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Relatore:
Dott.ssa
ALESSIA CATTABRIGA

Presentata da:
CLAUDIA MAGNAPERA

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Abstract

Minimal surfaces are of great interest in various fields of mathematics, and yet have lots of application in architecture and biology, for instance. It is possible to list different equivalent definitions for such surfaces, which correspond to different approaches. In the following thesis we will go through some of them, concerning: having mean curvature zero, being the solution of Lagrange's partial differential equation, having the harmonicity property, being the critical points of the area functional, being locally the least-area surface with a given boundary and proving the existence of a solution of the Plateau's problem.

Le superfici minime, sono di grande interesse in vari campi della matematica, e parecchie sono le applicazioni in architettura e in biologia, ad esempio. È possibile elencare diverse definizioni equivalenti per tali superfici, che corrispondono ad altrettanti approcci. Nella seguente tesi ne affronteremo alcuni, riguardanti: la curvatura media, l'equazione differenziale parziale di Lagrange, la proprietà di una funzione di essere armonica, i punti critici del funzionale di area, le superfici di area minima con bordo fissato e la soluzione del problema di Plateau.

Introduction

The aim of this thesis is to go through the classical minimal surface theory, in this specific case to state six equivalent definitions of minimal surface and to prove the equivalence between them.
It's interesting to recall some historical facts that brought to the born of the theory. Notice that while nowadays new technologies help us to visualize a larger number of examples, at the beginning of the study, in the 18th century, examples were rare.
The first known attempt of proving the existence of a minimal surface is ascribed to Euler. In 1741, he successfully found the catenoid, a surface of revolution generated by the rotation of a plane curve, namely the catenary. Given a circle γ on the xy-plane and translating it in the direction of the z-axis to the circle γ', the problem was to find a surface, with least area, having these two circles as boundaries. Not many years later he also proved that this is the only surface of revolution to be minimal, beside the plane which is a trivial example.
In 1776 J. B. Meusnier showed that there exists another minimal surface, which is the helicoid - its shape reminds us, for example, the shape of DNA and the screw that Archimedes used to pump water. Furthermore, he proved that the helicoid was the only ruled surface to be minimal, together with the plane. Meusnier even proved that both the catenoid and the helicoid were solutions of a special partial differential equation, called the Lagrange's equation.
From this moment this fascinating subject has been studied by different points of view and this thesis is trying to put in evidence this aspect. From a physical point of view, another important contribution was give by J. Plateau (1801 - 1883), via the so-called Plateau's problem. The Plateau's problem regards the existence of a surface with least area spanning a given boundary. The experimentations he did made use of soap (and glycerie) films, chosen thanks to their physical properties.
Concerning the field of the calculus of variations, the problem of minimal surfaces can be stated in terms of making stationary the area of a surface with respect to variations in the normal direction. So these surfaces are solutions of a variational problem, specifically they are critical points of the area functional.

Specifically the work is structured as follows. The first chapter foretells and provides the reader of the principal concepts about regular surfaces in $\mathbb{R}^3$. The second chapter deepens what the thesis is aiming to. It tries to flow coherently through minimal surface's equivalent definitions, giving a proof of the equivalence. Every definition is accompanied by an example and the dissertation is completed with all the tools, which are useful to understand the equivalences.

Contents

Chapter 1

Regular surfaces in $\mathbb{R}^3$

The following chapter is useful to be in touch with the argument we are treating, specifically we will recall concepts about regular surfaces in $\mathbb{R}^3$, referring to [3]. In order to make the following notions clearer, we will exemplify each one of them via a specific surface.

1.1 Definition and tangent plane

In this section we recall how to define a regular surface in $\mathbb{R}^3$ and some related notions.

Roughly speaking the geometric object we are considering, which is a regular surface in $\mathbb{R}^3$, is obtained by taking pieces of a plane and deforming them in such a way that the resulting figure has no sharp points, edges or self-intersections. First recall that a real function of a real variable is smooth if it has continuous derivatives of all orders.

Definition 1.1. A subset $S \subset \mathbb{R}^3$ is a regular surface if for every point $p \in S$ there exists an open set $V \subset \mathbb{R}^3$ containing p and a map $\mathbf{x} : U \to S \cap V$, where U is an open subset of $\mathbb{R}^2$ such that:

1. $\mathbf{x}$ is a homeomorphism;
2. $\mathbf{x}$ is a smooth map and its inverse $\mathbf{x}^{-1}$ is smooth, too. In particular, $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ with x, y, z smooth functions;
3. the partial derivatives $\mathbf{x}_u = \frac{\partial \mathbf{x}}{\partial u}$ and $\mathbf{x}_v = \frac{\partial \mathbf{x}}{\partial v}$ are linearly independent $\forall (u, v) \in U$.

The map $\mathbf{x}$ is called a local parametrisation of S at p.

To not fall into the problem of deciding whether a surface is regular or not, next proposition gives us a relation between the above definition of a regular surface and the concept of the graph of a function.

Proposition 1.1.1. *Let $U \subset \mathbb{R}^2$ be an open set and let $f : U \to \mathbb{R}$ be a smooth function. Then the graph of f,*

$$graph(f) := \{(u, v, f(u, v)) \in \mathbb{R}^3 \,|\, (u, v) \in U\}$$

is a regular surface in $\mathbb{R}^3$.

Next proposition states that any regular surface is locally the graph of a differentiable function and, therefore, it is a local converse of the above proposition.

Proposition 1.1.2. *Let $S \subset \mathbb{R}^3$ be a regular surface and $p \in S$. Then there exists an open set $V \subset \mathbb{R}^3$ containing p in S such that $V \cap S$ is the graph of a differentiable function.*

An example of a regular surface is the *catenoid*, depicted in Figure 1.1. This is a surface of revolution, which means that it is obtained by rotating a planar curve α, called the *generating curve*, about an axis in the plane which doesn't meet the curve. More precisely, we take the yz-plane as the plane of the curve and the z-axis as the rotation axis. Let

$$y = f(v) \;\; z = g(v),$$

with $a < v < b$ and $f(v) \neq 0$, be a parametrization of α. Thus, we obtain the map $\mathbf{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$ from the open set $U = \{(u, v) \in \mathbb{R}^2; 0 \leq u < 2\pi,\ a < v < b\}$ into the surface. The circles described by the points of α, while rotating so with v constant, are called *parallels* of the surface and the curves corresponding to u constant, which are "copies" of α, are called the *meridians* of S.

Given the generating curve $y = \frac{1}{k}\cosh(kz)$, with $k \neq 0$, called the catenary, and rotating it about the z-axis, we obtain the catenoid having the following parametrization

$$\mathbf{x}(u, v) = (\frac{1}{k}\cosh v \cos u, \frac{1}{k}\cosh v \sin u, kv)$$

with $0 \leq u < 2\pi$, $-\infty < v < \infty$.

