $t_p=\varphi_\infty(t)$ and $I_p = [\min(\phi_p/p,\varphi_\infty(t)),\max(\phi_p/p,\varphi_\infty(t))]$. By classical arguments on discrete trees, this implies that the genealogical distance rescaled by $\bar{\varepsilon}_p$ between ϕ_p and m_p converges to 0, i.e., $\bar{\varepsilon}_p(\mathscr{H}(\phi_p)-\mathscr{H}(m_p))\Rightarrow 0$. Therefore, for any $\eta>0$ we obtain

$$\limsup_{p\to\infty}\mathbb{P}_p\left(m_p\le\phi_p-[p\delta]\right) \le \limsup_{p\to\infty}\mathbb{P}_p\left(m_p\le\phi_p-[p\delta],\bar{\varepsilon}_p(\mathscr{H}(\phi_p)-\mathscr{H}(m_p))\le\eta\right).$$

Since $L(n-m)\circ\vartheta^m = 0$ if and only if $m = m\wedge n$, Proposition 5.2 implies that $\mathscr{H}(n)-\mathscr{H}(m) = \pi(\mathbb{S}^n_m)\circ\mathscr{G} = \pi(\rho^n_m)$ for any $0\le m\le n$ with $m\in\mathscr{A}(n)\cap\mathbb{R}_+$. In particular, it follows by definition of m_p that $\mathscr{H}(\phi_p)-\mathscr{H}(m_p) = \pi(\rho^{\phi_p}_{m_p})$. Since $\pi(\rho^n_m)$ is non-increasing in m by Lemma 6.6, this gives

$$\mathbb{P}_p\left(m_p\le\phi_p-[p\delta],\bar{\varepsilon}_p(\mathscr{H}(\phi_p)-\mathscr{H}(m_p))\le\eta\right) \le \mathbb{P}_p\left(\bar{\varepsilon}_p\pi\left(\rho^{\phi_p}_{\phi_p-[p\delta]}\right)\le\eta\right).$$

Since this term converges to $\mathbb{P}(\mathscr{H}_\infty(\delta)\le\eta)$ (for $\phi_p=\bar{\varphi}(pt)$ this comes from Corollary 5.7 and for $\phi_p=[\varphi_\infty(pt)]$ this is the convergence of the genealogical height process) we finally obtain

$$\limsup_{p\to\infty}\mathbb{P}_p\left(m_p\le\phi_p-[p\delta]\right) \le \mathbb{P}\left(\mathscr{H}_\infty(\delta)\le\eta\right).$$

Letting $\eta\to 0$ in the last display therefore concludes the proof.

6.3. **Proof of Proposition 6.1.** In order to prove this result, we introduce two intermediate height processes. We enrich the probability space with a random variable $\widetilde{\mathscr{P}}$ which under $\mathbb{P}_p$ is equal in distribution to $\mathscr{P}_1$ and independent from the sequence $(\mathscr{P}_{\bar\varphi(pt)+k}, k \geq 1)$, and we consider $\widetilde{\mathbb{S}}_{(p)} = (\widetilde{\mathbb{S}}^n_{(p)}, n \geq 0)$ the spine process defined from the sequence $(\widetilde{\mathscr{P}}, \mathscr{P}_{\bar\varphi(pt)+1}, \cdots)$. For $k \geq 0$ we then let

$$\widehat{\mathbb{H}}^p(k) = \pi\left(\mathbb{S}^{\bar\varphi(pt)+k}_{\bar\varphi(pt)}\right) \text{ and } \widetilde{\mathbb{H}}^p(k) = \pi\left(\widetilde{\mathbb{S}}^k_{(p)}\right).$$

Lemma 6.7. *$\widetilde{\mathbb{H}}^p$ under $\mathbb{P}_p$ is equal in distribution to $\mathbb{H}$ under $\mathbb{P}_p$. Moreover, we have $\varepsilon_p \sup_{k\geq 0}|\widetilde{\mathbb{H}}^p(k) - \widehat{\mathbb{H}}^p(k)| \Rightarrow 0$.*

Proof. The first part of the lemma directly follows from the strong Markov property. As for the second part, Lemma 6.6 gives

$$0 \leq \pi\left(\mathbb{S}^{\bar\varphi(pt)+k}_{\bar\varphi(pt)}\right) - \pi\left(\mathbb{S}^{\bar\varphi(pt)+k}_{\bar\varphi(pt)+1}\right) \leq \pi(\mathscr{P}_{\bar\varphi(pt)}) \text{ and } 0 \leq \pi\left(\widetilde{\mathbb{S}}^k_{(p)}\right) - \pi\left(\mathbb{S}^{\bar\varphi(pt)+k}_{\bar\varphi(pt)+1}\right) \leq \pi(\widetilde{\mathscr{P}})$$

which gives $|\widetilde{\mathbb{H}}^p(k) - \widehat{\mathbb{H}}^p(k)| \leq \pi(\widetilde{\mathscr{P}}) + \pi(\mathscr{P}_{\bar\varphi(pt)})$. Since this bound is uniform in k and both $\widetilde{\mathscr{P}}$ and $\mathscr{P}_{\bar\varphi(pt)}$ converge weakly (by Corollary 5.7), multiplying by ε_p and letting $p \to \infty$ gives the result. □

We now turn to the proof of Proposition 6.1. Let in the rest of the proof $\Delta_p = \varphi(pt) - \bar\varphi(pt)$. Since by definition

$$\varphi(pt) = \inf\{k \geq 1 : 2\mathcal{V}(k-1) - \mathbb{H}(k) \geq pt\} \text{ and } \bar\varphi(pt) = \inf\{k \geq 1 : 2\mathcal{V}(k) \geq pt\},$$

it follows that

$$\Delta_p = \inf\{k \geq 0 : 2\mathcal{V}(\bar\varphi(pt) + k - 1) - \mathbb{H}(\bar\varphi(pt) + k) \geq pt\}.$$

Defining $\bar{\mathcal{V}}_p(k) = \mathcal{V}(\bar\varphi(pt) + k) - \mathcal{V}(\bar\varphi(pt))$ for $k \geq -1$, we obtain

$$\Delta_p = \inf\{k \geq 0 : 2\bar{\mathcal{V}}_p(k-1) - \mathbb{H}(\bar\varphi(pt)) \geq \mathbb{H}(\bar\varphi(pt) + k) - \mathbb{H}(\bar\varphi(pt)) - (2\mathcal{V}(\bar\varphi(pt)) - pt)\}$$

and so according to Proposition 5.2,

$$\Delta_p = \inf\{k \geq 0 : \tag{6.5}$$
$$2\bar{\mathcal{V}}_p(k-1) - \mathbb{H}(\bar\varphi(pt)) \geq \pi\left(\mathbb{S}^{\bar\varphi(pt)+k}_{\bar\varphi(pt)}\right) - D_{L(k)\circ\vartheta^{\bar\varphi(pt)}}\left(\mathbb{S}^{\bar\varphi(pt)}_0\right) - (2\mathcal{V}(\bar\varphi(pt)) - pt)\Big\}.$$

Since $D_k(\nu) \geq 0$ and $2\mathcal{V}(\bar\varphi(pt)) \geq pt$, we obtain by definition of $\widehat{\mathbb{H}}^p$ that

$$\Delta_p \leq \inf\{k \geq 0 : 2\bar{\mathcal{V}}_p(k-1) - \mathbb{H}(\bar\varphi(pt)) \geq \widehat{\mathbb{H}}^p(k)\}.$$

In particular, if $\sigma_p = [\eta\mathbb{H}(\bar\varphi(pt))]$ then in order to prove the result it is enough to show that $\mathbb{P}_p\left(2\bar{\mathcal{V}}_p(\sigma_p - 1) - \mathbb{H}(\bar\varphi(pt)) \geq \widehat{\mathbb{H}}^p(\sigma_p)\right) \to 1$ which we rewrite as

$$\mathbb{P}_p\left(2\bar{\mathcal{V}}_p(\sigma_p - 1) - \sigma_p/\eta \geq \widehat{\mathbb{H}}^p(\sigma_p)\right) \underset{p\to\infty}{\longrightarrow} 1.$$

Since for any $\gamma > 0$, we have

$$\mathbb{P}_p\left(2\bar{\mathcal{V}}_p(\sigma_p - 1) - \sigma_p/\eta \geq \widehat{\mathbb{H}}^p(\sigma_p)\right) \geq \mathbb{P}_p\left(2\bar{\mathcal{V}}_p(\sigma_p - 1) - \sigma_p/\eta \geq \gamma/\varepsilon_p \geq \widehat{\mathbb{H}}^p(\sigma_p)\right)$$

the desired convergence is implied by the following two relations:

$$\varepsilon_p\widehat{\mathbb{H}}^p(\sigma_p) \Rightarrow 0 \text{ and } \liminf_{p\to\infty} \mathbb{P}_p\left(2\bar{\mathcal{V}}_p(\sigma_p - 1) - \sigma_p/\eta \geq \gamma/\varepsilon_p\right) \underset{\gamma\to 0}{\longrightarrow} 1. \tag{6.6}$$

Let us begin by proving the first relation $\varepsilon_p\widehat{\mathbb{H}}^p(\sigma_p) \Rightarrow 0$. Since $\mathbb{H}_p \overset{\text{fdd}}{\Rightarrow} \mathbb{H}_\infty$, Proposition 6.2 shows that $\varepsilon_p\sigma_p \Rightarrow \eta\mathbb{H}_\infty(\varphi_\infty(t))$, and since $p\varepsilon_p \to \infty$ it follows that $\sigma_p/p \Rightarrow 0$. Since $\widetilde{\mathbb{H}}^p$ is equal in distribution to $\mathbb{H}$ by Lemma 6.7 and σ_p is independent of $\widetilde{\mathbb{H}}^p$, we obtain in view of Remark 3.2 that $\varepsilon_p\widetilde{\mathbb{H}}^p(\sigma_p) \Rightarrow 0$. The second part of Lemma 6.7 finally entails the desired result $\varepsilon_p\widehat{\mathbb{H}}^p(\sigma_p) \Rightarrow 0$.

We now prove the second convergence in (6.6). By construction, $\bar{\mathcal{V}}_p$ is a renewal process independent of $\mathbb{H}(\bar\varphi(pt))$, and thus independent of σ_p. Combined with Lemma 6.4,

one thus obtains that $\bar{V}_p(\sigma_p-1)/\sigma_p \Rightarrow \beta^*$. Since, as already mentioned, $\varepsilon_p\sigma_p \Rightarrow \eta\mathbb{H}_\infty(\varphi_\infty(t))$, we get

$$\liminf_{p\to\infty}\mathbb{P}_p\left(2\bar{V}_p(\sigma_p-1)-\sigma_p/\eta \geq \gamma/\varepsilon_p\right) \geq \mathbb{P}\left((2\beta^*\eta-1)\mathbb{H}_\infty(\varphi_\infty(t)) \geq \gamma\right).$$

Since $(2\beta^*\eta-1)>0$ and $\mathscr{H}_\infty(\varphi_\infty(t))>0$ a.s., the result follows by letting $\gamma\to 0$.

