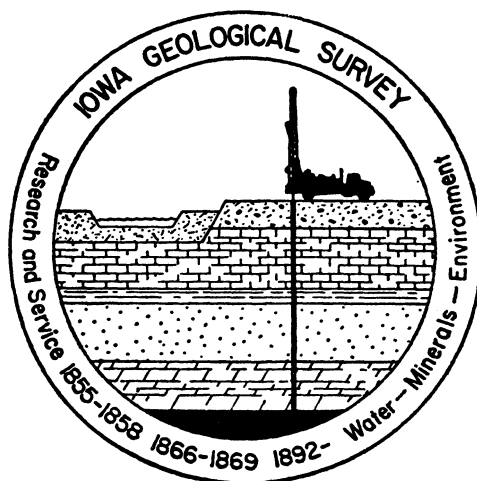


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A THERMAL MODEL FOR THE SURFACE TEMPERATURE OF MATERIALS ON THE EARTH'S SURFACE

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Abstract

A mathematical model for the determination of the surface temperature of materials on the surface of the earth is derived and solved numerically.

The material is assumed to be opaque and compact. Its surface is assumed to be flat and coincident with the plane of the earth's surface. The problem is mathematically represented by the following one dimensional heat conduction equation and boundary condition:

$$(k/\rho c) \partial^2 T(x,t) / \partial x^2 = \partial T(x,t) / \partial t \quad 0 \leq x$$

$$k \partial T(x,t) / \partial x \Big|_{x=0} = f[T(0,t), t]$$

$$T(x,0) = \text{constant}$$

The temperature of the material at distance x beneath its surface at time t is $T(x,t)$. The thermal conductivity, density, and specific heat are k , ρ , and c respectively and are assumed to be

constant. The function f is explicitly and nonlinearly dependent upon $T(0,t)$ and t , and is composed of the following heating or cooling effects: direct and scattered sunlight, radiant thermal energy emitted by the material, air convection, evaporation, and condensation at the material's surface. The function of f is dependent upon: the albedo, emissivity, and evaporation opportunity of the material, the temperature, relative humidity, and wind velocity of the air above the material, the declination of the sun, and the geographical latitude of the material. The temperature, relative humidity and wind velocity of the air above the material were assumed to be constant. Results for $T(0,t)$, $f[T(0,t),t]$, and the various components of $f[T(0,t),t]$ are presented in graphical form. Results for $T(0,t)$ at various significant items t during the solar day are given in tabular form. The solutions were carried out for t sufficiently large that $T(0,t)$ became approximately periodic. The model predicts that the surface temperature of a typical earth crust material at a given time is dependant only to a rather small degree upon the temperature of the material more than approximately 6 days prior to that time.

Introduction

The daily and seasonal variation of the surface temperature of materials on the earth's surface is a phenomena which is familiar to all of us. A quantitative understanding of this phenomena and its natural causes is important in temperature mapping of the earth's surface by means of remote sensors located in air or spacecraft.

The surface temperature of a given material at a given time of the year and day depends on a variety of natural heating and cooling effects and on the inherent properties of the material. Any reasonably practical mathematical model designed to quantitatively predict surface temperatures will not be accurate simply because most natural heating and cooling effects are dependent upon weather conditions, which are impossible to predict accurately. With certain assumptions about weather conditions, however, a mathematical model can be developed which lends considerable insight into the surface temperature phenomena and its causes, and which can be used as a guide for the quantitative prediction of surface temperatures.

Problem and Solution

The problem is to determine the surface temperature of an opaque, compact material on the earth's surface. Suppose that the material's surface, in contact with the atmosphere, is flat and coincident with the plane of the earth's surface. Further, assume that conditions are such that, at a given time t , the temperature T of the material is the same at all points at distance x vertically below the material's surface. The distance x is zero on the surface and increases from zero toward the earth's center. With this assumption regarding T it follows that the following form of the heat equation

is valid in the material:

$$(k/\rho c) \partial^2 T(x,t) / \partial x^2 = \partial T(x,t) / \partial t \quad 0 \leq x \quad (1)$$

The thermal conductivity, density, and specific heat are assumed to be constant and are respectively denoted by k , ρ , and c .

One also has:

$$k \partial T(x,t) / \partial x \Big|_{x=0} = f(t) \quad (2)$$

where $f(t)$ is defined as the rate of heat flow out of the material at its surface per unit surface area.

Dividing $f(t)$ into 5 separate terms:

$$f(t) = \sum_{i=1}^5 f_i(t) \quad (3)$$

$$-f_1(t) = (1-A) R_0 \cos \theta_0 \exp(-.365 \sec \theta_0) / .90 \quad (4)$$

A - Albedo of material.

R_0 - Irradiance received from the sun above earth's atmosphere ($.139 \text{ Wcm}^{-2}$).

θ_0 - Sun zenith angle.

$$\cos \theta_0 = \cos \theta \cos \phi \cos [2\pi t (24 \text{hr})^{-1}] + \sin \theta \sin \phi$$

θ - Declination of sun.

