RATIONALITY OF HILBERT SERIES
IN NONCOMMUTATIVE INVARIANT THEORY

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ABSTRACT. It is a fundamental result in commutative algebra and invariant theory that a finitely generated graded module over a commutative finitely generated graded algebra has a rational Hilbert series, and consequently the Hilbert series of the algebra of polynomial invariants of a group of linear transformations is rational, whenever this algebra is finitely generated. This basic principle is applied here to prove rationality of Hilbert series of algebras of invariants that are neither commutative nor finitely generated. Our main focus is on linear groups acting on certain factor algebras of the tensor algebra that arise naturally in the theory of polynomial identities.

1. INTRODUCTION

A natural possible framework to develop noncommutative invariant theory is to investigate the action of a linear transformation group $G \leq GL(V)$ on the factor of the tensor algebra $K\langle V\rangle$ modulo a $GL(V)$-stable ideal I, where V is a finite dimensional vector space over a base field K of characteristic 0. We shall assume that $\dim(V) \geq 2$. A special class of $GL(V)$-stable ideals in $K\langle V\rangle$ are the T-ideals. Recall that an ideal in $K\langle V\rangle$ is a *T-ideal* if it is stable with respect to all K-algebra endomorphisms of $K\langle V\rangle$. T-ideals are associated with varieties of associative K-algebras. The *variety* $\mathfrak{R}$ defined by a system of elements $J = \{f_j\}$ in the free associative algebra $K\langle x_1, x_2, \ldots\rangle$ of countable rank consists of all associative K-algebras A such that $f_j = 0$ is a polynomial identity on A for all $f_j \in J$. Identifying the tensor algebra $K\langle V\rangle$ with the free associative algebra $K\langle x_1, \ldots, x_n\rangle$ of rank $n = \dim(V)$, the T-ideal $I = I_n(\mathfrak{R})$ is recovered as the set of n-variable identities satisfied by all algebras in a suitable variety $\mathfrak{R}$. Therefore in this case we shall write $K\langle V\rangle/I = F_n(\mathfrak{R})$, and call it a *relatively free algebra*, since it is a free object generated by n elements in the variety $\mathfrak{R}$. For a background on T-ideals and polynomial identities of associative algebras see [13].

We shall assume that $I_n(\mathfrak{R})$ is non-zero, so $F_n(\mathfrak{R})$ is not the free algebra. In the special case when $\mathfrak{R}$ is the variety of commutative algebras, we recover $S(V) = K[x_1, \ldots, x_n]$, the symmetric tensor algebra of V, which can be thought of as the algebra of polynomial functions on the dual space V^* of V. For surveys on the

study of $F_n(\mathfrak{R})^G$ see [17] and [12]. Loosely speaking, the moral of these works is that one can expect results analogous to the commutative case only when $F_n(\mathfrak{R})$ is not very far from the commutative polynomial algebra $K[x_1, \ldots, x_n]$. For example, benchmark results of commutative invariant theory are the statements that the algebras of (commutative) polynomial invariants of finite groups or more generally, reductive groups are finitely generated. Kharchenko [22] characterized the class of varieties $\mathfrak{R}$ such that $F_n(\mathfrak{R})^G$ is finitely generated for all finite groups G, and this class turned out to be rather narrow. The present authors in [11] characterized the even narrower class of varieties such that $F_n(\mathfrak{R})^G$ is finitely generated for all reductive G. Furthermore, we mention that the first named author in [9] extended the Sheppard-Todd-Chevalley Theorem by showing that $F_n(\mathfrak{R})^G \cong F_n(\mathfrak{R})$ holds for some finite group G if and only if G is a pseudoreflection group and the T-ideal $I(\mathfrak{R})$ contains $[[x_1, x_2], x_2]$, where we write $[a, b] = ab - ba$.

In contrast with that, in the present paper we deal with a general fact from commutative invariant theory that remains valid for all noncommutative relatively free algebras. Namely, by the so-called Hilbert-Serre Theorem, whenever $K[x_1, \ldots, x_n]^G$ is finitely generated, its Hilbert series is rational. Notably this holds when G is reductive or G is a maximal unipotent subgroup of a reductive subgroup of $GL(V)$. Our main result Theorem 5.1 is that the Hilbert series of the graded algebra $F_n(\mathfrak{R})^G$ is rational, provided that $G \leq GL(V)$ is a group for which the finite generation property holds for subalgebras of G-invariants in finitely generated commutative graded algebras R on which $GL(V)$ acts rationally (in particular, for a closed reductive subgroup G of $GL(V)$ or for a maximal unipotent subgroup G of a closed reductive subgroup of $GL(V)$). Consequently, although $F_n(\mathfrak{R})^G$ is rarely finitely generated even for reductive groups, it always has rational Hilbert series for G reductive or a maximal unipotent subgroup of a reductive subgroup of GL_n.

A key ingredient of our proof is that extending a result of Belov [4], Berele [5] proved that the multivariate Hilbert series of a relatively free algebra is of special form (called *nice* by him). In Section 2 (working in a more general setup than [5]), we give in Proposition 2.2 a characterization of nice rational symmetric functions in terms of formal characters of the general linear group GL_n and finitely generated modules over commutative polynomial algebras, endowed with a compatible rational GL_n-action. This characterization allows us to deduce in Section 3 the general Theorem 3.4 about rationality of the Hilbert series of the subspace of G-invariants in a graded vector space on which GL_n acts rationally such that the formal character is a nice rational function, provided that the subgroup $G \leq GL_n$ satisfies a condition formulated in terms of commutative invariant theory. In Section 4 we discuss the problem for the rationality of the Hilbert series of non-finitely generated algebras of commutative polynomial invariants. We compute the Hilbert series of a textbook variant of the famous example of Nagata of an algebra of commutative invariants which is not finitely generated. It turns out that this algebra still has a nice rational Hilbert series. We also give an example of a normal graded subalgebra of $K[x_1, x_2]$ with transcendental Hilbert series, and formulate Problem 4.3 concerning rationality of Hilbert series in commutative invariant theory. In Section 5 we apply Theorem 3.4 for relatively free algebras, and deduce by the above mentioned result of Belov and Berele our main result Theorem 5.1 on rationality of the Hilbert series of the subalgebra of invariants $F_n(\mathfrak{R})^G$ for a notable class of groups $G \leq GL_n$. In Section 6 we give examples which show that the condition that the multivariate

Hilbert series of the relatively free algebra is a nice rational function cannot be relaxed to the weaker condition that it is a rational function. We also compute the Hilbert series of the algebra of invariants of the multiplicative group of the base field acting on the 2-generated relatively free algebra in the variety generated by the 2×2 matrix algebra.

