Alma Mater Studiorum · Università di Bologna

FACOLTÀ DI SCIENZE MATEMATICHE, FISICHE E NATURALI
Corso di Laurea Magistrale in Matematica

Representation functions in Signal Processing

Tesi di Laurea in Matematica - Indirizzo applicativo

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III Sessione
Anno Accademico 2016-2017

Contents

Introduction

The primary mathematical tool used in signal theory is the Fourier transform. It is applied to a signal, in the time domain, to obtain from it a representation in another form, in the frequency domain.
The theoretical domain of signal theory is very large. It involves the study of functions, and the study of partial differential operators as well, to pass through these two field it is necessary to introduce Wigner-Ville distribution (which we name W-V for short).
W-V distribution first appeared in quantum mechanics and its use in the applications is based on the useful properties this distribution satisfies. In the applications the positivity property is of interest because it removes interferences. Interferences are caused by the cross terms present in the formula, for all $(u, \xi) \in \mathbb{R}^2$:

$$W(\alpha f + \beta g)(u, \xi) = |\alpha|^2 W(f)(u, \xi) + |\beta|^2 W(g)(u, \xi) + \alpha\overline{\beta} W(f, g)(u, \xi) + \overline{\alpha}\beta W(g, f)(u, \xi),$$

which is obtained developing the expression for the W-V distribution W into autoterms, the former two, and cross terms, the latter two. To guarantee the positivity property is a complicate problem and it is resolved by averaging the W-V distribution with smoothing kernels. As a byproduct, this smoothing operation delocalize the support of the new averaged W-V distribution resulting in a loss of resolution.
Time-frequency analysis has been developed mainly because of the need of good resolution over the information we get from signals. This means to be able to distinguish between closed events in time or frequency space. Simultaneously, that is in the time-frequency space, it is not possible to obtain very good resolution above a fixed threshold. This due to the Uncertainty

Principle, it plays a role crucial in every area of Signal Analysis and it is a constraint superimposed by the theory over all the arguments following in this document.
In the following we will find instantaneous frequency, which is a mathematical concept which pretened to represent the sound intensity varying with time, that we perceive hearing sounds for example. In giving a definition of instantaneous frequency we adopted which one uses the analytic signal associated to the signal.
The tool we will use to explore signals is the Windowed Fourier Transform $\mathscr{F}^g_{win} f(u,\xi)$ defined in the time-frequency domain (u,ξ). In this new transform the signal f has to be integrated against a reference function g, the window. Firstly, is central the inversion formula, which provides a representation of a signal in terms of an integral expansion of vectors, in the discrete case we have a sum expansion and the vector are referred to as Gabor frame. In this context the smoothness and the decay of the signal affect these one of its transform and vice versa. In order to quantify the information given by the distribution of the transform coefficients we resort to modulation spaces defining Banach spaces of functions with a given time-freqeuency behaviour for which we can use operator theory. Ambiguity functions are also covered. They are of relevant utility in the applications and we use them as cross-correlation function between two signals in one pratical example.
The representation of a signal f whit the inversion formula is very redundant and not useful for the discretized case. Our approach consists rather in using frames. We will prove when the coefficients in the frame decomposition of f are unique, and this will corresponds to the case of having a Riesz bases. In the sampling-to-reconstruction process there is the need to recover any signal f from its sampled values $< f, e_j >_{j\in J}$. This lead to the iterative frame algorithms for implementing the reconstruction process of a signal f when samples of the signal are received. As a corollary of this part we will see how to implement an algorithm to reconstruct the signal and how its rate of convergence is related to the frame bounds of the frame.

The Wavelet Transform $\mathscr{F}_{wave}(f)(u,s)$ is introduced at the end. It has signed the beginning of a new era in signal processing since the easier computability with respect to the Windowed Fourier Transform. It is not a representation in the time-frequency plane (u,ξ) of a signal f, but it retain a property of localization thanks to the formula:

$$\mathscr{F}_{wave}(f)(u,s) = \int_{-\infty}^{\infty} f(t)\frac{1}{\sqrt{s}}\overline{\psi(\frac{t-u}{s})}dt = \int_{-\infty}^{+\infty} \mathscr{F}(f(\xi))e^{-iu\xi}\mathscr{F}\psi(s\xi)d\xi,$$

which implies that the frequency-support of the signal is restricted to

$$\text{supp } \mathscr{F}\psi(s\xi) = \frac{1}{s}\text{supp } \mathscr{F}\psi.$$

Nevertheless the s parameter gives the Wavelet transform the capacity of resizing the signal's support of an s factor. This property of multiresolution at different scales s links the Wavelet transform to Hoelder spaces.

Chapter 1

Fourier transforms

We will see that there is a sort of duality between a signal f and its transform $\mathscr{F}(f)$ and all matters covered in the sequel stem from the operation of Fourier transform. In particular, it is central the Uncertainty Principle.

1.1 Heisenberg's indeterminacy

Consider f as a function in $L^2(\mathbb{R}^d)$, this means it is a signal with finite energy. It is well defined the following:

Definition 1.1. Fourier transform of $f \in L^2(\mathbb{R}^2)$ is

$$\mathscr{F}(f)(\xi) := \int_{-\infty}^{\infty} f(t) e^{-i\xi t} dt. \tag{1.1}$$

We will use the definition above in the sequel. Nevertheless there exist other equivalent forms for the Fourier transform. We will need the following result by Riemann and Lebesgue:

Lemma 1.1. *(Riemann-Lebesgue) If $f \in L^1(\mathbb{R})$ then $\mathscr{F}f$ is uniformly continuous and*

$$\lim_{\xi \to \pm\infty} |\mathscr{F}f(\xi)| = 0$$

.

Observation 1. It should be proved that the definition (1.1) is well-defined in $L^2(\mathbb{R})$. Observe that the integral in the definition above is not defined pointwise, and furthermore, it does not associate to an L^1 function an L^1 function. For an example take the function $e^{ikt} \notin L^1(\mathbb{R})$, for $k > 0$. Or for example take $f(t) = \frac{sin(t)}{t} \notin L^1(\mathbb{R})$, the integral of this function doesn't converge, it can be shown simply using the contrapositive of lemma 1.1 and the fact that f is equal to $\mathscr{F}(\frac{1}{2}\chi_{[-1,1]})$. But both the functions are in $L^2(\mathbb{R})$.

Now we give a formula which says us how to move between these spaces with the Fourier integral operator. We give the inversion formula for the Fourier operator (1.1) defined above:

Theorem 1.1. *If $f \in L^2(\mathbb{R})$:*

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathscr{F}(f)(\xi) e^{it\xi} d\xi$$

It will be needed this:

Theorem 1.2. *(Plancherel formula) If $f \in L^2(\mathbb{R})$ then the operator $\mathscr{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is unitary, this is equivalent to:*

$$\|f\|_{L^2} = \|\mathscr{F}(f)\|_{L^2}$$

and this:

Theorem 1.3. *(Parseval formula) If $f \in L^2(\mathbb{R})$ then:*

$$\|f\overline{f}\|_{L^2} = \frac{1}{2\pi} \|\mathscr{F}(f)\overline{\mathscr{F}f}\|_{L^2}$$

It is useful to consider σ_t^2 and σ_ξ^2, the standard deviation respectively of a signal f and its spectrum $\mathscr{F}(f)$:

Definition 1.2.

$$\begin{array}{lll} {\sigma_t}^2 & := \int_{-\infty}^{\infty}(t-u)^2|f(t)|^2 dt & = \|\sigma_u\|_{L^2}^2 \\ {\sigma_\xi}^2 & := \frac{1}{2\pi}\int_{-\infty}^{\infty}(\xi-\omega)^2|\mathscr{F}(f)(\xi)|^2 d\xi & = \|\sigma_\omega\|_{L^2}^2 \end{array} \tag{1.2}$$

We are ready to state the uncertainty principle. Its usual formulation is: *"a realizable signal occupies a region of area at least a fixed constant in the time-frequency plane".*

Theorem 1.4. *(Heisenberg's principle) Given a normalized function $f \in L^2(\mathbb{R})$ then the following inequality holds:*

$$\sigma_t^2\sigma_\xi^2 \geq \frac{1}{4}$$

Proof. Consider A to be the following:

$$A = \int_{-\infty}^{\infty} f(t)e^{-i\xi t}dt.$$

We define F to be the following positive function:

$$\begin{array}{ll} F(\mu) & := \int_{-\infty}^{+\infty} |\frac{1}{\sqrt{2\pi}}(\mu\xi A - \frac{d}{d\xi}A)|^2 d\xi = \\ & = \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}(\mu\xi A - \frac{d}{d\xi}A)\overline{\frac{1}{\sqrt{2\pi}}(\mu\xi f - \frac{d}{d\xi}A)}d\xi \\ & = \mu^2\frac{1}{2\pi}\int_{-\infty}^{+\infty}(\xi|A|)^2 d\xi + \mu\frac{1}{2\pi}\int_{-\infty}^{+\infty} t(A\frac{d}{d\xi}\overline{f} + \overline{A}\frac{d}{d\xi}A)dt + \quad \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\frac{d}{d\xi}A|^2 d\xi \\ & \geq 0 \end{array}$$

We rewrite it applying integration by parts and using the hypothesis $\|f\| = 2\pi$ ($|f|$ must vanish at infinity):

$$\begin{array}{l} = \mu^2\sigma_\xi^2 + \frac{1}{2\pi}\mu([\xi|A|]_{-\infty}^{\infty} - \int_{-\infty}^{+\infty}|A|dt) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\frac{d}{d\xi}A|^2 d\xi \\ = \mu^2\sigma_\xi^2 + \frac{1}{2\pi}\mu(0-2\pi) + \frac{1}{2\pi}\int_{-\infty}^{+\infty}|\frac{d}{d\xi}A|^2 d\xi \end{array} \tag{1.3}$$

Observe that

$$A' = \frac{d}{d\xi}A = -i\int_{-\infty}^{\infty} tf(t)e^{-i\xi t}dt.$$

Substituting it in (1.3) and using reverse Parseval theorem 1.3:

$$\begin{aligned} &= \mu^2\sigma_\xi^2 \quad -\mu \quad + \textstyle\int_{-\infty}^{+\infty} |A'|^2 d\xi \\ &= \mu^2\sigma_\xi^2 \quad -\mu \quad + \tfrac{1}{2\pi}\textstyle\int_{-\infty}^{+\infty} |\mathscr{F}(tf)|^2 d\xi \\ &= \mu^2\sigma_\xi^2 \quad -\mu \quad + \textstyle\int_{-\infty}^{+\infty} |tf|^2 dt. \end{aligned} \tag{1.4}$$

We have obtained a polynomial with no real root. By using the discriminant formula, Heisenberg holds:

$$1 - 4\sigma_t^2\sigma_\xi^2 \leq 0$$

and

$$\sigma_t^2\sigma_\xi^2 \geq \tfrac{1}{4}$$

□

1.2 Instantaneous frequency

To analyze the time-frequency behaviour of a signal we first consider the instantenous frequency. This concept is undefinable in the sense of the Heisenberg indeterminacy's principle. In fact, consider a signal f, defined in a small interval I_t around a fixed instant of time t, of width ϵ. We associate to it $f \cdot g$ where g is a smooth cut-off function (a function zero everywhere outside an interval) and we Fourier transform it. Then invoking the Heisenberg principle 1.4, we have:

$$\sigma_\xi^2 \geq \frac{1}{4\sigma_t^2} \approx \frac{1}{4\epsilon}.$$

Which implies that for small ϵ we can not say that f has frequencies concentrated in a fixed bandwidth in ξ. Nevertheless, for many purposes is sufficient what we can do. So let us start with the following:

Definition 1.3. The instantaneous frequency is the derivative of the amplitude related to a signal. So if $f = a(t)\cos(\varphi(t))$ then the instantenous frequency is

$$\omega(t) = \varphi'(t).$$

We will give in theorem 2.2 in the next chapter a fundamental result on instantaneous frequency which well shown the connection to the Windowed Fourier Transform. In fact, in order to achieve better satisfying results, the structure of a signal is well investigated by the use of the following Windowed Fourier Transform introduced below, which is our second new type of transform.

