MAGYAR TUDOMÁNYOS AKADÉMIA
Közgazdaság- és Regionális Tudományi Kutatóközpont

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MŰHELYTANULMÁNYOK DISCUSSION PAPERS

MT-DP – 2017/25

Popular Edges and Dominant Matchings

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HUNGARIAN ACADEMY OF SCIENCES - BUDAPEST, 2017

Discussion papers
MT-DP – 2017/25

Institute of Economics, Centre for Economic and Regional Studies,
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September 2017

ISBN 978-615-5754-15-9
ISSN 1785 377X

Népszerű élek és domináns párosítások

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Összefoglaló

Adott egy páros gráf G = (A u B, E) szigorúan rendezett preferencialistákkal és adott egy él e*. A népszerű élproblémában kérdésünk az, hogy létezik-e G-ben olyan népszerű párosítás, amely tartalmazza e* élt. Egy M párosítást akkor nevezünk népszerűnek, ha nincsen olyan M' párosítás, amelyben több csúcs részesíti előnyben M'-t M-mel szemben, mint fordítva. Ismert eredmény, hogy minden stabil párosítás egyben népszerű is, azonban előfordulhat, hogy egyetlen stabil párosítás sem tartalmazza e*-ot. A tanulmányban leírjuk a népszerű párosítások halmazának egy természetes alosztályát, amit domináns párosításoknak kereszteltünk. Megmutatjuk, hogy pontosan akkor létezik e*-ot tartalmazó népszerű párosítás, ha létezik e*-ot tartalmazó stabil párosítás vagy e*-ot tartalmazó domináns párosítás. Erre építve bemutatunk egy lineáris időben futó algoritmust is, amellyel az összes népszerű élt kiszámíthatjuk. Egy O(n^3) algoritmust is mutatunk annak eldöntésére, hogy létezik-e egy adott élhalmazt tartalmazó népszerű párosítás teljes preferencialisták esetén.

Kulcsszavak: népszerű párosítás, párosítás preferenciákkal ellátott gráfon, domináns párosítás

JEL-kód: C63, C78

Popular Edges and Dominant Matchings

Ágnes Cseh · **Telikepalli Kavitha**

Abstract Given a bipartite graph $G = (A \cup B, E)$ with strict preference lists and given an edge $e^* \in E$, we ask if there exists a popular matching in G that contains e^*. We call this the *popular edge* problem. A matching M is popular if there is no matching M' such that the vertices that prefer M' to M outnumber those that prefer M to M'. It is known that every stable matching is popular; however G may have no stable matching with the edge e^*. In this paper we identify another natural subclass of popular matchings called "dominant matchings" and show that if there is a popular matching that contains the edge e^*, then there is either a stable matching that contains e^* or a dominant matching that contains e^*. This allows us to design a linear time algorithm for identifying the set of popular edges. When preference lists are complete, we show an $O(n^3)$ algorithm to find a popular matching containing a given set of edges or report that none exists, where $n = |A| + |B|$.

Keywords popular matching · matching under preferences · dominant matching

Mathematics Subject Classification (2000) 05C70 · 68W40 · 05C85

1 Introduction

Our input is an instance $G = (A \cup B, E)$ of the stable marriage problem with strict and possibly incomplete preference lists, along with an edge $e^* \in E$. A matching M is *stable* if there is no *blocking pair* with respect to M, in other words, there is no

Á. Cseh was supported by the Hungarian Academy of Sciences under its Momentum Programme (LP2016-3/2016), its János Bolyai Research Scholarship and OTKA grant K108383. Part of this work was carried out while she was visiting TIFR.

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pair (a,b) such that a is either unmatched or prefers b to $M(a)$ (a's partner in M) and similarly, b is either unmatched or prefers a to $M(b)$. The problem of deciding if there exists a stable matching that contains the edge e^* is an old and well-studied problem – this was first considered by Knuth [13] in 1976 who showed that a modified version of the Gale-Shapley algorithm solves this problem. Here we consider a related problem that we call the "popular edge" problem: is there a *popular matching* in G that contains the edge e^*?

The notion of popularity, introduced by Gärdenfors [8] in 1975, is a relaxation of stability. A popular matching allows blocking edges with respect to it, however there is *global* acceptance for this matching. We make this formal now.

A vertex $u \in A \cup B$ prefers matching M to matching M' if either u is matched in M and unmatched in M' or u is matched in both and it prefers $M(u)$ to $M'(u)$. For matchings M and M' in G, let $\phi(M,M')$ be the number of vertices that prefer M to M'. If $\phi(M',M) > \phi(M,M')$ then M' is *more popular than M*.

Definition 1 A matching M is *popular* if there is no matching that is more popular than M; in other words, $\phi(M,M') \geq \phi(M',M)$ for all matchings M' in G.

Thus in an election between any pair of matchings, where each vertex casts a vote for the matching that it prefers, a popular matching never loses. Popular matchings always exist in G since every stable matching is popular [8]. It is also known that every stable matching is a minimum size popular matching [10]. As stability is stricter than popularity, it may be the case that there is no stable matching that contains the given edge e^* while there is a popular matching that contains e^*. Figure 1 has such an example.

$a_1 : b_1 \; b_2$ $\qquad$ $b_1 : a_1 \; a_2$

$a_2 : b_1$ $\qquad$ $b_2 : a_1$

Fig. 1 The top-choice of both a_1 and of a_2 is b_1; the second choice of a_1 is b_2. The preference lists of the b_i's are symmetric. There is no edge between a_2 and b_2. The matching $S = \{(a_1,b_1)\}$ is the only stable matching here, while there is another popular matching $M = \{(a_1,b_2),(a_2,b_1)\}$. Thus every edge is a popular edge here, while there is only one stable edge, namely (a_1,b_1).

Stability is a very strong condition and there are several problems, for instance, in allocating projects to students or in assigning applicants to training posts, where the total absence of blocking edges may not be necessary. However the popularity of a matching is required, otherwise the vertices could vote to replace the current matching with a more popular one. The popular edge problem has applications in such a setting where the central authority seeks to pair $a \in A$ and $b \in B$ with each other and desires a matching M such that M is popular and $(a,b) \in M$.

A first attempt to solve this problem may be to ask for a stable matching S in the subgraph obtained by deleting the endpoints of e^* from G and add e^* to S. However $S \cup \{e^*\}$ need not be popular. Figure 2 has a simple example where $e^* = (a_2,b_2)$ and the subgraph induced by a_1,a_3,b_1,b_3 has a unique stable matching $\{(a_1,b_1)\}$.

However $\{(a_1,b_1),(a_2,b_2)\}$ is not popular in G as $\{(a_1,b_3),(a_2,b_1)\}$ is more popular. Note that there *is* a popular matching $M^* = \{(a_1,b_3),(a_2,b_2),(a_3,b_1)\}$ that contains e^*.

$a_1 : b_1\ b_3$ $\quad$ $b_1 : a_2\ a_1\ a_3$
$a_2 : b_1\ b_2$ $\quad$ $b_2 : a_2$
$a_3 : b_1$ $\quad$ $b_3 : a_1$

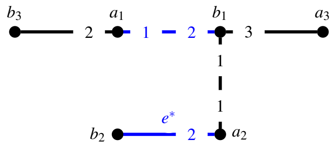

Fig. 2 Here we have $e^* = (a_2,b_2)$. The top choice for a_1 and a_2 is b_1, while b_3 is a_1's second choice and b_2 is a_2's second choice; b_1's top choice is a_2, second choice is a_1, and third choice is a_3. The vertices b_2,b_3, and a_3 have a_2,a_1, and b_1 as their only neighbors.

It would indeed be surprising if it was the rule that for every edge e^*, there is always a popular matching that can be decomposed as e^*+ a stable matching on the remaining vertices, as popularity is a far more flexible notion than stability; for instance, the set of vertices matched in every stable matching in G is the same [7] while there can be a large variation (up to a factor of 2) in the sizes of popular matchings in G. We need a larger palette than the set of stable matchings to solve the popular edge problem. We now identify another natural subclass of popular matchings called *dominant* popular matchings or dominant matchings, in short.