Finally, we change our topic in Section 7, and consider group actions on PI-algebras (algebras satisfying a polynomial identity), that are not necessarily relatively free algebras. In fact a finitely generated associative K-algebra is a PI-algebra if and only if it is a homomorphic image of some $F_n(\mathfrak{R})$ for some n and a proper subvariety $\mathfrak{R}$ of the variety of associative algebras. Vonessen [31] proved that if G is a linearly reductive group acting rationally on a finitely generated noetherian PI-algebra R, then the subalgebra R^G is finitely generated and noetherian. We note here that the transfer principle (a well-known tool in commutative invariant theory, see [19]) can be applied in this noncommutative setting as well, and assert in Theorem 7.2 that under the conditions in the theorem of Vonessen, R^H is finitely generated and noetherian for any subgroup H of G for which $K[G]^H$ is finitely generated, where the action of H on the coordinate ring $K[G]$ of G is induced by the right translation action of H on G.

We close the introduction by mentioning that a more general setup for developing noncommutative invariant theory that also received attention is provided by Hopf algebra actions on noncommutative rings, see for example the survey [3], the papers [20], [15], and the references therein. A typical problem considered in these works is whether the existence of a polynomial identity for the algebra of invariants implies the existence of an identity for the whole algebra.

2. A characterization of nice rational symmetric functions

Write GL_n for the general linear group $GL_n(K)$ over K, viewed as an affine algebraic group. In this paper by a *representation of GL_n* we shall always mean a *rational representation*, i.e., a K-vector space Z together with a group homomorphism $\rho : GL_n \to GL(Z)$ giving an action of GL_n on Z via linear transformations such that Z is spanned by finite dimensional GL_n-invariant subspaces and for any finite dimensional GL_n-invariant subspace W of Z, the map $GL_n \to GL(W)$, $g \mapsto \rho(g)|_W$ is a morphism of affine algebraic varieties. When Z is finite dimensional, define the *formal character* of Z as the element ch_Z in the Laurent polynomial ring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ satisfying that for any diagonal matrix $\mathrm{diag}(z_1, \ldots, z_n) \in GL_n$ with diagonal entries $z_1, \ldots, z_n \in K \setminus \{0\}$ we have the equality $\mathrm{Tr}(\rho(\mathrm{diag}(z_1, \ldots, z_n))) = \mathrm{ch}_V(z_1, \ldots, z_n)$. In fact $\mathrm{ch}_V \in \mathbb{N}_0[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$ has non-negative coefficients and is symmetric in the variables $t_1, \ldots, t_n$. Hence it belongs to the subring $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}$ of S_n-invariants in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$, where the symmetric group S_n acts by permuting the indeterminates. It is well known that for any sequence $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ there is a finite dimensional representation L_λ of GL_n whose formal character is given by the equality

$$\mathrm{ch}_{L_\lambda} \cdot \prod_{1 \leq i < j \leq n} (t_i - t_j) = \sum_{\sigma \in S_n} \mathrm{sign}(\sigma) \prod_{i=1}^{n} t_{\sigma(i)}^{\lambda_i + n - i}.$$

The set of all ch_{L_λ} forms a basis of the free $\mathbb{Z}$-module $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}$. Note that when $\lambda_n \geq 0$, the formal character ch_{L_λ} is the Schur function associated to the partition λ. Moreover, all representations of GL_n are completely reducible, and

$\{L_\lambda \mid \lambda \in \mathbb{Z}^n, \lambda_1 \geq \cdots \geq \lambda_n\}$ is a complete irredundant list of representatives of the isomorphism classes of irreducible representations of GL_n (see for example Section 9.8.1 in [29]). We shall deal with the situation when $Z = \bigoplus_{d=0}^{\infty} Z_d$ is graded, each homogeneous component Z_d is finite dimensional and GL_n-invariant. In this case we say that Z is a *graded representation of GL_n* and call the *formal character of the graded representation Z* the formal power series

$$\mathrm{ch}_Z(q) = \sum_{d=0}^{\infty} \mathrm{ch}_{Z_d} \cdot q^d \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[[q]].$$

Definition 2.1. Denote by $\mathcal{C}$ the class of graded representations Z of GL_n that have also the structure of a finitely generated graded module over the symmetric tensor algebra (polynomial algebra) $S(W)$ for some finite dimensional graded representation W of GL_n (the grading on W induces a grading on $S(W)$ in the standard way), such that the action of GL_n on Z and $S(W)$ is compatible with the $S(W)$-module structure of V (i.e., for $g \in GL_n$, $f \in S(W)$, and $m \in Z$, we have $g \cdot (fm) = (g \cdot f)(g \cdot m)$).

Proposition 2.2. *An element $f \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[[q]]$ belongs to the $\mathbb{Z}$-submodule of $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[[q]]$ generated by $\{\mathrm{ch}_V(q) \mid V \in \mathcal{C}\}$ if and only if f is of the form*

$$(1) \qquad\qquad f = \frac{P(t_1, \ldots, t_n, q)}{\displaystyle\prod_{(\alpha,k)} (1 - t_1^{\alpha_1} \cdots t_n^{\alpha_n} q^k)}$$

where $P \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}][q]$ is a polynomial in q and the product in the denominator ranges over a finite multiset of pairs (α, k) where $\alpha \in \mathbb{Z}^n$, $k \in \mathbb{N}_0$.

Proof. For $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$ and $k \in \mathbb{N}_0$ write

$$e_k^\lambda = \prod (1 - t_1^{\alpha_1} \cdots t_n^{\alpha_n} q^k)$$

where a factor $(1 - t_1^{\alpha_1} \cdots t_n^{\alpha_n} q^k)$ in the above product occurs exactly with multiplicity the coefficient of $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$ in $\mathrm{ch}_{L_\lambda} \in \mathbb{N}_0[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]$.