Chapter 2

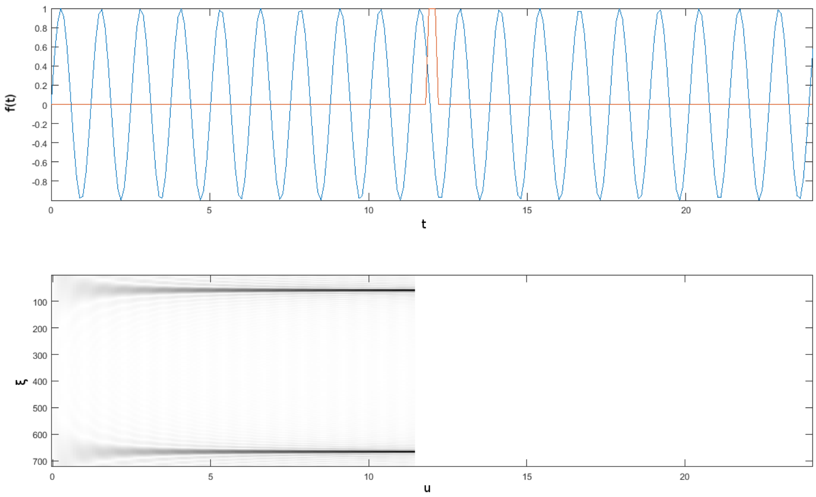
Windowed Fourier Transform

Previously, we have considered $e^{i\xi t}$ as a function with which we weighted the signal in the definition of Fourier transform (1.1). Now we consider the weights $g(t-u)e^{i\xi t}$. Furthermore, we obtain better approximations about the integral of the signal f considering the change of variable $t \to (t-u)$, a translation u in time under the integral sign.
The resulting transform $\mathscr{F}_{win} f$ we obtain so has the property that the intersecting supports of the functions under integral sign will give a more concentrated information about the signal f. This is the idea behind the following definition:

Definition 2.1. Let $L^2(\mathbb{R}) \ni g$ be an even and symmetric function with $||g|| = 1$ (g is our window). The Windowed Fourier Transfor is the:

$$\mathscr{F}_{win}(f)(u,\xi) := \int_{-\infty}^{\infty} f(t)\overline{g(t-u)}e^{-i\xi t}dt. \tag{2.1}$$

Observation 2. We now give an example of computation of the Windowed Fourier Transform of a sinusoidal signal f, with a characteristic function as window g. We will discuss about it in example 2.1. In the figure below is shown the plot of the transform obtained sliding the window along the time axis. We obtain very good results here, following Mallat:




```
freq=5;
n=24;
spec=.1;
t=[0:spec:n];
A=[];
m=length(t);
ampiezza=.3;
M=2*m;
s=zeros(1,M-amp-2);
for (i=m:1:M-1)
      s=[s(1:i-amp-1),fun((i-amp:1:(i+amp))),zeros(1,M+m-i-amp-1)];
      ss=fft(s);
      A=[A;ss];
end
```

```
subplot(2,1,1);plot(t,f(t,freq),t,g(t,n/2*ones(1,m),...
                                    ...ampiezza));axis tight;
for i=1:1:m
    subplot(2,1,2);colormap(1-gray(256));imagesc(t,(1:1:3*m-1),...
                    ...[abs(A(1:i,:));zeros(m-i,3*m-1)]');axis tight;
end
```

Next we define what is called an Heisenberg box.

Definition 2.2. We consider a fixed (u, ξ) point in the time-frequency plane, the Heisenberg box is a rectangle with edges the time spread and the frequency spread defined by (1.2).

These boxes are important because they define the area where the (support of the) window is concentrated. So boxes far from each other leave gaps in the time-frequency plane, we have that in these gaps the signal can not be well approximated by a system of functions (exempli gratia: taking the set $\{e^{i\xi t}\overline{g(t-u)}\}_{(u,\xi)}$, where we can take $g(t) = e^{-t^2}$, will gives birth to Gabor systems), so that we cannot expect that it forms, in a certain way, an acceptable frame. We will speak about these concepts below in chapter 3.
In literature the product of the frequency spread by the time spread corresponds to the resolution.

Observation 3. Scaling the window g by a dilation $g(t/s)/\sqrt{s}$, we obtain a new Heisenberg box with the same area. In fact we have from definitions

(1.2) by properties of the window g:

$$\begin{aligned}
\sigma_t^2 &= \int_{-\infty}^{\infty} t^2 |g(t)e^{i\xi t}|^2 dt = \int_{-\infty}^{\infty} t^2 |g(t)|^2 dt = \\
&= s\int_{-\infty}^{\infty} (\tfrac{t}{s})^2 |g(\tfrac{t}{s})\tfrac{1}{\sqrt{s}}|^2 d\tfrac{t}{s} \\
&= \tfrac{1}{s^2}\int_{-\infty}^{\infty} t^2 |g(\tfrac{t}{s})\tfrac{1}{\sqrt{s}}|^2 dt \\
&=: \tfrac{1}{s^2}\int_{-\infty}^{\infty} t^2 |g_s(t)|^2 dt \\
\sigma_\omega^2 &= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} (\omega-\xi)^2 |\int_{-\infty}^{\infty} g(t)e^{i\xi(t+u)}e^{-i\omega(t+u)} dt|^2 d\omega = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |e^{-iu(\omega-\xi)}\mathscr{F}_{\omega-\xi}(g)|^2 d\omega = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |e^{-iu(\omega-\xi)}\mathscr{F}_{\omega-\xi}(g)|^2 d\omega = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 |\mathscr{F}_{\omega}(g)|^2 d\omega = \\
&= \tfrac{1}{s2\pi}\int_{-\infty}^{\infty} (s\omega)^2 |\sqrt{s}\mathscr{F}_{s\omega}(g)|^2 d(s\omega) \\
&= \tfrac{s^2}{2\pi}\int_{-\infty}^{\infty} (\omega)^2 |\mathscr{F}(\tfrac{1}{\sqrt{s}}g(\tfrac{t}{s}))|^2 d(\omega) \\
&=: \tfrac{s^2}{2\pi}\int_{-\infty}^{\infty} (\omega)^2 |\mathscr{F}(g_s(t))|^2 d(\omega)
\end{aligned} \tag{2.2}$$

So the product $\sigma_t^2\sigma_\omega^2$ (the area of the Heisenberg box) is always the same (i.e. it is constant around the time-frequency plane). Note that the resolution in time and frequency respectively depends on s instead. This fact leaves open the way to Wavelet which we will explore in the last chapter.

2.1 Ambiguity function

In the applications, an analogue invariance property, like that of the area of the Heisenberg boxes above, is obtained by considering ambiguity functions.

Definition 2.3. An ambiguity function is the following:

$$Ag(u,\xi) := \int_{-\infty}^{\infty} g(\tau+u/2)g^*(\tau-u/2)e^{-i\tau\xi} d\tau \tag{2.3}$$

We note with * the complex conjugate. The definition above is expressed in the time domain. We prove here also the equivalence with the other equivalent form in the frequency domain:

Theorem 2.1.

$$\begin{aligned}
&Ag(u,\xi) := \\
&= \int_{-\infty}^{\infty} g(\tau+u/2)g^*(\tau-u/2)e^{-i\tau\xi}d\tau = \\
&= \int_{-\infty}^{\infty} g(\tau+u/2)g^*(\tau-u/2)e^{-i\frac{\xi}{2}(\tau+\frac{u}{2})}e^{-i\frac{\xi}{2}(\tau-\frac{u}{2})}d\tau = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(g(\tau+\tfrac{u}{2})e^{-i\frac{\xi}{2}(\tau+\frac{u}{2})})(\omega)\mathscr{F}(g(\tau-\tfrac{u}{2})e^{i\frac{\xi}{2}(\tau-\frac{u}{2})})^*(\omega)d\omega = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(g(\tau)e^{-i\frac{\xi}{2}\tau})(\omega)\mathscr{F}(g(\tau)e^{i\frac{\xi}{2}\tau})^*(\omega)e^{i\omega(\frac{u}{2}+\frac{u}{2})}d\omega = \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathscr{F}(g)(\omega+\xi/2)\mathscr{F}(g)^*(\omega-\xi/2)e^{iu\omega}d\omega =: A\hat{x}(u,\xi)
\end{aligned} \tag{2.4}$$

Proof. □

See that in the calculations (2.4) above we have used the Plancherel theorem 1.2. Observe that we did not use the Ambiguity function with an auxiliary window as in the Windowed Fourier transform. Furthermore we note that it is complex-valued and it do not constitute a probability density. Instead if we take two different functions $f \in L^2(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ instead of only g in the definition of Ambiguity function, namely:

$$A(f,g)(u,\xi) := \int_{-\infty}^{\infty} f(\tau+u/2)\overline{g(\tau-u/2)}e^{-i\tau\xi}d\tau, \tag{2.5}$$

we obtain the cross-ambiguity function, which is used as a cross-correlation function.

Observation 4. We look at the norm of the values of the ambiguity function as a kind of dependence of the signal energy on time and frequency. Supposing we have a radar emitting signals, we would determine the spacial values (by means of distance d and velocity v) of an object. To achieve this we use the echo signal produced by the object and so we measure it with a receiver. Consequently we have a returned signal reflected by the object incoming with a delay $t_0 = 2d/c$ in time and a Doppler shift $\xi_0 = -2\xi v/c$, where c is the speed of light. So the echo signal is

$$f_{ret}(t) = f(t-t_0)e^{-i\xi_0(t-t_0)}$$

and its spectrum:

$$\mathscr{F}(f_{ret})(\tau) = \int_{-\infty}^{+\infty} f_{ret}(t)e^{-i\tau t}dt.$$

If our receiver is an optimal receiver with frequency characteristics $F(\xi)$, then the incoming signal we have to treat is

$$\int_{-\infty}^{+\infty} F(\xi)\mathscr{F}(f_{ret})(\xi)e^{-i\tau\xi}d\tau,$$

which is the (Woodward) Ambiguity function (it can be shown with a bit of computations that it coincides with our Ambiguity A defined in (2.3)). Observe that the incoming signal is a function of the time delay t_0 and the Doppler shift ξ_0.
To determine the distance d and the velocity v we have to determine t_0 and ξ_0. In doing this we must to take into account this property (1): that $|Af(u,\xi)|$ takes on the maximum value $\|f\|^2$ at the origin $(0,0)$ in the (u,ξ)-plane.
So we necessarily have to determine experimentally the time-frequency values for which the following Ambiguity function

$$Af(t-t_0, \xi-\xi_0)$$

takes its maximum to use the property (1). The explanations of the role of the ambiguity function compared to the antenna pattern of the radar are not pursued here for brevity.

We obtain the following property that the volume of ambiguity is constant:

Observation 5.

$$
\begin{aligned}
&\tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|Af(u,\xi)|^2 du d\xi = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\int_{-\infty}^{\infty} g(\tau+u/2)g^*(\tau-u/2)e^{-i\tau\xi}d\tau|^2 du d\xi = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau+u/2)g^*(\tau-u/2)e^{-i\tau\xi}d\tau \\
&\qquad\qquad \int_{-\infty}^{\infty} g^*(\tau'+u/2)g(\tau'-u/2)e^{i\tau'\xi}d\tau' du d\xi = \\
&= \tfrac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i(\tau'-\tau)\xi} g(\tau+u/2)g^*(\tau-u/2) \\
&\qquad\qquad g^*(\tau'+u/2)g(\tau'-u/2)d\xi d\tau d\tau' du = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\tfrac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\tau'\xi}\mathscr{F}(g(\tau+u/2)g^*(\tau-u/2) \\
&\qquad\qquad g^*(\tau'+u/2)g(\tau'-u/2))(\xi)d\xi d\tau' du = \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(\tau'+u/2)g^*(\tau'-u/2)g^*(\tau'+u/2)g(\tau'-u/2)d\tau' du = \\
&= \int_{-\infty}^{\infty} g^*(x)g(x)dx \int_{-\infty}^{\infty} g(y)g^*(y)dy = \\
&= ||g||^2 \\
&= 1
\end{aligned}
\tag{2.6}
$$

which is another instance of the uncertainty property; in the sense that the 'amount of ambiguity' is constant as the energy of the signal g is.