Definition 2 Matching M is *dominant* if M is popular, moreover, for any matching M' we have: if $|M'| > |M|$, then M is more popular than M'.

When M and M' gather the same number of votes in the election between M and M', instead of declaring these matchings as incomparable, it seems natural to regard the larger of M and M' as the *superior* matching. Dominant matchings are those popular matchings that have no superior matchings. That is, a dominant matching M gets at least as many votes as any other matching M' in an election between them and if $|M'| > |M|$, then M gets more votes than M'.

Note that a dominant matching has to be a maximum size popular matching. However not every maximum size popular matching is a dominant matching, as the example (from [10]) in Figure 3 demonstrates.

$a_1 : b_1\ b_2\ b_3$ $\quad$ $b_1 : a_1\ a_2\ a_3$
$a_2 : b_1\ b_2$ $\quad$ $b_2 : a_1\ a_2$
$a_3 : b_1$ $\quad$ $b_3 : a_1$

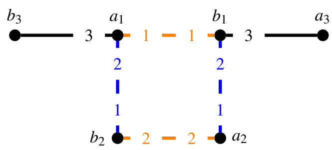

Fig. 3 The vertex b_1 is the top choice for all a_i's and b_2 is the second choice for a_1 and a_2 while b_3 is the third choice for a_1. The preference lists of the b_i's are symmetric. There are 2 maximum size popular matchings here: $M_1 = \{(a_1,b_1),(a_2,b_2)\}$ and $M_2 = \{(a_1,b_2),(a_2,b_1)\}$. The matching M_1 is not dominant since it is *not* more popular than the larger matching $M_3 = \{(a_1,b_3),(a_2,b_2),(a_3,b_1)\}$. The matching M_2 is dominant since M_2 is more popular than M_3.

Our contribution. Theorem 1 is our main result here. This enables us to solve the popular edge problem in linear time.

Theorem 1 *There exists a popular matching in $G = (A \cup B, E)$ that contains the edge e^* if and only if there exists either a stable matching in G that contains e^* or a dominant matching in G that contains e^*.*

Techniques. To prove Theorem 1, we show that any popular matching M can be decomposed as $M_0 \uplus M_1$, where M_0 is dominant in the subgraph induced by the vertices matched in M_0, and in the subgraph induced by the remaining vertices, M_1 is stable. If M contains e^*, then e^* is either in M_0 or in M_1. In the former case, we show a dominant matching in G that contains e^* and in the latter case, we show a stable matching in G that contains e^*.

We also show that every dominant matching in G can be realized as an image (under a simple and natural mapping) of a stable matching in a new graph G'. This allows us to determine in linear time if there is a dominant matching with the edge e^*. Moreover, we can find the set of popular edges in linear time. The above mapping between stable matchings in G' and dominant matchings in G can also be used to efficiently find a max-weight dominant matching in G, where each edge has a weight associated with it.

The graph G' that we construct here is closely related to the *2-level Gale-Shapley* algorithm from [12] – this algorithm computes a max-size popular matching in G in linear time. In fact, this algorithm computes a dominant matching in G as does the earlier polynomial time algorithm from [10] to find a max-size popular matching in G. Our decomposition of any popular matching M into a dominant part M_0 and a stable part M_1 is also inspired by the analysis in [12] proving the correctness of the 2-level Gale-Shapley algorithm.

When every vertex in $G = (A \cup B, E)$ has a complete preference list, then every popular matching is dominant. Thus in such instances, a max-weight popular matching can be efficiently computed and we use this to solve the "popular set" problem. In the popular set problem, we are given a set $\{e_1, e_2, \ldots, e_k\}$ of edges and we want to find a popular matching with *all* these edges, if one exists. We show an $O(n^3)$ algorithm for this problem (via max-weight popular matching) when preference lists are complete, where $n = |A| + |B|$. When preference lists are incomplete, the complexity of the popular set problem is open for $k \geq 2$.

1.1 Related Results

Stable matchings were defined by Gale and Shapley in their landmark paper [6]. The attention of the community was drawn very early to the characterization of *stable edges*: edges and sets of edges that can appear in a stable matching. In the seminal book of Knuth [13], stable edges first appeared under the term "arranged marriages". Knuth presented a linear time algorithm algorithm to find a stable matching with a given stable set of edges or report that none exists. Gusfield and Irving [9] provided a similar, simple method for the stable edge problem with the same running time.

The stable edge problem is a highly restricted case of the *max-weight stable matching problem*, where a stable matching that has the maximum edge weight among all stable matchings is sought. With the help of edge weights, various stable matching problems can be modeled, such as stable matchings with restricted edges [3] or egalitarian stable matchings [11]. A simple and elegant formulation of the stable matching polytope of $G = (A \cup B, E)$ is known [15] and using this, a max-weight stable matching can be computed in polynomial time via linear programming. When edge weights are non-negative integers, Feder [4,5] showed a max-weight stable matching algorithm with running time $O(n^2 \cdot \log(\frac{C}{n^2} + 2) \cdot \min\{n, \sqrt{C}\})$, where n is the number of vertices and C is the optimal weight computed based on the weight function represented as the sum of U-shaped weight functions at each vertex.

The popular matching problem is to decide if a given instance $G = (A \cup B, E)$ admits a popular matching or not. When ties are allowed in preference lists, this problem is NP-complete [1,2]. With strict preference lists, the popular matching problem becomes easy since every stable matching is popular [8]. The size of a stable matching in G can be as small as $|M_{\max}|/2$, where $M_{\max}$ is a maximum size matching in G. Relaxing stability to popularity yields larger matchings and it is easy to show that a largest popular matching has size at least $2|M_{\max}|/3$ in $G = (A \cup B, E)$ with strict preference lists. Efficient algorithms for computing a popular matching of maximum size were shown in [10,12]. The popular edge problem was solved by McDermid and Irving [14] for bipartite instances, where only one side has preferences and is allowed to vote.

Organization of the paper. A characterization of dominant matchings is given in Section 2. In Section 3 we show a surjective mapping between stable matchings in a larger graph G' and dominant matchings in G. Section 4 has our algorithm for the popular edge problem.

2 A characterization of dominant matchings

Let M be any matching in $G = (A \cup B, E)$. Recall that each $u \in A \cup B$ has a strict and possibly incomplete preference list and let $M(u)$ denote u's partner in M.

Definition 3 For any $u \in A \cup B$ and distinct neighbors x and y of u, define u's vote between x and y as follows.

$$\mathsf{vote}_u(x,y) = \begin{cases} + & \text{if } u \text{ prefers } x \text{ to } y \\ - & \text{if } u \text{ prefers } y \text{ to } x. \end{cases}$$

If a vertex u is unmatched in M, then $M(u)$ is undefined and this is the least preferred state for u, so $\mathsf{vote}_u(v, M(u)) = +$ for any neighbor v of u. Label each edge $e = (a,b)$ in $E \setminus M$ by the pair (α_e, β_e), where $\alpha_e = \mathsf{vote}_a(b, M(a))$ and $\beta_e = \mathsf{vote}_b(a, M(b))$, i.e., α_e is a's vote for b vs. $M(a)$ and β_e is b's vote for a vs. $M(b)$.

For any edge $(a,b) \notin M$, there are 4 possibilities for the label of edge (a,b):

- it is $(+,+)$ if (a,b) blocks M in the stable matching sense;
- it is $(+,-)$ if a prefers b to $M(a)$ while b prefers $M(b)$ to a;

- it is $(-,+)$ if a prefers $M(a)$ to b while b prefers a to $M(b)$;
- it is $(-,-)$ if both a and b prefer their respective partners in M to each other.

Let G_M be the subgraph of G obtained by deleting edges that are labeled $(-,-)$. The following theorem characterizes popular matchings.