Suppose that f is of the form (1). For any $\alpha \in \mathbb{Z}^n$ and $k \in \mathbb{N}_0$, $(1 - t_1^{\alpha_1} \cdots t_n^{\alpha_n} q^k)$ is a factor of e_k^λ, where λ is obtained by rearranging the elements in the sequence $(\alpha_1, \ldots, \alpha_n)$ in the decreasing order. Therefore we may assume that the denominator of f in (1) is $\displaystyle\prod_{(\lambda,k) \in A} e_k^\lambda$ for a finite multiset A of pairs (λ, k) where $k \in \mathbb{N}_0$, $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Then the denominator of f is symmetric, hence the numerator P of f belongs to $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[q]$ and can be written as an integral linear combination

$$P = \sum_{(\mu,r) \in B} m_{\mu,r} \mathrm{ch}_{L_\mu}(t_1, \ldots, t_n) q^r$$

where B is a finite multiset of pairs (μ, r), $r \in \mathbb{N}_0$, $\mu \in \mathbb{Z}^n$, $\mu_1 \geq \cdots \geq \mu_n$. Take the finite dimensional graded GL_n-representation

$$W = \bigoplus W_k, \quad W_k = \bigoplus_{\{\lambda \mid (\lambda,k) \in A\}} L_\lambda.$$

By construction of W the graded GL_n-representation $S(W)$ has formal character

$$\mathrm{ch}_{S(W)}(t) = \frac{1}{\prod_{(\lambda,k) \in A} e_k^{\lambda}}.$$

For $(\mu, r) \in B$ denote by $L_\mu^{[r]}$ the finite dimensional graded GL_n-representation whose degree r component is L_μ and all other homogeneous components are zero; the formal character of this graded representation is $q^r \mathrm{ch}_{L_\mu}(t_1, \ldots, t_n)$. Consider now the graded GL_n-representation $S(W) \otimes L_\mu^{[r]}$. It is naturally a graded $S(W)$-module, it is free and is generated by the finite dimensional subspace $1 \otimes L_\mu^{[r]}$. Moreover, the GL_n-action on $S(W)$ is compatible with the action on $S(W) \otimes L_\mu^{[r]}$. Thus $S(W) \otimes L_\mu^{[r]}$ belongs to $\mathcal{C}$, and

$$\mathrm{ch}_{S(W) \otimes L_\mu^{[r]}}(q) = \mathrm{ch}_{S(W)}(q) \cdot \mathrm{ch}_{L_\mu^{[r]}}(q) = \frac{\mathrm{ch}_{L_\mu}(t_1, \ldots, t_n) q^r}{\prod_{(\lambda,k) \in A} e_k^{\lambda}}.$$

Summarizing, we obtained

$$f = \frac{\sum_{(\mu,r) \in B} m_{\mu,r} \mathrm{ch}_{L_\mu}(t_1, \ldots, t_n) q^r}{\prod_{(\lambda,k) \in A} e_k^{\lambda}} = \sum_{(\mu,r) \in B} m_{\mu,r} \mathrm{ch}_{S(W) \otimes L_\mu^{[r]}}$$

showing that f belongs to $\sum_{Z \in \mathcal{C}} \mathbb{Z} \mathrm{ch}_Z$.

To see the converse, take $Z \in \mathcal{C}$. Then Z is a finitely generated graded module over the commutative polynomial algebra $S(W)$, where $W = \bigoplus_{k=0}^{\infty} W_k$ is a finite dimensional graded representation of GL_n. By the Hilbert Syzygy Theorem, there exists a $d \leq \dim(W)$ and an exact sequence

(2) $$0 \to F_d \to F_{d-1} \to \cdots \to F_1 \to F_0 \to Z \to 0$$

of graded $S(W)$-module homomorphisms, where the F_i are of the form $S(W) \otimes U_i$ for some finite dimensional graded representations U_i of GL_n (so they are free $S(W)$-modules belonging to $\mathcal{C}$), and the maps are GL_n-equivariant. Indeed, take for U_0 a GL_n-stable direct complement of $S(W)_+ Z$, where $S(W)_+$ stands for the sum of the positive degree homogeneous components of $S(W)$. Since $S(W)_+ Z$ is a graded subspace of Z, we may also assume that U_0 is spanned by homogeneous elements. Then U_0 minimally generates Z as an $S(W)$-module and is finite dimensional by the graded Nakayama Lemma (see for example Lemma 3.5.1 in [7]). So we get an $S(W)$-module surjection $\varphi_0 : S(W) \otimes U_0 \twoheadrightarrow Z$, which is GL_n-equivariant by construction. Moreover, identifying the subspace $1 \otimes U_0$ of $S(W) \otimes U_0$ with $U_0 \subset Z$, the free module $S(W) \otimes U_0$ becomes graded, and the surjection φ_0 is a homomorphism of graded modules. Next repeat the same construction for the kernel of the homomorphism φ_0 instead of Z, to come up with U_1 and $\varphi_1 : S(W) \otimes U_1 \twoheadrightarrow \ker(\varphi_0)$. Continue in the same way. This process stops in at most $\dim(W)$ steps by the Hilbert Syzygy Theorem (see for example Section 1.3.2 in [7]), and we obtain the desired exact sequence (2). It follows that $\mathrm{ch}_M(q) = \sum_{j=0}^{d} (-1)^j \mathrm{ch}_{F_i}(q)$. Note that

$\mathrm{ch}_{F_i} = \mathrm{ch}_{S(W)}(q) \cdot \mathrm{ch}_{U_i}(q)$. Here $\mathrm{ch}_{U_i}(q) \in \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[q]$ is a polynomial in q, whereas $\mathrm{ch}_{S(W)}(q) = \prod_{(\lambda,k)} (e_k^\lambda)^{-m_{\lambda,k}}$, where $m_{\lambda,k}$ stands for the multiplicity of L_λ as a summand in W_k. It follows that

$$\mathrm{ch}_Z(q) = \frac{\sum_{i=0}^d (-1)^i \mathrm{ch}_{U_i}(q)}{\prod_{(\lambda,k)} (e_k^\lambda)^{m_{\lambda,k}}}$$

is indeed of the form (1). $\hfill\square$

Definition 2.3. Following Berele [5] (but changing a bit the setup considered by him) we call an element of $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}][[q]]$ a *nice rational function* if it is of the form (1).

Remark 2.4. (i) With this terminology Proposition 2.2 asserts that the abelian group of symmetric nice rational functions is generated by $\{\mathrm{ch}_V(q) \mid V \in \mathcal{C}\}$.

(ii) Nice rational functions not depending on the indeterminates $t_1, \ldots, t_n$ are rational functions in $\mathbb{Z}[[q]]$ with denominators which are products of binomials $1 - q^k$. Recall that the Hilbert-Serre Theorem asserts that the Hilbert series of a finitely generated graded module over a finitely generated commutative graded K-algebra R with $R_0 = K$ is a nice rational function in $\mathbb{Z}[[q]]$.

(iii) If $f(t_1, \ldots, t_n)$ is a nice rational function in the sense of [5], then making the substitution $t_i \mapsto t_i q$ we get that $f(t_1 q, \ldots, t_n q)$ is a nice rational function in the sense of our Defininition 2.3.