Here we state, as promise some chapters ago, a result showing the connection between instantaneous frequency and Windowed Fourier Transform:

Theorem 2.2. *Consider* $f = a(t)cos(t)$*,* g *a normalized window and its rescaled* $g_s = \frac{1}{\sqrt{s}}g(\frac{t}{s})$*. If* $\xi \geq 0$*:*

$$\mathscr{F}_{win}(f)(u,\xi) = \frac{\sqrt{s}}{2}a(u)e^{i[\varphi(u)-\xi u]}(\hat{g}(s[\xi-\varphi'(u)]) + \epsilon_{u,\xi}),$$

where the Windowed Fourier transform is calculated with respect to g_s*, instead of* g*.*
We have that the corrective terms satisfy:

$$|\epsilon_{u,\xi}| \leq \epsilon_{a,1} + \epsilon_{a,2} + \epsilon_{\varphi,2} + \sup_{|\omega|\geq s\varphi'(u)} |\hat{g}(\omega)|$$

with:

$$\begin{cases} \epsilon_{a,1} & \leq \frac{s|a'(u)|}{|a(u)|} \\ \epsilon_{a,2} & \leq \sup\limits_{|t-u|\leq \frac{s}{2}} \frac{s^2|a''(u)|}{|a(u)|} \\ \epsilon_{\varphi,2} & \leq \sup\limits_{|t-u|\leq \frac{s}{2}} s^2|\varphi''(t)|, \quad \textit{if } \frac{s|a'(u)|}{|a(u)|} \leq 1. \end{cases} \tag{2.7}$$

Proof. By definition

$$\mathscr{F}_{win}(f)(u,\xi) = \int_{-\infty}^{+\infty} a(t)cos(\varphi(t))g_s(t-u)e^{-i\xi t}dt$$

which we rewrite as a sum of two terms:

$$= \int_{-\infty}^{+\infty}(\tfrac{1}{2}a(t)e^{i\varphi(t)}g_s(t-u)e^{-i\xi t} + \tfrac{1}{2}a(t)e^{-i\varphi(t)}g_s(t-u)e^{-i\xi t})dt \tag{2.8}$$

Reminding Taylor series expansion, we have here

$$a(u+t) = a(u) + a'(u)t + \frac{\alpha(t)}{2}t^2 \quad ,\text{with} \quad |\alpha(t)| \leq \sup_{[u,u+t]} |a''|$$

and

$$\varphi(u+t) = \varphi(u) + \varphi'(u)t + \frac{\beta(t)}{2}t^2 \quad ,\text{with} \quad |\beta(t)| \leq \sup_{[u,u+t]} |\varphi''|.$$

By the first summand in (2.8), which we name as $I(\varphi)$, we obtain:

$$I(\varphi) = \int_{-\infty}^{+\infty} \tfrac{1}{2}a(t+u)e^{i\varphi(t+u)}g_s(t)e^{-i\xi t}dt.$$

Multiplying it by $2e^{-i(\varphi(u)-\xi u)}$ we get:

$$= \int_{-\infty}^{+\infty} a(u)e^{it\varphi'(u)}e^{i\frac{t^2}{2}\beta(t)}e^{-i\xi t}g_s(t)dt +$$

$$+ \int_{-\infty}^{+\infty} a'(u)te^{it\varphi'(u)}e^{i\frac{t^2}{2}\beta(t)}e^{-i\xi t}g_s(t)dt +$$

$$+ \int_{-\infty}^{+\infty} \tfrac{\alpha(t)}{2}t^2e^{-i(\xi t+\varphi(u)-\varphi(u+t))}dt.$$

Now using first order Taylor expansion another time yields:

$$e^{i\frac{\beta(t)}{2}t^2} = 1 + \frac{\beta(t)}{2}t^2\gamma(t) \quad with \quad |\gamma(t)| \leq 1,$$

and finally observing that:

$$\int_{-\infty}^{+\infty} g_s(t)e^{-it(\xi-\varphi'(u))}dt \quad = \sqrt{s}\hat{g}(s[\xi - \varphi'(u)]),$$

we have that:

$$|I(\varphi) - \tfrac{\sqrt{s}}{2}a(u)e^{i(\varphi(u)-\xi u)}\hat{g}(\xi - \varphi'(u))| \leq \quad \tfrac{\sqrt{s}|a(u)|}{4}(\epsilon_{a,1}^{+} + \epsilon_{a,2} + \epsilon_{\varphi,2}) \tag{2.9}$$

with

$$\begin{cases} \epsilon_{a,1}^{+} &= \frac{2|a'(u)|}{|a(u)|}|\int_{-\infty}^{\infty} t\frac{g_s(t)}{\sqrt{s}}e^{-it(\xi-\varphi'(u))}dt| \\ \epsilon_{a,2} &= \int_{-\infty}^{\infty} t^2|\alpha(t)|\frac{|g_s(t)|}{\sqrt{s}}dt \\ \epsilon_{\varphi,2} &= \int_{-\infty}^{\infty} t^2|\beta(t)|\frac{|g_s(t)|}{\sqrt{s}}dt + \frac{|a'(u)|}{|a(u)|}\int_{-\infty}^{\infty}|t^3||\beta(t)|\frac{|g_s(t)|}{\sqrt{s}}dt. \end{cases} \tag{2.10}$$

Similarly for $I(-\varphi)$, we obtain:

$$|I(-\varphi)| \leq \quad \tfrac{\sqrt{s}|a(u)|}{2}|\hat{g}(\xi - \varphi'(u))| + \tfrac{\sqrt{s}|a(u)|}{4}(\epsilon_{a,1}^{-} + \epsilon_{a,2} + \epsilon_{\varphi,2}). \tag{2.11}$$

with

$$\epsilon_{a,1}^{-} = \frac{2|a'(u)|}{|a(u)|}|\int_{-\infty}^{\infty} t\frac{g_s(t)}{\sqrt{s}}e^{-it(\xi+\varphi'(u))}dt|.$$

By inequalities (2.9) and (2.11) we obtain:

$$I(\varphi)+I(-\varphi) = \frac{\sqrt{s}}{2}a(u)e^{(i(\varphi(u)-\xi u))} + \frac{\epsilon_{a,1}^{+} + \epsilon_{a,1}^{-}}{2} + \epsilon_{a,2} + \epsilon_{\varphi,2} + \sup_{|\omega|\geq s|\varphi'(u)|}|\hat{g}(\omega)|,$$

which is exactly the statement because

$$|\hat{g}(s[\xi + \varphi'(u)])| \leq \sup_{|\omega|\geq s|\varphi'(u)|}|\hat{g}(\omega)|$$

in the case

$$\xi \geq 0 \quad and \quad \varphi'(u) \geq 0$$

Finally, we prove the upper bounds of the statement, we set: $\epsilon_{a,1} = \frac{\epsilon^+_{a,1}+\epsilon^-_{a,1}}{2}$, since we can suppose supp $g = [-\frac{1}{2}, \frac{1}{2}]$, we get:

$$\int_{-\infty}^{+\infty} |t|^n \frac{|g_s(t)|}{\sqrt{s}} dt = s^n \int_{-\frac{1}{2}}^{\frac{1}{2}} |t|^n |g(t)| dt \leq \frac{s^n}{2^n}.$$

Applying the property for $n = 1$ gives the estimate

$$\epsilon_{a,1} \leq \frac{s|a'(u)|}{|a(u)|}.$$

Let's finally verify the upper bounds (2.7). The formers two are simple consequence of the fact that the Taylor remainder in each formula satisfy the property:

$$\sup_{|t| \leq \frac{s}{2}} |\alpha(t) \leq \sup_{|t-u| \leq \frac{s}{2}} |a''(t)| \quad , \quad \sup_{|t| \leq \frac{s}{2}} |\alpha(t) \leq \sup_{|t-u| \leq \frac{s}{2}} |a''(t)|.$$

Finally, in (2.10) above, replacing $|\beta(t)|$ by its upper bound and considering $s|a'(u)||a(u)|^{-1} \leq 1$ gives:

$$\epsilon_{\varphi,2} \leq \frac{1}{2}(1 + \frac{s|a'(u)|}{|a(u)|}) \sup_{|t-u| \leq \frac{s}{2}} s^2 |\varphi''(t)| \leq \sup_{|t-u| \leq \frac{s}{2}} s^2 |\varphi''(t)|$$

□

2.2 The inversion formula

The Windowed Fourier Transform fulfills the inversion formula as the Fourier Transform of the last chapter. First of all, Windowed Fourier Transform satisfies the following orthogonality relation.

Theorem 2.1. Let $f_1, f_2, g_1, g_2 \in L^2(\mathbb{R})$, then $\mathscr{F}_{winj}(f)(u,\xi) \in L^2(\mathbb{R})$ for

$j = 1, 2$, and:

$$< \mathscr{F}_{win}^{g_1}(f_1), \mathscr{F}_{win}^{g_2}(f_2) >_{L^2} = \frac{1}{2\pi} < f_1, f_2 >_{L^2} \overline{< g_1, g_2 >_{L^2}},$$

where $\mathscr{F}_{win}^{g_j}$ is applied to g_j.

Proof. Suppose for instance that g_1, g_2 are in $L^1 \cap L^\infty \subseteq L^2$. So we have $f_j\overline{g_j(\cdot - u)} \in L^2$. Therefore by Parseval formula theorem 1.3 we obtain

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} \mathscr{F}_{win}^{g_1}(f_1)\overline{\mathscr{F}_{win}^{g_2}(f_2)}d\xi dt =$$

$$= \int_{-\infty}^{+\infty}(\int_{-\infty}^{+\infty} \mathscr{F}(f_1(t)\overline{g_1(t-u)})\overline{\mathscr{F}(f_2(t)\overline{g_2(t-u)})}d\xi)dt$$

$$= 2\pi \int_{-\infty}^{+\infty}(\int_{-\infty}^{+\infty} f_1(t)\overline{f_2(t)}g_1\overline{g_1(t-u)}g_2(t-u)dt)du$$

next, applying Fubini to the products $f_1\overline{f_2} \in L^1_t$ and $g_1\overline{g_2} \in L^1_\xi$:

$$< \mathscr{F}_{win}^{g_1}(f_1), \mathscr{F}_{win}^{g_2}(f_2) >_{L^2} = 2\pi \int_{-\infty}^{+\infty} f_1\overline{f_2}(\int_{-\infty}^{+\infty}(g_2(t-u)\overline{g_1(t-u)}du)dt$$

$$= 2\pi < f_1, f_2 >_{L^2} < g_1, g_2 >_{L^2}$$

Finally we extend the relation, as usually, to $g_j \in L^2$ by density argument. Fixing $g_1 \in L^1 \cap L^\infty$, the mapping $g_2 \longrightarrow < \mathscr{F}_{win}^{g_1}(f_1), \mathscr{F}_{win}^{g_2}(f_2) >_{L^2}$ is a linear functional that coincides with $\frac{1}{2\pi} < f_1, f_2 >_{L^2} < g_2, g_1 >_{L^2}$ over $L^1 \cap L^\infty$, a dense subspace of L^2. It is therefore bounded and so it extends to all $g_2 \in L^2$. So now, considering arbitrary f_1, f_2 and g_2 in L^2, the linear functional $g_1 \longleftarrow < \mathscr{F}_{win}^{g_1}(f_1), \mathscr{F}_{win}^{g_2}(f_2) >_{L^2}$ equals $\frac{1}{2\pi} < f_1, f_2 >_{L^2} \overline{< g_1, g_2 >_{L^2}}$ on $L^1 \cap L^\infty$ and extends to all $g_1 \in L^2$. So the orthogonality relations are established on all $L^2(\mathbb{R})$. □