The condition 3 of Definition 1.1 of a regular surface is the so-called regularity condition and it is equivalent to say that, for each $q \in U$ the differential $\mathrm{d}\mathbf{x}_q : \mathbb{R}^2 \to \mathbb{R}^3$ is injective. Furthermore this condition guarantees the existence of a tangent plane at every point of the surface. Let's state the definition of tangent plane and specify some notions about it.

Definition 1.2. Let S be a regular surface and let $p \in S$. Then:

1. a tangent vector to S at $p \in S$ is a tangent vector $\alpha'(0) \in \mathbb{R}^3$, where $\alpha : (-\epsilon, \epsilon) \to S \subset \mathbb{R}^3$ with $\alpha(0) = p$, and α smooth;

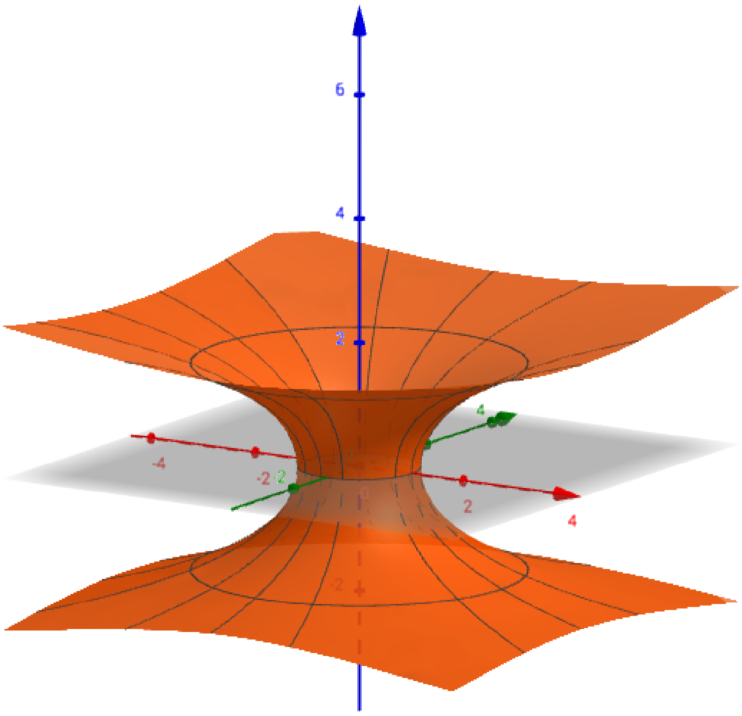

Figure 1.1: The catenoid.

2. the tangent plane to S at p is defined as the set of all tangent vectors to S at p, i.e.,

$$\begin{aligned} T_pS &:= \{\mathbf{w} \in \mathbb{R}^3 \,|\, \mathbf{w} \text{ is a tangent vector to } S \text{ at } p\} \\ &= \{\alpha'(0) \,|\, \alpha : (-\epsilon, \epsilon) \to S \text{ smooth, } \alpha(0) = p\}. \end{aligned}$$

Proposition 1.1.3. *Let $\mathbf{x} : U \to S$ be a local parametrization of a regular surface S with $U \subset \mathbb{R}^2$ open and let $q \in U$ with $\mathbf{x}(q) = p$, then $\mathrm{d}_q\mathbf{x}(\mathbb{R}^2) = T_pS$.*

In particular, the tangent plane is a two dimensional vector subspace of $\mathbb{R}^3$ spanned by $\mathbf{x}_u(q)$ and $\mathbf{x}_v(q)$.

The tangent plane $d_q\mathbf{x}(\mathbb{R}^2)$ which passes through $\mathbf{x}(q) = p$ doesn't depend on the local parametrization $\mathbf{x}$, but the choice of a parametrization determines a basis associated to $\mathbf{x}$, which is $\{\mathbf{x}_u(q), \mathbf{x}_v(q)\}$.

Given the parametrization of the catenoid as before with $k = 1$, let us calculate its partial derivatives. We have

$$\mathbf{x}_u(u, v) = (-\cosh v \sin u, \cosh v \cos u, 0)$$

$$\mathbf{x}_v(u, v) = (\sinh v \cos u, \sinh v \sin u, 1)$$

and so, the tangent plane T_pS is spanned by $\{\mathbf{x}_u, \mathbf{x}_v\}$.

1.2 First fundamental form

On each tangent plane it is possible to define an inner product as follows. Specifically, the canonical inner product of $\mathbb{R}^3$ induces on every tangent plane T_pS of a regular surface S an inner product, denoted by $<, >_p$ by taking the inner product $\mathbf{w}_1 \cdot \mathbf{w}_2$ of $\mathbf{w}_1, \mathbf{w}_2$ as vectors in $\mathbb{R}^3$ for $\mathbf{w}_1, \mathbf{w}_2 \in T_pS \subset \mathbb{R}^3$

$$<, >_p : T_pS \times T_pS \to \mathbb{R}$$

$$(\mathbf{w}_1, \mathbf{w}_2) \mapsto \mathbf{w}_1 \cdot \mathbf{w}_2$$

Furthermore, to this inner product corresponds a quadratic form, denoted by I_p, as defined below.

Definition 1.3. Let S be a regular surface and let $p \in S$, then the quadratic form $I_p : T_pS \to \mathbb{R}$ with $I_p(\mathbf{w}) :=< \mathbf{w}, \mathbf{w} >_p= ||\mathbf{w}||_p^2$ is called the first fundamental form (shorten 1FF) at $p \in S$ of a regular surface S.

Definition 1.4. The functions $E, F, G : U \to \mathbb{R}$ defined by

$$E :=< \mathbf{x}_u, \mathbf{x}_u >_p, \; F :=< \mathbf{x}_u, \mathbf{x}_v >_p, \; G :=< \mathbf{x}_v, \mathbf{x}_v >_p$$

are called the coefficients of the 1FF in the local parametrization $\mathbf{x} : U \to S$.

Between all the possible parametrizations, one of the most interesting is the so-called isothermal parametrization, which is useful as we will see later. For the proof of the existence of a local isothermal coordinate system on any regular surface see [5].

Definition 1.5. Let $\mathbf{x} = \mathbf{x}(u, v)$ be a parametrization of a regular surface, then $\mathbf{x}$ is said to be isothermal if $< \mathbf{x}_u, \mathbf{x}_u >=< \mathbf{x}_v, \mathbf{x}_v >$ and $< \mathbf{x}_u, \mathbf{x}_v >= 0$, that is to say $E = G$ and $F = 0$.

Using the previous coefficients, it is possible to express the 1FF in terms of a local parametrization $\mathbf{x}$ as follows.