Theorem 2 (from [10]) *A matching M is popular if and only if the following three conditions are satisfied in the subgraph G_M:*

(i) There is no alternating cycle with respect to M that contains a $(+,+)$ edge.
(ii) There is no alternating path starting from an unmatched vertex with respect to M that contains a $(+,+)$ edge.
(iii) There is no alternating path with respect to M that contains two or more $(+,+)$ edges.

Lemma 1 characterizes those popular matchings that are dominant. The "if" side of Lemma 1 was shown in [12]: it was shown that if there is no augmenting path with respect to a popular matching M in G_M then M is more popular than all larger matchings.

Here we show that the converse holds as well, i.e., if M is a popular matching such that M is more popular than all larger matchings, in other words, if M is a dominant matching, then there is no augmenting path with respect to M in G_M.

Lemma 1 *A popular matching M is dominant if and only if there is no augmenting path with respect to M in G_M.*

Proof Let M be a popular matching in G. Suppose there is an augmenting path ρ with respect to M in G_M. Let us use $M \approx M'$ to denote both matchings getting the same number of votes in an election between them, i.e., $\phi(M,M') = \phi(M',M)$. We will now show that $M \oplus \rho \approx M$. Since $M \oplus \rho$ is a larger matching than M, $M \oplus \rho \approx M$ means that M is not dominant.

Consider $M \oplus \rho$ versus M: every vertex that does not belong to the path ρ gets the same partner in both these matchings. Hence vertices outside ρ are indifferent between these two matchings. Consider the vertices on ρ. In the first place, there is no edge in $\rho \setminus M$ that is labeled $(+,+)$, otherwise that would contradict condition (ii) of Theorem 2. Since the path ρ belongs to G_M, no edge is labeled $(-,-)$ either. Hence every edge in $\rho \setminus M$ is labeled either $(+,-)$ or $(-,+)$. Note that the $+$ signs count the number of votes for $M \oplus \rho$ while the $-$ signs count the number of votes for M. Thus the number of votes for $M \oplus \rho$ equals the number of votes for M on vertices of ρ, and thus in the entire graph G. Hence $M \oplus \rho \approx M$.

Now we show the other direction: if there is no augmenting path with respect to a popular matching M in G_M then M is dominant. Let M' be a larger matching. Consider $M \oplus M'$ in G: this is a collection of alternating paths and alternating cycles and since $|M'| > |M|$, there is at least one augmenting path with respect to M here. Call this path p, running from vertex u to vertex v. Let us count the number of votes for M versus M' among the vertices of p.

No edge in p is labeled $(+,+)$ as that would contradict condition (ii) of Theorem 2, thus all the edges of M' in p are labeled $(-,+)$, $(+,-)$, or $(-,-)$. Since p

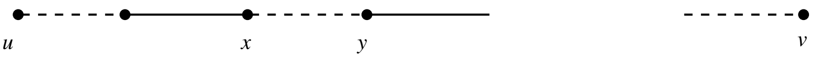

Fig. 4 The u-v augmenting path p in G where the bold edges are in M; at least one edge here (say, (x,y)) is labeled $(-,-)$.

does not exist in G_M, there is at least one edge that is labeled $(-,-)$ here (see Figure 4): thus among the vertices of p, matching M gets more votes than matching M' (recall that $+$'s are votes for M' and $-$'s are votes for M). Thus M is more popular than M' among the vertices of p.

By the popularity of M, we know that M gets at least as many votes as M' over all other paths and cycles in $M \oplus M'$; this is because if ρ is an alternating path/cycle in $M \oplus M'$ such that the number of vertices on ρ that prefer M' to M is more than the number that prefer M to M', then $M \oplus \rho$ is more popular than M, a contradiction to the popularity of M. Thus adding up over all the vertices in G, it follows that $\phi(M,M') > \phi(M',M)$. Hence M is more popular than any larger matching and so M is a dominant matching. $\square$

Corollary 1 is a characterization of dominant matchings. This follows immediately from Lemma 1 and Theorem 2.

Corollary 1 *Matching M is a dominant matching if and only if M satisfies conditions (i)-(iii) of Theorem 2 and condition (iv): there is no augmenting path with respect to M in G_M.*

3 The set of dominant matchings

In this section we show a surjective mapping from the set of stable matchings in a new instance $G' = (A' \cup B', E')$ to the set of dominant matchings in $G = (A \cup B, E)$. It will be convenient to refer to vertices in A and A' as *men* and vertices in B and B' as *women*. The construction of $G' = (A' \cup B', E')$, depicted in Figure 5, is as follows.

- Corresponding to every man $a \in A$, there will be two men a_0 and a_1 in A' and one woman $d(a)$ in B'. The vertex $d(a)$ will be referred to as the dummy woman corresponding to a. Corresponding to every woman $b \in B$, there will be exactly one woman in B' – for the sake of simplicity, we will use b to refer to this woman as well. Thus $B' = B \cup d(A)$, where $d(A) = \{d(a) : a \in A\}$ is the set of dummy women.
- Regarding the other side of the graph, $A' = A_0 \cup A_1$, where $A_i = \{a_i : a \in A\}$ for $i = 0, 1$, and vertices in A_0 are called level 0 vertices, while vertices in A_1 are called level 1 vertices.

We now describe the edge set E' of G'. For each $a \in A$, the vertex $d(a)$ has exactly two neighbors: these are a_0 and a_1; $d(a)$'s preference order is a_0 followed by a_1. The dummy woman $d(a)$ is a_1's most preferred neighbor and a_0's least preferred neighbor. The preference list of a_0 is all the neighbors of a (in a's preference order) followed by $d(a)$. The preference list of a_1 is $d(a)$ followed by the neighbors of a (in a's preference order) in G.

For any $b \in B$, its preference list in G' is level 1 neighbors in the same order of preference as in G followed by level 0 neighbors in the same order of preference as in G. For instance, if b's preference list in G is a followed by a', then b's preference list in G' is top-choice a_1, then a'_1, and then a_0, and the last-choice is a'_0. We show an example in Figure 5.

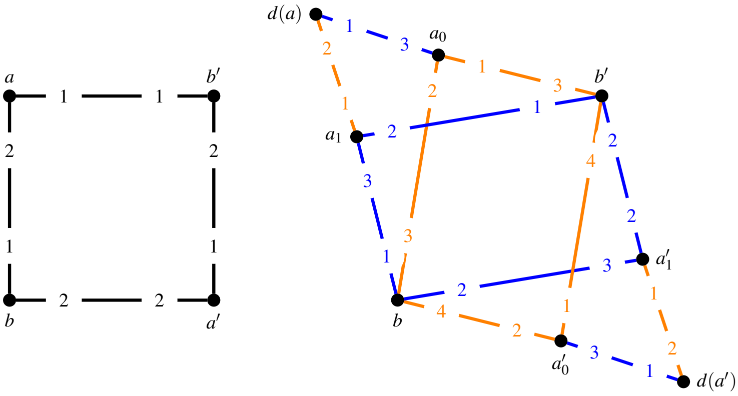

Fig. 5 The graph G' on the right corresponding to G on the left. We used blue to color edges in $(A_1 \times B) \cup (A_0 \times d(a))$ and orange to color edges in $(A_0 \times B) \cup (A_1 \times d(a))$.

We now define the mapping T : {stable matchings in G'} $\to$ {dominant matchings in G}. Let M' be any stable matching in G.

- $T(M')$ is the set of edges obtained by deleting all edges involving vertices in $d(A)$ (i.e., dummy women) from M' and replacing every edge $(a_i, b) \in M'$, where $b \in B$ and $i \in \{0,1\}$, by the edge (a,b).

It is easy to see that $T(M')$ is a valid matching in G. This is because M' has to match $d(a)$, for every $a \in A$, since $d(a)$ is the top-choice for a_1. Thus for each $a \in A$, one of a_0, a_1 has to be matched to $d(a)$. Hence at most one of a_0, a_1 is matched to a non-dummy woman b and thus $M = T(M')$ is a matching in G.