6.4. **Proof of Proposition 6.3.** Let as in the previous subsection $\Delta_p = \varphi(pt)-\bar{\varphi}(pt)$. Proposition 5.2 gives

$$\mathbb{H}_p(\varphi_p(t))-\mathbb{H}_p(\bar{\varphi}_p(t)) = \varepsilon_p\pi\left(\mathbb{S}^{\varphi(pt)}_{\bar{\varphi}(pt)}\right)-\varepsilon_p D_{L'_p(\Delta_p)}\left(\mathbb{S}^{\bar{\varphi}(pt)}_0\right), \tag{6.7}$$

where we have defined $L'_p(k)=L(k)\circ\vartheta^{\bar{\varphi}(pt)}$. We now show that each term of the right-hand side of (6.7) vanishes, and we start with the second one, i.e., we show that

$$\varepsilon_p D_{L'_p(\Delta_p)}\left(\mathbb{S}^{\bar{\varphi}(pt)}\right)\Rightarrow 0. \tag{6.8}$$

Since $D_{L'_p(k)}(\mathbb{S}^{\bar{\varphi}(pt)}_0)$ is non-decreasing in k and the sequence $(\varepsilon_p\Delta_p, p\geq 1)$ is tight, it is enough to show that

$$\varepsilon_p D_{L(t_p)\circ\vartheta^{\bar{\varphi}(pt)}}\left(\mathbb{S}^{\bar{\varphi}(pt)}_0\right)\Rightarrow 0 \tag{6.9}$$

for some deterministic integer-valued sequence (t_p) with $\varepsilon_p t_p\to\infty$: we will consider $t_p=[(p/\varepsilon_p)^{1/2}]$, which satisfies in addition $t_p/p\to 0$. In order to prove (6.9), we fix until further notice $\gamma,\gamma'>0$ and two integer-valued sequences (γ_p), (γ'_p) such that $\gamma_p/p\to\gamma$ (in particular $t_p/\gamma_p\to 0$) and $\gamma'_p/(p\bar{\varepsilon}_p)\to\gamma'$. Since both $D_k(\mathbb{S}^{\bar{\varphi}(pt)}_0)$ and $L(k)\circ\vartheta^{\bar{\varphi}(pt)}$ are non-decreasing with k, it follows that for p large enough such that $t_p\leq\gamma_p$, we have

$$\mathbb{P}_p\left(\varepsilon_p D_{L(t_p)\circ\vartheta^{\bar{\varphi}(pt)}}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\geq\eta\right)\leq\mathbb{P}_p\left(L(\gamma_p)\circ\vartheta^{\bar{\varphi}(pt)}\geq\gamma'_p\right)+\mathbb{P}_p\left(\varepsilon_p D_{\gamma'_p}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\geq\eta\right).$$

By definition of L and S, the first term is equal to

$$\mathbb{P}_p\left(L(\gamma_p)\circ\vartheta^{\bar{\varphi}(pt)}\geq\gamma'_p\right)=\mathbb{P}_p\left(\min_{i=0,\dots,\gamma_p}\sum_{k=\bar{\varphi}(pt)}^{\bar{\varphi}(pt)+i}(|\mathscr{P}_k|-1)\leq-\gamma'_p\right).$$

Isolating the term $|\mathscr{P}_{\bar{\varphi}(pt)}|-1$ and using that the $\mathscr{P}_k$'s for $k\geq\bar{\varphi}(pt)+1$ are i.i.d., we further get

$$\mathbb{P}_p\left(L(\gamma_p)\circ\vartheta^{\bar{\varphi}(pt)}\geq\gamma'_p\right)\leq\mathbb{P}_p\left(|\mathscr{P}_{\bar{\varphi}(pt)}|\leq-\frac{\gamma'_p}{2}+1\right)+\mathbb{P}_p\left(\min_{i=1,\dots,\gamma_p}\sum_{k=1}^{i}(|\mathscr{P}_k|-1)\leq-\frac{\gamma'_p}{2}\right).$$

The first term vanishes by Corollary 5.7 and since $p\varepsilon_p\to\infty$, and so dividing the second term by $p\bar{\varepsilon}_p$ and using (H1), we obtain

$$\limsup_{p\to\infty}\mathbb{P}_p\left(L(\gamma_p)\circ\vartheta^{\bar{\varphi}(pt)}\geq\gamma'_p\right)\leq\mathbb{P}\left(\inf_{0\leq t\leq\gamma}S_\infty(t)\leq-\frac{\gamma'}{2}\right).$$

By letting first $p\to\infty$ and then $\gamma\downarrow 0$, we thus have at this point

$$\limsup_{p\to\infty}\mathbb{P}_p\left(\varepsilon_p D_{L(t_p)\circ\vartheta^{\bar{\varphi}(pt)}}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\geq\eta\right)\leq\limsup_{p\to\infty}\mathbb{P}_p\left(\varepsilon_p D_{\gamma'_p}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\geq\eta\right).$$

Fix now some $0<\delta<t/(2\beta^*)$: by definition (5.2) of D,

$$D_{\gamma'_p}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\leq\left(\sum_{i:0<T(i)\leq\tau_{\gamma'_p}}\mathbb{Y}(i)\right)\circ\vartheta^{\bar{\varphi}(pt)}$$

and so in the event $\{\tau_{\gamma'_p}\circ\vartheta^{\bar{\varphi}(pt)}\leq[p\delta]\}$, we get

$$D_{\gamma'_p}(\mathbb{S}^{\bar{\varphi}(pt)}_0)\leq\left(\sum_{i=1}^{\tilde{T}^{-1}([p\delta])}\mathbb{Y}(i)\right)\circ\vartheta^{\bar{\varphi}(pt)}=\pi\left(\mathbb{S}^{\bar{\varphi}(pt)}_{\bar{\varphi}(pt)-[p\delta]}\right),$$

where we have used (2.10) to derive the last equality. In particular,

$$\mathbb{P}_p\left(\varepsilon_p D_{\gamma'_p}(\mathbb{S}_0^{\bar\varphi(pt)}) \geq \eta\right) \leq \mathbb{P}_p\left(\tau_{\gamma'_p} \circ \vartheta^{\bar\varphi(pt)} > [p\delta]\right) + \mathbb{P}_p\left(\varepsilon_p \pi\left(\mathbb{S}^{\bar\varphi(pt)}_{\bar\varphi(pt)-[p\delta]}\right) > \eta\right)$$

and since by definition we have

$$\mathbb{P}_p\left(\tau_{\gamma'_p} \circ \vartheta^{\bar\varphi(pt)} > [p\delta]\right) = \mathbb{P}_p\left(\sup_{k=0,\dots,[p\delta]} S(k) \circ \vartheta^{\bar\varphi(pt)} \leq \gamma'_p\right),$$

Corollary 5.7 implies that

$$\limsup_{p\to\infty} \mathbb{P}_p\left(\varepsilon_p D_{\gamma'_p}(\mathbb{S}_0^{\bar\varphi(pt)}) \geq \eta\right) \leq \mathbb{P}\left(\sup_{0\leq t\leq\delta} S_\infty(t) \leq \gamma'\right) + \mathbb{P}\left(\mathbb{H}_\infty(\delta) \geq \eta\right).$$

Letting first $\gamma' \to 0$ and then $\delta \to 0$ concludes the proof of (6.9), and so also of (6.8).

We now show that the first term in the right-hand side of (6.7) also vanishes. In view of (6.5) and using $2\mathcal{V}(\bar\varphi(pt)) - pt \leq 2V_{\bar\varphi(pt)}$, we obtain

$$\varepsilon_p \pi\left(\mathbb{S}^{\varphi(pt)}_{\bar\varphi(pt)}\right) \leq \varepsilon_p\left(2\bar{\mathcal{V}}_p(\Delta_p - 1) - \mathbb{H}(\bar\varphi(pt))\right) + \varepsilon_p D_{L(\Delta_p)\circ\vartheta^{\bar\varphi(pt)}}\left(\mathbb{S}_0^{\bar\varphi(pt)}\right) + 2\varepsilon_p V_{\bar\varphi(pt)}.$$

We have just proved that the second term vanishes (in law), and since the third term also vanishes by Corollary 5.7 it only remains to control the first term. Since $\bar{\mathcal{V}}$ is an increasing sequence, for any $\gamma, \eta > 0$ we have

$$\begin{aligned}\mathbb{P}_p\left(\varepsilon_p\left(2\bar{\mathcal{V}}_p(\Delta_p - 1) - \mathbb{H}(\bar\varphi(pt))\right) \geq \gamma\right) &\leq \mathbb{P}_p\left(\Delta_p > \eta\mathbb{H}(\bar\varphi(pt))\right) \\ &\quad + \mathbb{P}_p\left(\varepsilon_p\left(2\bar{\mathcal{V}}_p([\eta\mathbb{H}(\bar\varphi(pt))]) - \mathbb{H}(\bar\varphi(pt))\right) \geq \gamma\right).\end{aligned}$$

Choose now $\eta > 1/(2\beta^*)$, so that the first term vanishes by Proposition 6.1. For the second term, we note that $\bar{\mathcal{V}}$ is independent from $\mathbb{H}(\bar\varphi(pt))$ to obtain with similar arguments as in the proof of Proposition 6.1

$$\limsup_{p\to\infty} \mathbb{P}_p\left(\varepsilon_p\left(2\bar{\mathcal{V}}_p([\eta\mathbb{H}(\bar\varphi(pt))]) - \mathbb{H}(\bar\varphi(pt))\right) \geq \gamma\right) \leq \mathbb{P}\left((2\beta^*\eta - 1)\mathbb{H}_\infty(\varphi_\infty(t)) \geq \gamma\right).$$

Since $\mathbb{P}(\mathbb{H}_\infty(\varphi_\infty(t)) > 0) = 1$, letting $\eta \to 1/(2\beta^*)$ concludes the proof.

7. Proof of Theorem 1.9

We assume in this section that conditions G, Y, VP and V hold and we prove Theorem 1.9.

Lemma 7.1. *Let $(\ell(p), p \geq 0)$ be a deterministic sequence in $\mathbb{R}_+$ going to ∞. Then for every $t > 0$ we have*

$$\varepsilon_p\left(D_{\ell(p)}(\mathbb{S}_0^{[pt]}) - \alpha^* D_{\ell(p)}(\mathbb{S}_0^{[pt]} \circ \mathscr{G})\right) \Rightarrow 0.$$

Proof. Let $\widetilde{T}_p^{-1} = \widetilde{T}^{-1}(\min(\tau_{\ell(p)}, [pt]))$ and $R_p = \mathbb{1}(0 < \tau_{\ell(p)} \leq [pt])\pi(\mu_{\ell(p)})$, so that by definition (5.2) of D we have

$$D_{\ell(p)}(\mathbb{S}_0^{[pt]}) = \left(\sum_{i=1}^{\widetilde{T}_p^{-1}} \mathbb{Y}(i)\right) \circ \vartheta^{[pt]} - R_p \circ \vartheta^{[pt]}.$$

Using the various facts that $D_{\ell(p)}(\mathbb{S}_0^{[pt]} \circ \mathscr{G}) = D_{\ell(p)}(\mathbb{S}_0^{[pt]}) \circ \mathscr{G}$, that $\mathbb{Y}(i) \circ \mathscr{G} = \pi(\mu_\ell) \circ \mathscr{G} = 1$, that $\widetilde{T}_p^{-1}$ and $\tau_{\ell(p)}$ are genealogical quantities and finally that $\vartheta^{[pt]}$ and $\mathscr{G}$ commute, composing on the right with $\mathscr{G}$ in the previous display gives

$$D_{\ell(p)}(\mathbb{S}_0^{[pt]} \circ \mathscr{G}) = \left(\widetilde{T}_p^{-1} - \mathbb{1}(\tau_{\ell(p)} \leq [pt])\right) \circ \vartheta^{[pt]}.$$

By duality, we therefore only have to show that the three quantities

$$\varepsilon_p R_p,\ \varepsilon_p \mathbb{1}(\tau_{\ell(p)} \leq [pt]) \text{ and } \varepsilon_p \sum_{k=1}^{\widetilde{T}_p^{-1}} \left(\mathbb{Y}(k) - \mathbb{E}(\mathbb{Y}^*_p)\right)$$

converge weakly to 0. The second one obviously does since $\varepsilon_p \to 0$. For the third one we proceed similarly as in the proof of Theorem 3.1: indeed, $\varepsilon_p \widetilde{T}_p^{-1}$ is tight (because it is smaller than $\varepsilon_p \widetilde{T}^{-1}([pt])$ by monotonicity of $\widetilde{T}^{-1}$, which is equal in distribution to $\mathscr{H}_p(t)$), which is the only assumption necessary for the proof of Theorem 3.1 to go through.