Remark 1.1. Let $p \in S$ and let $\mathbf{x} : U \to S$, with $U \subset \mathbb{R}^2$ open, be a local parametrization of a regular surface S at $p = \mathbf{x}(q)$ with $q \in U$. It is possible to write $\mathbf{w} \in T_pS$ in terms of the basis $(\mathbf{x}_u, \mathbf{x}_v)$ as

$$\mathbf{w} = a\mathbf{x}_u(q) + b\mathbf{x}_v(q),$$

where $(a, b) \in \mathbb{R}^2$ are the coordinates of $\mathbf{w}$ with respect to this basis. So,

$$I_p(\mathbf{w}) =< \mathbf{w}, \mathbf{w} >_p=$$

$$= a^2 < \mathbf{x}_u(q), \mathbf{x}_u(q) >_p +2ab < \mathbf{x}_u(q), \mathbf{x}_v(q) >_p +b^2 < \mathbf{x}_v(q), \mathbf{x}_v(q) >_p=$$

$$= a^2E(q) + 2abF(q) + b^2G(q).$$

Let us compute the coefficients of the 1FF for the catenoid, which are:

$$E = \cosh^2 v(\sin^2 u + \cos^2 u) = \cosh^2 v,$$

$$F = -\cosh v \sinh v \sin u \cos u + \cosh v \sinh v \sin u \cos u = 0,$$

$$G = \sinh^2 v(\sin^2 u + \cos^2 u) + 1 = \cosh^2 v.$$

On a regular surface, the 1FF allows us to compute, for example, the length of a curve, the angle between curves and the area of a bounded region of it. Focusing on how to calculate the area of a subset of a regular surface, we need to understand how this subset is defined: let R be a region of a regular surface S and let $\mathbf{x} : U \to S$ be a local parametrization. We assume that R is the image by $\mathbf{x}$ of a bounded region $\widetilde{R} \subset U$. Now we can end up to the definition of area.

Definition 1.6. With the previous assumptions, the non negative number

$$A(R) := \int\int_R \mathrm{d}A = \int\int_{\widetilde{R}} |\mathbf{x}_u \wedge \mathbf{x}_v| \,\mathrm{d}u\mathrm{d}v, \quad \widetilde{R} = \mathbf{x}^{-1}(R),$$

is called the area of R.

Note that $|\mathbf{x}_u \wedge \mathbf{x}_v| = \sqrt{EG - F^2}$.
For what concerns the catenoid, we have

$$\mathrm{d}A = \cosh^2 v \,\mathrm{d}u\mathrm{d}v.$$

1.3 Second fundamental form and curvatures

Next steps will lead us to the definition of the second fundamental form (shorten 2FF) and eventually to introduce the notion of curvature, which is essential to study surfaces and to measure the bending of a surface in the space.

Let's introduce a function $\mathbf{n}$, named Gauss map, which assigns to each point of the surface its unit normal vector. This map allows us to investigate how different a surface S is from the sphere S^2. Given a parametrization $\mathbf{x}$ of a regular surface S at a point $p \in S$, a unit normal vector in the point p can be chosen by using one of the two possibilities of $\mathbf{n}(p) = \pm \frac{\mathbf{x}_u \wedge \mathbf{x}_v}{|\mathbf{x}_u \wedge \mathbf{x}_v|}(p)$. This local definition of $\mathbf{n}(p)$ can't always be extended coherently to the whole surface. Whenever it is possible we say that the surface is orientable. Note that if a surface is orientable, then there are two different possible choices for $\mathbf{n}$ and we call orientation a choice between one of these two. Moreover, we say that a surface is oriented whenever we've chosen an orientation for it.

Definition 1.7. Let $S \subset \mathbb{R}^3$ be a surface with an orientation $\mathbf{n}$. The map $\mathbf{n} : S \to \mathbb{R}^3$ takes its values in the unit sphere $S^2 := \{(x, y, z) \in \mathbb{R}^3 |\, x^2 + y^2 + z^2 = 1\}$. The smooth map $\mathbf{n} : S \to S^2$, thus defined, is called the Gauss map of S.

The catenoid is an orientable surface and we have

$$(\mathbf{x}_u \wedge \mathbf{x}_v)(u, v) = (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v),$$

so that the Gauss map of the catenoid in our local parametrization is:

$$\mathbf{n}(\mathbf{x}(u, v)) = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).$$

From now on we will always suppose that S is an oriented surface.

By the fact that $\mathbf{n}$ is smooth, we are interested in considering its differential $\mathrm{d}_p\mathbf{n}$ at $p \in S$. It turns out that $\mathrm{d}_p\mathbf{n}$ is symmetric, which means that $< \mathrm{d}_p\mathbf{n}(\mathbf{w}_1), \mathbf{w}_2 >=< \mathbf{w}_1, \mathrm{d}_p\mathbf{n}(\mathbf{w}_2) > \forall \mathbf{w}_1, \mathbf{w}_2 \in T_pS$.

Lemma 1.3.1. *Let S be a surface in $\mathbb{R}^3$ and let $\mathbf{n} : S \to S^2$ be its Gauss map. Then $\mathrm{d}_p\mathbf{n}(\mathbf{w})$ is normal to $\mathbf{n}(p)$ for each $\mathbf{w} \in T_pS$. In particular, we can consider $d_p\mathbf{n}$ as the following map*

$$\mathrm{d}_p\mathbf{n} : T_pS \to T_pS.$$

The map $-\mathrm{d}_p\mathbf{n} : T_pS \to T_pS$ is symmetric and is called the Weingarten map of the surface $S \subset \mathbb{R}^3$ at $p \in S$. We call the quadratic form $II_p : T_pS \to \mathbb{R}$, $II_p(\mathbf{w}) :=< -\mathrm{d}_p\mathbf{n}(\mathbf{w}), \mathbf{w} >$, the second fundamental form of S at p, (shorten 2FF).

Definition 1.8. Let S be a regular surface in $\mathbb{R}^3$ with Gauss map $\mathbf{n} : S \to S^2$ and let $\mathbf{x} : U \to S$ be a local parametrization. We call

$$L :=< \mathbf{x}_u, -\mathrm{d}_p\mathbf{n}(\mathbf{x}_u) >_p =< \mathbf{x}_{uu}, \mathbf{n} >_p,$$

$$M :=< \mathbf{x}_u, -\mathrm{d}_p\mathbf{n}(\mathbf{x}_v) >_p =< \mathbf{x}_v, -\mathrm{d}_p\mathbf{n}(\mathbf{x}_u) >_p =< \mathbf{x}_{uv}, \mathbf{n} >_p,$$

$$N :=< \mathbf{x}_v, -\mathrm{d}_p\mathbf{n}(\mathbf{x}_v) >_p =< \mathbf{x}_{vv}, \mathbf{n} >_p$$

the coefficients of the 2FF in the given parametrisation.