3.1 The proof that M is a dominant matching in G

This proof is similar to the proof of correctness of the maximum size popular matching algorithm in [12]. As described in Section 2, in the graph G, label each edge $e = (a,b)$ in $E \setminus M$ by the pair (α_e, β_e), where $\alpha_e \in \{+,-\}$ is a's vote for b vs. $M(a)$ and $\beta_e \in \{+,-\}$ is b's vote for a vs. $M(b)$.

- It will be useful to assign a value in $\{0,1\}$ to each $a \in A$. If $M'(a_1) = d(a)$, then $f(a) = 0$ else $f(a) = 1$. So if $a \in A$ is unmatched in M then $(a_0, d(a)) \in M'$ and so $f(a) = 1$.

– We will now define f-values for vertices in B as well. If $M'(b) \in A_1$ then $f(b) = 1$, else $f(b) = 0$. In particular, if $b \in B$ is unmatched in M' (and thus in M) then $f(b) = 0$.

Claim 1 *The following statements hold on the edge labels if $a, y \in A$ and $b, z \in B$:*

(1) If the edge (a,b) is labeled $(+,+)$, then $f(a) = 0$ and $f(b) = 1$.
(2) If (y,z) is an edge such that $f(y) = 1$ and $f(z) = 0$, then (y,z) has to be labeled $(-,-)$.

Proof We show part (1) first, see Figure 6. The edge (a,b) is labeled $(+,+)$. Let $M(a) = z$ and $M(b) = y$. Thus in a's preference list, b ranks better than z and similarly, in b's preference list, a ranks better than y. We know from the definition of our function T that $M'(z) \in \{a_0, a_1\}$ and $M'(b) \in \{y_0, y_1\}$. So there are 4 possibilities: M' contains (1) (a_0, z) and (y_0, b), (2) (a_1, z) and (y_0, b), (3) (a_1, z) and (y_1, b), (4) (a_0, z) and (y_1, b).

We know that M' has no blocking pairs in G' since it is a stable matching. In (1), the pair (a_0, b) blocks M', and in (2) and (3), the pair (a_1, b) blocks M'. Thus the only possibility is (4). That is, $M'(b) \in A_1$ and $M'(a_1) = d(a)$. In other words, $f(a) = 0$ and $f(b) = 1$.

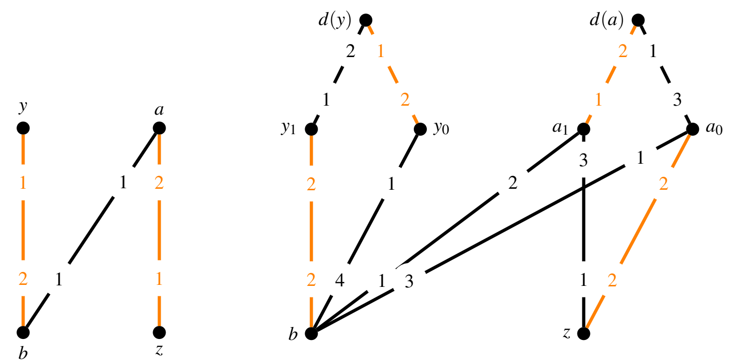

Fig. 6 The orange matching on the left is blocked by edge (a,b). The corresponding stable matching in G' is shown by orange edges on the right.

We now show part (2) of Claim 1. We are given that $f(y) = 1$, so $M'(y_0) = d(y)$. We know that $d(y)$ is y_0's last choice and y_0 is adjacent to z, thus z must have a better partner than y_0. Since we are given that $f(z) = 0$, i.e., $M'(z) \in A_0$, it follows that $M'(z) = u_0$, where u ranks better than y in z's preference list in G.

In the graph G', the vertex z prefers y_1 to u_0 since it prefers any level 1 neighbor to a level 0 neighbor. Thus y_1 is matched to a neighbor that is ranked better than z in y's preference list, i.e., $M'(y_1) = v$, where y prefers v to z. We have the edges (y,v) and (u,z) in M, thus both y and z prefer their respective partners in M to each other. Hence the edge (y,z) has to be labeled $(-,-)$. □

Lemmas 2 and 3 shown below, along with Lemma 1, imply that M is a dominant matching in G.

Lemma 2 *There is no augmenting path with respect to M in G_M.*

Proof Let $a \in A$ and $b \in B$ be unmatched in M. Then $f(a) = 1$ and $f(b) = 0$. If there is an augmenting path $\rho = \langle a, \cdots, b\rangle$ with respect to M in G_M, then in ρ we move from a man whose f-value is 1 to a woman whose f-value is 0. Thus there have to be two consecutive vertices $y \in A$ and $z \in B$ on ρ such that $f(y) = 1$ and $f(z) = 0$. However part (2) of Claim 1 tells us that such an edge (y,z) has to be labeled $(-,-)$. In other words, G_M does not contain the edge (y,z) or equivalently, there is no such augmenting path ρ in G_M. $\square$

Lemma 3 *M is a popular matching in G.*

Proof We will show that M satisfies conditions (i)-(iii) of Theorem 2.

Condition (i). Consider any alternating cycle C with respect to M in G_M and let a be any vertex in C: if $f(a) = 0$ then its partner $b = M(a)$ also satisfies $f(b) = 0$ and part (2) of Claim 1 tells us that there is no edge in G_M between b and any a' such that $f(a') = 1$. Similarly, if $f(a) = 1$ then its partner $b = M(a)$ also satisfies $f(b) = 1$ and though there can be an edge (y,b) labeled $(+,+)$ incident on b, part (1) of Claim 1 tells us that $f(y) = 0$ and thus there is no way the cycle C can return to a, whose f-value is 1. Hence if G_M contains an alternating cycle C with respect to M, then all vertices in C have the same f-value. Since there can be no edge labeled $(+,+)$ between two vertices whose f-value is the same (by part (1) of Claim 1), it follows that C has no edge labeled $(+,+)$.

Condition (ii). Consider any alternating path p with respect to M in G_M and suppose first that the starting vertex in p is $a \in A$. Since a is unmatched in M, we have $f(a) = 1$ and we know from part (2) of Claim 1 that there is no edge in G_M between such a man and a woman whose f-value is 0. Thus a's neighbor is p is a woman b' such that $f(b') = 1$. Since $f(b') = 1$, its partner $a' = M(b')$ satisfies $f(a') = 1$ and part (2) of Claim 1 tells us that there is no edge in G_M between a' and any b'' such that $f(b'') = 0$. Iterating this argument we deduce that all vertices of p have f-value 1 and thus there is no edge labeled $(+,+)$ in p.

Suppose now that the starting vertex in p is $b \in B$. Since b is unmatched in M, we have $f(b) = 0$ and we again know from part (2) of Claim 1 that there is no edge in G_M between such a woman and a man whose f-value is 1. Thus b's neighbor in p is a woman a' such that $f(a') = 0$. Since $f(a') = 0$, its partner $b' = M(a')$ also satisfies $f(b') = 0$ and part (2) of Claim 1 tells us that there is no edge in G_M between b' and any a'' such that $f(a'') = 1$, thus all vertices of p have f-value 0 and thus there is no edge labeled $(+,+)$ in p.

Condition (iii). Consider any alternating path ρ with respect to M in G_M. We can assume that the starting vertex in ρ is matched in M (as condition (ii) has dealt with the case when this vertex is unmatched). Suppose the starting vertex is $a \in A$. If $f(a) = 0$ then its partner $b = M(a)$ also satisfies $f(b) = 0$ and part (2) of Claim 1 tells us that there is no edge in G_M between b and any a' such that $f(a') = 1$, thus all vertices of ρ have f-value 0 and thus there is no edge labeled $(+,+)$ in ρ. If $f(a) = 1$ then after traversing some vertices whose f-value is 1, we can encounter an edge (y,z) that is labeled $(+,+)$ where $f(z) = 1$ and $f(y) = 0$. However once we reach y, we get stuck in vertices whose f-value is 0 and thus we can see no more edges labeled $(+,+)$.