We now prove that $\varepsilon_p R_p \Rightarrow 0$, which will conclude the proof. First of all, let Γ such that $\tau_{\ell(p)} = T(\Gamma)$: then by definition, $\mu_{\ell(p)} = \mu_0 \circ \theta_{T(\Gamma-1)}$ and so $\pi(\mu_{\ell(p)}) = \pi \circ \mu_0 \circ \theta_{T(\Gamma-1)} = \mathbb{Y}(\Gamma-1)$. In addition, if $\tau_{\ell(p)} \le [pt]$ then $\Gamma \le \widetilde{T}^{-1}([pt])$ and so $R_p \le \max_{k=1,\dots,\widetilde{T}^{-1}([pt])} \mathbb{Y}(k)$. Next, we fix some $N \ge 0$, consider $N_p = [N/\varepsilon_p]$ and use the previous inequality to write

$$\mathbb{P}_p\left(\varepsilon_p \max_{k=1,\dots,\widetilde{T}^{-1}([pt])} \mathbb{Y}(k) \ge \eta\right) \le \mathbb{P}_p\left(\widetilde{T}^{-1}([pt]) \ge N_p\right) + \mathbb{P}_p\left(\max_{k=1,\dots,\min(N_p,G-1)} \mathbb{Y}(k) \ge \eta_p\right) \tag{7.1}$$

where $G = \inf\{k \ge 0 : T(k) = \infty\}$ and $\eta_p = \eta/\varepsilon_p$. For the first term of the right-hand side, we note that $\widetilde{T}^{-1}([pt])$ is by duality equal in distribution to $\mathscr{H}(pt)$ to get

$$\limsup_{p\to\infty} \mathbb{P}_p\left(\widetilde{T}^{-1}([pt]) \ge N_p\right) = \limsup_{p\to\infty} \mathbb{P}_p\left(\mathscr{H}_p(t) \ge \varepsilon_p N_p\right) \underset{N\to\infty}{\longrightarrow} 0.$$

It remains to control the second term in the right-hand side of (7.1): since the $(\mathbb{Y}(k), k = 1,\dots,G-1)$ are i.i.d. by Lemma 2.12, we have

$$\mathbb{P}_p\left(\max_{k=1,\dots,\min(N_p,G-1)} \mathbb{Y}(k) \ge \eta_p\right) \le 1 - \left[1 - \mathbb{P}\left(\mathbb{Y}_p^* \ge \eta_p\right)\right]^{N_p}.$$

This last bound vanishes because $N_p \mathbb{P}(\mathbb{Y}_p^* \ge \eta_p) \to 0$ as a direct consequence of the uniform integrability of the $\mathbb{Y}_p^*$ together with the following bound:

$$N_p \mathbb{P}\left(\mathbb{Y}_p^* \ge \eta_p\right) \le \frac{N}{\eta} \mathbb{E}\left(\mathbb{Y}_p^*; \mathbb{Y}_p^* \ge \frac{\eta}{\varepsilon_p}\right).$$

The proof is complete. □

In the sequel for $0 \le u \le v$ we define

$$\mathscr{M}(u,v) = \inf_{u\le t\le v} \mathscr{C}(t) \text{ and } \mathbb{M}(u,v) = \inf_{u\le t\le v} \mathbb{C}(t).$$

Corollary 7.2. *For any $0 < a < b$ we have*

$$\varepsilon_p\left(\mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) - \alpha^* \mathscr{M}(2pa, 2pb)\right) \Rightarrow 0.$$

Proof. First of all, we note that

$$\varepsilon_p\left(\mathscr{M}(2pa,2pb) - \mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) \circ \mathscr{G}\right) \Rightarrow 0.$$

Indeed, this follows from rewriting $\mathscr{M}(2\beta^* pa, 2\beta^* pb) = \inf\{\mathscr{C}_p(t) : 2a \le t \le 2b\}$ and

$$\mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) \circ \mathscr{G} = \inf\left\{\mathscr{C}_p(t) : \frac{1}{p} K_{\lfloor pa\rfloor} \circ \mathscr{G} \le t \le \frac{1}{p} K_{\lfloor pb\rfloor} \circ \mathscr{G}\right\},$$

together with the following two facts: 1) $\mathscr{C}_p \Rightarrow \mathscr{C}_\infty$ with $\mathscr{C}_\infty$ continuous and 2) $p^{-1} K_{\lfloor pa\rfloor} \circ \mathscr{G} \Rightarrow 2a$. Therefore, in order to prove the result we only have to prove that

$$\varepsilon_p\left(\mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) - \alpha^* \mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) \circ \mathscr{G}\right) \Rightarrow 0.$$

To prove this, we define $L_p = L([pb] - [pa]) \circ \vartheta^{[pa]}$ and apply Corollary 5.4 to write

$$\varepsilon_p\left(\mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) - \alpha^* \mathbb{M}(K_{\lfloor pa\rfloor}, K_{\lfloor pb\rfloor}) \circ \mathscr{G}\right) = \varepsilon_p\left(\mathbb{H}([pa]) - \alpha^* \mathscr{H}([pa])\right) - \varepsilon_p\left(D_{L_p}(\mathbb{S}_0^{[pa]}) - \alpha^* D_{L_p}(\mathbb{S}_0^{[pa]}) \circ \mathscr{G}\right).$$

The first term on the right-hand side vanishes by Theorem 3.1, so we are left with the second term. Since L_p is a genealogical quantity, this term is equal to

$$\varepsilon_p\left(D_{L_p}(\mathbb{S}_0^{[pa]}) - \alpha^* D_{L_p}(\mathbb{S}_0^{[pa]}) \circ \mathscr{G}\right) = \varepsilon_p\left(D_{L_p}(\mathbb{S}_0^{[pa]}) - \alpha^* D_{L_p}(\mathbb{S}_0^{[pa]} \circ \mathscr{G})\right)$$

and we can now invoke Lemma 7.1 to conclude that this term vanishes, as L_p is independent of $\mathbb{S}_0^{[pa]}$ and converges weakly to ∞. This proves the result. □

Proof of Theorem 1.9. In order to prove Theorem 1.9 we have to prove that

$$\varepsilon_p\left(\mathbb{M}(ps,pt) - \alpha^* \mathscr{M}(2\varphi_\infty(ps), 2\varphi_\infty(pt))\right) \Rightarrow 0.$$

Since for any $t \in \mathbb{R}_+$ we have $p^{-1}K_{[pt]} \Rightarrow 2\beta^* t$, for any $0 < \gamma < t$ we have $\mathbb{P}_p(E_p(t,\gamma)) \to 1$ as $p \to \infty$ where $E_p(t,\gamma)$ is the event

$$E_p(t,\gamma) = \left\{K_{[\varphi_\infty(pt-p\gamma)]} \le pt \le K_{[\varphi_\infty(pt+p\gamma)]}\right\}.$$

Thus in the sequel, for any $0 < \gamma < s < t$ we can assume that the event $E_p(s,\gamma) \cap E_p(t,\gamma)$ holds. By monotonicity, in this event we have

$$\mathbb{M}\left(K_{[\varphi_\infty(ps-p\gamma)]}, K_{[\varphi_\infty(pt+p\gamma)]}\right) \le \mathbb{M}\left(ps,pt\right) \le \mathbb{M}\left(K_{[\varphi_\infty(ps+p\gamma)]}, K_{[\varphi_\infty(pt-p\gamma)]}\right).$$

Thus defining $a = \varphi_\infty(ps)$, $b = \varphi_\infty(pt)$, $a^\pm = [\varphi_\infty(ps \pm p\gamma)]$ and $b^\pm = \varphi_\infty(pt \pm p\gamma)$, we have

$$\begin{aligned}&\left|\mathbb{M}(ps,pt) - \alpha^* \mathscr{M}\left(2\varphi_\infty(ps), 2\varphi_\infty(pt)\right)\right| \\ &\qquad \le \left|\mathbb{M}\left(K_{a^+}, K_{b^-}\right) - \alpha^* \mathscr{M}\left(2a, 2b\right)\right| + \left|\mathbb{M}\left(K_{a^-}, K_{b^+}\right) - \alpha^* \mathscr{M}\left(2a, 2b\right)\right|\end{aligned}$$

and pursuing with the triangular inequality, we obtain

$$\begin{aligned}&\left|\mathbb{M}(ps,pt) - \alpha^* \mathscr{M}\left(2\varphi_\infty(ps), 2\varphi_\infty(pt)\right)\right| \\ &\quad \le \left|\mathbb{M}\left(K_{a^+}, K_{b^-}\right) - \alpha^* \mathscr{M}\left(2a^+, 2b^-\right)\right| + \left|\mathbb{M}\left(K_{a^-}, K_{b^+}\right) - \alpha^* \mathscr{M}\left(2a^-, 2b^+\right)\right| \\ &\qquad + \alpha^* \left|\mathscr{M}\left(2a^+, 2b^-\right) - \mathscr{M}\left(2a, 2b\right)\right| + \alpha^* \left|\mathscr{M}\left(2a^-, 2b^+\right) - \mathscr{M}\left(2a, 2b\right)\right|.\end{aligned}$$

Multiplying by ε_p, the two terms of the second line vanish as $p \to \infty$ by Corollary 7.2; letting then $\gamma \to 0$ makes the terms of the third line disappear by virtue of the convergence $\mathscr{C}_p \Rightarrow \mathscr{C}_\infty$ with $\mathscr{C}_\infty$ continuous. The proof of Theorem 1.9 is complete. □

8. Some examples where tightness fails

We show in this section that, for the chronological processes, the gap between convergence of finite-dimensional distributions and functional convergence is more significant than in the genealogical case. In particular, the chronological processes may converge in the sense of finite-dimensional distributions but not in a functional sense in a non-triangular setting.

We consider a simple family of Crump–Mode–Jagers processes which are caracterized by the offspring distribution ξ, namely

$$\left(V_p^*, \mathscr{P}_p^*\right) = \left(1+\xi,\ (\xi-1)^+\epsilon_1 + \mathbb{1}(\xi \ge 1)\epsilon_\xi\right).$$

The corresponding CMJ tree is then almost a Galton-Watson tree with offspring distribution the distribution of ξ, except that:

- each edge is extended by a length equal to one plus the number of children of the corresponding individual;
- for each individual, its children are born at time 1 except for one child born at a time equal to the number of children.

Assume that $\mathbb{E}(\xi) = 1$ and $\mathbb{P}(\xi \geq x) \sim cx^{-\alpha}$ as $x \to \infty$ for some constant $c \in (0,\infty)$. Then it is known that condition G holds for the choice $\varepsilon_p = p^{-(1-1/\alpha)}$. In particular, S_p has jumps of the order of one which means that, typically, some nodes have of the order of $p\varepsilon_p = p^{1/\alpha}$ children: these nodes are called macroscopic.

Moreover, $\mathbb{E}(V_p^*) = 2$ and

$$\mathbb{E}(\mathbb{Y}_p^*) = \mathbb{E}\left(\int_0^\infty u\mathscr{P}_p^*(\mathrm{d}u)\right) = \mathbb{E}\left(\xi - 1 + \xi; \xi \geq 1\right) = 1 + \mathbb{P}(\xi = 0).$$

Since in addition we are not in a triangular setting, this implies that conditions Y, VP and V holds. In particular, all the results of the paper hold and $\mathbb{H}_p$ and $\mathbb{C}_p$ converge in the sense of finite-dimensional distributions: we now show that they cannot converge in a functional sense.

By construction, macroscopic nodes have edges with length of the order of $p^{1/\alpha}$. When the particle traveling along the edges meets such an edge, this makes $\mathbb{C}$ go up and then down at rate ± 1 for a duration $p^{1/\alpha}$, so that during this time interval $\mathbb{C}$ has variation of the order of $p^{1/\alpha}$. Because of the scaling $\mathbb{C}_p(t) = p^{-(1-1/\alpha)}\mathbb{C}(pt)$, such a time interval corresponds for $\mathbb{C}_p$ to a time interval of size $p^{1/\alpha} \times (1/p) = p^{-(1-1/\alpha)}$, during which $\mathbb{C}_p$ has variation of the order of $p^{1/\alpha} \times p^{-(1-1/\alpha)} = p^{2/\alpha-1}$. Since $\alpha \in (1,2)$, in the limit we see that each macroscopic node should induce an infinite jump of $\mathbb{C}_p$. Since macroscopic nodes are dense, this strongly proscribes the tightness of $\mathbb{C}_p$.