3. Hilbert series of fixed point subspaces

Fix now a subgroup G of GL_n. We write Z^G for the subspace of G-fixed points in a representation Z of GL_n. Define

$$D : \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n} \to \mathbb{Z}$$

as the abelian group homomorphism with $D(\mathrm{ch}_{L_\lambda}) = \dim(L_\lambda^G)$. This definition makes sense, since the ch_{L_λ} constitute a free $\mathbb{Z}$-module basis in $\mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}$. Furthermore, we keep the notation D for the abelian group homomorphism

$$D : \mathbb{Z}[t_1^{\pm 1}, \ldots, t_n^{\pm 1}]^{S_n}[[q]] \to \mathbb{Z}[[q]] \text{ with } D\left(\sum_{d=0}^\infty c_d q^d\right) = \sum_{d=0}^\infty D(c_d) q^d.$$

Since every representation of GL_n is completely reducible, for any finite dimensional GL_n-representation W we have $D(\mathrm{ch}_W) = \dim W^G$, and consequently for a graded representation $Z = \bigoplus_{d=0}^\infty Z_d$ of GL_n we have that

$$(3) \qquad\qquad D(\mathrm{ch}_Z(q)) = \sum_{d=0}^\infty \dim(Z_d^G) q^d = H(Z^G, q)$$

is the *Hilbert series of the graded vector space* Z^G.

We need the following technical lemma (we provide a proof, since though the arguments are well known, we did not find a reference where this is stated exactly in the form as below):

Lemma 3.1. *The following conditions are equivalent for a subgroup G of GL_n:*

(i) *For any $Z \in \mathcal{C}$, the subspace Z^G is finitely generated as an $S(W)^G$-module (where W is as in Definition 2.1 of the class $\mathcal{C}$).*

(ii) *For any finitely generated graded commutative K-algebra R on which GL_n acts rationally via graded K-algebra automorphisms, the subalgebra R^G of G-invariants is a finitely generated K-algebra.*

Proof. Suppose (i) holds, and take a K-algebra R as in (ii). Denote by R_+ the sum of the positive degree homogeneous components in R. Then R_+ contains a finite dimensional GL_n-submodule W such that W is spanned by homogeneous elements, and W generates R as a K-algebra. The identity map $W \to W$ extends to a surjection $S(W) \to R$ of graded K-algebras, and this surjection is G-equivariant. So in this way R_+ becomes a graded $S(W)$-module in the class $\mathcal{C}$. Take a finite $S(W)^G$-module generating system of R_+^G (it exists by assumption). This is easily seen to be a finite K-algebra generating system of R^G by the graded Nakayama Lemma (see for example Lemma 3.5.1 in [7]).

Assume next that (ii) holds, and let $Z \in \mathcal{C}$. We repeat the proof of Theorem 16.9 in [19]: make the direct sum $R = S(W) \oplus Z$ a graded K-algebra by imposing the multiplication $(r,v) \cdot (s,w) = (rs, rw + sv)$ for $r, s \in S(W)$ and $v, w \in Z$. By assumption the algebra $R^G = S(W)^G \oplus Z^G$ is finitely generated, implying in turn that the $S(W)^G$-module Z^G is finitely generated. $\qquad\square$

Remark 3.2. The equivalent conditions (i) and (ii) of Lemma 3.1 hold for a subgroup G of GL_n if and only if they hold for its Zariski closure $\bar{G}$ in GL_n. Indeed, for a given element f of an algebra R as in (ii), the stabilizer of f in GL_n is Zariski closed, since GL_n acts rationally on R. Therefore we have $R^G = R^{\bar{G}}$.

We mention two important and well-known sufficient conditions that imply condition (ii) (and hence condition (i)) of Lemma 3.1:

Proposition 3.3. *The conditions (i) and (ii) of Lemma 3.1 hold for G in each of the following two cases:*

(1) *G is a Zariski closed reductive subgroup of GL_n.*
(2) *G is the unipotent radical of a Borel subgroup of a Zariski closed reductive subgroup of GL_n.*

Proof. (1) is classical, see for example Theorem A (a) on page 3 of [19] and the reference therein. (2) is due to Hadžiev [21] and Grosshans (in arbitrary characteristic, see Theorem 9.4 in [19]). $\qquad\square$

Theorem 3.4. *Suppose that the subgroup G of GL_n satisfies condition (i) (and hence (ii)) of Lemma 3.1, and let Z be a graded representation of GL_n for which $\mathrm{ch}_Z(q)$ is a nice rational function in the sense of Definition 2.3. Then the Hilbert series $H(Z^G, q) \in \mathbb{Z}[[q]]$ of the subspace of G-invariants in Z is a rational function of the form*

$$H(Z^G, q) = \frac{P(q)}{\prod_{j=1}^{m}(1 - q^{d_j})}$$

for some positive integers $m, d_1, \ldots, d_m$ and a polynomial $P \in \mathbb{Z}[q]$.

Proof. Since $\mathrm{ch}_Z(q)$ is a nice rational function and it is also symmetric in $t_1, \ldots, t_n$, by Proposition 2.2 there exists a positive integer r, $Z_1, \ldots, Z_r \in \mathcal{C}$ and integers

$a_1, \ldots, a_r$ such that $\mathrm{ch}_Z(q) = \sum_{i=1}^{r} a_i \mathrm{ch}_{Z_i}$. By (3) we have

$$(4) \qquad H(Z^G, q) = D(\mathrm{ch}_Z(q)) = \sum_{i=1}^{r} a_i D(\mathrm{ch}_{Z_i}) = \sum_{i=1}^{r} a_i H(Z_i^G, q).$$

Here Z_i is a finitely generated graded $S(W_i)$-module (with compatible GL_n-action) for some finite dimensional graded representation W_i of GL_n, $i = 1, \ldots, r$. Since conditions (i) and (ii) of Lemma 3.1 hold for G, we have that the algebra $S(W_i)^G$ is finitely generated, and Z_i^G is a finitely generated graded $S(W_i)^G$-module for $i = 1, \ldots, r$. By the Hilbert-Serre Theorem (see for example Theorem 11.1 in [2]) the finitely generated $S(W_i)^G$-module Z_i^G has a rational Hilbert series $H(Z_i^G, q)$ of the stated form, implying in turn by (4) that $H(Z^G, q)$ is a rational function of the stated form (i.e. a nice rational function in $\mathbb{Z}[[q]]$). $\qquad\qquad\square$

4. Examples in commutative algebra

We do not know any example of an algebra of commutative polynomial invariants with Hilbert series which is not a rational function. In this section we compute the Hilbert series of a famous algebra of invariants which is not finitely generated but as it turns out, still has a nice rational Hilbert series. The first example of a finite dimensional G-module V such that the corresponding algebra $K[V]^G = K[x_1, \ldots, x_n]^G$ of invariants is not finitely generated was found by Nagata (see [27]). A simplification of Nagata's argument (resulting in an example with a group of smaller dimension) was given by Steinberg [30]. Here we shall compute the Hilbert series of this latter algebra of invariants. We refer to the books [8], [26] for facts about this algebra of invariants.