Suppose the starting vertex in ρ is $b \in B$. If $f(b) = 1$ then its partner $a = M(b)$ also satisfies $f(a) = 1$ and part (2) of Claim 1 tells us that there is no edge in G_M between a and any b' such that $f(b') = 0$, thus all vertices of ρ have f-value 1 and thus there is no edge labeled $(+,+)$ in ρ. If $f(b) = 0$ then after traversing some vertices whose f-value is 0, we can encounter an edge (y,z) labeled $(+,+)$ where $f(y) = 0$ and $f(z) = 1$. However once we reach z, we get stuck in vertices whose f-value is 1 and thus we can see no more edges labeled $(+,+)$. Thus in all cases there is at most one edge labeled $(+,+)$ in ρ. □

With this we have proved that every stable matching M' in G' projects to a popular matching $M = T(M')$ in G.

3.2 Mapping T is surjective

We now show that corresponding to any dominant matching M in G, there is a stable matching M' in G' such that $T(M') = M$. We will work in G_M, the subgraph of G obtained by deleting all edges labeled $(-,-)$. We now construct sets $A_0, A_1 \subseteq A$ and $B_0, B_1 \subseteq B$ as described in the algorithm below. These sets will be useful in constructing the matching M'.

0. Let $A_0 = B_1 = \emptyset$, $A_1 = \{\text{unmatched men in } M\}$, $B_0 = \{\text{unmatched women in } M\}$.
1. For every edge $(y,z) \in M$ that is labeled $(+,+)$ do:
   - let $A_0 = A_0 \cup \{y\}$, $B_0 = B_0 \cup \{M(y)\}$, $B_1 = B_1 \cup \{z\}$, and $A_1 = A_1 \cup \{M(z)\}$.
2. While there exists a matched man $a \notin A_0$ adjacent in G_M to a woman in B_0 do:
   - $A_0 = A_0 \cup \{a\}$ and $B_0 = B_0 \cup \{M(a)\}$.
3. While there exists a matched woman $b \notin B_1$ adjacent in G_M to a man in A_1 do:
   - $B_1 = B_1 \cup \{b\}$ and $A_1 = A_1 \cup \{M(b)\}$.

At start, all unmatched men are in A_1 and all unmatched women are in B_0. For every edge (y,z) that is labeled $(+,+)$, we add y and its partner to A_0 and B_0, respectively while z and its partner are added to B_1 and A_1, respectively. For any man a, if a is adjacent to a vertex in B_0 and a is not in A_0, then a and its partner get added to A_0 and B_0, respectively. Similarly, for any woman b, if b is adjacent to a vertex in A_1 and b is not in B_1, then b and its partner get added to B_1 and A_1, respectively.

The following observations are easy to see (refer to Figure 7). Every $a \in A_1$ has an even length alternating path in G_M to either:

(1) a man unmatched in M (by Step 0 and Step 3) or
(2) a man $M(z)$ where the woman z has an edge labeled $(+,+)$ incident on it (by Step 1 and Step 3).

Similarly, every $a \in A_0$ has an odd length alternating path in G_M to either:

(3) a woman unmatched in M (by Step 0 and Step 2) or
(4) a woman $M(y)$ where the man y has an edge labeled $(+,+)$ incident on it (by Step 1 and Step 2).

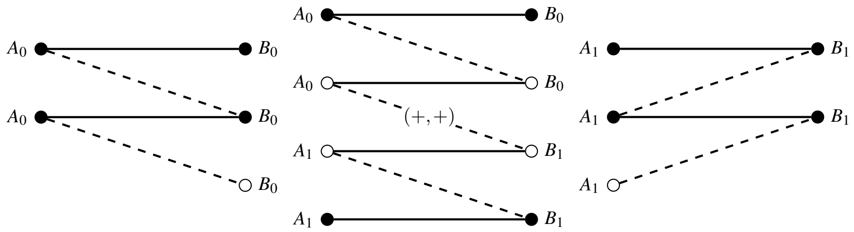

Fig. 7 Vertices get added to A_1 and A_0 by alternating paths in G_M from either unmatched vertices (first and third paths) or endpoints of edges labeled $(+,+)$ (middle path). The solid black edges are in M, and the white vertices get added to their respective sets in Steps 0 and 1.

We show the following lemma here and its proof is based on the characterization of dominant matchings in terms of conditions (i)-(iv) as given by Corollary 1. We will also use (1)-(4) observed above in our proof.

Lemma 4 $A_0 \cap A_1 = \emptyset$.

Proof We distinguish four cases here, based on the reason for a being added to A_1 (reason (1) or (2) from above) and to A_0 (reason (3) or (4) from above) simultaneously.

Case 1. Suppose a satisfies reasons (1) and (3) for its inclusion in A_1 and in A_0, respectively. So a is in A_1 because it is reachable via an even alternating path in G_M from an unmatched man u; also a is in A_0 because it is reachable via an odd length alternating path in G_M from an unmatched woman v. Then there is an augmenting path $\langle u,\dots,v\rangle$ with respect to M in G_M – a contradiction to the fact that M is dominant (by Lemma 1).

Case 2. Suppose a satisfies reasons (1) and (4) for its inclusion in A_1 and in A_0, respectively. So a is in A_1 because it is reachable via an even alternating path with respect to M in G_M from an unmatched man u; also a is in A_0 because it is reachable via an odd length alternating path in G_M from z, where edge (y,z) is labeled $(+,+)$. Then there is an alternating path with respect to M in G_M from an unmatched man u to the edge (y,z) labeled $(+,+)$ and this is a contradiction to condition (ii) of popularity.

Case 3. Suppose a satisfies reasons (2) and (3) for its inclusion in A_1 and in A_0, respectively. This case is absolutely similar to Case 2. This will cause an alternating path with respect to M in G_M from an unmatched woman to an edge labeled $(+,+)$, a contradiction again to condition (ii) of popularity.

Case 4. Suppose a satisfies reasons (2) and (4) for its inclusion in A_1 and in A_0, respectively. So a is reachable via an even length alternating path in G_M from an edge labeled $(+,+)$ and $M(a)$ is also reachable via an even length alternating path in G_M from an edge labeled $(+,+)$. If it is the same edge labeled $(+,+)$ that both a and

$M(a)$ are reachable from, then there is an alternating cycle in G_M with a $(+,+)$ edge – a contradiction to condition (i) of popularity. If these are two different edges labeled $(+,+)$, then we have an alternating path in G_M with two edges labeled $(+,+)$ – a contradiction to condition (iii) of popularity.

These four cases finish the proof that $A_0 \cap A_1 = \emptyset$. □

We now describe the construction of the matching M'. Initially $M' = \emptyset$.

- For each $a \in A_0$: add the edges $(a_0, M(a))$ and $(a_1, d(a))$ to M'.
- For each $a \in A_1$: add the edge $(a_0, d(a))$ to M' and if a is matched in M then add $(a_1, M(a))$ to M'.
- For $a \notin (A_0 \cup A_1)$: add the edges $(a_0, M(a))$ and $(a_1, d(a))$ to M'.
  (Note that the men outside $A_0 \cup A_1$ are not reachable from either unmatched vertices or edges labeled $(+,+)$ via alternating paths in G_M.)

Lemma 5 *M' is a stable matching in G'.*

Proof Suppose M' is not stable in G'. Then there are edges (u_i, v) and (a_j, b) in M', where $i, j \in \{0,1\}$, such that in the graph G', the vertices v and a_j prefer each other to u_i and b, respectively. There cannot be a blocking pair involving a dummy woman, thus the edges (u,v) and (a,b) are in M.

If $i = j$, then the pair (a,v) blocks M in G. However, from the construction of the sets A_0, A_1, B_0, B_1, we know that all the blocking pairs with respect to M are in $A_0 \times B_1$. Thus there is no blocking pair in $A_0 \times B_0$ or in $A_1 \times B_1$ with respect to M and so $i \neq j$. Since v prefers a_j to u_i in G', the only possibility is $i = 0$ and $j = 1$. It has to be the case that a prefers v to b, so there is an edge labeled $(+,-)$ between $a \in A_1$ and $v \in B_0$ (see Figure 8).