Moreover, because of the last child born at time ξ, the exact same phenomenon affects $\mathbb{H}_p$.

Appendix A. Proof of Lemma 2.3

In this section we prove Lemma 2.3: first consider the following lemma.

Lemma A.1. *For any $n \geq 0$ with $L(n) \circ \theta_n > 0$, we have*

$$T(T^{-1}(n)) = n + \tau_{L(n)} \circ \theta_n \text{ and } \mathscr{Q}(T^{-1}(n)) = \mu_{L(n)} \circ \theta_n. \tag{A.1}$$

Considering this lemma with $n = n - m$, composing to the right with ϑ^n and using $\theta_{n-m} \circ \vartheta^n = \vartheta^m$ by (1.12), this lemma gives Lemma 2.3 except for the fact that $n - T(T^{-1}(n-m)) \circ \vartheta^n$ (or $m - \tau_{L(n-m)} \circ \vartheta^m$) is equal to $m \wedge n$. Thus, in order to prove Lemma 2.3 we first prove Lemma A.1 and then prove that $m \wedge n = n - T(T^{-1}(n-m)) \circ \vartheta^n$.

A.1. **Proof of Lemma A.1.** Let $n \geq 0$ with $L(n) \circ \theta_n > 0$. Simple computation shows that

$$L(n) \circ \theta_n = \max_{i=0,\dots,n} S(-i) \circ \theta_n = \max_{\{0,\dots,n\}} S - S(n) \tag{A.2}$$

and so $L(n) \circ \theta_n > 0$ means that n is not a weak ascending ladder height time of S, in which case by definition of $T^{-1}(n)$ we have

$$T(T^{-1}(n)) = \inf\left\{k > n : S(k) \geq \max_{\{0,\dots,n\}} S\right\}.$$

The right-hand side is always equal to $n + \tau_{L(n)} \circ \theta_n$: indeed,

$$\begin{aligned}
\tau_{L(n)} \circ \theta_n &= \inf\{k > 0 : S(k) \geq L(n)\} \circ \theta_n \\
&= \inf\{k > 0 : S(k) \circ \theta_n \geq L(n) \circ \theta_n\} \\
&= \inf\left\{k > 0 : S(n+k) - S(n) \geq \max_{\{0,\dots,n\}} S - S(n)\right\} \\
&= \inf\left\{k > n : S(k) \geq \max_{\{0,\dots,n\}} S\right\} - n.
\end{aligned}$$

This proves the first identity in (A.1), and we now prove the second one. Define the random time $\Gamma = T(T^{-1}(n) - 1)$: recalling the definition $\mathscr{Q}(k) = \mu_0 \circ \theta_{T(k-1)}$, we see that

we have to prove that $\mu_0 \circ \theta_\Gamma = \mu_{L(n)} \circ \theta_n$ (under the assumption $L(n) \circ \theta_n > 0$). Going back to the definition of $\mu_k = \Upsilon_{\zeta_k}(\mathscr{P}_{\tau_k - 1})$, we see that

$$\mu_0 \circ \theta_\Gamma = \Upsilon_{\zeta_0 \circ \theta_\Gamma}\left(\mathscr{P}_{\tau_0 \circ \theta_\Gamma + \Gamma - 1}\right) \text{ and } \mu_{L(n)} \circ \theta_n = \Upsilon_{\zeta_{L(n)} \circ \theta_n}\left(\mathscr{P}_{\tau_{L(n)} \circ \theta_n + n - 1}\right)$$

and so it is enough to show that

$$\zeta_0 \circ \theta_\Gamma = \zeta_{L(n)} \circ \theta_n \text{ and } \tau_0 \circ \theta_\Gamma + \Gamma = \tau_{L(n)} \circ \theta_n + n.$$

We first show the second identity. Since $\tau_0 = T(1)$ and $T(1) \circ \theta_{T(k)} + T(k) = T(k+1)$ for any $k \geq 0$, considering $k = T^{-1}(n) - 1$ yields $\tau_0 \circ \theta_\Gamma + \Gamma = T(T^{-1}(n))$ which is equal to $\tau_{L(n)} \circ \theta_n + n$ as has been argued above.

Using this equality, we now prove that $\zeta_0 \circ \theta_\Gamma = \zeta_{L(n)} \circ \theta_n$ which will conclude the proof of Lemma A.1. Since $\zeta_{L(n)} = L(n) - S(\tau_{L(n)} - 1)$, $L(n) \circ \theta_n = \max_{\{0,\dots,n\}} S - S(n)$ by (A.2) and

$$S(\tau_{L(n)} - 1) \circ \theta_n = S\big(\tau_{L(n)} \circ \theta_n + n - 1\big) - S(n) = S\big(T(T^{-1}(n)) - 1\big) - S(n),$$

we get

$$\zeta_{L(n)} \circ \theta_n = \max_{\{0,\dots,n\}} S - S\big(T(T^{-1}(n)) - 1\big).$$

Moreover,

$$\zeta_0 \circ \theta_\Gamma = -S(\tau_0 - 1) \circ \theta_\Gamma = S(\Gamma) - S(\tau_0 \circ \theta_\Gamma + \Gamma - 1) = S(\Gamma) - S\big(T(T^{-1}(n)) - 1\big).$$

Since the condition $L(n) \circ \theta_n > 0$ means that n is not a weak ascending ladder height time of S, we have $T^{-1}(n) = \widetilde{T}^{-1}(n) + 1$ and in particular, $\Gamma = T(\widetilde{T}^{-1}(n))$. Thus, $S(\Gamma) = \max_{\{0,\dots,n\}} S$ by definition of $\widetilde{T}^{-1}(n)$ which concludes the proof.

A.2. **Proof of** $m \wedge n = n - T(T^{-1}(n-m)) \circ \vartheta^n$**.** Let $0 \leq m \leq n$ with $L(n-m) \circ \vartheta^m > 0$ and define $\kappa = n - T(T^{-1}(n-m)) \circ \vartheta^n$: in order to conclude the proof of Lemma 2.3, we now prove that $m \wedge n = \kappa$. Since on the one hand $\kappa = n - T(T^{-1}(n-m)) \circ \vartheta^n$, it follows from the definition of $\mathscr{A}(n)$ that $\kappa \in \mathscr{A}(n)$. Moreover, $\tau_{L(n-m)}$ is by definition a weak ascending ladder height time, i.e., for every $k \geq 0$ there exists Γ such that $\tau_k = T(\Gamma)$: in particular, $\kappa = m - \tau_{L(n-m)} \circ \vartheta^m$ also belongs to $\mathscr{A}(m)$. In order to conclude the proof it remains to show that $\kappa \geq \alpha$ for any $\alpha \in \mathscr{A}(m) \cap \mathscr{A}(n)$. By definition, we can write such an α as

$$\alpha = n - T(\Gamma) \circ \vartheta^n = m - T(\Gamma') \circ \vartheta^m$$

for some $\Gamma, \Gamma' \geq 0$. In particular,

$$T(\Gamma) \circ \vartheta^n = n - m + T(\Gamma') \circ \vartheta^m \geq n - m$$

and so by definition of T^{-1}, we have $\Gamma \circ \vartheta^n \geq T^{-1}(n-m) \circ \vartheta^n$. Since the weak ascending ladder height times form an increasing sequence, this implies $T(\Gamma) \circ \vartheta^n \geq T(T^{-1}(n-m)) \circ \vartheta^n$ and so $\alpha \leq \kappa$, which concludes the proof.

Appendix B. Proof of Lemma 2.5

Let $n \geq 0$, $m = n - \tau_0 \circ \vartheta^n$, $i = \zeta_0 \circ \vartheta^n$ and assume that $m \geq 0$: we have to prove that $i \in \{0, \dots, |\mathscr{P}_m| - 1\}$ and $\chi(m,i) = n$. Let us first prove that $i \in \{0, \dots, |\mathscr{P}_m| - 1\}$. Since $\mathscr{P}_m = \mathscr{P}_{n - \tau_0 \circ \vartheta^n} = \mathscr{P}_{\tau_0 - 1} \circ \vartheta^n$, this follows from the fact that $\zeta_0 = -S(\tau_0 - 1) \leq S(\tau_0) - S(\tau_0 - 1) = |\mathscr{P}_{\tau_0 - 1}| - 1$ and then composing on the right with ϑ^n.

Let us now prove that $\chi(m,i) = n$. By definition of χ and since S only makes negative jumps of size -1, we have to prove that

$$S(n) = S(m+1) - i \tag{B.1}$$

and that

$$S(\ell) > S(m+1) - i, \ \ell = m+1, \dots, n-1. \tag{B.2}$$

Let us first prove (B.1). By definition of m and i we have

$$S(m+1) - i = S\big(n - \tau_0 \circ \vartheta^n + 1\big) - \zeta_0 \circ \vartheta^n = S\big(n - \tau_0 \circ \vartheta^n + 1\big) + S(\tau_0 - 1) \circ \vartheta^n$$

which by (2.3) (applied with $\Gamma = \tau_0 - 1$) implies (B.1). Let us now prove (B.2): in view of (B.1) we have to prove that

$$\min\{S(k) : k = m+1, \dots, n-1\} > S(n)$$

which directly follows from the fact that

$$\min\{S(k) : k = m+1, \dots, n-1\} = S(n) - \max\{S(k) : k = 1, \dots, T(1)-1\} \circ \vartheta^n.$$

Appendix C. Proof of Propositions 5.5 and 5.6

C.1. **Coupling between random walks.** We present here the coupling of Lemma 9.21 in Kallenberg [13] between two random walks with the same step distribution and possibly different initial distributions. The coupling starts from the following stochastic primitives, which are assumed to be mutually independent:

- α and α', two independent real-valued random variables;
- (ξ_k), i.i.d. sequence of real-valued random variables;
- (ϱ_k), i.i.d. sequence with $\mathbb{P}(\varrho_k = \pm 1) = 1/2$.

Let

$$\widetilde{W}(n) = \alpha' - \alpha + \sum_{k=1}^{n} \varrho_k \xi_k,\ n \geq 0,$$

so that $\widetilde{W}$ is a critical random walk with initial distribution $\alpha' - \alpha$ and step distribution $\varrho_1 \xi_1$. Fix in the rest of this subsection $\varepsilon > 0$ and define the following quantities:

- $A_\varepsilon = \inf\{n \geq 0 : \widetilde{W}(n) \in [0, \varepsilon]\}$;
- $\varrho'_k = (-1)^{\mathbb{1}(k \leq A_\varepsilon)} \varrho_k$;
- $\kappa_1 < \kappa_2 < \cdots$ the values of k with $\varrho_k = 1$ and $\kappa'_1 < \kappa'_2 < \cdots$ the values of k with $\varrho'_k = 1$;
- and finally

$$W(n) = \alpha + \sum_{j=1}^{n} \xi_{\kappa_j} \text{ and } W'(n) = \alpha' + \sum_{j=1}^{n} \xi_{\kappa'_j}.$$

Lemma C.1. *W, respectively W', is a random walk with step distribution ξ_1 and initial distribution α, respectively α'.*

Proof. See the proof of Lemma 9.21 in Kallenberg [13]. □

Thus we have constructed a coupling of two random walks with the same step distribution, as promised. The interest of this coupling lies in the following result, which exhibits an event in which many increments of W and W' are equal. In the sequel we define:

- $\sigma = |\{j : \kappa_j \leq A_\varepsilon\}|$, $\sigma' = |\{j : \kappa'_j \leq A_\varepsilon\}|$ and

$$\gamma = \max\left(\max_{k=0,\dots,\sigma} W(k),\ \max_{k=0,\dots,\sigma'} W'(k) \right);$$

- $\psi(t) = \inf\{n \geq 0 : W(n) \geq t\}$ and $\psi'(t) = \inf\{n \geq 0 : W'(n) \geq t\}$ for $t \geq 0$;
- $\Delta_k = W(k) - W(k-1)$ and $\Delta'_k = W'(k) - W'(k-1)$ for $k \geq 1$.