Let C be an irreducible cubic algebraic curve in the complex projective plane $\mathbb{P}^2$. The set of non-singular points in C has a well-known abelian group structure. Choose 9 distinct points $(a_{i1} : a_{i2} : a_{i3})$ in this abelian group such that their sum is not a torsion element and the matrix $(a_{ij})_{i=1,\ldots,9}^{j=1,2,3}$ has full rank 3. Let G_1 be the subgroup of $\mathbb{C}^9$ (the direct sum of 9 copies of the additive group of $\mathbb{C}$) given by

$$G_1 = \{s = (s_1, \ldots, s_9) \mid \sum_{i=1}^{9} s_i a_{ij} = 0, \quad j = 1, 2, 3\} \subset \mathbb{C}^9$$

so $G \cong \mathbb{C}^6$. Consider also the torus

$$T = \{t = (t_1, \ldots, t_9) \mid \prod_{i=1}^{9} t_i = 1\} \subset (\mathbb{C}^\times)^9$$

so $T \cong (\mathbb{C}^\times)^8$. The group $G = G_1 \times T \subset \mathbb{C}^9 \times (\mathbb{C}^\times)^9$ acts on the 18-variable polynomial algebra $S = \mathbb{C}[x_1, \ldots, x_9, y_1, \ldots, y_9]$ via

$$(s, t) \cdot x_i = t_i x_i, \quad (s, t) \cdot y_i = t_i(s_i x_i + y_i), \quad i = 1, \ldots, 9.$$

Set

$$A_j = \sum_{i=1}^{9} a_{ij} \frac{y_i}{x_i}, \quad j = 1, 2, 3, \quad D = x_1 \cdots x_9.$$

Then A_1, A_2, A_3, D generate the subfield of G-invariants in the field of fractions of S, and in fact

$$R = S^G = S \cap \mathbb{C}[A_1, A_2, A_3, D, D^{-1}].$$

Thus R has a bigrading $R = \bigoplus_{d \in \mathbb{N}_0, m \in \mathbb{Z}} R_{(d,m)}$, where

$$R_{(d,m)} = S \cap \{D^{d-m} f(A_1, A_2, A_3) \mid f \in \mathbb{C}[x, y, z] \text{ is homogeneous of degree } d\}.$$

Theorem 4.1. (i) (Proposition 2.49 in [26] or page 60 in [8]) *The $\mathbb{C}$-algebra $R = S^G$ is not finitely generated.*

(ii) (Lemma 2.50 in [26] and Lemma 4.5 in [8]) *For $d \in \mathbb{N}_0$ and $m \in \mathbb{Z}$ we have*

$$\dim R_{(d,m)} = \begin{cases} \binom{d+2}{2}, & \text{if} \quad m \leq 0; \\ \binom{d+2}{2} - 9\binom{m+1}{2}, & \text{if} \quad d \geq 3m > 0; \\ 0, & \text{if} \quad 3m > d. \end{cases}$$

Observe that $R_{(d,m)}$ is contained in the homogeneous component of degree $9(d-m)$ of the polynomial algebra of S, hence we have

$$H(S, q) = \sum_{n=0}^{\infty} c_n q^{9n} \text{ where } c_n = \sum_{m \geq -n} \dim R_{(m+n,m)}.$$

We infer from Theorem 4.1 (ii) that

$$c_n = \sum_{m=1}^{\lfloor \frac{n}{2} \rfloor} \left(\binom{n+m+2}{2} - 9\binom{m+1}{2} \right) + \sum_{m=-n}^{0} \binom{n+m+2}{2}$$

where $\lfloor \frac{n}{2} \rfloor$ stands for the lower integer part of $\frac{n}{2}$. Now we have

$$\sum_{m=-n}^{0} \binom{n+m+2}{2} = \binom{n+3}{3},$$

for $n = 2s > 0$ even we have

$$\sum_{m=1}^{s} \dim R_{(2s+m,m)} = \frac{1}{6}(10s^3 + 3s^2 - 7s) = 10\binom{s}{3} + 11\binom{s}{2} + s,$$

and for $n = 2s + 1$ odd we have

$$\sum_{m=1}^{s} \dim R_{(2s+1+m,m)} = \frac{1}{6}(10s^3 + 18s^2 + 8s) = 10\binom{s}{3} + 16\binom{s}{2} + 6s.$$

We obtain

$$\begin{aligned} \sum_{n=0}^{\infty} c_n q^n &= \sum_{n=0}^{\infty} \binom{n+3}{3} q^n \\ &+ \sum_{s=1}^{\infty} (10\binom{s}{3} + 11\binom{s}{2} + s) q^{2s} \\ &+ \sum_{s=0}^{\infty} (10\binom{s}{3} + 16\binom{s}{2} + 6s) q^{2s+1} \\ &= \frac{1}{(1-q)^4} + \frac{10q^6 + 10q^7}{(1-q^2)^4} + \frac{11q^4 + 16q^5}{(1-q^2)^3} + \frac{q^2 + 6q^3}{(1-q^2)^2} \\ &= \frac{1 + 4q + 7q^2 + 10q^3 + 10q^4 + 4q^5}{(1-q^2)^4}. \end{aligned}$$

Summarizing, we found the following:

Proposition 4.2. *We have the equality*

$$H(S^G, q) = \frac{1 + 4q^9 + 7q^{18} + 10q^{27} + 10q^{36} + 4q^{45}}{(1 - q^{18})^4}.$$

In particular, $H(S^G, q)$ is a nice rational function.

Problem 4.3. *Does there exist a G-module V such that $K[V]^G = K[x_1, \ldots, x_n]^G$ has a non-rational Hilbert series?*

There are many examples of subalgebras of the polynomial algebra in several variables which are not finitely generated. See e.g., [28] for a family of subalgebras between $K[x_1]$ and $K[x_1, x_2]$ which are not finitely generated as $K[x_1]$-algebras. It is also easy to construct a graded subalgebra of $K[x_1, x_2]$ with transcendental Hilbert series. One of the few general abstract properties that all subalgebras of invariants in the polynomial algebra do have is that they are normal (integrally closed in their field of fractions). Below we give an example showing that normality does not imply rationality of the Hilbert series.