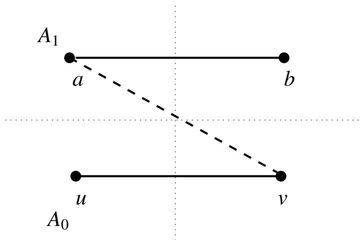

Fig. 8 If the vertex a_1 prefers v to b in G', then a prefers v to b in G; thus the edge (a,v) has to be present in G_M.

So once v got added to B_0, since a is adjacent in G_M to a vertex in B_0, vertex a satisfied Step 2 of our algorithm to construct the sets A_0, A_1, B_0, and B_1. So a would have got added to A_0 as well, i.e., $a \in A_0 \cap A_1$, a contradiction to Lemma 4. Thus there is no blocking pair with respect to M' in G'. □

This concludes the proof that every dominant matching in G can be realized as an image under T of some stable matching in G'. Thus T is surjective.

3.3 The max-weight dominant matching problem

Here we are given a weight function $w : E \to Q$ and the problem is to find a dominant matching in G whose sum of edge weights is the highest. We will use the surjective mapping T established from $\{$stable matchings in $G'\}$ to $\{$dominant matchings in $G\}$ to solve the max-weight dominant matching problem in G.

It is easy to extend w to the edge set of G'. For each edge (a,b) in G, we will assign $w(a_0,b) = w(a_1,b) = w(a,b)$ and we will set $w(a_0,d(a)) = w(a_1,d(a)) = 0$. Thus the weight of any stable matching M' in G' is the same is the weight of the dominant matching $T(M')$ in G.

Since every dominant matching M in G equals $T(M')$ for some stable matching M' in G', it follows that the max-weight dominant matching problem in G is the same as the max-weight stable matching problem in G'. Since a max-weight stable matching in G' can be computed in polynomial time, we can conclude Theorem 3 stated in Section 3.

If every vertex in $G = (A \cup B, E)$ has a complete preference list, then every popular matching M is dominant. This is because M is A-perfect (assuming $|A| \le |B|$). So every vertex in A is matched in M, thus there is no augmenting path with respect to M in G (and thus in G_M). It now follows from Lemma 1 that M is dominant. Thus we can deduce Theorem 3.

Theorem 3 *Given a graph $G = (A \cup B, E)$ with strict and complete preference lists and a weight function $w : E \to \mathbb{Q}$, the problem of computing a max-weight popular matching can be solved in polynomial time.*

We now use the above result on max-weight popular matchings to efficiently solve the *popular set* problem in complete bipartite graphs. In the popular set problem, we are given a set $\{e_1, \dots, e_k\}$ and we need to find a popular matching containing these k edges, if one exists. Else we seek a popular matching that contains as many of these edges as possible. The problem of determining if there exists a popular matching containing all these k edges can be easily posed as a max-weight popular matching problem by assigning edge weights as follows: for $1 \le i \le k$, set $w(e_i) = 1$ and set the weight of every other edge to be 0.

It is easy to see that under the above assignment of weights, a max-weight popular matching is exactly a popular matching that contains the largest number of edges in $\{e_1, \dots, e_k\}$. In particular, if the weight of this popular matching is k, then there exists a popular matching that contains all these k edges. Using the max-weight stable matching algorithm of Feder [4,5] here, we can deduce the following theorem.

Theorem 4 *The popular set problem in $G = (A \cup B, E)$ with strict and complete preference lists can be solved in $O(n^3)$ time, where $|A| + |B| = n$.*

4 The popular edge problem

In this section we show a decomposition for any popular matching in terms of a stable matching and a dominant matching. We use this result to design a linear time algorithm for the *popular edge* problem. Here we are given an edge $e^* = (u,v)$ in

$G = (A \cup B, E)$ (with strict and possibly incomplete preference lists) and we would like to know if there exists a popular matching in G that contains e^*. We claim the following algorithm solves the above problem.

1. If there is a stable matching M_{e^*} in G that contains edge e^*, then return M_{e^*}.
2. If there is a dominant matching M'_{e^*} in G that contains edge e^*, then return M'_{e^*}.
3. Return "there is no popular matching that contains edge e^* in G".

Running time of the above algorithm. In step 1 of our algorithm, we have to determine if there exists a stable matching M_{e^*} in G that contains $e^* = (u, v)$. We modify the Gale-Shapley algorithm so that the woman v rejects all proposals from anyone worse than u. If the modified Gale-Shapley algorithm produces a matching M containing e^*, then it will be a man-optimal matching among stable matchings in G that contain e^*. Else no stable matching in G contains e^*. We refer the reader to [9, Section 2.2.2] for the correctness of the modified Gale-Shapley algorithm; it is based on the following fact:

– If G admits a stable matching that contains $e^* = (u, v)$, then exactly one of (i), (ii), (iii) occurs in any stable matching M of G: *(i) $e^* \in M$, (ii) v is matched to a neighbor better than u, (iii) u is matched to a neighbor better than v.*

In step 2 of our algorithm for the popular edge problem, we have to determine if there exists a dominant matching in G that contains $e^* = (u, v)$. This is equivalent to checking if there exists a stable matching in G' that contains either the edge (u_0, v) or the edge (u_1, v). This can be determined by using the same modified Gale-Shapley algorithm as given in the previous paragraph. Thus both steps 1 and 2 of our algorithm can be implemented in $O(m)$ time, where $m = |E|$.

We now show the correctness of our algorithm. Let M be a popular matching in G that contains edge e^*. We will partition M into two sets M_0 and M_1 to show that there is either a stable matching or a dominant matching that contains e^*. As before, label each edge $e = (a, b)$ outside M by the pair of votes (α_e, β_e), where α_e is a's vote for b vs. $M(a)$ and β_e is b's vote for a vs. $M(b)$.

We run the following algorithm now – this is similar to the algorithm in the previous section (where we showed T to be surjective) to build the subsets A_0, A_1 of A and B_0, B_1 of B, except that all the sets A_0, A_1, B_0, B_1 are initialized to empty sets here.

0. Initialize $A_0 = A_1 = B_0 = B_1 = \emptyset$.
1. For every edge $(a, b) \in M$ that is labeled $(+, +)$:
   – let $A_0 = A_0 \cup \{a\}$, $B_1 = B_1 \cup \{b\}$, $A_1 = A_1 \cup \{M(b)\}$, and $B_0 = B_0 \cup \{M(a)\}$.
2. While there exists a man $a' \notin A_0$ that is adjacent in G_M to a woman in B_0 do:
   – $A_0 = A_0 \cup \{a'\}$ and $B_0 = B_0 \cup \{M(a')\}$.
3. While there exists a woman $b' \notin B_1$ that is adjacent in G_M to a man in A_1 do:
   – $B_1 = B_1 \cup \{b'\}$ and $A_1 = A_1 \cup \{M(b)\}$.

All vertices added to the sets A_0 and B_1 are matched in M – otherwise there would be an alternating path from an unmatched vertex to an edge labeled $(+, +)$ and this contradicts condition (ii) of popularity of M (see Theorem 2). Note that every vertex in A_1 is reachable via an even length alternating path with respect to M in G_M from some man $M(b)$ whose partner b has an edge labeled $(+,+)$ incident on it. Similarly, every vertex in A_0 is reachable via an odd length alternating path with respect to M in G_M from some woman $M(a)$ whose partner a has an edge labeled $(+,+)$ incident on it. The proof of Case 4 of Lemma 4 shows that $A_0 \cap A_1 = \emptyset$.

We have $B_1 = M(A_1)$ and $B_0 = M(A_0)$ (see Figure 9). All edges labeled $(+,+)$ are in $A_0 \times B_1$ (from our algorithm) and all edges in $A_1 \times B_0$ have to be labeled $(-,-)$ (otherwise we would contradict either condition (i) or (iii) of popularity of M).