Observation 6. It is a corollary that

$$\|f\|_2 = \frac{1}{2\pi}\|\mathscr{F}_{win}(f)\|_2$$

for all $f \in L^2$ (which is the isometry property of the Windowed Fourier Transform). So it follows that f is completely determined by its windowed Fourier transform $\mathscr{F}_{win}(f)$. But furthermore, the implication

$$\mathscr{F}_{win}(f) = < f(\cdot), e^{i\xi \cdot} g(\cdot - u) > = 0 \quad \forall u, \xi \in \mathbb{R}^2 \quad \Rightarrow f = 0$$

is equivalent to say that for each fixed $g \in L^2$ the set

$$\{e^{i\xi t}\overline{g(t-u)} : (u, \xi) \in \mathbb{R}^2\}$$

spans a dense subspace of L^2. The matter of how recover f from $\mathscr{F}_{win}(f)$ is shown in the next theorem:

Theorem 2.2. (Reconstruction formula) We suppose that f is in $L^2(\mathbb{R})$. Then

$$f = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathscr{F}_{win}(f)(t, \xi) g(t-u) e^{it\xi} d\xi du.$$

Proof. Observe that

$$\mathscr{F}(f)(u, \xi) = e^{-iu\xi} \int_{-\infty}^{+\infty} f(t) g(t-u) e^{i\xi(u-t)} dt$$

$$= (e^{-iu\xi}) f \star [g(-t) e^{i\xi t}]$$

$$= (e^{-iu\xi}) f \star [g(t) e^{i\xi t}]$$

where the convolution is a function of the u variable and by its property g is even, so $g(-t) = g(t)$. So its Fourier transform is

$$\hat{\mathscr{F}}(f)(\omega) = \hat{f}(\omega + \xi)\hat{g}(\omega).$$

Consider that the Fourier transform of $g(t-u)$ with respect to u is $\hat{g}(-\omega)e^{-it\xi}$. We have finally, by Parseval theorem 1.3 applied to the integral formula in the statement of our theorem:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{F}(f)(u,\xi)g(t-u)e^{it\xi}dud\xi$$

$$=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\hat{f}(\omega+\xi)|\hat{g}(\omega)|^2e^{it(\omega+\xi)}d\omega d\xi.$$

But $\hat{f}\in L^1$, so we can apply Fubini's theorem to reverse the integration order. From the formula we obtain, using the inverse Fourier transform theorem 1.1, which we recall is:

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\hat{f}(\omega+\xi)e^{it(\omega+\xi)}d\xi$$

which results in

$$=\int_{-\infty}^{+\infty}f(t)|\hat{g}(\omega)|^2dt.$$

But since we have

$$\int_{-\infty}^{+\infty}|\hat{g}|^2d\omega=1$$

it finally results the statement above. For a more general $\hat{f}\in L^2(\mathbb{R})$ a density argument is used and the proof is complete. □

The key ingredient in this proof is the Parseval formula. in fact we can prove a more general result:

Theorem 2.3. *We suppose that* g,γ *are in* $L^2(\mathbb{R})$ *and* $<g,\gamma>\neq 0$. *Then for all* $f\in L^2$

$$f(t)=\frac{1}{2\pi<g,\gamma>}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{F}_{win}(f)(u,\xi)\gamma(t-u)e^{it\xi}d\xi du. \quad (2.12)$$

Proof. Since $\mathscr{F}_{win}(f)\in L^2$ by observation 6, the integral

$$\tilde{f}:=\frac{1}{2\pi<\gamma,g>}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\mathscr{F}_{win}(f)(u,\xi)\gamma(t-u)e^{i\xi t}du$$

is well-defined in L^2 as we will see in the observation 8.

Consider g function on $\mathbb{R}$ with values in the Banach space L^2, then $f = \int_{-\infty}^{+\infty} g(u)du$, operator-valued integral, for us means that

$$< f, h >= \int_{-\infty}^{+\infty} < g(u), h > du \quad for \quad all \quad h \in L^{2*}.$$

If the mapping

$$h \longrightarrow \int_{-\infty}^{+\infty} < g(u), h > du$$

is a bounded (conjugate-)linear function on L^2 (where the (conjugate-) is applied on h), then the mapping defines a unique element $f \in (L^{2*})^*$. Although in general we can only say that the integral $< f, h >$ is in the bidual $(L^{2*})^*$. But this problem don't worry us, because we the spaces we deal with are all reflexive Banach spaces, L^2 included. Using the orthogonality relation theorem 2.1 it yelds

$$< \tilde{f}, h >= \frac{1}{2\pi < \gamma, g >} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \mathscr{F}_{win}(f)(u, \xi) \overline{< h, \gamma(t-u)e^{i\xi t} >} du d\xi$$

$$= \frac{1}{2\pi < \gamma, g >} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} < \mathscr{F}^{g}_{win}(f), \mathscr{F}^{\gamma}_{win}(h) >=< f, h > .$$

So $\tilde{f} = f$, and the inversion formula is proved. □

Definition 2.4. We call γ the reconstruction function.

Observation 7. The inversion formula (2.12) shows that f can be expressed as a continuous superposition of time-frequency shifts with the Windowed Fourier Transform as weight function. In this sense, (2.12) is similar to the inversion formula for the Fourier transform, that is, $f(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{f}(\xi) e^{i\xi t} d\xi$. However, in Fourier inversion the elementary functions $e^{i\xi t}$ are not in L^2, whereas in theorem 2.3, the elementary functions $\gamma(t-u)e^{i\xi t}$ are instead particularly nice functions in L^2, in fact they are used as the starting point for the reconstruction of a signal f. We will speak about this in chapter 3.

Observation 8. The integral (2.12) in the theorem above is a superposition of time-frequency shifts of the form:

$$f = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,\xi) g(t-u) e^{i\xi t} du d\xi. \tag{2.13}$$

Let us now specify well this integral in the more general setting of Banach spaces. For example, if $F \in L^2(\mathbb{R})$, then the (conjugate-)linear functional

$$l(h) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,\xi) \overline{< h, g(\cdot - u) e^{i\xi \cdot} >} du d\xi$$

is a bounded functional on L^2. To see this, we apply the Cauchy-Schwartz inequality and use that (it follows from the orthogonality relations above)

$$\|\mathscr{F}_{win}(f)\|_{L^2} = \|f\|_2 \|g\|_2,$$

where the Windowed Fourier transform is applied with g. So the following holds

$$|l(h)| \leq \|F\|_2 \|\mathscr{F}_{win}(h)\|_2 = \|F\|_2 \|g\|_2 \|h\|_2. \tag{2.14}$$

This means that l defines a unique function

$$f = \int_{-\infty}^{+\infty} F(u,\xi) g(t-u) e^{i\xi t} du d\xi \in L^2$$

with norm $\|f\|_2 \leq \|f\|_2 \|g\|_2$ and satisfying $l(h) =< f, h >$.
We show now how the integral (2.13) gives a relation for the Windowed Fourier transform. Let A_g be the linear operator defined by

$$A_g F = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,\xi) \overline{g(t-u)} e^{i\xi t} du d\xi.$$

By the estimate (2.14), A_g is a bounded operator from L^2 onto L^2. Moreover, by

$$< A_g F, h >= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} F(u,\xi) < g(t-u) e^{-i\xi t}, h > du d\xi$$

$$< F, \mathscr{F}_{win}(h) >=< \mathscr{F}_{win}^*(F), h >,$$

with $h \in L^2$ and $F \in L^2$, so A_g is exactly the adjoint operator (the conjugate transpose is the same as the inverse) of the Windowed Fourier transform viewed as an operator from L^2 to L^2. Thus indeed $\mathscr{F}_{win}^* = A_g$, where the Windowed transform is computed by g.
Thus the inversion formula reads as

$$\frac{1}{< \gamma, g >} \mathscr{F}_{win}^{\gamma}{}^* \mathscr{F}_{win}^{g} = \mathbb{I}$$

This inversion formula will lead to important results about modulations spaces. The definition of these spaces is based over an extension of the inversion formula as:

$$\{f \in \mathscr{S}(\mathbb{R})' | (\int_{-\infty}^{+\infty} (\int_{-\infty}^{+\infty} |\mathscr{F}_{wing} f(u,\xi)|^p \gamma(u,\xi)^p du)^{\frac{q}{p}})^{\frac{1}{q}} < \infty\},$$

where $g \in \mathscr{S}(\mathbb{R})$ and $1 \leq p, q \leq \infty$. Modulation spaces are spaces of functions that are better suited to describe the action of the Windowed Fourier transform, and they give a general framework for the definition of admissible window. In the applications is important to choose a Window such that both g and $\hat{g}$ decay very rapidly, that is, for example for Schwartz functions or C_0^∞ functions.
For completeness we show an:

Example 2.1. One signal processing application known as signal segmentation amounts to using a characteristic function as window. We have that

$$\{\chi_{[0,1]}(t-k) e^{2\pi i n t}\}_{k,n}$$

is an orthonormal bases in $L^2(\mathbb{R})$. Indeed, if $g = \chi_{[0,1]}$, the $\mathscr{F}_{win}(f)$ respect to g provides an accurate picture of the temporal behaviour of f since $\mathscr{F}_{win}(f)(u,0) = \int_{u+[0,1]} f(t)dt$ is the average value of f in a neighborhood of u. But on the frequency side, since $\mathscr{F}\chi_{[0,1]}(\xi) = \frac{1-e^{-i\xi}}{i\xi}$ decays slowly and is not even in L^1. This gives a bad frequency localization and the Windowed

Fourier transform

$$\mathscr{F}^{g}_{win}(f)(u,\xi) = \mathscr{F}^{\hat{g}}_{win}(\hat{f})(\xi,-u)e^{i\xi u}$$

provides a completely inadequate frequency resolution.

Chapter 3

Frames

In this chapter we start considering sequences $\{e_j : j \in J\}$ in a separable Hilbert space $\mathfrak{H}$ which we call frame

Definition 3.1. We call $\{e_j : j \in J\}$ a frame in case there exist positive constants $A, B > 0$ such that

$$A\|f\|^2 \leq \sum_{j \in J} | < f, e_j > |^2 \leq B\|f\|^2, \quad for \quad all \quad f \in \mathscr{H}. \tag{3.1}$$

If the frame bounds A, B satisfying (3.1) are equals then $\{e_j : j \in J\}$ is called a tight frame. We will see that in case of tight frames, both the frame and its dual defined in (3.4) coincide.
When we considered the Windowed Fourier transform (2.1) for the Inversion formula in theorem 2.3, we used it to write f as a continuous expansion of f with respect to the uncountable system of functions

$$\{\gamma(t - u)e^{i\xi t} : (u, \xi) \in \mathbb{R}^2\}.$$

Since L^2 is a separable Hilbert space, only a countable subset of them suffice to represent every signal $f \in L^2$. In fact the representation of f by the Inversion formula is highly redundant, hence in the case the supports $\{\text{supp} < \gamma(\ \cdot\ - u_i)e^{i\xi_i\ \cdot}, h >\}$, overlap when (u_i, ξ_i) varies in a countable subset of $\mathbb{R} \times \mathbb{R}$. This because the coefficients

$$\{\mathscr{F}^g_{win}(f)(u_i, \xi_i)\}_{(u_i,\xi_i)}$$

in the Inversion formula are almost equal and so they represent the same time-frequency behaviour of f varying (u_i, ξ_i).
The formal idea behind this is that requiring to have a frame is less than requiring the invertibility of the Windowed Fourier transform operator.
We now give an interesting geometrical interpretation of the formula (3.1). We cover the time-frequency plane by a lattice $(\alpha n, \beta k)$ with $(n, k) \in \mathbb{Z} \times \mathbb{Z}$. Our window function g has support essentially concentrated in a rectangle R over the lattice. Its size by the uncertainty principle can not be larger than a costant. So we have a covering of the time-frequency plane given by $R + (\alpha\mathbb{Z}, \beta\mathbb{Z})$, which is a countable set of shifted rectangles in time and frequency. If the product $\alpha\beta > 1$, then the rectangles R do not overlap leaving gaps. The signal in the gaps can not be approximated, so giving a set of vectors $\{e_j, j \in J\}$ which do not constitute a frame. This is a theorem and is formulated as:

Theorem 3.1. *If $\{g(t-\alpha n)e^{i\beta nt}\}$ is a (Gabor) frame in $L^2(\mathbb{R})$, then $\alpha\beta \leq 1$.*

The converse is not necessarily true. This is the case also when the signal is oversampled but f is not completely determined by its frame coefficients $< f, e_j >$. We can take for example $\mathscr{G}(g = \chi, \alpha = 2, \beta) = \{\chi_{[0,1]}(t-2n)e^{ik\beta t}\}$; they do not form a frame because the functions with supports in $\bigcup_{k\in\mathbb{Z}}[2k + 1, 2k + 2)$ are not even in the span of $\mathscr{G}(\chi, 2, \beta)$. We conclude this brief introduction with an example of frame:

Example 3.1. Take an orthonormal bases $\{g_1, g_2, g_3\}$ in a three dimensional Hilbert space $\mathfrak{H}$. Considering a cube the corresponding tetrahedron can be given by the coordinates

$$(1,1,1),(1,-1,1),(-1,-1,1),(1,-1,-1).$$

Considering the following rotation matrix

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} & \sqrt{\frac{2}{3}} & -\frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

applying it to the tetrahedron above we obtain the following four vectors:

$$\phi_1 = g_1, \phi_2 = -\frac{g_1}{3} - \frac{\sqrt{2}}{3} g_2 + \sqrt{\frac{2}{3}} g_3,$$

$$\phi_3 = -\frac{g_1}{3} - \frac{\sqrt{2}}{3} g_2 - \sqrt{\frac{2}{3}} g_3, \phi_4 = -\frac{g_1}{3} + \frac{2\sqrt{2}}{3} g_2.$$

By simple computations we obtain that $\sum | < f, \phi_n > |^2 = \frac{4}{3} \|f\|^2$. So they form a tight frame with bounds $A = B = 4/3$.

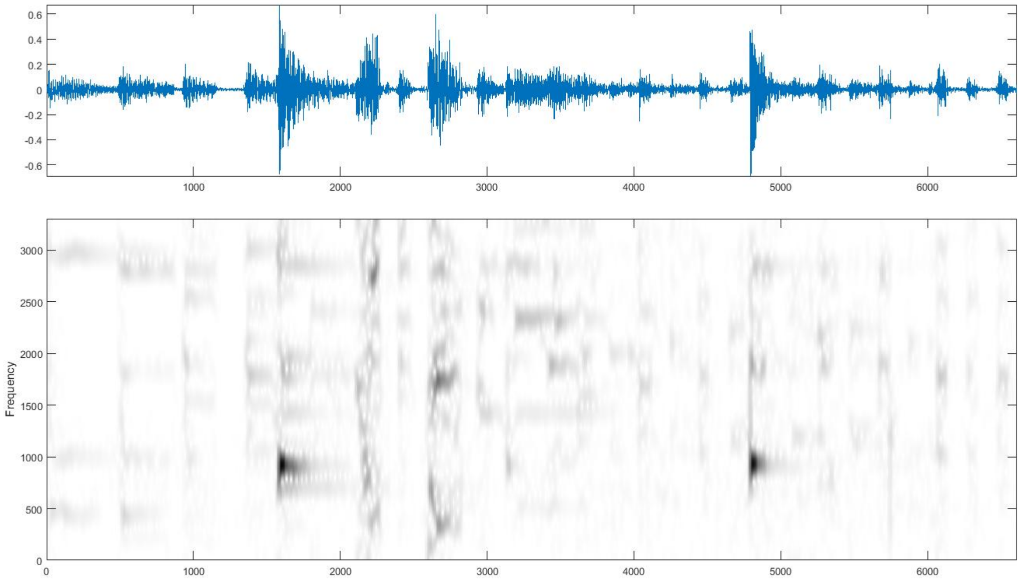
Observation 9. Here is an example of computation made using Mallat code. It is a representation of the Windowed Fourier coefficients of signal, a (compressed for logistic necessity) piece of music taken as input (its time profile the graph immediately above, in blue). The logarithm of the coefficients of the spectrogram $log(\mathscr{F})f$ are calculated using a Hanning windowed, tipically used in musical recordings.

3.1 Riesz bases

We call

$$|\mathscr{F}(f)(u,\xi)|^2$$

the energy of f in a time-frequency box centered at (u,ξ).
A way to collect both the property that the sampling operation is continuous



on L^2 (stability) and that f is uniquely determined by the samples of the Windowed Fourier transform (completeness) is considering:

$$A\|f\|_2^2 \leq \sum_k \sum_n |\mathscr{F}(f)(k,n)|^2 \leq B\|f\|_2^2$$

fulfilled for all $f \in L^2$. This is more or less to say that the energy of the signal f is preserved under discretization. Observe that the samples of the Windowed Fourier transform are just inner products of f with a given collection of functions:

$$\mathscr{F}(f)(k,n) =< f, g(h-k)e^{inh} >= e^{-ikn} < f, g(h-k)e^{in(h-k)} > .$$

We have that f is unique under the representation of the Windowed Fourier coefficients if $\{g(h-k)e^{in(h-k)} : k, n \in \mathbb{Z}\}$ spans a dense subspace of L^2. How this is realized will be the content of proposition 3.3, which states when the particular case of uniqueness of the coefficients in the Windowed Fourier expansion of f happens. We will use different types of operators in the sequel.

Definition 3.2. Consider $\{e_j : j \in J\} \subseteq \mathfrak{H}$ subset and $\{c_j\}_{j\in J}$ a finite sequence .

- The analysis operator C is given by

$$Cf = \{< f, e_j >: j \in J\}$$

- The synthesis operator D is defined by $Dc = \sum c_j e_j \in \mathfrak{H}$
- The frame operator S is defined on $\mathfrak{H}$ by

$$S(f) = \sum < f, e_j > e_j$$

Proposition 3.1. Given $\{e_j : j \in J\}$ a frame for $\mathfrak{H}$, the following holds:

a) C is bounded from $\mathfrak{H}$ into $\ell^2(J)$ with closed range.

b) The operators C and D are adjoint to each other; that is, $D = C^*$. Consequently, D extends to a bounded operator from $\ell^2(J)$ into $\mathfrak{H}$ and satisfies

$$\| \sum c_j e_j \| \leq \sqrt{B} \|c\|_{\ell^2}.$$

c) The frame operator $S = C^*C = DD^*$ (here * is the involution operator) maps $\mathfrak{H}$ onto $\mathfrak{H}$ and is a positive invertible operator satisfying $A\mathbb{I}_{\mathfrak{H}} \leq S \leq B\mathbb{I}_{\mathfrak{H}}$ and $B^{-1}\mathbb{I}_{\mathfrak{H}} \leq S^{-1} \leq A^{-1}\mathbb{I}_{\mathfrak{H}}$. In particular, $\{e_j : j \in J\}$ is a tight frame if and only if $S = A\mathbb{I}_{\mathfrak{H}}$.

d) The optimal frame bounds are $B_{opt} = \|S\|$ and $A_{opt} = \|S^{-1}\|^{-1}$, here $\| \cdot \|$ is the operator norm of S.

Proof.

a) We have that

$$\|Cf\|_2 \geq A\|f\|_2,$$

so taking the sequence

$$Cf_n \to y$$

we obtain that $\{x_n\}_n$ is a Cauchy sequence by the fact that $\|x_n - x_m\|_2 \leq \frac{1}{A}\|Cf_n - Cf_m\|_2$. So $C \lim f_n = y$ and hence, y is in the image of C. Hence it is a closed subspace of ℓ^2.

b) Take a finite sequence of coefficients $\{c_j\}_{j\in J}$. Then

$$< C^*c, f >=< c, Cf >= \sum c_j \overline{< f, e_j >} =< \sum c_j e_j, f >=< Dc, f > .$$

Now, since C is bounded on $\mathfrak{H}$ and has operator norm $\|C\| \leq \sqrt{B}$ by (3.1), it follows that $D = C^* : \ell^2(J) \to \mathfrak{H}$ is also bounded with the same operator norm. Thus b) follows.

c) Since $S = C^*C = D^*D$ we have that S is self-adjoint and positive definite. Since

$$< Sf, f >= \sum | < f, e_j > |^2$$

it's immediate that $A\mathbb{I} \leq S \leq B\mathbb{I}$. Further, S is invertible on $\mathfrak{H}$ because $A > 0$. Since S is a positive definite operator and it commute (i. e. $[S, S^{-1}] = SS^{-1} - S^{-1}S = 0$) therefore $AS^{-1} \leq SS^{-1} \leq BS^{-1}$, as desired.

d) We remember that the operator norm is defined by $\|S\| = \sup\limits_{\|f\|\leq 1} < Sf, f >$. Then from inequalities (3.1) the statement follows. □

We observe that point b) above is remarkable because it says to us that $\sum c_j e_j$ is well defined for arbitrary ℓ^2 sequences even if we are not claiming that the frame vectors in the sequence are not even orthogonal. We can explain better this phenomenon using the following:

Corollary 3.2. *Let $\{e_j : j \in J\}$ be a frame for $\mathfrak{H}$. If $f = \sum\limits_{j\in J} c_j e_j$ for some $\{c_j\}_j \in \ell^2(J)$, then for every $\epsilon > 0$ there exists a finite subset $F_0 = F_0(\epsilon) \subseteq J$ such that*

$$\|f - \sum c_j e_j\| \leq \epsilon \quad \textit{for all finite subsets} \quad F \supseteq F_0.$$

In this case we say that the series converges unconditionally to $f \in \mathfrak{H}$.

Proof. Choose $F_0 \subseteq J$ such that $|c_j|^2 \underset{n \notin F}{\leq} \epsilon/\sqrt{B}$ for $F \supseteq F_0$. Let $c_F = c\chi_F \in \ell^2(J)$ be the finite sequence with terms $c_{F,j} = c_j$ if $j \in F$ and $c_{F,j} = 0$ if $j \notin F$. Then $\underset{j \in F}{c_j e_j} = Dc_F$ and

$$\|f - \underset{j \in F}{c_j e_j}\| = \|Dc - Dc_F\|$$

and by proposition 3.1 b)

$$= \|D(c - c_F)\|$$

$$= \sqrt{B}\|c - c_F\|_{\ell^2} \leq \epsilon$$

□

Corollary 3.3. *If $\{e_j : j \in J\}$ is a frame with positive bound coefficients A and B, then $\{S^{-1}e_j : j \in J\}$ is a frame with bounds A^{-1}, B^{-1}. Every $f \in \mathfrak{H}$ has non-orthogonal expansions*

$$f = \sum_{j \in J} < f, S^{-1}e_j > e_j \tag{3.2}$$

and

$$f = \sum_{j \in J} < f, e_j > S^{-1}e_j, \tag{3.3}$$

where both sums converge unconditionally in $\mathfrak{H}$.

Proof. First of all we have that

$$\sum_{j \in J} | < f, S^{-1}e_j > |^2 = \sum_{j \in J} | < S^{-1}f, e_j > |^2 =$$

because is self-adjoint and by definition:

$$=< S(S^{-1}f), S^{-1}, f >=< S^{-1}f, f > .$$

Therefore by proposition 3.1, c):

$$B^{-1}\|f\|^2 \leq < S^{-1}f, f >= \sum_{j\in J} | < f, S^{-1}e_j > |^2 \leq A^{-1\|f\|^2}.$$

Thus the collection $\{S^{-1}e_j : j \in J\}$ is a frame with bounds B^{-1} and A^{-1}. Using the factorizations $\mathbb{I}_{\mathfrak{H}} = S^{-1}S = SS^{-1}$, we obtain the series expansions

$$f = S(S^{-1}f) = \sum_{j\in J} < S^{-1}f, e_j > e_j = \sum_{j\in J} < f, S^{-1}e_j > e_j$$

and

$$f = S^{-1}Sf = \sum_{j\in J} < f, e_j > S^{-1}e_j.$$

Because both $\{< f, e_j >\}$ and $\{< f, S^{-1}e_j >\}$ are in $\ell^2(J)$, both series converge unconditionally by the corollary above. □

Observation 10. If the frame is tight, that is the bound coefficients are identical, then both the decompositions are identical. So from

$$\forall f \in \mathfrak{H} \quad \frac{1}{B}\|f\|^2 \leq \sum_{j\in J} | < f, S^{-1}e_j > |^2 \leq \frac{1}{A}\|f\|^2, \tag{3.4}$$

we have that also:

$$f = \frac{1}{A}\sum_{j\in J} < f, e_j > e_j.$$

Definition 3.3. The frame $\{S^{-1}e_j : j \in J\}$ in the statement above is called the dual frame.