Lemma C.2. *For any $m \in \mathbb{N}$ and $t, \varepsilon \in \mathbb{R}_+$, in the event*

$$\{\gamma < t\} \cap \{\psi(t) > A_\varepsilon + m\} \cap \{W'(\psi'(t)) \geq t + \varepsilon\},$$

we have $\Delta_{\psi(t)-k} = \Delta'_{\psi'(t)-k}$ for any $k = 0, \dots, m$.

Proof. Let $\overline{W}$ and $\overline{W}'$ be the processes W and W' shifted at time σ and σ', respectively, i.e., defined by $\overline{W}(n) = W(\sigma + n)$ and $\overline{W}'(n) = W'(\sigma' + n)$. Then for any $n \geq 0$, we have

$$\overline{W}(n) = \overline{W}'(n) - \widetilde{W}(A_\varepsilon), \tag{C.1}$$

see [13, Lemma 9.21] for details. Assume in the rest of the proof that $\gamma < t$, $\psi(t) > A_\varepsilon + m$ and $W'(\psi'(t)) \geq t + \varepsilon$. By definition, $\gamma < t$ implies that $\psi(t) \geq \sigma$ and so we can write

$$\psi(t) = \inf\{n \geq \sigma : W(n) \geq t\} = \sigma + \inf\left\{n \geq 0 : \overline{W}(n) \geq t\right\}.$$

In particular, using (C.1) we obtain

$$\psi(t) = \sigma + \inf\left\{n \geq 0 : \overline{W}'(n) \geq t + \widetilde{W}(A_\varepsilon)\right\}$$

and since $\gamma < t$ implies $\psi'(t) \geq \sigma'$ as well, a symmetric reasoning finally entails

$$\psi(t) - \sigma = \psi'\left(t + \widetilde{W}(A_\varepsilon)\right) - \sigma'.$$

Since by definition $\widetilde{W}(A_\varepsilon) \leq \varepsilon$ and since we assume $W'(\psi'(t)) \geq t + \varepsilon$, we further get that $\psi'(t + \widetilde{W}(A_\varepsilon)) = \psi'(t)$, which finally proves that $\psi(t) - \sigma = \psi'(t) - \sigma'$. Consider now any $k = 0, \ldots, m+1$, so that $\psi(t) \geq k + \sigma$ and $\psi'(t) \geq k + \sigma'$ as a consequence of the assumption $\psi(t) > \sigma + m$ and the fact that $\psi(t) - \sigma = \psi'(t) - \sigma'$: then we have

$$\begin{aligned} W(\psi(t) - k) &= \overline{W}(\psi(t) - \sigma - k) && \text{(by definition of } \overline{W}) \\ &= \overline{W}(\psi'(t) - \sigma' - k) && \text{(by } \psi(t) - \sigma = \psi'(t) - \sigma') \\ &= \overline{W}'(\psi'(t) - \sigma' - k) - \widetilde{W}(A_\varepsilon) && \text{(by (C.1))}, \end{aligned}$$

which finally gives

$$W(\psi(t) - k) = W'(\psi'(t) - k) - \widetilde{W}(A_\varepsilon)$$

by definition of $\overline{W}'$. This last equality readily implies the desired result. □

C.2. **Stationary renewal processes on $\mathbb{R}$.** In this subsection and the following one, we fix some $p \geq 1$. We enrich the probability space $\mathbb{L}^{\mathbb{Z}}$ to $\mathbb{L}^{\mathbb{Z}} \times \mathbb{R}_+ \times (0, \infty)$ and denote by $(\omega, d_-, d_+) \in \mathbb{L}^{\mathbb{Z}} \times \mathbb{R}_+ \times (0, \infty)$ the canonical sequence. We then define

$$W_\pm(n) = d_\pm + \sum_{k=1}^{n} V_{\pm k},\ n \geq 0,$$

as well as the following point process on $\mathbb{R} \times \mathscr{M}$:

$$Z = \sum_{n \geq 0} \epsilon_{(W_+(n), \mathscr{P}_n)} + \sum_{n \geq 0} \epsilon_{(-W_-(n), \mathscr{P}_{-n-1})}.$$

For χ a probability distribution on $\mathbb{R}_+ \times (0, \infty) \times \mathscr{M}$, let $\mathbb{P}_p^\chi$ be the probability measure under which:

- $((V_n, \mathscr{P}_n), n \in \mathbb{Z} \setminus \{0\})$ are i.i.d. with common distribution $(2V_p^*, \mathscr{P}_p^*)$;
- $(d_-, d_+, \mathscr{P}_0)$ is independent from this sequence and has distribution χ.

Under $\mathbb{P}_p^\chi$, V_0 will not play a role. We consider Θ_t the shift operator acting on measures on $\mathbb{R} \times \mathscr{M}$ as follows: for any measure ν on $\mathbb{R} \times \mathscr{M}$ and any Borel sets $B \subset \mathbb{R}$ and $M \subset \mathscr{M}$,

$$\Theta_t \nu(B \times M) = \nu((t + B) \times M).$$

Note that Z uniquely characterizes the canonical sequence (ω, d_-, d_+) and so with a slight abuse of notation, we will sometimes consider that we are working on the canonical space of locally finite point measures on $\mathbb{R} \times \mathscr{M}$, that Z is the canonical measure and that ω and $d_\pm$ are functional thereof, e.g., $d_+ = \inf\{t > 0 : Z(\{t\} \times \mathscr{M}) > 0\}$. In particular, the notation $\mathbb{P}_p^\chi \circ \Theta_p^{-1}$ makes sense, which is rigorously to be understood as the law of $\Theta_t Z$ under $\mathbb{P}_p^\chi$.

For a random variable $V > 0$ with finite mean, we define $\hat{V}$ as follows:

$$\mathbb{P}(\hat{V} \geq x) = \frac{1}{\mathbb{E}(V)} \int_x^\infty \mathbb{P}(V \geq y) \mathrm{d}y,\ x \geq 0.$$

Let $(\hat{V}_p^*, U_p^*, \hat{\mathscr{P}}_p^*)$ be such that, conditionally on $\hat{V}_p^* = v$:

- $\hat{\mathscr{P}}_p^*$ is independent from U_p^* and is distributed like $\mathscr{P}_p^*$ conditionally on $V_p^* = v$;

- if V_p^* is non-arithmetic, U_p^* is a uniform random variable on $[0, v]$;
- if V_p^* is arithmetic with span h, U_p^* is a uniform random variable on $\{0, h, \dots, v\}$.

Let χ_p be the law of $(0, 2V_p^*, \mathscr{P}_p^*)$ and $\hat{\chi}_p$ the law of $(2(\hat{V}_p^* - U_p^*), 2U_p^*, \hat{\mathscr{P}}_p^*)$. The following result corresponds to Theorem 2.1 in Miller [20].

Theorem C.3. *The measure $\mathbb{P}_p^{\hat{\chi}_p}$ is shift invariant, i.e.:*

- *if V_p^* is non-arithmetic, then $\mathbb{P}_p^{\hat{\chi}_p} \circ \Theta_t^{-1} = \mathbb{P}_p^{\hat{\chi}_p}$ for every $t \in \mathbb{R}$;*
- *if V_p^* is arithmetic with span h, then $\mathbb{P}_p^{\hat{\chi}_p} \circ \Theta_{ih}^{-1} = \mathbb{P}_p^{\hat{\chi}_p}$ for every $i \in \mathbb{Z}$.*

The coupling presented in the previous section can be extended to the case of a marked random walk to give the following result. In the sequel, let $h_p \in \mathbb{R}_+$ be the span of V_p^* in the arithmetic case, and with a slight abuse in notation let $h_p : \mathbb{R}_+ \to \mathbb{R}_+$ be the function such that $h_p(t) = t$ in the non-arithmetic case and $h_p(t) = h_p[t/h_p]$ in the arithmetic case.

Lemma C.4. *For any $m \in \mathbb{N}$, $t \in \mathbb{R}_+$ and $f : \mathbb{R} \times \mathscr{M}^{m+1} \to [0,1]$ measurable, the inequality*

$$\begin{aligned}\left|\mathbb{E}_p^{\chi_p} \circ \Theta_t^{-1}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right] - \mathbb{E}_p^{\hat{\chi}_p}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right]\right| \\ \leq \mathbb{P}\left(U_p^* < \varepsilon/2\right) + \mathbb{P}\left(U_p^* \geq t'\right) + 3\mathbb{P}_p\big(\mathcal{V}(m+n) \geq h_p(t)/2 - t'\big) + 2\mathbb{P}\left(A_\varepsilon^p \geq n\right),\end{aligned}$$

holds for any $n \in \mathbb{N}$ and $t', \varepsilon \in \mathbb{R}_+$, where A_ε^p is the random variable defined in Section C.1 for α and ξ_1 equal in distribution to $2V_p^*$ and α' to $2U_p^*$.

Proof. First of all, note that by definition of $h_p(t)$ the law of $(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})$ is the same under $\mathbb{P}_p^{\chi_p} \circ \Theta_t^{-1}$ and $\mathbb{P}_p^{\chi_p} \circ \Theta_{h_p(t)}^{-1}$. In particular, we can assume without loss of generality that $t = h_p(t)$, which allows us to use Theorem C.3 to get $\mathbb{P}_p^{\hat{\chi}_p} \circ \Theta_t^{-1} = \mathbb{P}_p^{\hat{\chi}_p}$.

Next, considering the notation of Section C.1, we consider the coupling described there with α and ξ_1 equal in distribution to $2V_p^*$ and α' to $2U_p^*$. We modify this coupling in two ways: (1) we extend $W(n)$ and $W'(n)$ for $n \leq -1$ arbitrarily; (2) we consider an additional sequence (ν_k) of marks, whereby $W(n)$, resp. $W'(n)$, is given the mark $m_n = \nu_{\kappa_n}$, resp. $m'_n = \nu_{\kappa'_n}$. This way, in addition to the conclusions of Lemma C.2 we obtain that $m_{\psi(t)-k} = m'_{\psi'(t)-k}$ for any $k = 0, \dots, m$ in the event described there. In particular, if marks take value in $\mathscr{M}$ then for any measurable function $f : \mathbb{R} \times \mathscr{M}^{m+1} \to [0,1]$ we obtain

$$\begin{aligned}\left|\mathbb{E}\left[f(\Delta_{\psi(t)}, \nu_{\psi(t)}, \dots, \nu_{\psi(t)-m})\right] - \mathbb{E}\left[f(\Delta'_{\psi'(t)}, \nu'_{\psi'(t)}, \dots, \nu'_{\psi'(t)-m})\right]\right| \\ \leq \mathbb{P}\left(\gamma \geq t\right) + \mathbb{P}\left(\psi(t) \leq A_\varepsilon^p + m + 1\right) + \mathbb{P}\left(W'(\psi'(t)) < t + \varepsilon\right).\end{aligned}$$

When α, α' and ξ_1 are as prescribed above and the $((\xi_k, \nu_k), k \in \mathbb{N})$ are i.i.d. with common distribution $(V_p^*, \mathscr{P}_p^*)$, we get the identities

$$\mathbb{E}\left[f(\Delta_{\psi(t)}, \nu_{\psi(t)}, \dots, \nu_{\psi(t)-m})\right] = \mathbb{E}_p^{\chi_p} \circ \Theta_t^{-1}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right]$$

and

$$\mathbb{E}\left[f(\Delta'_{\psi'(t)}, \nu'_{\psi'(t)}, \dots, \nu'_{\psi'(t)-m})\right] = \mathbb{E}_p^{\hat{\chi}_p} \circ \Theta_t^{-1}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right].$$

Since $\mathbb{P}_p^{\hat{\chi}_p}$ is shift-invariant, we thus get the bound

$$\begin{aligned}\left|\mathbb{E}_p^{\chi_p} \circ \Theta_t^{-1}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right] - \mathbb{E}_p^{\hat{\chi}_p}\left[f(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \dots, \mathscr{P}_{-m})\right]\right| \\ \leq \mathbb{P}\left(\gamma \geq t\right) + \mathbb{P}\left(\psi(t) \leq A_\varepsilon^p + m\right) + \mathbb{P}\left(W'(\psi'(t)) < t + \varepsilon\right)\end{aligned}$$

and so in order to conclude the proof, it remains to show that

$$\mathbb{P}\left(\gamma \geq t\right) + \mathbb{P}\left(\psi(t) \leq A_\varepsilon^p + m\right) + \mathbb{P}\left(W'(\psi'(t)) < t + \varepsilon\right) \\ \leq \mathbb{P}\left(U_p^* < \varepsilon\right) + \mathbb{P}\left(U_p^* \geq t'\right) + 3\mathbb{P}_p\left(\mathcal{V}(m+n) \geq t/2 - t'\right) + 2\mathbb{P}\left(A_\varepsilon^p \geq n\right).$$