Let us compute these coefficients for the catenoid:

$$\mathbf{x}_{uu}(u, v) = (-\cosh v \cos u, -\cosh v \sin u, 0),$$

$$\mathbf{x}_{uv}(u, v) = (-\sinh v \sin u, \sinh v \cos u, 0),$$

$$\mathbf{x}_{vv}(u, v) = (\cosh v \cos u, \cosh v \cos u, 0).$$

Recalling that

$$\mathbf{n}(\mathbf{x}(u, v)) = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v),$$

we have

$$L = -1, \ \ M = 0, \ \ N = 1.$$

We are ready to introduce the concept of curvature. The curvature of a surface is described by two numbers at each point, called the principal curvatures, so that combination of the principal curvatures give us what we call the Gaussian curvature and the mean curvature.

Definition 1.9. Let S be regular surface in $\mathbb{R}^3$ with Gauss map and Weingarten map as above defined.

- The eigenvalues $k_1(p), k_2(p)$ of $-\mathrm{d}_p\mathbf{n}$ are called principal curvatures of S at p;
- The eigenvectors $e_1(p), e_2(p)$ of $-\mathrm{d}_p\mathbf{n}$ are called principal directions of S at p.
- $K(p) := det(-\mathrm{d}_p\mathbf{n}) = k_1(p) \cdot k_2(p)$ is called Gauss curvature of S at p;
- $H(p) := \frac{1}{2}trace(-\mathrm{d}_p\mathbf{n}) = \frac{1}{2}(k_1(p) + k_2(p))$ is called mean curvature of S at p.

In the next proposition we are going to see how we can re-write all the above curvatures with respect to a local parametrization.

Proposition 1.3.1. *Let E, F, G and L, M, N be the coefficients, respectively, of the 1FF and the 2FF of a surface S with respect to a local parametrization $\mathbf{x}$, then*

1. *the Gauss curvature is $K = \frac{LN-M^2}{EG-F^2}$;*

2. *the mean curvature is $H = \frac{EN-2FM+GL}{2(EG-F^2)}$;*

3. *the principal curvatures are the roots of $k^2 - 2Hk + K = 0$;*

4. *the principal directions are given by $a\mathbf{x}_u + b\mathbf{x}_v$, where the coefficients (a, b) are the solutions of*
$$det \begin{bmatrix} b^2 & -ab & a^2 \\ E & F & G \\ L & M & N \end{bmatrix} = 0.$$

Finally, we are ready to calculate all the curvatures in the case of the catenoid. Neatly,

- the Gauss curvature $K = -\frac{1}{\cosh^4 v}$;
- the mean curvature $H = 0$;
- the principal curvatures are $k_1(p) = \frac{1}{\cosh^2 v}$ and $k_2(p) = -\frac{1}{\cosh^2 v}$.

The next definition is the one of lines of curvature, which are curves on a surface whose tangents are always in the direction of the principal curvatures.

Definition 1.10. Let S be a regular surface. A smooth curve $\alpha : I \subset \mathbb{R}^3 \to S$ in a surface $S \subset \mathbb{R}^3$ is called a line of curvature if $\alpha'(t)$ is a principal direction at $\alpha(t)\, \forall t \in I$; i.e. $\alpha'(t)$ is an eigenvector of the Weingarten map at $\alpha(t)\, \forall t \in I$.

Definition 1.11. Let S be a regular surface and $K(p)$ its Gauss curvature at the point $p \in S$. We say that p is:

- elliptic if $K(p) > 0$;
- parabolic if $K(p) = 0$
- hyperbolic if $K(p) < 0$;
- flat or planar if $k_1(p) = k_2(p) = 0$.

The last definition we give is the one of local isometry and what it means to be locally isometric.

Definition 1.12. Let $f : S \to \widetilde{S}$ be a smooth map between two surfaces. Then the map f is called a local isometry if

$$< \mathrm{d}_p f(\mathbf{w}_1), \mathrm{d}_p f(\mathbf{w}_2) >_{f(p)} = < \mathbf{w}_1, \mathbf{w}_2 >_p$$

$\forall \mathbf{w}_1, \mathbf{w}_2 \in T_p S$ and $p \in S$. The surfaces S and $\widetilde{S}$ are said to be locally isometric.

Despite the principal curvatures and the mean curvature are not intrinsic properties (i.e. not preserved under local isometries), the Gaussian curvature is. We end this section by stating the so-called Theorema Egregium, proved by Gauss, which, in substance, formalises what we have just said.

Theorem 1.3.2 (Theorema Egregium)**.** *The Gaussian curvature on a surface in $\mathbb{R}^3$ depends on E, F, G and their derivatives only (in a local parametrisation) and so it is invariant under local isometries.*

Chapter 2

Different approaches to minimal surfaces theory

This is the main chapter of the thesis: it flows between the different definitions of what a minimal surface is and tries to give a general point of view of the subject in question. For further references we refer to [1, 2].

2.1 Via mean curvature

The first definition of a minimal surface we deal with is the one that comes further from the notions introduced in the first chapter.

Definition 2.1 (1). A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if its mean curvature vanishes identically.

The condition given by the mean curvature implies that all the points of a minimal surface are hyperbolic or planar. In fact, if $H = 0$, then $k_1(p) = -k_2(p)$ and so the Gaussian curvature is always $K \leq 0$.

In the previous chapter we saw that the mean curvature of the catenoid is equal to zero and therefore we can say that the catenoid is a minimal surface.

Proposition 2.1.1. *Among the surfaces of revolution, the catenoid is, together with the plane, the only one being minimal.*

Proof. The proof of the statement consists in characterising the curves $y = f(z)$ which describe a minimal surface while rotating around the z-axis. Since the parallels and the meridians of a surface of revolution are the lines of curvature of it, we must have that the curvature of the curve $y = f(z)$ is the negative of the normal curvature of the circle

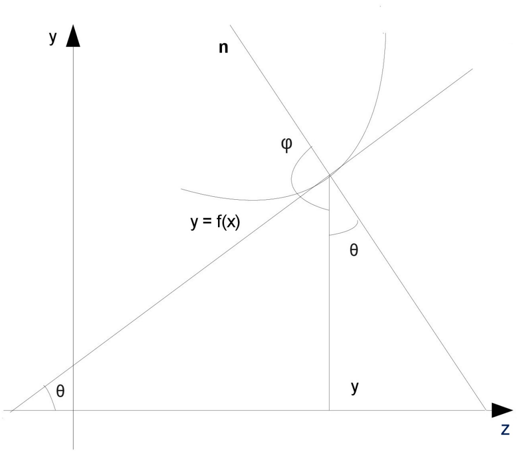

Figure 2.1: .

generated by the point $f(z)$ (both are principal curvatures).