Observation 11. In the applications our synthesis operator is discretized as a pseudo inverse. The (continuous) linear pseudo inverse C^+ is defined as the left inverse that is zero on $(R(C))^\perp$:

$$\forall f \in \mathfrak{H} \quad C^+Cf = f \quad and \quad \forall a \in (R(C))^\perp \quad C^+a = 0. \tag{3.5}$$

The pseudo inverse of the analysis operator C, also called frame analysis

operator in literature, allows a reconstruction with the dual frame just defined above. This reconstruction from inner products (3.2) is the counterpart of the series expansion with respect to a set of vectors (3.3).
So the signal f is reconstructed from the frame coefficients $Cf(e_j) =< f, e_j >$ with the dual frame coefficients $\tilde{e_j} = S^{-1}e_j$ as expanding functions. For orthonormal bases these two aspects, (3.2) and (3.3), coincide. Note however that we have to be able to compute the dual coefficients $\tilde{e_j}$ in advance to make effectively computations with these formulae. In general this is not the case, to provide for this situation we will prove Richardson iteration procedure below.
Further, our decomposition is not unique in general. This in contrast as it is in the case of orthonormal bases. The following proposition says us when that uniqueness of the coefficients $< f, S^{-1}e_j >$ happens. Here is a preliminary result, it says us that the coefficients are canonical in a certain sense:

Proposition 3.2. Suppose $\{e_j : j \in J\}$ is a frame for $\mathfrak{X}$ and $f = \sum_{j\in J} c_j e_j$ for some coefficients $c \in \ell^2(J)$, then

$$\sum_{j\in J} |c_j|^2 \geq \sum_{j\in J} | < f, S^{-1}e_j > |^2,$$

with equality only if $c_j =< f, S^{-1}e_j >$ for all $j \in J$.

Proof. Set $a_j =< f, S^{-1}e_j >$. Then $f = \sum_{j\in J} a_j e_j$ and

$$< f, S^{-1}f >= \underset{j\in J}{a_j} < e_j, S^{-1}f >= \sum_{j\in J} |a_j|^2.$$

On the other hand, using that S is self-adjoint we have,

$$< f, S^{-1}f >= \sum_{j\in J} c_j < e_j, S^{-1}f >= \sum_{j\in J} c_j \overline{a_j} =< c, a > .$$

Therefore $\|a\|_{\ell^2}^2 =< c, a >$, and we see that

$$\|c\|_{\ell^2}^2 = \|c - a + a\|_{\ell^2}^2$$

$$= \|c - a\|_{\ell^2}^2 + \|a\|_{\ell^2}^2 + < c - a, a > + < a, c - a >$$

$$= \|c - a\|_{\ell^2}^2 + \|a\|_{\ell^2}^2 \geq \|a\|_{\ell^2}^2,$$

with equality only if $c = a$. □

So the question when the coefficients are uniquely determined is settled by the following statement:

Proposition 3.3. Suppose that $\{e_j : j \in J\}$ is a frame for $\mathfrak{H}$. Then the following conditions are equivalent:

i) The coefficients $c \in \ell^2(J)$ in the series expansion (3.2) are unique;

ii) The analysis operator C maps $\mathfrak{H}$ surjectively onto $\ell^2(J)$;

iii) There exist positive constants A', B' such that the inequalities

$$A'\|c\|_{\ell^2} \leq \|\sum_{j \in J} c_j e_j\| \leq B'\|c\|_{\ell^2} \tag{3.6}$$

hold for all finite sequences $c = \{c_j\}_{j \in J}$.

iv) $\{e_j : j \in J\}$ is the image of an orthonormal bases $\{g_j : j \in J\}$ under an invertible operator $T \in Bound(\mathfrak{H})$.

v) The Gram matrix G, given by $G_{j,m} =< e_j, e_m >_{j,m \in J}$, defines a positive invertible operator on $\ell^2(J)$.

Proof. Consider $\{e_j : j \in J\}$ a frame, we remember that from proposition 3.1 and equation (3.2) we have that C is one-to-one with closed range and that D is onto.

We recall also that a bounded operator is one-to-one if and only if its adjoint operator has dense range.

$i) \iff ii)$ The coefficients are uniwue if and only if D is one-to-one if and only if $D^* = C$ is onto (id est its range, $R(C)$, is closed and dense in $\ell^2(J)$)

$i) \Rightarrow iii)$ The continuity of D, by proposition 3.1, implies the existence of a constant B' such that

$$\sum_{j\in J} \|c_j e_j\| \leq B'\|c\|_2.$$

Since D is bijective, D^{-1} is continuous by the open mapping theorem, from which the lower estimate follows and $iii)$ is proved.

$iii) \Rightarrow iv)$ Let $\{f_j : j \in J\}$ be an orthonormal bases of $\mathfrak{H}$ For $f = \sum_{j\in J} c_j f_j$, we define $Tf = \sum_{j\in J} c_j e_j$. Then $\|f\| = \|c\|_{\ell^2}$ and

$$\|Tf\| = \|\sum_{j\in J} c_j e_j\| \geq A\|c\|_{\ell^2} = A\|f\|,$$

and similarly, $\|Tf\| \leq B\|f\|$ for all $f \in \mathfrak{H}$. Thus T is well defined, invertible operator on $\mathfrak{H}$ and $Tf_j = e_j$, as desired.

$iv) \Rightarrow i)$ If $Tf_j = e_j$, $j \in J$ for an orthonormal bases $\{f_j\}_j$ and an invertible operator $T \in Bound(\mathfrak{H})$, then

$$\sum_{j\in J} c_j e_j = T(\sum_{j\in J} c_j f_j) = 0$$

if and only if

$$\sum_{j\in J} c_j f_j = 0$$

if and only if

$$c_j = 0, \quad for \quad all \quad j \in J$$

$iii) \iff v)$ For any finite sequence $c = \{c_j\}_{j\in J}$,

$$< Gc, c >= \sum_{m,j\in J} < e_m, e_j > c_m \overline{c_j} = \|\sum_{m\in J} c_m e_m\|^2.$$

Therefore (3.6) is equivalent to saying that G is a positive invertible operator on $\ell^2(J)$. □

Definition 3.4. A frame $\{e_j : j \in J\}$ that satisfies the conditions of proposition 3.3 is called a Riesz bases of $\mathfrak{H}$.

Observation 12. A Riesz bases is a frame of vectors that are linearly independent, which implies that $R(C) = \ell^2(J)$, so the vectors of the dual frame are also linearly independent. Inserting $f = e_k$ in (3.2) and (3.3) above yelds

$$e_k = \sum_{j \in J} < e_k, S^{-1}e_j > e_j \tag{3.7}$$

and by linear indenpendence we have that

$$< e_k, S^{-1}e_j >= \delta_{k,j}.$$

Thus dual Riesz bases are biorthogonal families of vectors. If we take a normalized bases ($\|e_j\| = 1$), substituting $f = e_j$ in the frame inequality of corollary 3.3:

$$B^{-1}\|f\|^2 \leq < S^{-1}f, f >= \sum_{j \in J} | < f, S^{-1}e_j > |^2 \leq A^{-1\|f\|^2},$$

we have that

$$A \leq 1 \leq B.$$

3.2 Richardson algorithm

In signal processing we have to represent a signal f with as possibly as less coefficients in our frame $\{e_j : j \in J\}$. So the problem at the end of the approximation process is how to well reconstruct f. We devote this section to the computations needed in order to achieve the reconstruction of a function approximated by only sparse coeffients.

The best linear approximation of a function f by a subspace spanned by $\ell^2(J)$ functions is the orthogonal projection of f in the subspace. When it is

not this case, id est we do not make use of dual coefficients, we will use the Richardson algorithm.
However, it is possible to compute the orthogonal projection with the dual frame stated in the previous section. This is proved exactly by the following result:

Proposition 3.4. Let $\{e_j\}_{j\in J}$ be a frame of $\ell^2(J)$, and $\{S^{-1}e_j\}_{j\in J}$ its dual frame. The orthogonal projection of $f \in \mathfrak{H}$ in $\ell^2(J)$ is

$$Pf = \sum_{j\in J} < f, e_j > S^{-1}e_j = \sum_{j\in J} < f, S^{-1}e_j > e_j. \tag{3.8}$$

Proof. Since both frames are dual, by corollary 3.3, in the case $f \in \ell^2(J)$, then the operator Pf satisfies trivially $Pf = f$.
To prove that it is an orthogonal projection it is sufficient to verify that if $f \in \mathfrak{H}$ then $< f - Pf, e_k >= 0 \ \forall k \in J$. Indeed,

$$< f - Pf, e_k >=< f, e_k > - \sum_{j\in J} < f, e_j >< S^{-1}e_j, e_k >,$$

because by the dual frame property (3.7) we have that finally

$$< f - Pf, e_k >= 0.$$

□

This result is particularly important for approximating signals from a finite set of vectors. In fact in the case that J is a finite set, $\{e_j : j \in J\}$ is a frame of the space it generates.
But our situation is not the case: $f \in \mathfrak{H}$ and our pseudo inverse (3.5) is only invertible on $\ell^2(J)$, the definition of the pseudo inverse changes in this:

Definition 3.5. A pseudo-inverse on the subspace $\ell^2(J) \subseteq \mathfrak{H}$ is:

$$\forall f \in \ell^2(J) \quad C^+C = f \quad and \quad \forall a \in (R(C))^\perp \quad C^+a = 0.$$

If the frame $\mathscr{E} = \{e_j : j \in J\}$ does not depend on the signal f, then the dual frame vectors $\tilde{e_j} = S^{-1}e_j$ are precomputed and the dual reconstruction is solved directly with our projection formula (3.8).
In many applications however, the frame vectors $\mathscr{E}$ depend on the signal f, in which case the dual frame vectors cannot be computed in advance. This is the case, for example, when coefficients $\{< f, e_j >\}_{j\in J}$ are selected in a redundant transform in building a sparse signal representation. Thus, the transform coefficients Cf are known and we must compute

$$Pf = (C^* C_{_{\ell^2(J)}})^{-1} C^* Cf = (C^* C_{_{\ell^2(J)}})^{-1} Sf.$$

A dual synthesis algorithm computes first

$$y = C^* Cf = \sum_{j\in J} < f, e_j > e_j \in \ell^2(J)$$

and then derives $Pf = L^{-1}y = z$ by applying the inverse of the symmetric operator $L = C^* C_{_{\ell^2(J)}}$ to y, with

$$\forall h \in \ell^2(J), \quad Lh = \sum_{j\in J} < h, e_j > e_j.$$

Note that the eigenvalues of L are between A and B.
Observe that the operator L is symmetric:

$$< Lh, e_k >= \sum_{j\in J} < h, e_j >< e_j, e_k >= \sum_{j\in J} < h, e_j >< e_j, e_k >=$$

$$= \sum_{j\in J} < h, e_j >< e_k, e_j > 1 =< h, Le_k >, \quad \forall e_k \in \mathscr{E}.$$

The step requiring more effort in the algorithm above is the inversion of L to compute $z = L^{-1}y$, where the eigenvalues of L are between A and B. The first algorithm we see requires knowing the frame bounds A and B.