First of all, by definition we have $\mathbb{P}(W'(\psi'(t)) < t+\varepsilon) = \mathbb{P}_p^{\hat{\chi}_p} \circ \Theta_t^{-1}(d_+ < \varepsilon)$ and so since $\mathbb{P}_p^{\hat{\chi}_p}$ is shift-invariant, we obtain

$$\mathbb{P}\left(W'(\psi'(t)) < t+\varepsilon\right) = \mathbb{P}_p^{\hat{\chi}_p}(d_+ < \varepsilon) = \mathbb{P}\left(2U_p^* < \varepsilon\right) = \mathbb{P}\left(U_p^* < \varepsilon/2\right).$$

Further, since in the present case W and W' are increasing and $\sigma, \sigma' \leq A_\epsilon$ by construction, we get

$$\mathbb{P}\left(\gamma \geq t\right) = \mathbb{P}\left(W(\sigma) \geq t \text{ or } W'(\sigma') \geq t\right) \leq \mathbb{P}\left(A_\varepsilon^p \geq n\right) + \mathbb{P}\left(W(n) \geq t\right) + \mathbb{P}\left(W'(n) \geq t\right).$$

Since $W(n)$ is equal in distribution to $2\mathcal{V}(n)$ (under $\mathbb{P}_p$), $W'(0)$ is equal in distribution to $2U_p^*$ and $W'(n) - W'(0)$ is equal in distribution to $2\mathcal{V}(n-1)$, we obtain

$$\mathbb{P}\left(W(n) \geq t\right) + \mathbb{P}\left(W'(n) \geq t\right) \leq \mathbb{P}\left(U_p^* \geq t'\right) + 2\mathbb{P}_p\left(\mathcal{V}(m+n) \geq t/2 - t'\right).$$

Finally, since

$$\mathbb{P}\left(\psi(t) \leq A_\varepsilon^p + m\right) \leq \mathbb{P}\left(\psi(t) \leq m+n\right) + \mathbb{P}\left(A_\varepsilon^p \geq n\right)$$

and $\mathbb{P}\left(\psi(t) \leq m+n\right) = \mathbb{P}(W(m+n) \geq t) = \mathbb{P}_p(2\mathcal{V}(m+n) \geq t) \leq \mathbb{P}_p(\mathcal{V}(m+n) \geq t/2 - t')$, gathering the previous inequalities gives the desired result. □

C.3. **Proof of Propositions 5.5 and 5.6.** Let

$$Y_p = \left(2V_{\bar{\varphi}(pt)}, \mathscr{P}_{\bar{\varphi}(pt)}, \ldots, \mathscr{P}_{\bar{\varphi}(pt)-[p\delta]}\right) \text{ and } \hat{Y}_p = \left(2\hat{V}_p^*, \hat{\mathscr{P}}_p^*, \mathscr{P}_p^*(1), \ldots, \mathscr{P}_p^*([p\delta])\right).$$

Then by definition of $\mathbb{P}_p^{\chi_p}$ and $\mathbb{P}_p^{\hat{\chi}_p}$, we have

$$\mathbb{E}_p\left[f\left(Y_p\right)\right] = \mathbb{E}_p^{\chi_p} \circ \Theta_{pt}^{-1}\left[f\left(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \ldots, \mathscr{P}_{-[p\delta]}\right)\right]$$

and

$$\mathbb{E}\left[f\left(\hat{Y}_p\right)\right] = \mathbb{E}_p^{\hat{\chi}_p}\left[f\left(d_+ + d_-, \mathscr{P}_0, \mathscr{P}_{-1}, \ldots, \mathscr{P}_{-[p\delta]}\right)\right]$$

and so for any $n \in \mathbb{N}$, $t', \varepsilon \in \mathbb{R}_+$ and $f : \mathbb{R}_+ \times \mathscr{M}^{[p\delta]+1} \to [0,1]$ measurable, Lemma C.4 gives

$$\left|\mathbb{E}_p\left[f\left(Y_p\right)\right] - \mathbb{E}\left[f\left(\hat{Y}_p\right)\right]\right| \leq \mathbb{P}\left(U_p^* < \varepsilon/2\right) + \mathbb{P}\left(U_p^* \geq t'\right) \\ + 3\mathbb{P}_p\left(\mathcal{V}([p\delta]+n) \geq h_p(pt)/2 - t'\right) + 2\mathbb{P}\left(A_\varepsilon^p \geq n\right). \tag{C.2}$$

Let $p \to \infty$, and assume for a moment that the previous upper bound vanishes by suitably playing on the free parameters ε, t' and n (after having taken the limit $p \to \infty$): by considering

$$f(v, \nu_0, \ldots, \nu_{[p\delta]}) = f(v/2, \nu_0)$$

with $f : \mathbb{R}_+ \times \mathscr{M} \to \mathbb{R}_+$ continuous bounded for Proposition 5.5, and $f = g \circ \Xi_{[p\delta]}$ with $g : \mathbb{R} \to \mathbb{R}$ continuous and bounded for Proposition 5.6, this would give the desired result. We now explain how to make the upper bound in (C.2) vanish.

First of all, note that $h_p(t) \sim pt$ as $p \to \infty$: in the non-arithmetic case this is trivial, while in the arithmetic case, this follows from the fact that $\sup_p h_p < \infty$ (which follows from the assumption $V_p^* \Rightarrow V_\infty^*$ with V_∞^* arithmetic). Therefore, the weak triangular law of large numbers implies that $\mathbb{P}_p\left(\mathcal{V}([p\delta]+n) \geq h_p(pt)/2 - t'\right) \to 0$ as $p \to \infty$, for $\delta < t/(2\beta^*)$ and fixed n and t'.

To deal with the other terms, define U^*_∞ and A^∞_ε from V^*_∞ similarly as U^*_p as A^p_ε from V^*_p, respectively. Under conditions VP and V we have $\hat{V}^*_p \Rightarrow \hat{V}^*_\infty$. Thus we obtain that $U^*_p \Rightarrow U^*_\infty$ and so letting $t' \to \infty$ after $p \to \infty$, we obtain

$$\left|\mathbb{E}_p\left[f\left(Y_p\right)\right] - \mathbb{E}\left[f\left(\hat{Y}_p\right)\right]\right| \leq \limsup_{p\to\infty} \mathbb{P}\left(U^*_p < \varepsilon/2\right) + 2\limsup_{p\to\infty} \mathbb{P}\left(A^p_\varepsilon \geq n\right). \tag{C.3}$$

We further distinguish the arithmetic and non-arithmetic cases.

Arithmetic case. In this case, we have $A^p_0 \Rightarrow A^\infty_0$ and since $U^*_p \geq 0$, considering (C.3) with $\varepsilon = 0$ gives

$$\limsup_{p\to\infty} \left|\mathbb{E}_p\left[f_p\left(Y_p\right)\right] - \mathbb{E}\left[f_p\left(\hat{Y}_p\right)\right]\right| \leq 2\mathbb{P}\left(A^\infty_0 \geq n\right).$$

Since A^∞_0 is almost surely finite, letting $n \to \infty$ gives the result.

Non-arithmetic case. In this case, we have $A^p_\varepsilon \Rightarrow A^\infty_\varepsilon$ for any $\varepsilon > 0$ and since U^*_∞ is absolutely continuous with respect to Lebesgue measure, (C.3) with $\varepsilon > 0$ gives

$$\limsup_{p\to\infty} \left|\mathbb{E}_p\left[f_p\left(Y_p\right)\right] - \mathbb{E}\left[f_p\left(\hat{Y}_p\right)\right]\right| \leq \mathbb{P}\left(U^*_\infty < \varepsilon/2\right) + 2\mathbb{P}\left(A^\infty_\varepsilon \geq n\right).$$

Since A^∞_ε is almost surely finite (for $\varepsilon > 0$) and U^*_∞ does not put mass at 0, letting first $n \to \infty$ and then $\varepsilon \to 0$ finally achieves the proof.

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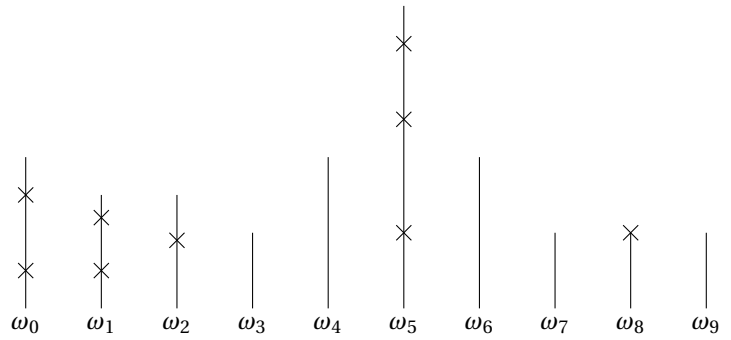


FIGURE 1. Sequence of sticks used in the next figures: this sequence corresponds to one chronological tree.

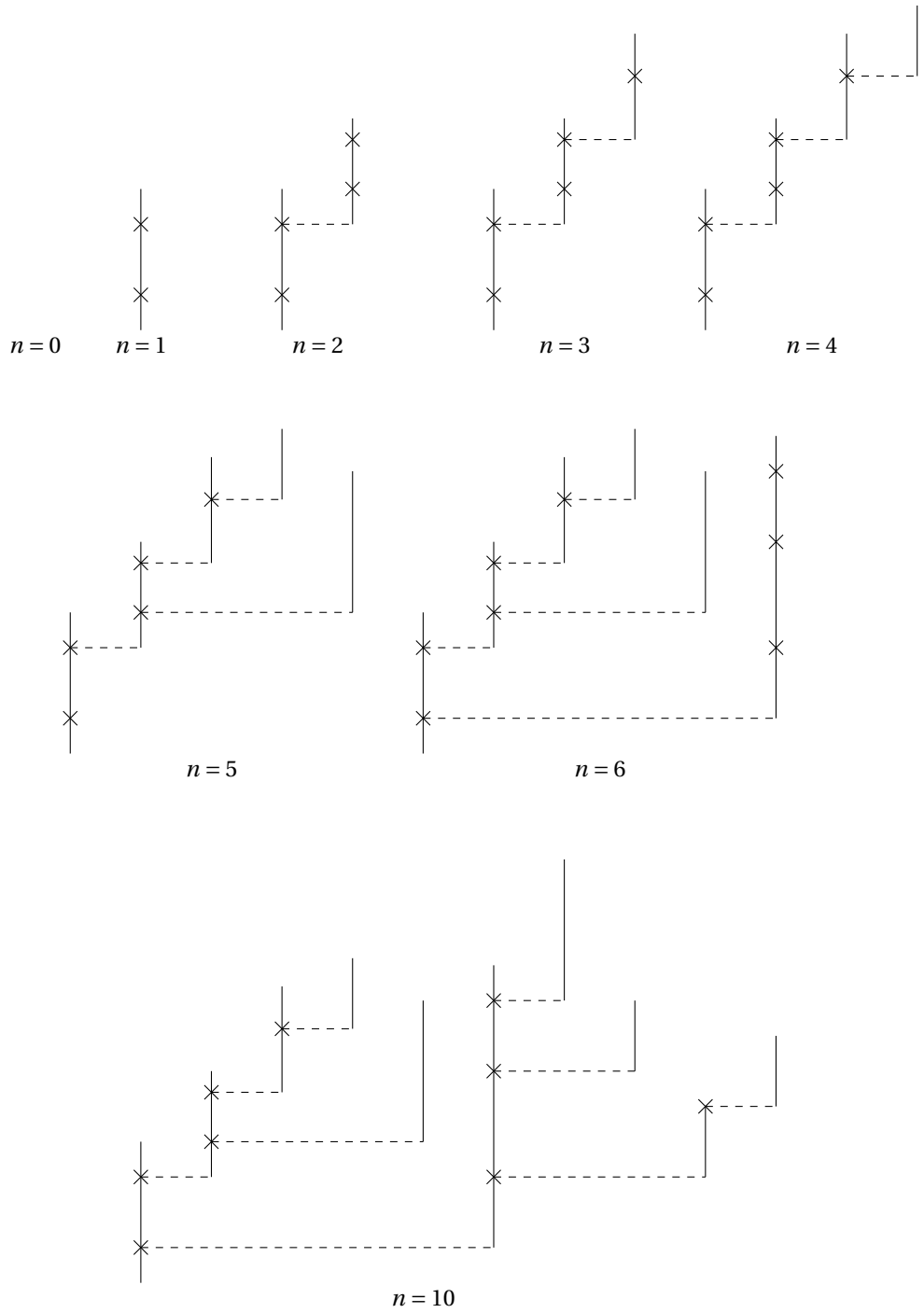


FIGURE 2. Sequential construction of the chronological tree from the sequence of sticks of Figure 1: as long as there is a stub availabe, we graft the next stick at the highest one. At $n = 10$ the construction is complete (there is no more stub available) and the next stick will therefore start the next tree in the forest.