Let $A' = A_0 \cup A_1$ and $B' = B_0 \cup B_1$. Let M_0 be the matching M restricted to $A' \cup B'$. The matching M_0 is popular on $A' \cup B'$. Suppose not and there is a matching N_0 on $A' \cup B'$ that is more popular. Then the matching $N_0 \cup (M \setminus M_0)$ is more popular than M, a contradiction to the popularity of M. Since M_0 matches all vertices in $A' \cup B'$, it follows that M_0 is dominant on $A' \cup B'$.

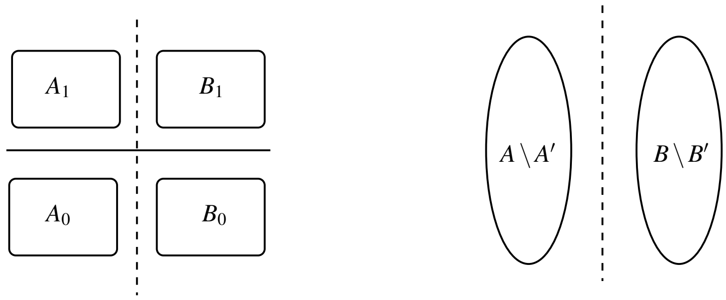

Fig. 9 M_0 is the matching M restricted to $A' \cup B'$. All unmatched vertices are in $(A \setminus A') \cup (B \setminus B')$.

Let $M_1 = M \setminus M_0$ and let $Y = A \setminus A'$ and $Z = B \setminus B'$. The matching M_1 is stable on $Y \cup Z$ as there is no edge labeled $(+,+)$ in $Y \times Z$ (all such edges are in $A_0 \times B_1$ by Step 1 of our algorithm above).

The subgraph G_M contains no edge in $A_1 \times Z$ – otherwise such a woman $z \in Z$ should have been in B_1 (by Step 3 of the algorithm above) and similarly, G_M contains no edge in $Y \times B_0$ – otherwise such a man $y \in Y$ should have been in A_0 (by Step 2 of this algorithm). We will now show Lemmas 6 and 7. These lemmas prove the correctness of our algorithm.

Lemma 6 *If $e^* \in M_0$ then there exists a dominant matching in G that contains e^*.*

Proof Let H be the induced subgraph of G on $Y \cup Z$. We will transform the stable matching M_1 in H to a dominant matching M_1^* in H. We do this by computing a stable matching in the graph $H' = (Y' \cup Z', E')$ – the definition of H' (with respect to H) is analogous to the definition of G' (with respect to G) in Section 3. So for each man $y \in Y$, we have two men y_0 and y_1 in Y' and one dummy woman $d(y)$ in Z'; the set $Z' = Z \cup d(Y)$ and the preference lists of the vertices in $Y' \cup Z'$ are exactly as given in Section 2 for the vertices in G'.

We wish to compute a dominant matching in H, equivalently, a stable matching in H'. However we will not compute a stable matching in H' from scratch since we want to obtain a dominant matching in H using M_1. So we compute a stable matching in H' by starting with the following matching in H' (this is essentially the same as M_1):

- for each edge (y,z) in M_1, include the edges (y_0,z) and $(y_1,d(y))$ in this initial matching and for each unmatched man y in M_1, include the edge $(y_0,d(y))$ in this matching. This is a feasible starting matching as there is no blocking pair with respect to this matching.

Now run the Gale-Shapley algorithm in H' with unmatched men proposing and women disposing. Note that the starting set of unmatched men is the set of all men y_1 where y is unmatched in M_1. However as the algorithm progresses, other men could also get unmatched and propose. Let M_1' be the resulting stable matching in H'. Let M_1^* be the dominant matching in H corresponding to the stable matching M_1' in H'.

Observe that M_0 is untouched by the transformation $M_1 \rightsquigarrow M_1^*$. Let $M^* = M_0 \cup M_1^*$. Since $e^* \in M_0$, the matching M^* contains e^*.

Claim 2 *M^* is a dominant matching in G.*

Proof We need to show that $M^* = M_0 \cup M_1^*$ is a dominant matching, where M_1^* is the dominant matching in H corresponding to the stable matching M_1' in H'.

Let Y_0 be the set of men $y \in Y$ such that $(y_1,d(y)) \in M_1'$ and let Y_1 be the set of men $y \in Y$ such that $(y_0,d(y)) \in M_1'$. Let Z_1 be the set of those women in Z that are matched in M_1' to men in Y_1 and let $Z_0 = Z \setminus Z_1$.

The following properties will be useful to us:

(i) If $y \in Y_1$, then $M_1^*(y)$ ranks at least as good as $M_1(y)$ in y's preference list. This is because $y \in Y_1$ and note that Y_1 is a *promoted* set when compared to Y_0. Thus y_1 gets at least as good a partner in M_1^* as in the men-optimal stable matching in H, which is at least as good as $M_1(y)$, as M_1 is a stable matching in H.

(ii) If $z \in Z_0$, then $M_1^*(z)$ ranks at least as good as $M_1(z)$ in z's preference list. This is because in the computation of the stable matching M_1', if the vertex z rejects $M_1(z)$, then it was upon receiving a better proposal from a neighbor in Y_0 (since $z \in Z_0$). Thus z's final partner in M_1', and hence in M_1^*, ranks at least as good as $M_1(z)$ in her preference list.

Claim Every edge in $(A_1 \cup Y_1) \times (B_0 \cup Z_0)$ is labeled $(-,-)$ with respect to M^*.

We already know that every edge in $A_1 \times B_0$ is labeled $(-,-)$ with respect to M_0 and as shown in part (2) of Claim 1, it is easy to see that every edge in $Y_1 \times Z_0$ is labeled $(-,-)$ with respect to M_1^*. We will now show that all edges in $(Y_1 \times B_0) \cup (A_1 \times Z_0)$ are labeled $(-,-)$ with respect to M.

- Consider any edge $(y,b) \in Y_1 \times B_0$. We know that (y,b) was labeled $(-,-)$ with respect to M. We have $M^*(b) = M_0(b) = M(b)$. Thus b prefers $M^*(b)$ to y. The man y preferred $M(y)$ to b and since $y \in Y_1$, we know from (i) above that y ranks $M_1^*(y)$ at least as good as $M_1(y) = M(y)$. Thus the edge (y,b) is labeled $(-,-)$ with respect to M^* as well.

– Consider any edge in $(a,z) \in A_1 \times Z_0$. We know that (a,z) was labeled $(-,-)$ with respect to M. We have $M^*(a) = M_0(a) = M(a)$. Thus a prefers $M^*(a)$ to z. The woman z preferred $M_1(z)$ to a and we know from (ii) above that z ranks $M_1^*(z)$ at least as good as $M_1(z)$. Thus the edge (a,z) is labeled $(-,-)$ with respect to M^* as well.

Thus we have shown that every edge in $(A_1 \cup Y_1) \times (B_0 \cup Z_0)$ is labeled $(-,-)$. We will now show the following claim.

Claim Any edge labeled $(+,+)$ with respect to M^* has to be in $(A_0 \cup Y_0) \times (B_1 \cup Z_1)$.

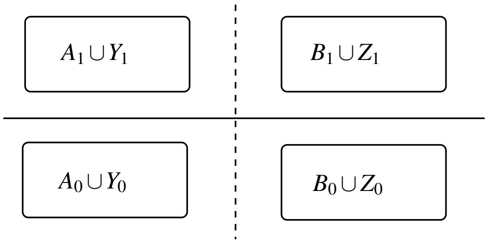

Fig. 10 All edges in $(A_1 \cup Y_1) \times (B_0 \cup Z_0)$ are labeled $(-,-)$ with respect to M^* and all edges labeled $(+,+)$ with respect to M^* are in $(A_0 \cup Y_0) \times (B_1 \cup Z_1)$.

Note that we already know that no edge in $A_i \times B_i$ is labeled $(+,+)$ with respect to M_0 and no edge in $Y_i \times Z_i$ is labeled $(+,+)$ with respect to M_1^*, for $i = 0,1$. We will now show that no edge in $\cup_{i=0}^{1}(A_i \times Z_i) \cup (Y_i \times B_i)$ is labeled $(+,+)$ (see Figure 10).