Proposition 3.5. To compute $z = L^{-1}y$, let z_0 be an initial value and $\gamma > 0$

be a relaxation parameter. For any $k > 0$, define

$$z_k = z_{k-1} + \gamma(y - Lz_{k-1}). \tag{3.9}$$

If

$$\delta = \max\{|1 - \gamma A|, |1 - \gamma B|\} < 1,$$

then

$$\|z - z_k\| \leq \delta^k \|z - z_0\| \tag{3.10}$$

and therefore

$$\lim_{k \to +\infty} z_k = z.$$

Proof. We rewrite equation (3.9):

$$z - z_k = z - z_{k-1} - \gamma L(z - z_{k-1}).$$

Let

$$R = \mathbb{I} - \gamma L,$$

and

$$z - z_k = R(z - z_{k-1}) = R^k(z - z_0).$$

Since the eigenvalues of L are between A and B,

$$A\|z\|^2 \leq \langle Lz, z\rangle \leq B\|z\|^2.$$

This implies that $R = \mathbb{I} - \gamma L$ satisfies

$$| < Rz, z > | \leq \delta \|z\|^2,$$

where δ is defined as in the statement. Since R is symmetric (L was), this inequality proves that $\|R\| \leq \delta$. Thus iterating we derive (3.10). Finally, the error converges to zero in the case $\delta < 1$. □

Observation 13. We note that convergence is guaranteed for all possible initial values of z_0. The convergence rate to the term $L^{-1}y$ is maximized when δ is minimum. This is the case if we choose $\gamma = \frac{2}{A+B}$, because then

$$|1 - \gamma A| \approx |1 - \gamma B|,$$

so $\delta = \frac{B-A}{B+A} = \frac{1-A/B}{1+A/B}$ is optimum.
We now derive an estimate on the velocity of convergence. From the error estimate (3.10) above we obtain an error smaller than ϵ for a number n of iterations, which satisfies

$$\frac{\|z - z_k\|}{\|z - z_0\|} \leq \delta^k = \epsilon.$$

Inserting the value of γ in $\delta = |1 - \gamma A| = |1 - \frac{2}{A+B} A| \approx 1 - 2\frac{A}{B}$, by Taylor series it yelds:

$$k \approx \frac{\ln \epsilon}{\ln(1 - 2\frac{A}{B})} \approx \frac{-B}{2A} \ln \epsilon.$$

Therefore, the number n of iterations increases proportionally to the frame bound ratio $\frac{B}{A}$. Usually in the applications, the exact values of A and B are often not known. We have that A is generally more difficult to compute than B and $B = \|CC^*\|$. By proposition above for

$$\gamma < \|CC^*\|,$$

the algorithm is guaranteed to converge, but the convergence rate still depends on A.
The optimal relaxation parameter γ is in the range $\|CC^*\|^{-1} \leq \gamma \leq 2\|CC^*\|^{-1}$

The difficulty in finding the bounds coefficients A and B, often found far from an optimal ratio $\frac{B_{opt}}{A_{opt}}$, leads to an implementation using conjugate gradient's method. It is an alternative approach in finding $Pf = z$ using iterative algorithms. In computing, $z = L^{-1}y$ we follow a gradient descent along orthogonal directions with respect to the norm (and its related scalar product) induced by the symmetric operator L:

$$\|z\|_L^2 = \|Lz\|^2.$$

This L norm is used to estimate the error. The implementation is given by the following:

Proposition 3.6. To compute $z = L^{-1}y$, we have the initial data

$$z_0 = 0, \quad r_0 = p_0 = y, \quad p_{-1} = 0.$$

For any $k \geq 0$, we define by induction:

$$\lambda_k = \frac{\langle r_k, p_k\rangle}{\langle p_k, Lp_k\rangle}$$

$$z_{k+1} = z_k + \lambda_k p_k$$

$$r_{k+1} = r_k - \lambda_k Lp_k$$

$$p_{k+1} = Lp_k - \frac{\langle Lp_k, Lp_k\rangle}{\langle p_k, Lp_k\rangle} p_k - \frac{\langle Lp_k, Lp_{k-1}\rangle}{\langle p_{k-1}, Lp_{k-1}\rangle} p_{k-1} \tag{3.11}$$

If $\sigma = \frac{\sqrt{B}-\sqrt{A}}{\sqrt{B}+\sqrt{A}}$, then

$$\|z - z_k\|_L \leq \frac{2\sigma^k}{1+\sigma^{2k}} \|z\|_L, \tag{3.12}$$

and therefore:

$$\lim_{k\to+\infty} z_k = z$$

Proof. Following the Groechenig implementation of the proof we outline the following important steps:

$i)$ Let $\mathbf{U}_k$ be the subspace generated by $\{L^j z\}_{1\leq j\leq k}$. By the induction formula (3.11) on k, we have that $p_j \in \mathbf{U}_k$ for $j < k$.

$ii)$ By induction we can prove that $\{p_j\}_{0\leq j\leq k}$ is an orthogonal bases of $\mathbf{U}_k$ with respect to the inner product $\langle z, h\rangle_L := \langle z, Lh\rangle$. Assuming that $\langle p_k, Lp_j\rangle = 0$, for $j \leq k-1$, it can be shown that $\langle p_{k+1} Lp_j\rangle = 0$ for $j \leq k$.

iii) We can verify that z_k is the orthogonal projection of z onto $\mathbf{U}_k$ with respect to $\langle .,. \rangle_L$, which means that

$$\forall g \in \mathbf{U}_k, \quad \|z-g\|_L \geq \|z-z_k\|_L.$$

Since $z_k \in \mathbf{U}_k$, this requires proving that $\langle z-z_k, p_j\rangle_L = 0$, for $j<k$.

iv) We can compute so the orthogonal projection of z in embedded spaces $\mathbf{U}_k$ of dimension k, and one can verify that $\lim\limits_{k\leftarrow+\infty} \|z-z_k\|_L = 0$. The exponential convergence formula (3.12)also can be proved. □

Observation 14. Note here that we must choose $z_0 = 0$ to start the algorithm. The convergence is slower when $\frac{A}{B}$ is small. In this case,

$$\sigma = \frac{1-\sqrt{A/B}}{1+\sqrt{A/B}} \approx 1 - 2\sqrt{\frac{A}{B}}.$$

The exponential convergence (3.12) proves that we obtain a relative error

$$\frac{\|z-z_k\|_L}{\|z\|_L} \leq \epsilon$$

for a number of iterations

$$k \approx \frac{\ln\frac{\epsilon}{2}}{\ln\sigma} \approx -\frac{1}{2}\sqrt{\frac{B}{A}} \ln\frac{\epsilon}{2}.$$

Comparing this result with the previous one, we observe that when B/A in σ above is big, the conjugate gradient algorithm is more faster than the Richardson iteration algorithm to compute $z = L^{-1}y$ at a fixed precision.

Chapter 4

Wigner-Ville Distributions

The mathematical approach to time-frequency quadratic distributions consists in looking for sesquilinear forms $G(f,g)(x,\omega)$; that is, G is linear in f and conjugate linear in g. Then there are two ways to make G quadratic in f. To take $Cf = |G(f,g)|^2$ and $Cf = G(f,f)$. In both cases we have:

$$C(\alpha f + \beta h) = |\alpha|^2 Cf + |\beta|^2 Ch + \alpha\overline{\beta}G(f,h) + \overline{\alpha}\beta G(h,f), \tag{4.1}$$

where $\alpha, \beta \in \mathbb{C}$. For the last decades, the more effort has been spending explaining the non linear formula (4.1) above relative to the two cross terms $G(f,h)$ and $G(h,f)$. Plotting it gives figure 4. We observe immediately the typical phenomenon of interferences. Interferences are shading created in unexpected regions of the time-frequency plane. They are caused not by a property of the signal but by the transform's quadratic property. The result is that Wigner-Ville distribution do not always reveal the exact pattern of the signal's spectrum or energy. We define the Wigner-Ville distribution of a signal:

Definition 4.1. The Wigner-Ville distribution Wf of a function $f \in L^2(\mathbb{R})$

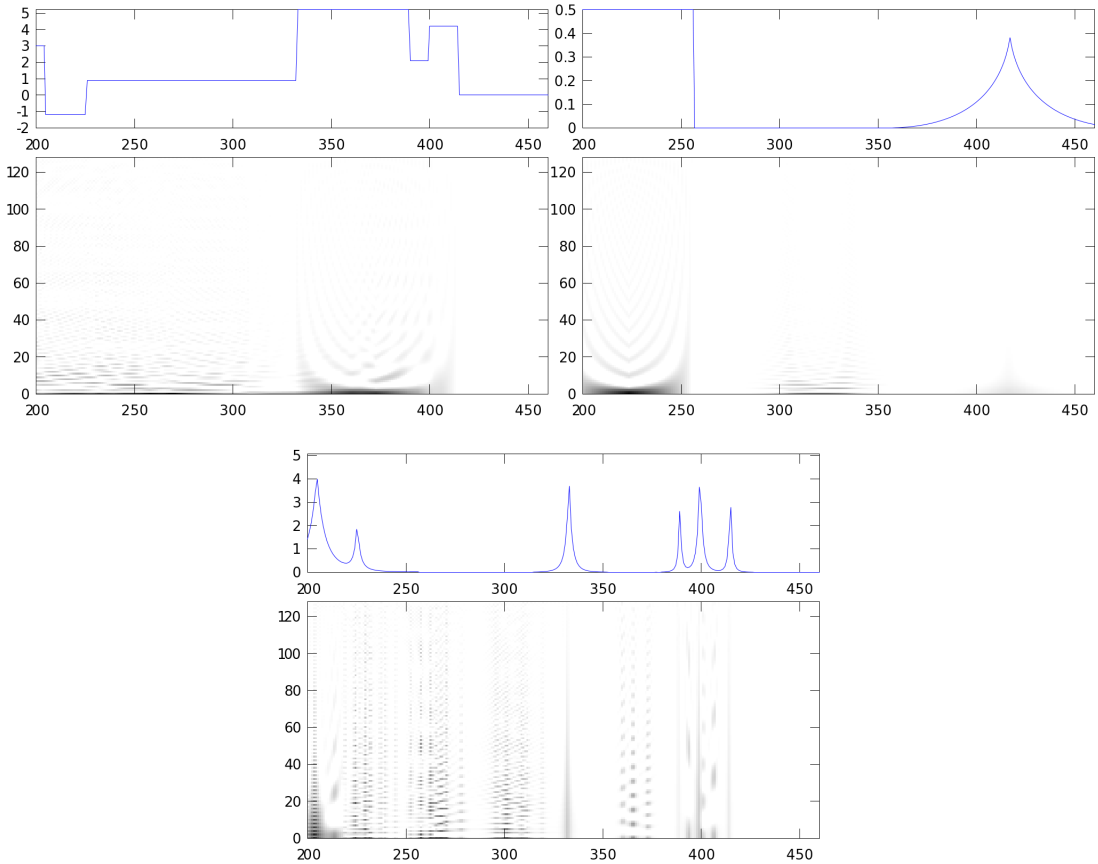


Figure 4.1: Three pairs of different signals and their corresponding Wigner-Ville distributions plot

is defined to be

$$Wf(u,\xi) = \int_{-\infty}^{\infty} f(u+\frac{t}{2})\overline{f(u-\frac{t}{2})}e^{-it\xi}dt \tag{4.2}$$

Observation 15. Our defined distribution in (4.2) is a function which takes real values because $w(f)(t,\xi) = f(\xi + t/2)\overline{f(\xi - t/2)}$ is hermitian in t (id est $w(f)(t,\xi) = \overline{w(f)(-t,\xi)}$).

Observation 16. The Wigner-Ville distribution, by polarization formulas, becomes:

$$W(f,g)(u,\xi) = \int_{-\infty}^{+\infty} f(u+\frac{t}{2})\overline{g(u-\frac{t}{2})}e^{-iu\xi}dt,$$

$\forall f, g \in L^2(\mathbb{R})$, which is named Cross-Wigner distribution.