From now on we refer to Figure 2.1. Calculating the curvature of the curve $y = f(z)$, we have

$$\frac{y''}{(1+(y')^2)^{\frac{3}{2}}} = -\frac{\cos\phi}{y},$$

with the curvature of the circle equals to the projection of its curvature (i.e. $\frac{1}{y}$) over the normal **n** to the surface.
But $-\cos\phi = \cos\theta$ and $\tan\theta = y'$ and we have

$$\frac{y''}{(1+(y')^2)^{\frac{3}{2}}} = \frac{1}{y}\frac{1}{(1+(y')^2))^{\frac{1}{2}}}$$

as the equation to be satisfied by the curve. We exclude this surface is a plane, so there exists a point z where $f'(z) \neq 0$, we obtain

$$\frac{2y''y'}{1+(y')^2} = \frac{2y'}{y}.$$

Set $1+(y')^2 = w$, we have

$$\frac{w'}{w} = \frac{2y'}{y}$$

and integrating this, we have

$$\log w = \log y^2 + \log k^2 = \log(yk)^2$$

with k constant. This expression is, after substituting $1 + (y')^2 = w$, separating the variables and keeping in mind that $y' = \frac{\mathrm{d}y}{\mathrm{d}z}$,

$$\frac{\mathrm{d}y}{\sqrt{(yk)^2 - 1}} = \mathrm{d}z.$$

Again by integrating, the last expression gives

$$\cosh^{-1}(yk) = kz + c,$$

with c constant. Equivalently,

$$y = \frac{1}{k}\cosh(kz + c).$$

Thus, in the neighbourhood of a point where $f' \neq 0$, the curve $y = f(z)$ is a catenary. But then y' can only be zero at $z = 0$ and if the surface has to be connected, it is by continuity a catenoid. □

2.2 Via Lagrange's equation

In 1776 the French mathematician Meusnier verified that the catenoid is a solution of an important equation that we will introduce as a second definition of minimal surface. In fact, at the beginning of the study of this theory, minimal surfaces were seen mostly as solutions to a special second order partial differential equation, but only later it was possible to understand that satisfying this equation was equivalent to have mean curvature zero. So, what we are going to do is, firstly to give the definition, secondly to state the equivalence between Definition 2.1 and Definition 2.2 and in the end to prove it.

Recall that the notation that follows now on is:

$$f_u = \frac{\partial f}{\partial u}, \ \ f_v = \frac{\partial f}{\partial v}.$$

Definition 2.2 (2)**.** A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if it can be locally expressed as the graph of a solution of the following equation, called Lagrange's equation,

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0.$$

Proposition 2.2.1. *Definition 2.1 and Definition 2.2 are equivalent.*

Proof. Let $\mathbf{x}(u, v) = (u, v, f(u, v))$ be a parametrization for the surface S. Then we have

$$\mathbf{x}_u = (1, 0, f_u),$$

$$\mathbf{x}_v = (0, 1, f_v),$$

and

$$\mathbf{n}(\mathbf{x}(u, v)) = \frac{(-f_u, f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}.$$

The 1FF and the 2FF coefficients are, respectively:

$$E = (1 + f_u^2),\ F = f_u f_v,\ G = (1 + f_v^2),$$

$$L = \frac{f_{uu}}{1 + f_u^2 + f_v^2},\ M = \frac{f_{uv}}{1 + f_u^2 + f_v^2},\ N = \frac{f_{vv}}{1 + f_u^2 + f_v^2}.$$

Let us substitute these into the mean curvature formula, we have:

$$H = \frac{(1 + f_v^2)f_{vv} - 2f_{uv}f_u f_v + (1 + f_u^2)f_{vv}}{2(1 + f_u^2 + f_v^2)\sqrt{1 + f_u^2 + f_v^2}}$$

we have that S is a minimal surface if and only if $H = 0$ if and only if

$$(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0,$$

which is exactly the differential equation we were looking for. □

Using the Lagrange's equation we can check that the surface, called the helicoid and depicted in Figure 2.2, is minimal. It can be expressed as the following graph of function

$$f(u, v) = \frac{1}{\tan(\frac{v}{u})} = \arctan\left(\frac{v}{u}\right).$$

Computing the partial derivatives

$$f_u = -\frac{v}{u^2 + v^2},\ f_v = \frac{u}{u^2 + v^2},$$

$$f_{uu} = \frac{2uv}{(u^2 + v^2)^2},$$

$$f_{uv} = \frac{v^2 - u^2}{(u^2 + v^2)^2},$$

$$f_{vv} = -\frac{2uv}{(u^2 + v^2)^2}$$

and putting these results into the Lagrange's equation we see that it is satisfied,

$$\left(1+\frac{v^2}{(u^2+v^2)^2}\right)\left(-\frac{2uv}{(u^2+v^2)^2}\right)-2\left(-\frac{v}{u^2+v^2}\right)\left(\frac{u}{u^2+v^2}\right)\left(\frac{v^2-u^2}{(u^2+v^2)^2}\right)+$$

$$+\left(1+\frac{u^2}{(u^2+v^2)^2}\right)\left(\frac{2uv}{(u^2+v^2)^2}\right)=0$$

So this is a solution of the Lagrange's equation.

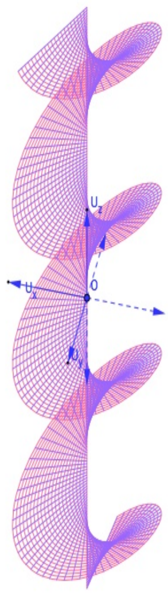

Figure 2.2: The helicoid.

A ruled surface $S \subset \mathbb{R}^3$ is a surface which contains at least one 1-parameter family of straight lines. A ruled surface has a parametrization $\mathbf{x} : I \to S$ of the form $\mathbf{x}(u,v) = \alpha(u) + v\beta(u)$ where α is a smooth regular space curve and β is a never zero smooth map. We call such $\mathbf{x}$ a ruled coordinate system, α the directrix and β the director curve. The rulings are the straight lines $u \mapsto \alpha(u) + v\beta(u)$. Specifically, a local parametrization of the helicoid is

$$\mathbf{x}(u,v) = (\sinh v \cos u, \sinh v \sin u, u),$$

with $0 < u \le 2\pi$, $-\infty < v < \infty$.