(1) Consider any edge in $(a,z) \in A_1 \times Z_1$. We know that (a,z) was labeled $(-,-)$ with respect to M. Since $M^*(a) = M_0(a) = M(a)$, the first coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.

(2) Consider any edge in $(y,b) \in Y_1 \times B_1$: there was no edge labeled $(+,+)$ with respect to M in $Y \times B_1$.
– Suppose (y,b) was labeled $(-,-)$ or $(+,-)$ with respect to M. Since $M^*(b) = M_0(b) = M(b)$, the second coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.
– Suppose (y,b) was labeled $(-,+)$ with respect to M. Since $y \in Y_1$, we know from (i) above that y ranks $M_1^*(y)$ at least as good as $M_1(y)$. Hence the first coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.

(3) Consider any edge $(y,b) \in Y_0 \times B_0$: we know that (y,b) was labeled $(-,-)$ with respect to M. Since $M^*(b) = M_0(b) = M(b)$, the second coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.

(4) Consider any edge in $(a,z) \in A_0 \times Z_0$: there was no edge labeled $(+,+)$ with respect to M in $A_0 \times Z$.

– Suppose (a,z) was labeled $(-,-)$ or $(-,+)$ with respect to M. Since $M^*(a) = M_0(a) = M(a)$, the first coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.
– Suppose (a,z) was labeled $(+,-)$ with respect to M. Since $z \in Z_0$, we know from (ii) above that z ranks $M_1^*(z)$ at least as good as $M_1(z)$. Hence the second coordinate in this edge label with respect to M^* is still $-$. Thus this edge is not labeled $(+,+)$ with respect to M^*.

Thus any edge labeled $(+,+)$ has to be in $(A_0 \cup Y_0) \times (B_1 \cup Z_1)$. This fact along with the earlier claim that all edges in $(A_1 \cup Y_1) \times (B_0 \cup Z_0)$ are labeled $(-,-)$, immediately implies that Claim 1 holds here, where we assign f-values to all vertices in $A \cup B$ as follows: if $a \in A_1 \cup Y_1$ then $f(a) = 1$ else $f(a) = 0$; similarly, if $b \in B_1 \cup Z_1$ then $f(b) = 1$ else $f(b) = 0$.

Thus if the edge (a,b) is labeled $(+,+)$, then $f(a) = 0$ and $f(b) = 1$, and if (y,z) is an edge such that $f(y) = 1$ and $f(z) = 0$, then (y,z) has to be labeled $(-,-)$. Lemmas 2 and 3 with M^* replacing M follow now (since all they need is Claim 1). We can conclude that M^* is dominant in G. Thus there is a dominant matching in G that contains e^*. $\square$

This finishes the proof of Lemma 6. $\square$

Lemma 7 *If the edge $e^* \in M_1$ then there exists a stable matching in G that contains e^*.*

Proof Here we leave M_1 untouched and transform the dominant matching M_0 on $A' \cup B'$ to a stable matching M_0' on $A' \cup B'$. We do this by *demoting* all men in A_1. That is, we run the stable matching algorithm on $A' \cup B'$ with preference lists as in the original graph G, i.e., men in A_1 are not promoted over the ones in A_0. Our starting matching is M_0 restricted to edges in $A_1 \times B_1$. Since there is no blocking pair with respect to M_0 in $A_1 \times B_1$, this is a feasible starting matching.

Now unmatched men (all those in A_0) propose in decreasing order of preference to the women in B' and when a woman receives a better proposal than what she currently has, she discards her current partner and accepts the new proposal. This may make men in A_1 single and so they too propose. This is the Gale-Shapley algorithm with the only difference that our starting matching is not empty but M_0 restricted to the edges of $A_1 \times B_1$. Let M_0' be the resulting matching on $A' \cup B'$. Let $M' = M_0' \cup M_1$. This is a matching that contains the edge e^* since $e^* \in M_1$.

Claim 3 *M' is a stable matching in G.*

Proof We will now show that $M' = M_0' \cup M_1$ is a stable matching. We already know that there is no edge labeled $(+,+)$ in $A' \times B'$ with respect to M_0' and there is no edge labeled $(+,+)$ in $Y \times Z$ with respect to M_1. Now we need to show that there is no edge labeled $(+,+)$ either in $A' \times Z$ or in $Y \times B'$.

We will first show that there is no edge labeled $(+,+)$ in $A' \times Z$, i.e., in $(A_0 \cup A_1) \times Z$.

(1) Consider any $(a,z) \in A_1 \times Z$: this edge was labeled $(-,-)$ with respect to M. Since $M'(z) = M_1(z) = M(z)$, the second coordinate of the label of this edge with respect to M' is $-$. Thus this edge cannot be labeled $(+,+)$ with respect to M'.

(2) Consider any $(a,z) \in A_0 \times Z$: there was no edge labeled $(+,+)$ with respect to M in $A' \times Z$.
– Suppose (a,z) was labeled $(+,-)$ or $(-,-)$ with respect to M. Since $M'(z) = M_1(z) = M(z)$, the second coordinate of the label of this edge with respect to M' is $-$.
– Suppose (a,z) was labeled $(-,+)$. Since $a \in A_0$, his neighbor $M'_0(a)$ is ranked at least as good as $M_0(a)$ in his preference list. This is because women in B_0 are unmatched in our starting matching and no woman $b \in B_0$ prefers any neighbor in A_1 to $M_0(b)$ (all edges in $A_1 \times B_0$ are labeled $(-,-)$ with respect to M_0). Thus in our algorithm that computes M'_0, a will get accepted either by $M_0(a)$ or a better neighbor. Hence the first coordinate of this edge label with respect to M' is still $-$.

We will now show that there is no edge labeled $(+,+)$ with respect to M' in $Y \times B'$, i.e., in $Y \times (B_0 \cup B_1)$.

(3) Consider any $(y,b) \in Y \times B_0$: the edge (y,b) was labeled $(-,-)$ with respect to M. Since $M'(y) = M_1(y) = M(y)$, the first coordinate of the label of this edge with respect to M' is $-$. Thus this edge cannot be labeled $(+,+)$ with respect to M'.
(4) Consider any $(y,b) \in Y \times B_1$: there was no edge labeled $(+,+)$ with respect to M in $Y \times B'$.
– Suppose (y,b) was labeled $(-,+)$ or $(-,-)$ with respect to M. Since $M'(y) = M_1(y) = M(y)$, the first coordinate of the label of this edge with respect to M' is $-$.
– Suppose (y,b) was labeled $(+,-)$. Since $b \in B_1$, her neighbor $M'_0(b)$ is ranked at least as good as $M_0(b)$ in her preference list. This is because our starting matching matched b to $M_0(b)$ and b would reject $M_0(b)$ only upon receiving a better proposal. Thus the second coordinate of the label of this edge with respect to M' is $-$.

This completes the proof that there is no edge labeled $(+,+)$ with respect to M' in G. In other words, M' is a stable matching in G. □

This finishes the proof of Lemma 7. □

We have thus shown the correctness of our algorithm. Theorem 5 now follows.

Theorem 5 *Given a stable marriage instance $G = (A \cup B, E)$ with strict preference lists and an edge $e^* \in E$, we can determine in linear time if there exists a popular matching in G that contains e^*.*

We remark that computing the entire set of popular edges in an instance also takes linear time. The proof of Theorem 5 shows that an edge is popular if and only if it corresponds to a stable edge either in G or in G'. It follows from [9] that all stable edges in G (and similarly, in G') can be computed in linear time.

Open problems. When vertices in $G = (A \cup B, E)$ have incomplete preference lists, the complexity of the popular set problem is open. That is, given $e_1, \ldots, e_k$, for $k \geq 2$, we would like to find a popular matching that contains all these edges, if one exists. On a similar note, if we know that an instance has only dominant popular matchings, then optimizing over the set of popular matchings is tractable. Thus a relevant problem is to identify instances where all popular matchings are dominant.

Acknowledgment. Thanks to Chien-Chung Huang for useful discussions which led to the definition of dominant matchings.

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