ϕ - Geographical latitude of material on earth's surface.

t - Local solar time.

$$-f_2(t) = \epsilon \sigma T_a^4 (.37 + .019 x_a m^3 gm^{-1}) \quad (5)$$

ϵ - Absorbance (emissivity) of material.

σ - Stephan-Boltzmann constant (5.67×10^{-12} $Wcm^{-2} K^{-4}$).

T_a - Air temperature above the material's surface in degrees Kelvin.

x_a - Absolute humidity of air above material's surface.

$$f_3(t) = \epsilon \sigma T^4(0, t) \quad (6)$$

$$f_4(t) = [T(0, t) - T_a] (1.09 + .23 V_a sec ft^{-1}) \times (.000568 Wcm^{-2} K^{-1}) \quad (7)$$

V_a - Velocity of air assumed to be moving parallel to and above the materials surface.

$T(0, t)$ - Surface temperature of material in degrees Kelvin.

$$f_5(t) = a \{ e[T(0, t)] - e[T_a] R_a \} (mm of Hg)^{-1} \times (.44 + .173 V_a sec ft^{-1}) (.00183 Wcm^{-2}) \quad (8)$$

a - Evaporation opportunity of the material.

R_a - Relative humidity of air above material's surface.

$e[T]$ - Saturated vapor pressure of water at temperature T .

$$e[T] \approx [5.80 + .0295 (T^\circ K^{-1} - 273.2)^2] (mm of Hg)$$

T - Temperature in degrees Kelvin.

The first term $f_1(t)$ approximates the radiant energy from the sun and in addition that radiant energy received from the sun but scattered by the atmosphere before arriving at the material's surface. Most of this radiant energy will have wavelengths in the .3-3 μ region. The semiempirical expression given for $f_1(t)$ was determined from measurements taken during the summer in Colorado (Hulstrom, 1970). It is intended to be a reasonable approximation for clear atmospheric conditions only.

The second term $f_2(t)$ approximates the thermal radiant energy received from the atmosphere. Most of this radiant energy will have wavelengths in the 3-14 μ region and originates principally from the water vapor in the atmosphere. Part of the expression given for $f_2(t)$ was derived from graphically presented measured data taken by Sloan et al. (1956).

The third term gives the thermal radiant energy emitted by the material as dictated by the Stephan-Boltzmann law.

The fourth term approximates the convective heat loss (gain) due to the passage of air assumed to be moving parallel to and above the material's surface. It is an example of the so-called Newton heat exchange boundary condition. The numerical coefficients in the term were determined (McAdams, 1954) by laboratory experiments measuring the temperature of a rough copper plate being cooled (heated) by a wind of velocity v_a parallel to the plate's surface.

The fifth term approximates the heat loss due to the evaporation of water from the material's surface. This term was obtained by multiplying the evaporation rate of water from the material by the heat of vaporization of water. Several semiempirical formulae for the evaporation rate of water from materials exist in the literature (Wisler, 1949). The one chosen was determined by Rohwer, and sea level barometric pressure was used in his formula. The constant a is the so-called evaporation opportunity (Wisler, 1949) of the material. For materials containing no water a would be zero. For highly porous materials which are saturated with water a would be approximately 1.

The absolute humidity x_a in term $f_2(t)$ is obtained by means of the ideal law with temperature T_a and pressure $e[T_a]R_a$. The approximation given for $e[T]$ was determined empirically from tabulations for saturated water vapor pressure and is reasonably accurate for $273.2^\circ K < T < 310^\circ K$.

In terms $f_2(t)$, $f_4(t)$, and $f_5(t)$ the quantities T_a , V_a , and R_a occur. These quantities depend upon weather conditions and thus change with time. At first assume that they are constant, and note that $f_1(t)$ depends explicitly upon t , that $f_2(t)$ is a constant, and that $f_3(t)$, $f_4(t)$, and $f_5(t)$ depend explicitly upon $T(0,t)$

Thus:

$$f(t) = f[T(0,t), t]$$

In (1) and (2) $T(x, t)$, x , and t have the dimensions of temperature, distance, and time respectively.

Transform these three quantities into dimensionless form by means of the following substitutions:

$$T(x, t) = K_0 u(y, \tau)$$

$$x = c_0 y$$

$$t = S_0 \tau$$

where

$$K_0 = R_0^{1/4\sigma - 1/4}$$

$$c_0 = kR_0^{-3/4\sigma - 1/4}$$

$$S_0 = R_0^{-3/2\sigma - 1/2k\rho c}$$

The quantities $u(y, \tau)$, y , and τ are dimensionless temperature, distance, and time respectively. In terms of these new quantities (1) and (2) become:

$$\partial u^2(y, \tau) / \partial y^2 = \partial u(y, \tau) / \partial \tau \quad 0 \leq y \quad (9)$$

$$\partial u(y, \tau) / \partial y \Big|_{y=0} = f[K_0 u(0, \tau), S_0 \tau] R_0^{-1} \equiv F[u(0, \tau), \tau] \quad (10)$$