Proposition 2.2.2. *Among the ruled surfaces, the helicoid is, together with the plane, the only one being minimal.*

For the proof of the above proposition see [2, 3].
There's a deep connection between the helicoid and the catenoid. Indeed it is possible to continuously deformate the helicoid into the catenoid through locally isometric minimal surfaces, as depicted in Figure 2.3. Simply, the deformation consists of cutting the catenoid vertically and twisting it round itself and it can be written as follows

$$\begin{cases} x = \sin(\theta)\cosh(v)\cos(u) + \cos(\theta)\sinh(v)\cos(u), \\ y = \sin(\theta)\cosh(v)\sin(u) - \cos(\theta)\sinh(v)\sin(u), \\ z = v\sin(\theta) + u\cos(\theta) \end{cases}$$

where $(u, v) \in [0, \frac{\pi}{2}] \times (-\infty, \infty)$ and θ is the deformation parameter in the interval.

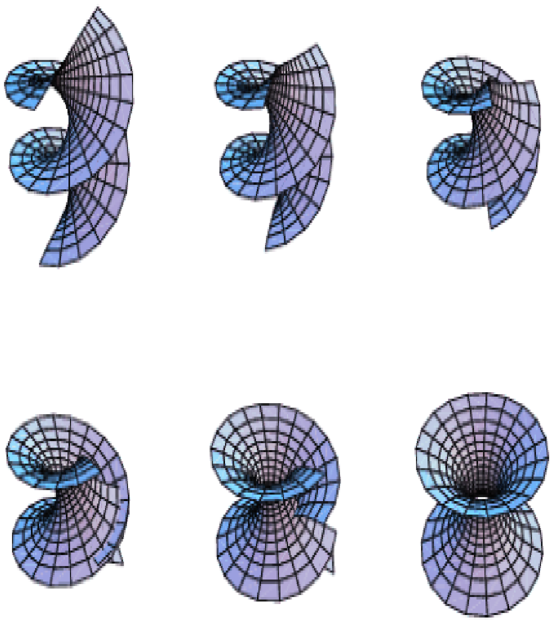

Figure 2.3: Deformation of the helicoid to the catenoid.

Both the catenoid and the helicoid are harmonic functions in the sense of the following definition (for more references see [4]).

Definition 2.3. A function $f : D \to \mathbb{R}$ is said to be harmonic in a domain $D \subseteq \mathbb{R}^2$ if the partial derivatives

$$f_x, f_y, f_{xx}, f_{yy}$$

exist and are continuous, and if

$$\Delta f := f_{xx} + f_{yy} = 0$$

at all point of D.

Computing these partial derivatives for the helicoid, parametrized by $\mathbf{x}$, we have

$$\mathbf{x}_{uu} = (-\cos u \sinh v, -\sin u \sinh v, 0)$$

and

$$\mathbf{x}_{vv} = (\cos u \sinh v, \sin u \sinh v, 0)$$

and the sum of these two is 0.
Doing the same computation for the catenoid, parametrized by $\mathbf{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$ we have

$$\mathbf{x}_{uu} = (-\cosh v \cos u, -\cosh v \sin u, 0)$$

$$\mathbf{x}_{vv} = (\cosh v \cos u, \cosh v \sin u, 0)$$

and the sum is 0, too.

As we will see in the next definition the property of being expressed through harmonic functions concerns all minimal surfaces.

Definition 2.4 (3)**.** Let $\mathbf{x}(u, v) = (x(u, v), y(u, v), z(u, v))$ be a parametrized surface and assume that $\mathbf{x}$ is isothermal. Then $\mathbf{x}$ is *minimal* if and only if its coordinate functions x, y, z are harmonic.

The equivalence among Definition 2.1 and Definition 2.4 follows directly from the following proposition.

Proposition 2.2.3. *Let $\mathbf{x} = \mathbf{x}(u,\ v)$ be a parametrization of a regular surface and let $\mathbf{x}$ be isothermal. Then*

$$|\mathbf{x}_{uu} + \mathbf{x}_{vv}| = 2\lambda^2 H,$$

where $\lambda^2 =< \mathbf{x}_u, \mathbf{x}_u >=< \mathbf{x}_v, \mathbf{x}_v >$ *and* $< \mathbf{x}_u, \mathbf{x}_v >= 0$.

Proof. Since $\mathbf{x}$ is isothermal, we have that $< \mathbf{x}_u, \mathbf{x}_u >=< \mathbf{x}_v, \mathbf{x}_v >$ and $< \mathbf{x}_u, \mathbf{x}_v >= 0$. Differentiating, we have

$$< \mathbf{x}_{uu}, \mathbf{x}_u >=< \mathbf{x}_{vu}, \mathbf{x}_v >= - < \mathbf{x}_u, \mathbf{x}_{vv} > .$$

Thus,

$$< \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_u >= 0,$$

and analogously

$$< \mathbf{x}_{uu} + \mathbf{x}_{vv}, \mathbf{x}_v >= 0.$$

It follows that $\mathbf{x}_{uu} + \mathbf{x}_{vv}$ is parallel to $\mathbf{n}$. Now,

$$H = \frac{1}{2} \frac{N + L}{\lambda^2}$$

and

$$2\lambda^2 H = N + L =< \mathbf{n}, \mathbf{x}_{uu} + \mathbf{x}_{vv} >,$$

hence,

$$|\mathbf{x}_{uu} + \mathbf{x}_{vv}| = 2\lambda^2 H.$$

□

Beside the most known examples, which were the only ones known for a long period of time, we introduce another minimal surface, named Enneper's minimal surface. It is notable that it's a minimal surface which has self-intersections, as it can be seen in Figure 2.4. It is parametrized by

$$\mathbf{x}(u, v) = \left(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right), \quad (u, v) \in \mathbb{R}^2.$$

Let us compute the partial derivatives of $\mathbf{x}$ and the 1FF and 2FF coefficients for the Enneper's minimal surface:

$$\mathbf{x}_u = (1 - u^2 + v^2, 2uv, 2u),$$

$$\mathbf{x}_v = (2uv, 1 - v^2 + u^2, -2v),$$

$$\mathbf{x}_{uu} = (-2u, 2v, 2), \ \mathbf{x}_{uv} = (2v, 2u, 0), \ \mathbf{x}_{vv} = (2u, -2v, -2),$$

thus, we have that

$$\mathbf{x}_{uu} + \mathbf{x}_{vv} = 0.$$

So that the Enneper's surface is minimal.