The Cross-Wigner distribution is just a Windowed Fourier transform in disguise:

Proposition 4.1. For all $f, g \in L^2(\mathbb{R})$,

$$W(f,g)(u,\xi) = 2^d e^{2iu\xi} \int_{-\infty}^{+\infty} f(t)\overline{g(u-t)}e^{-it\xi}dt$$

Proof. We make the substitution $\eta = t + \frac{u}{2}$ in definition (4.2) and obtain

$$W(f,g)(u,\xi) = \int_{-\infty}^{+\infty} f(u+\frac{t}{2})\overline{g(u-\frac{t}{2})}e^{-iu\xi}dt$$

$$= 2^d \int_{-\infty}^{+\infty} f(\eta)\overline{g(-(\eta-2u))}e^{-2i\xi(u-t)}du$$

$$= 2^d e^{2iu\xi}\mathscr{F}_{win}f(2u, 2\xi),$$

where the Windowed Fourier transform is computed with $g(\,\cdot)$. □

There is here an orthogonality property for the Wigner-Ville distribution corresponding to which one already seen in theorem 2.1, and it implies also that the Wigner-Ville distribution is unitary (which implies that energy is conserved).

Proposition 4.2. (Moyal's formula) For every f and g in $L^2(\mathbb{R})$:

$$\frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(f)(u,\xi)W(g)(u,\xi)dud\xi = |\langle f, g\rangle|^2,$$

where $\langle f, g\rangle = \int_{-\infty}^{+\infty} f(t)\overline{g(t)}dt$.

In general, Wigner-Ville distributions are not positive. So it has been proposed, as a remedy for its negative values, to take averages at each point. The standard averaging procedure in maths is the convolution of Wf with a smoothing function σ which is centered at $(0,0)$. Then the convolution

$$([Wf]\star\sigma)(t,\xi)=\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}Wf(u,\omega)\sigma(t-u,\xi-\omega)dud\omega$$

can be seen as a local average of Wf at (t,ξ) According to our discussion of section 1.1, a region of area $\Delta t\Delta\xi<1$ in phase space does not have any physical meaning. For such small regions in phase space paradoxical conclusions may be deduced. On a formal level, these can be expressed in the form of new uncertainty principles for quadratic time-frequency representations given below, in proposition 4.3, in this particular case for the Wigner-Ville distribution. We may think that since only regions of size larger than 1 are relevant, the oscillations, caused by the cross terms in eq. (4.1), will cancel out and that $(Wf)\star g$ is non-negative for all $f\in L^2(\mathbb{R})$, whenever the support of σ is large enough. This beautiful conjecture however is not true in general. It is difficult to determine those kernels σ for which the averaged Wigner distribution $(Wf)\star g$ is always positive. It can be seen that this question is equivalent to characterizing the positivity of pseudodifferential operators by their symbol. Nevertheless, if σ is a gaussian, our intuition on the uncertainty principle is true.

Proposition 4.3. Let $\sigma_{a,b}(t,\xi)=e^{-(\frac{t^2}{a}+\frac{\xi^2}{b})}=\varphi_{\frac{a}{2}}(t)\varphi_{\frac{b}{2}}(\xi)$.

i) If $ab=1$, then $(Wf)\star\sigma_{a,b}\geq 0$ for all $f\in L^2(\mathbb{R})$.

ii) If $ab>1$, then $(Wf)\star\sigma_{a,b}>0$ for all $f\in L^2(\mathbb{R})$.

iii) If $ab<1$, then $(Wf)\star\sigma_{a,b}$ may assume negative values.

To prove this result we need the following semigroup property of Gaussians:

Lemma 4.1. *For $a>0$ we have*

$$W\varphi_a(u,\xi)=\varphi_{\frac{a}{2}}(u)\varphi_{\frac{1}{2a}}(\xi)\sqrt{2\pi a}.$$

Proof. We take a Fourier transform and we apply the same Fourier transform property used in the previous lemma:

$$W\varphi_a(u,\xi) = \int_{-\infty}^{+\infty} e^{-\frac{2}{a}[(u+\frac{t}{2})^2+(u-\frac{t}{2})^2]} e^{-it\xi} dt$$

$$= e^{-\frac{u^2}{a}} \int_{-\infty}^{+\infty} e^{-\frac{t^2}{2a}} e^{-it\xi} dt$$

$$= \varphi_{\frac{a}{2}}(u)\hat{\varphi_{2a}}(\xi)$$

$$= \sqrt{2\pi a}\varphi_{\frac{a}{2}}(u)\varphi_{\frac{1}{2a}}(\xi).$$

□

Lemma 4.2. *For $a, b > 0$ we have*

$$\varphi_a \star \varphi_b = \varphi_{2\pi(a+b)}\sqrt{\frac{ab}{a+b}}.$$

Proof. We take a Fourier transform and apply this Fourier transform property:

$$\mathscr{F}(f \star g) = (\mathscr{F}f)(\mathscr{F}g) \quad for \quad all \quad f, g \in L^2,$$

to have

$$\mathscr{F}(\varphi_a \star \varphi_b)(\xi) = \hat{\varphi}_a(\xi)\hat{\varphi}_b(\xi)$$

$$= \pi\sqrt{ab}\varphi_{\frac{2}{a}}(\xi)\varphi_{\frac{2}{b}}(\xi)$$

$$= \pi e^{-\frac{1}{4}(a+b)\xi^2}\sqrt{ab}$$

$$= \pi\sqrt{\frac{ab}{a+b}}\sqrt{a+b}\varphi_{\frac{2}{a+b}}(\xi)$$

$$= \sqrt{\frac{ab}{a+b}}\mathscr{F}(\varphi_{2\pi(a+b)})(\xi).$$

□

We are now ready to prove proposition 4.3

Proof. The trick is to write $\sigma_{a,b}$ as a Wigner distribution and the convolution as an inner product as in

$$f \star g(u) =< f, \overline{g(t-u)} >, \quad for \quad all \quad f, g \in L^2$$

where convolution is now an inner product with a translation and an involution.

Furthermore the following identity hold.

$$\begin{aligned} Wf(-u,-\xi) &= \int_{-\infty}^{+\infty} f(-u+\frac{t}{2})\overline{f(-u-\frac{t}{2})}e^{i\xi t}dt \\ &= \int_{-\infty}^{+\infty} f(-u-\frac{t}{2})\overline{f(-u+\frac{t}{2})}e^{-i\xi t}dt \\ &= W(f(- \quad \cdot))(u,\xi). \end{aligned} \tag{4.3}$$

Now assume that $ab = 1$. Then by lemma 4.1:

$$\sigma_{a,b}(u,\xi) = \varphi_{\frac{a}{2}}(u)\varphi_{\frac{1}{2a}}(\xi) = \sqrt{2\pi a}^{-1} W\varphi_a(u,\xi).$$

Using involution identity (4.3), the covariance of Wf, and Moyal's formula proposition 4.2, we obtain:

$$\begin{aligned} (Wf \star \sigma_{a,\frac{1}{a}})(u,\xi) &= \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} Wf(u-t,\xi-\eta)\sigma_{a,\frac{1}{a}}(t,\eta)dtd\eta \\ &= \int_{-\infty}^{+\infty} W(f(- \quad \cdot))(t-u,\eta-\xi)\sigma_{a,\frac{1}{a}}(t,\eta)dt, d\eta \\ &= \frac{1}{\sqrt{2a\pi}}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} W(e^{it\xi}f(-(\cdot-u))(t,\eta)\overline{W\varphi_a(t,\eta)}dtd\eta \\ &= \sqrt{2a\pi}|\langle e^{it\xi}f(-(\cdot-u)),\varphi_a\rangle|^2 \geq 0. \end{aligned}$$

In the case $ab > 1$, we can choose $0 < c < a$ and $0 < d < b$ such that $cd = 1$, and by lemma 4.2 we can write $\sigma_{a,b} = \sigma_{c,d} \star \sigma_{a-c,b-d}$. Therefore

$$Wf \star \sigma_{a,b} = (Wf \star \sigma_{c,d}) \star \sigma_{a-c,b-d} > 0$$

is strictly positive since it is a convolution of a non-negative function with a positive function.

Finally, in the last case $ab < 1$ we have that $f(t) = te^{-t^2}$ will give

$$(Wf \star \sigma_{a,b})(0,0) < 0.$$

$\square$

Chapter 5

Wavelet transform

Now we introduce the wavelet transform.

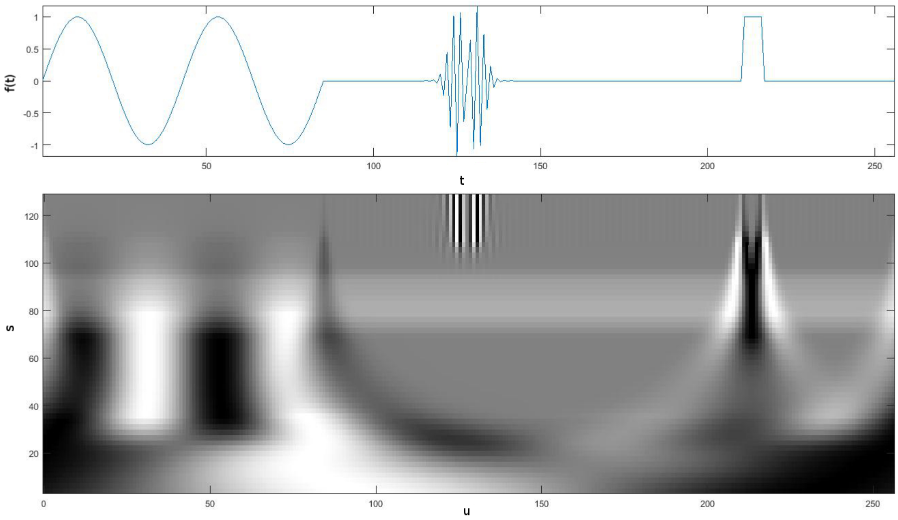
Definition 5.1. A wavelet is a function $\psi \in L^2(\mathbb{R})$, symmetric, with a zero average property ($\int_{-\infty}^{+\infty} \psi(t)dt = 0$) and $||\psi|| = 1$ (not necessarily with compact support). We translate and scale the wavelet in order to weight the signal in the integral below with $\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})$. The wavelet transform is:

$$\mathscr{F}_{wave}(f)(u,s) := \int_{-\infty}^{+\infty} f(t)\frac{1}{\sqrt{s}}\psi^*(\frac{t-u}{s})dt \tag{5.1}$$

The symbol * in the definition is the complex conjugate.

Observation 17. In the figure below there is the wavelet transform applied to a signal varying by time. The wavelet is not shown because changes in width as the parameter s change.

We could define here Heisenberg boxes and Ambiguity functions mentioned above identically as for the windowed Fourier transform case.



Observation 18. In $L^2(\mathbb{R})$, Hilbert space, we decompose the signal f in the subspace generated by the following family of vectors:

$$\{g(t-u)e^{it\xi}\}_{u,\xi}$$

making the windowed Fourier transform. We can observe that

$$g(t-u-v)e^{i(t-v)\xi} = g((t-v)-u)e^{i(t-v)\xi}$$

and

$$g(t-u)e^{it\xi}e^{it\omega} = g(t-u)e^{it(\xi+\omega)}.$$

So the family of vectors is closed under time and frequency translation. So the windowed Fourier transform (2.1) above is particularly useful in analyzing patterns that are translated in time and frequency

The aforementioned wavelet transform (5.1), instead is useful to analyze patterns translated and scaled. In fact, considering the family of vectors $\{\frac{1}{\sqrt{s}}\psi(\frac{t-u}{s})\}_{u,s}$, we have:

$$\frac{1}{\sqrt{s}}\psi(\frac{t-u-v}{s}) = \frac{1}{\sqrt{s}}\psi(\frac{(t-v)-u}{s})$$

and

$$\frac{1}{\sqrt{s}}\frac{1}{\sqrt{r}}\psi(\frac{t-u}{rs}) = \frac{1}{\sqrt{rs}}\psi(\frac{t-u}{rs}).$$

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