Example 4.4. Fix a positive irrational real number α and consider the subalgebra $A = \mathrm{Span}_K\{x_1^i x_2^j \mid j < \alpha i\}$ of $K[x_1, x_2]$. The algebra A is not finitely generated (see for example [28]). The number of degree $d \geq 1$ monomials in A is $|\{i \mid i \leq d, \quad d - i < \alpha i\}|$, which is the number of integers in the interval $(\dfrac{d}{1+\alpha}, d]$. Set $\beta = 1 - \dfrac{1}{1+\alpha} = \dfrac{\alpha}{\alpha+1}$. The Hilbert series of A is

$$H(A, q) = 1 + \sum_{d=1}^{\infty}(1 + \lfloor d\beta \rfloor)q^d = \frac{1}{1-q} + \sum_{d=1}^{\infty}\lfloor d\beta \rfloor q^d,$$

where $\lfloor d\beta \rfloor$ stands for the lower integer part of $d\beta$. Suppose that this power series is algebraic over $\mathbb{Z}(q)$. Then $H_1(q) = \displaystyle\sum_{d=1}^{\infty}\lfloor d\beta \rfloor q^d$ is algebraic as well, implying in turn that $H_2(q) = (1-q)H_1(q) = \displaystyle\sum_{d=1}^{\infty}(\lfloor d\beta \rfloor - \lfloor (d-1)\beta \rfloor)q^d$ is algebraic. Since $0 < \beta < 1$, we have $c_d = \lfloor d\beta \rfloor - \lfloor (d-1)\beta \rfloor \in \{0, 1\}$, so the coefficients of $H_2(q)$ take only two integer values. By the classical theorem of Fatou [16], see Borwein and Coons [6] for an easy proof, $H_2(q)$ must be a rational function. As in the proof of Theorem 2 in [6], this implies that the sequence $(c_d \mid d = 1, 2, \ldots)$ is eventually periodic. This contradicts the assumption that β is irrational. Indeed, suppose that there exist positive integers N, p such that $c_{d+p} = c_d$ for all $d \geq N$. Set $s = |\{1 \leq i \leq p \mid c_{N+i} = 1\}|$. Then for any positive integer k we have

$$\lfloor (N + kp)\beta \rfloor - \lfloor N\beta \rfloor = \sum_{i=N+1}^{N+kp} c_i = ks.$$

We conclude that $|kp\beta - ks| \leq 1$, and hence $|\beta - \dfrac{s}{p}| \leq \dfrac{1}{kp}$ holds for all k. This leads to the contradiction $\beta = \dfrac{s}{p}$.

We claim that A is normal. Indeed, note that A is the semigroup algebra of the subsemigroup $S = \{(i,j) \mid j \leq \alpha i\}$ of the additive semigroup $\mathbb{N}_0^2$. The semigroup S is saturated in $\mathbb{N}_0^2$, i.e., if $ms \in S$ for some $s \in \mathbb{N}_0^2$ and $m \in \mathbb{N}$ then $s \in S$. Suppose that some element f from the field of fractions of A is integral over A. Then there exists a finitely generated subsemigroup $S' \subset S$ such that f is integral over the subalgebra $K[S'] = \mathrm{Span}_K\{x_1^i x_2^j \mid (i,j) \in S'\}$ of A (take all monomials that occur in the coefficients of the monic polynomial in $A[x]$ satisfied by f). So f belongs to the normalization of $K[S']$, which is $K[S'_{\mathrm{sat}}]$, where $S'_{\mathrm{sat}} = \mathbb{Q}_{\geq 0}S' \cap \mathbb{N}_0^2$ is the saturation of S' (see for example Proposition 7.25 in [25]). Now since S is saturated, we have $S'_{\mathrm{sat}} \subset S$, so $f \in K[S] = A$, showing that A is integrally closed in its field of fractions.

5. Applications in noncommutative invariant theory

Let $\mathfrak{R}$ be a proper subvariety of the variety of all associative K-algebras. Denote by $F_n(\mathfrak{R})$ the relatively free algebra of rank n in the variety $\mathfrak{R}$. Then $F_n(\mathfrak{R}) = K\langle V\rangle/I$ is the factor algebra of the tensor algebra $K\langle V\rangle$ of an n-dimensional vector space V modulo a non-zero T-ideal I. Choosing a basis $\{x_1,\ldots,x_n\}$ in V we identify GL_n with $GL(V)$ and the tensor algebra $K\langle V\rangle$ with the free associative algebra $K\langle x_1,\ldots,x_n\rangle$. This is graded as usual, so the generators x_i have degree 1. Since I is necessarily homogeneous, there is an induced grading on $F_n(\mathfrak{R})$, and $F_n(\mathfrak{R}) = \bigoplus_{d=0}^{\infty} F_n(\mathfrak{R})_d$ is a graded representation of GL_n in the sense of Section 2.

For any subgroup $G \leq GL_n$, the subalgebra $F_n(\mathfrak{R})^G = \bigoplus_{d=0}^{\infty} F_n(\mathfrak{R})_d^G$ is spanned by homogeneous elements, so it is also graded. Its Hilbert series is defined as

$$H(F_n(\mathfrak{R})^G, q) = \sum_{d=0}^{\infty} \dim(F_n(\mathfrak{R})_d^G)q^d \in \mathbb{Z}[[q]].$$

Theorem 5.1. *Suppose that condition (i) (and hence (ii)) of Lemma 3.1 holds for the subgroup G of GL_n. Then the Hilbert series $H(F_n(\mathfrak{R})^G, q)$ of the subalgebra of G-invariants in $F_n(\mathfrak{R})$ is rational of the form*

$$H(F_n(\mathfrak{R})^G, q) = \frac{P(q)}{\displaystyle\prod_{j=1}^{m}(1 - q^{d_j})}$$

for some positive integers $m, d_1, \ldots, d_m$ and a polynomial $P \in \mathbb{Z}[q]$.