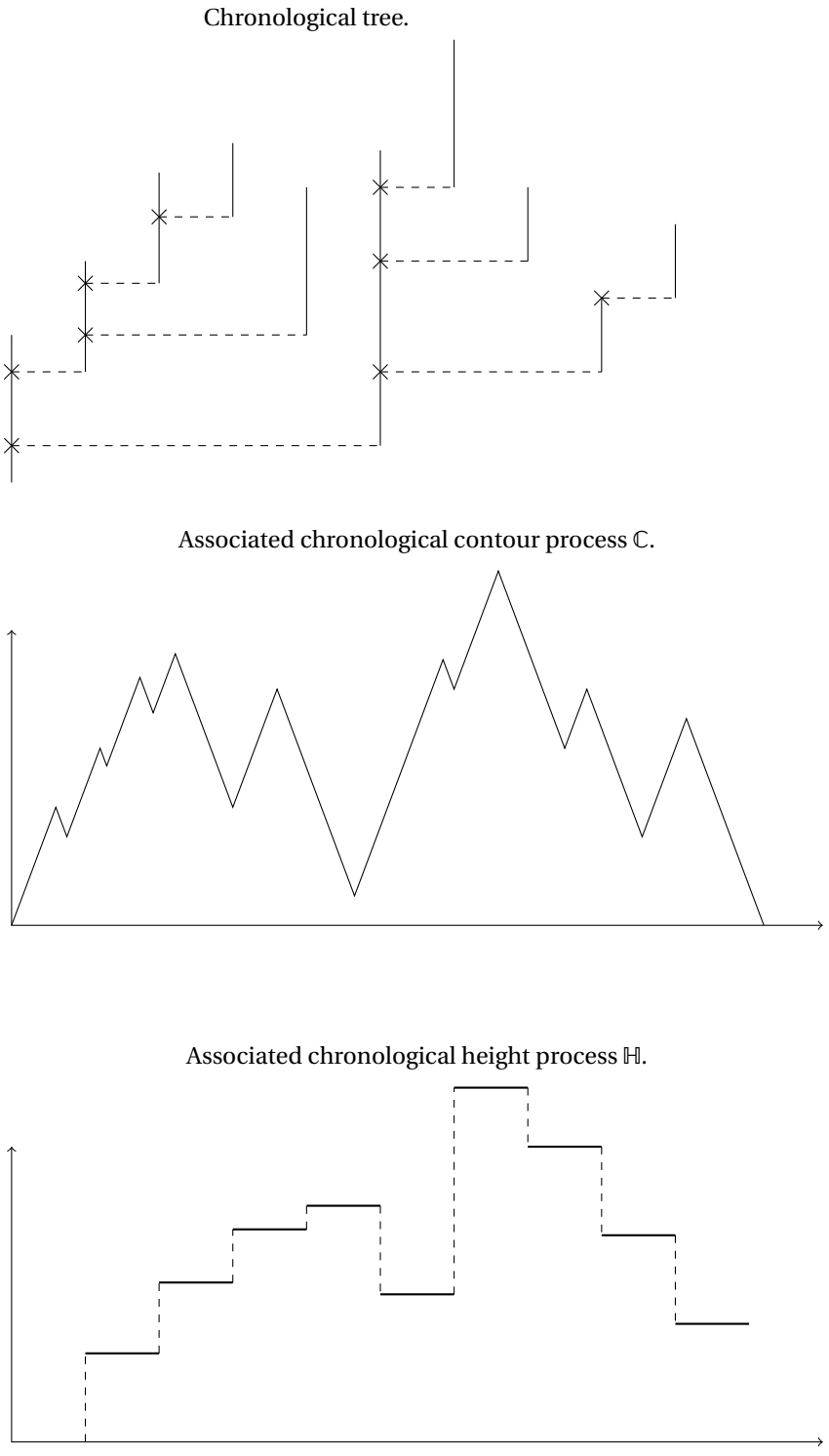


FIGURE 3. Chronological height and contour processes associated to the chronological tree constructed from the sequence of sticks of Figure 1.

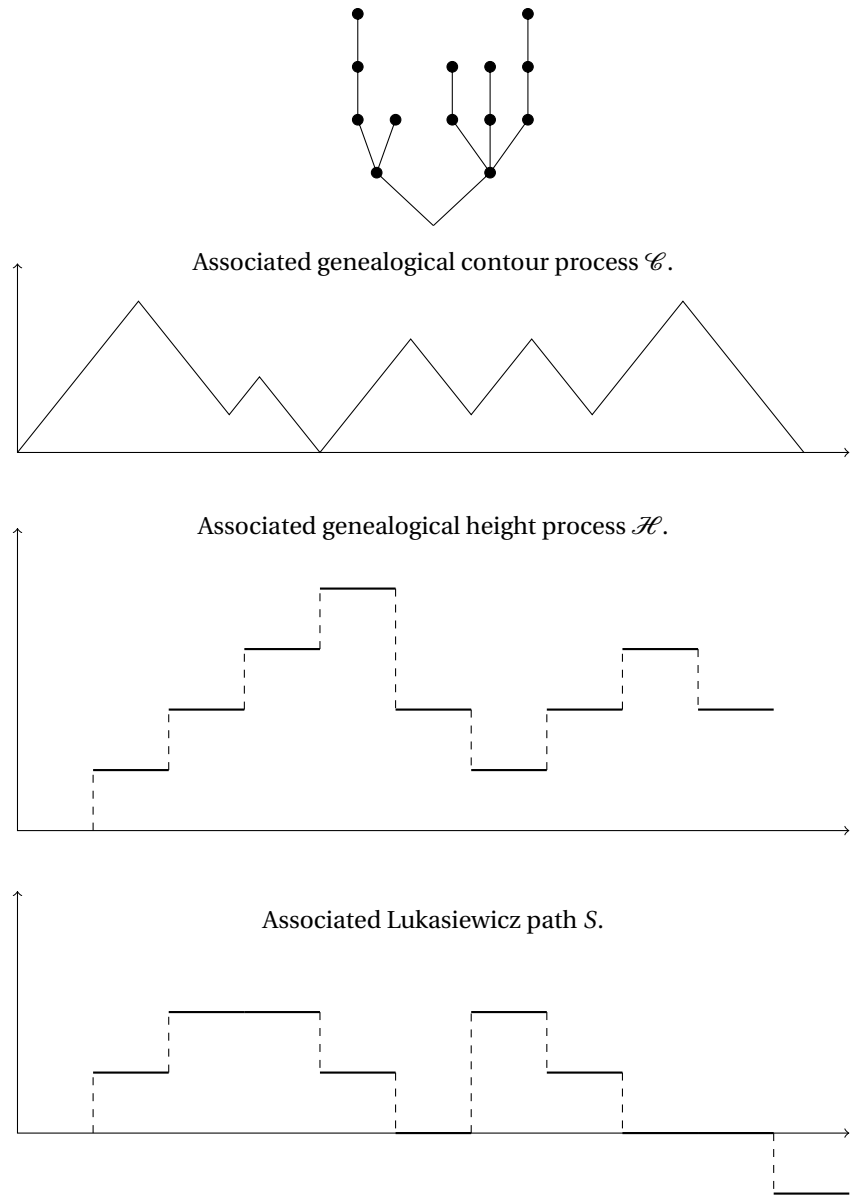
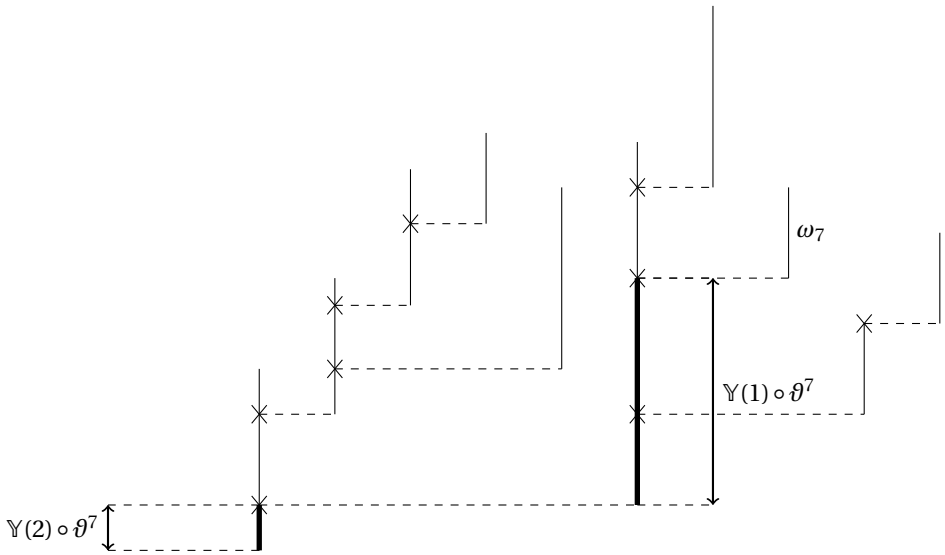
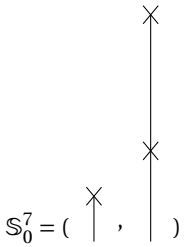


FIGURE 4. The genalogical tree of the chronological tree of the previous figure, together with the genealogical processes S, $\mathscr{H}$ and $\mathscr{C}$. The genealogical tree is obtained by applying the mapping $\mathscr{G}$ to the initial sequence of sticks, which amounts to resizing all the sticks to unit size and putting all the atoms at one. The (genealogical) height and contour processes are then obtained as before, but from the genealogical tree.



Chronological tree with the spine of the 7th individual in thick lines.



Value of the spine process at time $n = 7$.

FIGURE 5. Illustration of the random variables $\mathbb{Y}(k)\circ\vartheta^n$ and of the spine process $\mathbb{S}_0^n$: the figure presents these objects for $n = 7$. The 7th individual has two ancestors, so the spine process at time n is of length 2 and is made, according to Proposition 2.4, of the two measures corresponding to the thick lines in this figure. The random variable $\mathbb{Y}(1)\circ\vartheta^n = \mathbb{S}_0^n(2)$ records the part of the life of the first ancestor that is currently or has not been visited yet, $\mathbb{Y}(2)\circ\vartheta^n = \mathbb{S}_0^n(1)$ the part of the life of the second ancestor that is currently or has not been visited yet.