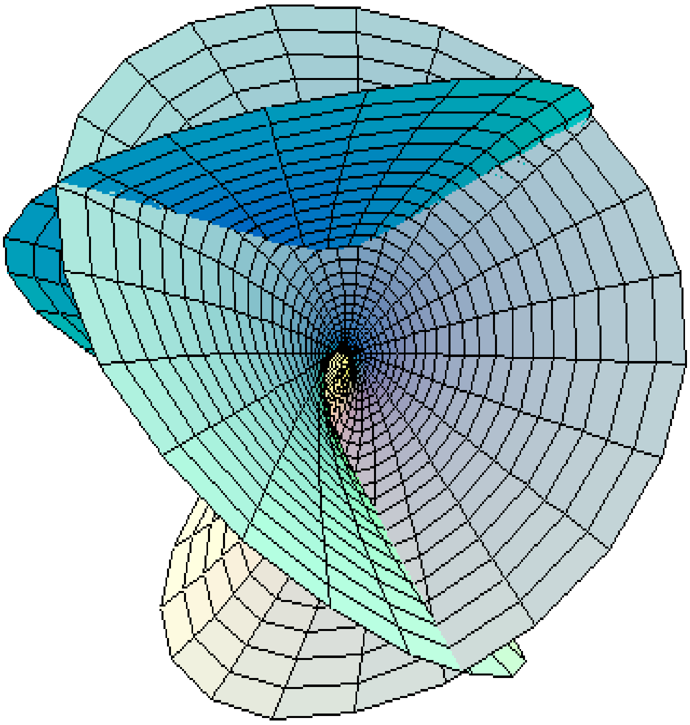

Figure 2.4: Enneper's minimal surface.

2.3 Via area functional

The next definitions will explain the meaning of the word minimal for such surfaces. Briefly, what we are going to look for is a surface which makes the area stationary with respect to variations in the normal direction of the surface itself and, moreover, we are choosing the one with least-area among all surfaces with the same boundary.
Let

$$\mathbf{x} : U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$$

be a parametrization of a surface, let $D \subset U$ be a bounded domain and let $h : \widetilde{D} \to \mathbb{R}$ be a differentiable function, where $\widetilde{D}$ is the union of the domain D with its boundary ∂D. The *normal variation* of $\mathbf{x}(\widetilde{D})$, determined by h, is the map

$$\varphi : \widetilde{D} \times (-\epsilon, \epsilon) \to \mathbb{R}^3$$

for which

$$\varphi(u, v, t) = \mathbf{x}(u, v) + th(u, v), \mathbf{n}(u, v), \ \ (u, v) \in \widetilde{D}, \ \ t \in (-\epsilon, \epsilon).$$

Then, for each fixed $t \in (-\epsilon, \epsilon)$, the map $\mathbf{x}^t : D \to \mathbb{R}^3$,

$$\mathbf{x}^t(u, v) = \varphi(u, v, t)$$

is still a parametrized regular surface, for small ϵ, with

$$\mathbf{x}^t_u = \mathbf{x}_u + th\mathbf{n}_u + th_u\mathbf{n},$$

$$\mathbf{x}^t_v = \mathbf{x}_v + th\mathbf{n}_v + th_v\mathbf{n}.$$

The 1FF and 2FF coefficients for this parametrization are respectively:

$$E^t = E + th(2 < \mathbf{x}_u, \mathbf{n}_u >) + t^2h^2 < \mathbf{n}_u, \mathbf{n}_u > +t^2h_u^2,$$

$$F^t = F + th(< \mathbf{x}_u, \mathbf{n}_v > + < \mathbf{x}_v, \mathbf{n}_u >) + t^2h^2 < \mathbf{n}_u, \mathbf{n}_v > +t^2h_uh_v,$$

$$G^t = G + th(2 < \mathbf{x}_v, \mathbf{n}_u >) + t^2h^2 < \mathbf{n}_v, \mathbf{n}_v > +t^2h_v^2,$$

$$< \mathbf{x}_u, \mathbf{n}_u >= -L, < \mathbf{x}_u, \mathbf{n}_v > + < \mathbf{x}_v, \mathbf{n}_u >= -2M, < \mathbf{x}_v, \mathbf{n}_u >= -N.$$

To express the quantity $E^tG^t - (F^t)^2$, recall the mean curvature and get:

$$E^tG^t - (F^t)^2 = EG - F^2 - 2th(EN - 2FM + GL) =$$

$$= (EG - F^2)(1 - 4thH).$$

The area functional

$$A(t) = \int_{\widetilde{D}} \sqrt{E^tG^t - (F^t)^2}\, \mathrm{d}u\mathrm{d}v = \int_{\widetilde{D}} \sqrt{1 - 4thH}\sqrt{EG - F^2}\, \mathrm{d}u\mathrm{d}v.$$

If ϵ is small, A is a differentiable function and its derivative at $t = 0$ is

$$A'(0) = -\int_{\widetilde{D}} 2hH\sqrt{EG - F^2}\, \mathrm{d}u\mathrm{d}v.$$

Definition 2.5 (4)**.** A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if its a critical point of the area functional $A(t)$ for all the compactly supported variations.

Proposition 2.3.1. *Definition 2.1 and Definition 2.5 are equivalent.*

Proof. Let $\mathbf{x}$ be a parametrization of a minimal surface.
$\longrightarrow$ By Definition 2.1 $H \equiv 0$ and so $A'(0) = 0$ easily.
$\longleftarrow$ Assume that $A'(0) = 0$ for all such domain D and all the normal variation of $\mathbf{x}(\widetilde{D})$ and suppose that $H(q) \neq 0$ for some $q \in D$. Choose $h : \widetilde{D} \to \mathbb{R}$ such that $h(q) = H(q)$ and h is identically zero outside a small neighbourhood of q. Then $A'(0) < 0$ for the variation determined by this h and this is actually a contradiction. $\square$

By above Definition 2.4, any bounded region $\mathbf{x}(D)$ of a minimal surface parametrized by $\mathbf{x}$ is a critical point for the area functional of any normal variation of $\mathbf{x}(\widetilde{D})$, but the critical point is not necessairly a minimun. Hence, a way to justify the use of the word minimal comes with the next definition, which imposes a global minimization of area on every compact subdomain, that is a stronger condition for a complete minimal surface to be satisfied.

The way to impose this condition is by asking that the surface S is stable. In fact, a stable surface S is area - minimizing relative to nearby surfaces with the same boundary. Stability need to be studied with respect to the second variation which is defined by

$$A''(0) = -\int_{\widetilde{D}} h(\Delta h - 2Kh)\sqrt{EG - F^2}\,\mathrm{d}u\mathrm{d}v.$$

A consequence of the second variation of area is that any point in a minimal surface has a neighbourhood with least-area relative to its boundary, as we see in the definition immediately below.

Definition 2.6 (5)**.** A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if every point $p \in S$ has a neighbourhood with least-area relative to its boundary.

Before giving a proof of the equivalence between Definition 2.2 and Definition 2.5, we need to recall one important theorem from analysis.