Proof. To simplify notation set $R = F_n(\mathfrak{R})$. An $\mathbb{N}_0^n$-grading on the free associative algebra $K\langle x_1,\ldots,x_n\rangle$ is defined by setting the multidegree of x_i the ith standard basis vector $(0,\ldots,0,1,0,\ldots,0)$ in $\mathbb{Z}^n$. This induces a multigrading $R = \bigoplus_{\alpha\in\mathbb{N}_0^n} R_\alpha$, and the *multivariate Hilbert series* of R is

$$H(R, t_1,\ldots,t_n) = \sum_{\alpha\in\mathbb{N}_0^n} \dim(R_\alpha)t_1^{\alpha_1}\cdots t_n^{\alpha_n}.$$

Keep the notation x_i for the image of x_i under the natural surjection $K\langle x_1,\ldots,x_n\rangle \to R$. Then there is a subset $S \subset \{1,\ldots,n\}^d$ such that the monomials $\{x_{i_1}\cdots x_{i_d} \mid$

$(i_1, \ldots, i_d) \in S\}$ constitute a basis of R_α. Note that this basis of R_α consists of joint eigenvectors of the subgroup of diagonal matrices in GL_n, since we have

$$\mathrm{diag}(z_1, \ldots, z_n) \cdot x_i = z_i x_i,$$

and consequently

$$\mathrm{diag}(z_1, \ldots, z_n) \cdot x_{i_1} \cdots x_{i_d} = z_1^{\alpha_1} \cdots z_n^{\alpha_n} x_{i_1} \cdots x_{i_d}.$$

For a fixed $d \in \mathbb{N}_0$ the degree d homogeneous component R_d of R is GL_n-stable, and by the above considerations the sum of the terms of $H(R, t_1, \ldots, t_n)$ of total degree d is the formal character ch_{R_d} of R_d. It follows that substituting $t_i \mapsto t_i q$ in the multivariate Hilbert series of R we obtain

$$H(R, qt_1, \ldots, qt_n) = \mathrm{ch}_R(q),$$

the formal character of the graded GL_n-representation R. Belov [4] proved that the univariate Hilbert series $H(R, t)$ of the graded algebra R is rational. Berele strengthened this statement in Theorem 1 of [5] by showing that the multigraded Hilbert series $H(R, t_1, \ldots, t_n)$ of R is nice (in the sense of [5]), which implies that $H(R, qt_1, \ldots, qt_n)$ is a nice rational function in the sense of Definition 2.3. Summarizing, the formal character $\mathrm{ch}_R(q)$ is a nice rational function in the sense of Definition 2.3, thus the statement of our theorem now follows from Theorem 3.4.   □

Remark 5.2. In contrast with commutative invariant theory, the algebra $F_n(\mathfrak{R})^G$ is typically not finitely generated, even if condition (i) and (ii) of Lemma 3.1 hold for G. For example, by the proof of Theorem 3.1 in [11], if $\mathfrak{R}$ is not a subvariety of a variety of Lie nilpotent algebras with a given index of Lie nilpotency, then for any $n \geq 2$ there is a Zariski closed subgroup G of GL_n isomorphic to the multiplicative group of K such that $F_n(\mathfrak{R})^G$ is not finitely generated. Therefore Theorem 5.1 yields interesting examples of naturally occurring graded algebras, that are not finitely generated, but have rational Hilbert series like finitely generated commutative graded algebras.

6. EXAMPLES

Example 6.1. We provide two examples showing that in Theorem 3.4 the condition that $\mathrm{ch}_Z(q)$ is a *nice* rational function can not be relaxed to the weaker condition that $\mathrm{ch}_Z(q)$ is a rational function.

(i) Let V be 2-dimensional and $G = SL(V)$, the special linear subgroup of $GL(V)$. In this case G is a Zariski closed reductive subgroup of $GL(V)$. The tensor algebra $K\langle V \rangle$ has rational Hilbert series

$$H(K\langle V \rangle, t_1, t_2) = \frac{1}{1 - t_1 - t_2},$$

whereas by Example 5.10 in [1] we have that the subalgebra of $SL(V)$-invariants has non-rational Hilbert series

$$H(K\langle V \rangle^{SL(V)}, q) = \frac{1}{2q^2}(1 - \sqrt{1 - 4q^2}) = \sum_{n=0}^{\infty} \frac{1}{n+1}\binom{2n}{n}q^n.$$

Further examples of subalgebras of SL_2-invariants in free algebras $K\langle x_1, \ldots, x_n \rangle$ having non-rational Hilbert series can be found in [1].

(ii) Again let $V = \mathrm{Span}_K\{x, y\}$ be 2-dimensional, and take

$$G = \{x \mapsto x, \ y \mapsto cx + y \mid c \in K\} \subset GL(V).$$

In this case G is the unipotent radical of a Borel subgroup of $GL(V)$, and in fact G is isomorphic to the additive group of K. By Proposition 5.3 in [14] we have that

$$H((K\langle V\rangle)^G, q) = \sum_{p=0}^{\infty} \left(\binom{2p}{p} q^{2p} + \binom{2p+1}{p} q^{2p+1} \right)$$

$$= \frac{1 - \sqrt{1 - 4q^2}}{2q^2} \cdot \frac{1}{1 - \dfrac{1 - \sqrt{1 - 4q^2}}{2q}}.$$

In particular, $H(K\langle V\rangle^G, q)$ is not rational.

Example 6.2. Here we compute the Hilbert series of a particular algebra of invariants in a relatively free algebra along the lines of the proof of Theorem 5.1. Let $V = \mathrm{Span}_K\{x, y\}$ be 2-dimensional, let I be the T-ideal in $K\langle V\rangle$ of 2-variable identities of the algebra $K^{2\times 2}$ of 2×2 matrices over K, and $G = K^\times$, the multiplicative group of the base field acting on V via $z \cdot (ax + by) = zax + z^{-1}by$ ($z \in K^\times$, $a, b \in K$). The Hilbert series of $R = K\langle V\rangle/I$ was computed in [18]:

(5) $$H(R, t_1, t_2) = \frac{1}{(1 - t_1)(1 - t_2)} + \frac{t_1 t_2}{(1 - t_1)^2(1 - t_2)^2(1 - t_1 t_2)}.$$

Taking into account that $\mathrm{ch}_R(q) = H(R, t_1 q, t_2 q)$ we deduce from (5) the equality

(6) $$\mathrm{ch}_R(q) = \mathrm{ch}_Z(q)$$

where Z is the polynomial $GL(V)$-module

$$Z = S(V) \oplus (\overset{2}{\bigwedge}(V) \otimes S(V \oplus V) \otimes S(\overset{2}{\bigwedge}(V))).$$