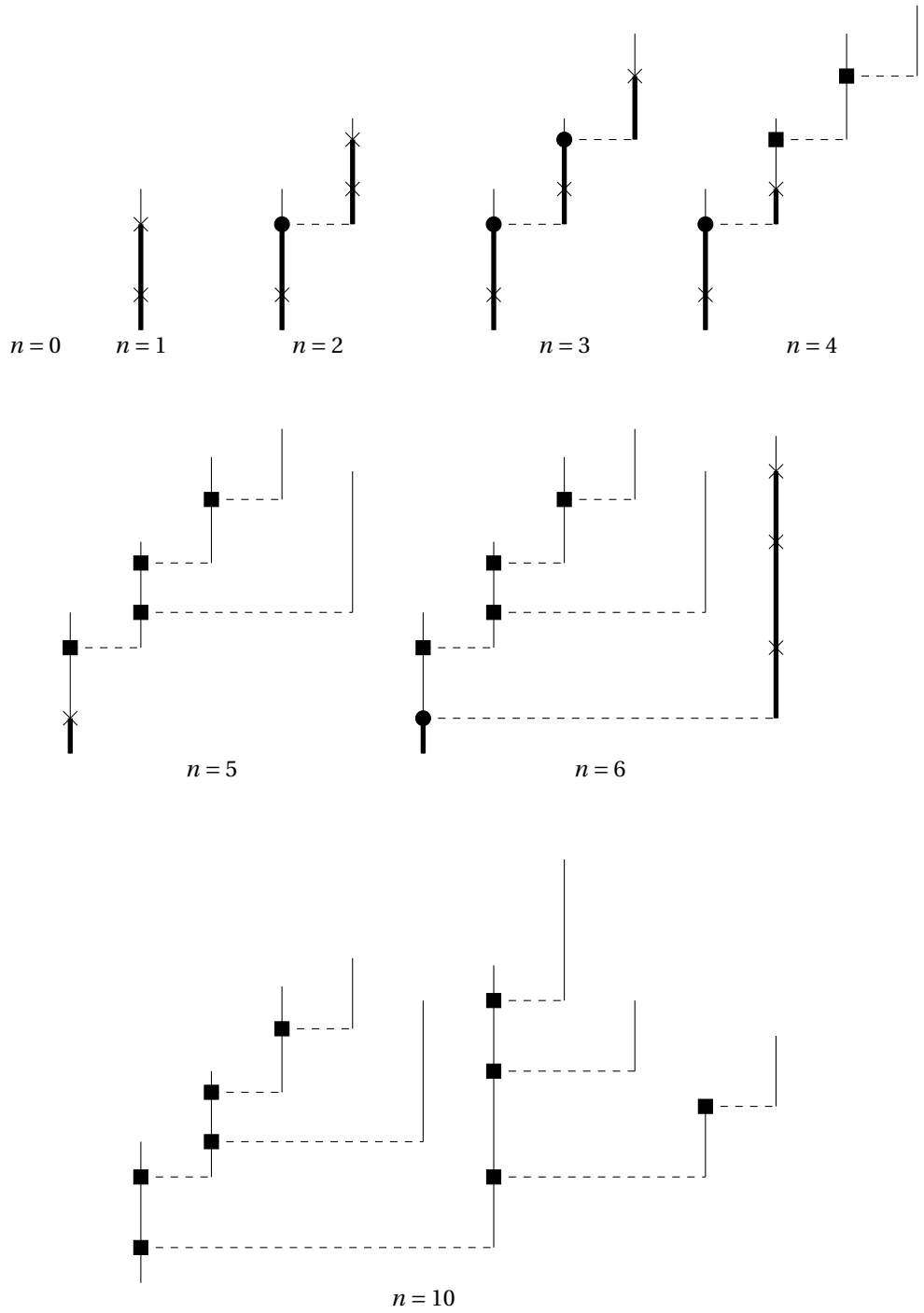


FIGURE 6. Same construction as in Figure 2, but now with the spine highlighted in thick line. This allows to differentiate three kinds of atoms:

Cross: represents a stub and corresponds to an atom on the spine whose subtree has not been explored yet;

Circle: represents an atom on the spine whose subtree is being explored;

Square: represents an atom whose subtree has been explored and that is no longer on the spine.

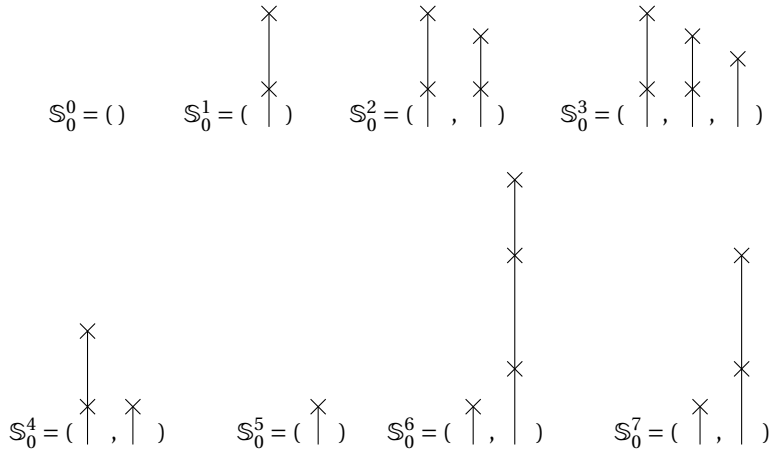


FIGURE 7. Evolution of the spine process for $n = 0, \ldots, 6$. At each time, the sequence of sticks (for instance, making up $\mathbb{S}_0^4$) is made up of the thick lines, together with their atoms, of Figure 6, and so $\mathbb{S}_0^n$ indeed encodes the spine of n.

This sequence can also be obtained by iteration of the dynamic (2.1) to the initial sequence of sticks of Figure 1: each time, either we add a new stick, or if the next stick has no atom, we remove the highest stub of the current last stick (possibly iteratively, thereby removing several sticks at once).

In the classical exploration process of Le Gall and Le Jan, one only counts the number of stubs (minus one, since we know that there must be at least one): one does not need to record their positions since they are all at the deterministic location one.

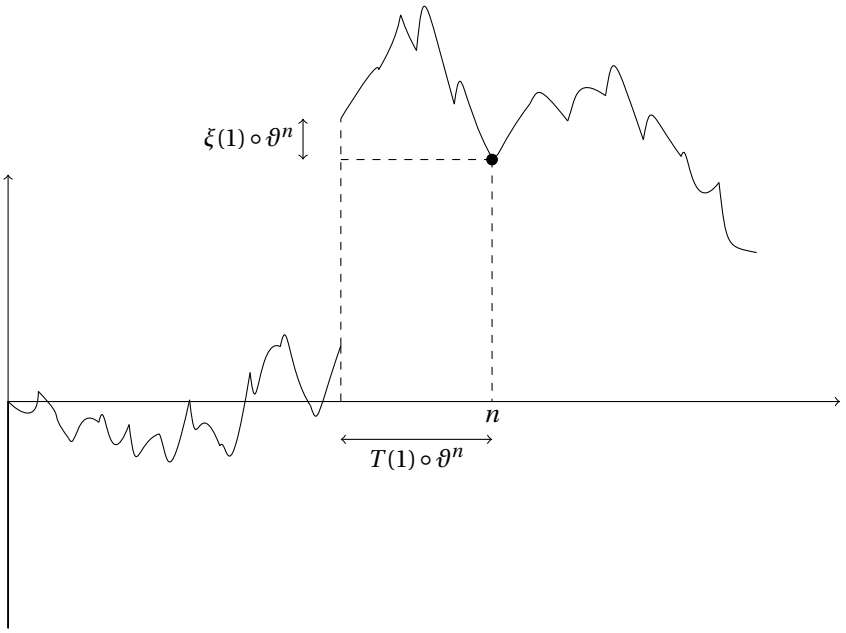


FIGURE 8. The $n - T(k)\circ\vartheta^n$ are n's ancestors. To compute the contribution of the kth ancestor to the spine of n, i.e., to compute $\mathbb{S}_0^n(\mathscr{H}(n)-k) = \mathbb{Y}(k)\circ\vartheta^n$, we do as follows:

- look at the Lukasiewicz path backward in time from n and stop at the kth record time $T(k)\circ\vartheta^n$;
- in the construction of the chronological tree, this time corresponds to the addition of the stick $\mathscr{P}_{T(k)-1}\circ\vartheta^n$;
- the overshoot (for the process forward in time) $\xi(k)\circ\vartheta^n$ represents the number of children of the kth ancestor of n that have already been explored;
- thus, the remaining contribution of this ancestor to the spine is obtained by deleting this number of atoms from $\mathscr{P}_{T(k)-1}\circ\vartheta^n$.

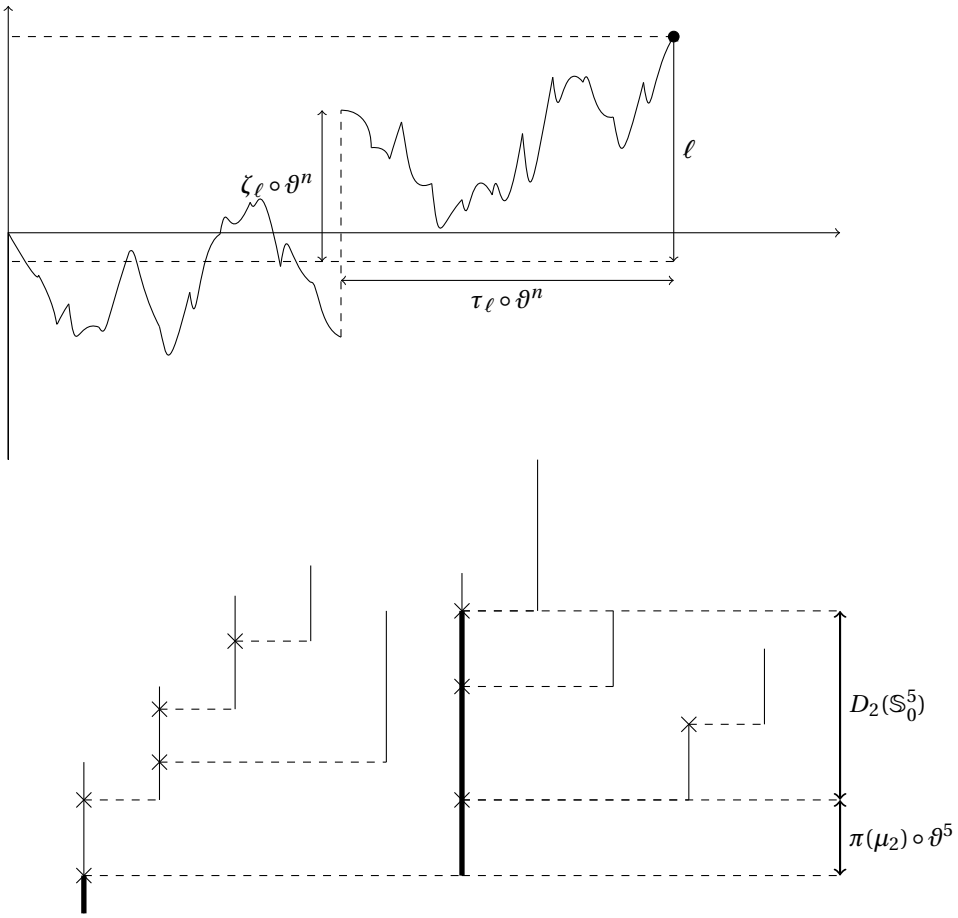


FIGURE 9. Two graphical representations when $\tau_\ell \circ \vartheta^n \leq n$ of the random variable $D_\ell(\mathbb{S}_0^n)$ defined in (5.2). The ℓth stub on the spine of n belongs to the ancestor, say α, corresponding to the jump at time $n - \tau_\ell \circ \vartheta^n$. Thus to compute the distance of the ℓth stub to the end of the spine, one needs to:

(1) add up the lengths of the first $\tau_\ell \circ \vartheta^n$ sticks on the spine process, which is what the sum $(\sum_{i:0<T(i)\leq\tau_\ell} \mathbb{Y}(i)) \circ \vartheta^n$ does;
(2) this brings us to the bottom of α's stick, and so we need to compensate for the stubs of α that have already been explored: there are $\zeta_\ell \circ \vartheta$ such stubs, and so we need to add back $A_{\zeta_\ell}(\mathscr{P}_{\tau_\ell - 1}) \circ \vartheta$ which is exactly the term $\pi(\mu_\ell) \circ \vartheta^n$.