Theorem 2.3.1 (Green's formula)**.** *Let $C \subseteq \mathbb{R}^2$ be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve, that is $\partial D = C$. If $P, Q : D \to \mathbb{R}$ have continuous first order partial derivatives on D, then*

$$\int_C P\mathrm{d}x + Q\mathrm{d}y = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathrm{d}x\mathrm{d}y.$$

Proposition 2.3.2. *Definition 2.2 and Definition 2.5 are equivalent.*

Proof. Suppose that S, locally expressed as the graph of function $z = f(u, v)$, is a surface with boundary C. The nearby surfaces look like deformation of the surface S, that is $S^t : z^t(u, v) = (u, v, f(u, v)) + th(u, v))$, where h is a function on the same domain of f which translates S of an ϵ factor, never modifying the boundary. Let's parametrize S^t by

$$\mathbf{x}^t(u, v) = (u, v, f(u, v) + th(u, v)).$$

The surface area of S^t is given by

$$A(t) = \int\int_D \sqrt{1 + f_u^2 + f_v^2 + 2t(f_u h_u + f_v h_v) + t^2(h_u^2 + h_v^2)}\,\mathrm{d}u\mathrm{d}v.$$

Taking the derivative with respect to t, we have

$$A'(t) = \int\int_D \frac{f_u h_u + f_v h_v + t(h_u^2 + h_v^2)}{\sqrt{1 + f_u^2 + f_v^2 + 2t(f_u h_u + f_v h_v) + t^2(h_u^2 + h_v^2)}} dudv$$

Let's call $P = \frac{f_u h_u}{\sqrt{1+f_u^2+f_v^2}}$ and $Q = \frac{f_v h_v}{\sqrt{1+f_u^2+f_v^2}}$ and compute P_u and Q_v. Now, applying the Green's theorem we get

$$\int\int_D \frac{f_u h_u + f_v h_v}{\sqrt{1 + f_u^2 + f_v^2}} \,\mathrm{d}u\mathrm{d}v +$$

$$+ \int\int_D \frac{h[(1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu}]}{(1 + f_u^2 + f_v^2)^{\frac{3}{2}}} \,\mathrm{d}u\mathrm{d}v =$$

$$= \int\int_D (Q_v + P_u)\,\mathrm{d}u\mathrm{d}v =$$

$$= \int_C P\,\mathrm{d}v - Q\,\mathrm{d}u = 0,$$

by the fact that $h = 0$ on ∂C, boundary of the domain of f. All these equalities are true for all such h, then by the fundamental lemma of the calculus of variations, we have

$$A'(0) = 0 \Leftrightarrow (1 + f_u^2)f_{vv} - 2f_u f_v f_{uv} + (1 + f_v^2)f_{uu} = 0,$$

which is exactly the Lagrange's equation. □

2.4 Via soap films

The link between minimal surfaces and a physical realization of them comes from the study of the so-called Plateau's problem and actually this is another version of the problem of finding a least area surface having a given boundary (as saw in the last section). The Plateau's problem consists in proving the existence of a regular simply-connected minimal surface S bounded by γ, where γ is a Jordan curve (that is a plane simple closed curve).
Around 1850, Plateau empirically described a multitude of experiments with soap films. A soap film can be obtained by dipping a wire frame into a soap solution and withdrawing it paying attention in the meantime to not break it. In order to study soap films from a mathematical and a physical point of view we should neglect the force of gravity. We see that the shape obtained will have the chosen frame as boundary.

Note that a soap film is separated by a homogeneous membrane. If the difference of the pressures at the two sides of the membrane is zero, then the soap film/surface is minimal, since this difference is equal to the value of the mean curvature.

Every minimal surfaces we have seen in this chapter can be practically constructed with a soap film, with the exception of the Enneper's minimal surface, whose construction by soap film is pretty hard, since it's a surface with self-intersections. In fact,

- the catenoid is obtained by taking two circle shaped wires. First, when they match exactly together, they are in a soap solution and then they are carefully turned away, but paying attention to keep the two wire always on parallel planes;
- the helicoid is obtaied by taking a wire frame of a helix into a soap and glycerine solution.

Definition 2.7 (6)**.** A surface $S \subset \mathbb{R}^3$ is *minimal* if and only if every point $p \in M$ has a neighbourhood D_p which is equal to the unique idealized soap film with boundary ∂D_p.

We introduce a last example of a surface with respect to Definition 2.7. The Scherkâs doubly periodic surface, depicted in Figure 2.6 is obtained by taking a wire frame like a box, but missing two parallel edges on the top and two parallel edges on the bottom, as you can see in Figure 2.5.

Furthermore, here we give the parametrization of the Scherk's surface, which is

$$\mathbf{x}(u,v) = \left(\arg \frac{z+i}{z-i}, \arg \frac{z+1}{z-1}, \log \left| \frac{z^2+1}{z^2-1} \right| \right), \ z \neq 1, -1, i, -i,$$

where $z = u + iv \in \mathbb{C}$ and $\arg z$ is the angle that the positive x semiaxis forms with z.

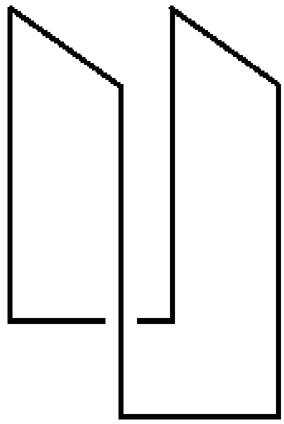

Figure 2.5: Scherk's wire frame.

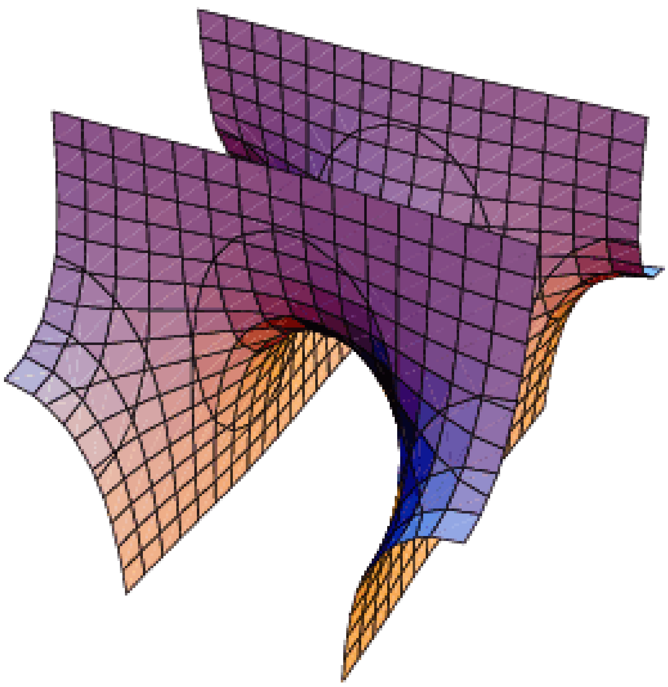

Figure 2.6: Scherk's minimal surface.

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