Here $\overset{2}{\bigwedge}(V)$ is the exterior tensor square of V, whereas $S(V)$ and $S(V \oplus V)$ are the symmetric tensor algebras of V and $V \oplus V$. The grading is given by the decomposition into isotypic components with respect to the action of the subgroup of scalar transformations in $GL(V)$. Note that G acts trivially on $\overset{2}{\bigwedge}(V)$, hence

$$Z^G = S(V)^G \oplus (\overset{2}{\bigwedge}(V) \otimes S(V \oplus V)^G \otimes S(\overset{2}{\bigwedge}(V))).$$

Clearly, $S(V)^G = K[xy]$ is a polynomial algebra generated by a degree 2 element, and $S(V \oplus V)^G = K[x_1, y_1, x_2, y_2]^G = K[x_1 y_1, x_1 y_2, x_2 y_1, x_2 y_2]$. It is well known that the ideal of relations between the generators $x_i y_j$, $1 \le i, j \le 2$, of $S(V \oplus V)^G$ is a principal ideal generated by the relation $(x_1 y_1)(x_2 y_2) = (x_1 y_2)(x_2 y_1)$. It follows that $H(S(V \oplus V)^G, q) = \dfrac{1 - q^4}{(1 - q^2)^4}$. By (3) and (6) we obtain

$$H(R^G, q) = D(\mathrm{ch}_R(q)) = D(\mathrm{ch}_Z(q)) = H(Z^G, q) = \frac{1}{1 - q^2} + \frac{q^2(1 + q^2)}{(1 - q^2)^4}.$$

Note that R^G is not a finitely generated K-algebra. Indeed, I is contained in C^2, the square of the commutator ideal of $K\langle V\rangle$, since C^2 is known to be the T-ideal of 2-variable identities of the subalgebra of upper triangular matrices in $K^{2\times 2}$ (see [23]). Thus $K\langle V\rangle/C^2$ is a homomorphic image of R, and hence $(K\langle V\rangle/C^2)^G$ is a homomorphic image of R^G (recall that G is linearly reductive). Moreover, using

an example from [22] it was pointed out in the proof of Theorem 3.1 in [11] that $(K\langle V\rangle/C^2)^G$ is not finitely generated, implying in turn that R^G is not finitely generated.

Example 6.3. In some cases by a non-commutative variant of the Molien-Weyl theorem the operator D from Section 3 can be evaluated using the Weyl integration formula and residue calculus. An instance where the Hilbert series of the subalgebra of invariants in a relatively free algebra was computed that way occurs in [10]. Namely, take for V the space of 2×2 matrices over K, on which $G = GL_2$ acts by conjugation, and let I be the T-ideal in $K\langle V\rangle = K\langle x_1, x_2, x_3, x_4\rangle$ generated by $[[x_1, x_2], x_2]$. It was shown in Lemma 2.1 of [10] that

$$H((K\langle V\rangle/I)^{GL_2}, t) = \frac{1 + 2t^3 + t^4}{(1 - t)(1 - t^2)}.$$

In that case the algebra $K\langle V\rangle/I$ is Noetherian and hence $(K\langle V\rangle/I)^{GL_2}$ is finitely generated (see [11]). Explicit generators are also found in [10].

7. Transfer principle

In this section the characteristic of the base field K can be positive, but we assume that K is algebraically closed. Let G be a linear algebraic group with coordinate ring $K[G]$, and let H be a subgroup of G. Let R be an associative K-algebra on which G acts rationally via K-algebra automorphisms. Then $G \times H$ acts on $K[G] \otimes R$ via K-algebra automorphisms as follows:

$$(g, h) \cdot \sum_i f_i \otimes r_i = \sum_i f_i(g^{-1}xh) \otimes gr_i \text{ for } g \in G, h \in H, \ f_i \in K[G], \ r_i \in R,$$

where we write $f_i(g^{-1}xh) \in K[G]$ for the function $G \to K$, $x \mapsto f_i(g^{-1}xh)$. Using the embeddings $G \to G \times H$, $g \mapsto (g, 1)$ and $H \to G \times H$, $h \mapsto (1, h)$, the $G \times H$-action on $K[G] \otimes R$ restricts to commuting actions of G and H on $K[G] \otimes R$. Recall that the subalgebra $(K[G] \otimes R)^G$ of G-invariants is stable under the H-action, and similarly, the subalgebra $(K[G] \otimes R)^H = K[G]^H \otimes R$ of H-invariants is G-stable. Clearly we have

$$(K[G]^H \otimes R)^G = ((K[G] \otimes R)^H)^G = (K[G] \otimes R)^{G \times H} = ((K[G] \otimes R)^G)^H.$$

Lemma 7.1. *Denote by* $\Phi : (K[G] \otimes R)^G \to R$ *the restriction to* $(K[G] \otimes R)^G$ *of the linear map* $K[G] \otimes R \to R$ *given by* $f \otimes r \mapsto f(1_G)r$. *Then we have the following:*

   (i) Φ *is an H-equivariant K-algebra isomorphism.*
   (ii) Φ *restricts to a K-algebra isomorphism* $(K[G]^H \otimes R)^G \to R^H$.

Proof. This is Theorem 9.1 in [19] (in [19] R is assumed to be a commutative K-algebra, but commutativity is nowhere used in the proof, which works verbatim in the present generality). □

Theorem 7.2. *Let R be a left noetherian finitely generated K-algebra satisfying a polynomial identity, and G a linearly reductive linear algebraic group acting rationally on R via K-algebra automorphisms. Suppose that H is a subgroup of G such that $K[G]^H$ is a finitely generated K-algebra (for example, H can be a maximal unipotent subgroup of G). Then the subalgebra R^H of H-invariants in R is finitely generated and is left noetherian.*

Proof. Recall that the polynomial algebra $S[x]$ over a left noetherian ring S is left noetherian (see for example [24]). An iterated use of this yields that since $K[G]^H$ is a finitely generated commutative K-algebra and R is left noetherian, the algebra $K[G]^H \otimes R$ is left noetherian. As both $K[G]^H$ and R are finitely generated K-algebras, the same holds for $K[G]^H \otimes R$. Moreover, $K[G]^H \otimes R$ is a PI-algebra, since it satisfies the same multilinear polynomial identities as R, and every PI-algebra over any field K necessarily satisfies a multilinear polynomial identity. A theorem of Vonessen [31] asserts that the subalgebra of invariants of the linearly reductive group G acting rationally on an affine left noetherian PI-algebra is finitely generated and left noetherian. By this theorem we conclude that $(K[G]^H \otimes R)^G$ is finitely generated and is left noetherian. The statement of the theorem now follows from the isomorphism $R^H \cong (K[G]^H \otimes R)^G$ in Lemma 7.1. $\qquad\square$

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