The only place where t explicitly occurs in $f(t)$ is in the $\cos[2\pi t(24hr)^{-1}]$ term for $\cos\theta_0$ in $f_1(t)$. This term becomes $\cos[2\pi\eta^{-1}\tau]$ where $\eta \equiv (24hr)S_0^{-1}$ when one replaces t by $S_0\tau$. The quantity η is dimensionless and is inversely proportional to the square of the thermal inertia $(k\rho c)^{1/2}$.

$$u(y,0) = u_0 \quad 0 \leq y \quad (11)$$

The solution for $u(0,\tau)$ as determined from (9), (10), and (11) is (Carslaw and Jaeger, 1959):

$$u(0,\tau) = u_0 - \pi^{-\frac{1}{2}} \int_0^\tau (\tau-z)^{-\frac{1}{2}} F[u(0,z),z] dz \quad (12)$$

Equation (12) was solved numerically. The procedure was to approximate $F[u(0,z),z]$ in the usual manner by a series of polynomials of 1st or 2nd order. An indication of the accuracy of this procedure may be obtained by comparing the results for $u(0,\tau)$ for the two orders. The discrepancy between $u(0,\tau)$ using 2nd order polynomials was less than .01% for $\tau > 0$. The discrepancy tended to become less for larger τ . These solutions show that $u(0,\tau) \approx u(0,\tau+n)$ for sufficiently large values of τ . This is particularly true for an appropriate choice for u_0 . This result is physically reasonable since one would expect that $u(0,\tau)$ would be mainly determined by the forcing function F for sufficiently large τ , rather than the material's initial temperature at some large time in the past.

The numerical solutions were time phased in a manner such that τ and t were zero at 18hrs (6 PM) local solar time. Solutions for $u(0,\tau)$ were found for $0 \leq \tau \leq 6\eta$, or equivalently $T(0,t)$ was found for $0 \leq t \leq 6$ days. Values for $T(0,t) - T_a$ at 6 hrs local solar time, $T[0,12hr+n(24hr)] - T_a$, in degrees Kelvin are given in Table 1 for two values of $T_0 \equiv K_0 u_0 = T(x,0)$. The values taken for the constants in F

were: $A=.05, \epsilon=1.0, R_a=.25, V_a=5 \text{ ft/sec}, T_a=285^\circ \text{K}, \eta=.18,$
 $a=.1, \theta=0,$ and $\phi=45$.

Note that the values listed for $T-T_a$ in Table 1 increase asymptotically with n for $T_0=279^\circ \text{K}$ but decrease asymptotically with n for $T_0=285^\circ \text{K}$. For sufficiently large n both should converge to the same value. Previous treatments of this surface temperature problem (Watson, 1971; Jaeger, 1953) have used a periodic solution for $T(0,t)$. For these solutions the initial transient behavior of $T(0,t)$, which depends upon one's choice of $T(0,t)$, which depends upon one's choice of T_0 , is not present. Such solutions are less complex but contain less information particularly if one supposes that T_0 is an appropriate earth subsurface temperature.

Referring to the previously listed values for the constants in F , together with $T_0=282^\circ \text{K}$ as the "standard choice" of constants, figure 1 shows $T(0,t)$, 5 days $\leq t \leq 6$ days, for the "standard choice" of constants. Figure 2 shows $F(\tau) = \sum_{i=1}^5 F_i(\tau)$ and $F_i(\tau) \equiv f_i(t)R_0^{-1}$, $i=1,2,3,4,5$, 5 days $\leq t \leq 6$ days again with the "standard choice" of constants.

TABLE 1. $T[0, 12hr + n(24hr)] - T_a$

n	T_0 (K)	
	279	285
0	-10.94	-9.030
1	-10.44	-9.243

3	-10.25	-9.441
4	-10.21	-9.495
5	-10.18	-9.536

TABLE 2. $a_1, a_2,$ and a_3 of $T(0, t)$ for 5 days $\leq t \leq$ 6 days.

Constants	a_1 ($^{\circ}$ K)	a_2 ($^{\circ}$ K)	a_3 ($^{\circ}$ K)
standard	5.07	9.86	7.07
A=.15	5.76	10.1	5.14
$\epsilon=.9$	4.05	8.92	8.16
$R_a=.75$	3.66	8.37	8.44
$V_a=15$ ft/sec	4.57	6.91	5.15
$T_a=295^{\circ}$ K	7.57	11.9	4.38
$n=.12$	4.57	9.33	5.88
$a=.5$	7.44	11.7	3.74
$\theta=.3$ radians	-1.10	8.29	15.3
$T_0=285^{\circ}$ K	4.75	9.54	7.36

Table 2 shows the values found for $a_1, a_2,$ and a_3 as depicted in Figure 1:

$$-a_1 \equiv T(0, 5 \text{ days}) - T_a$$

$$-a_2 \equiv T(0, t)_{\min} - T_a$$

$$a_3 \equiv T(0, t)_{\max} - T_a$$

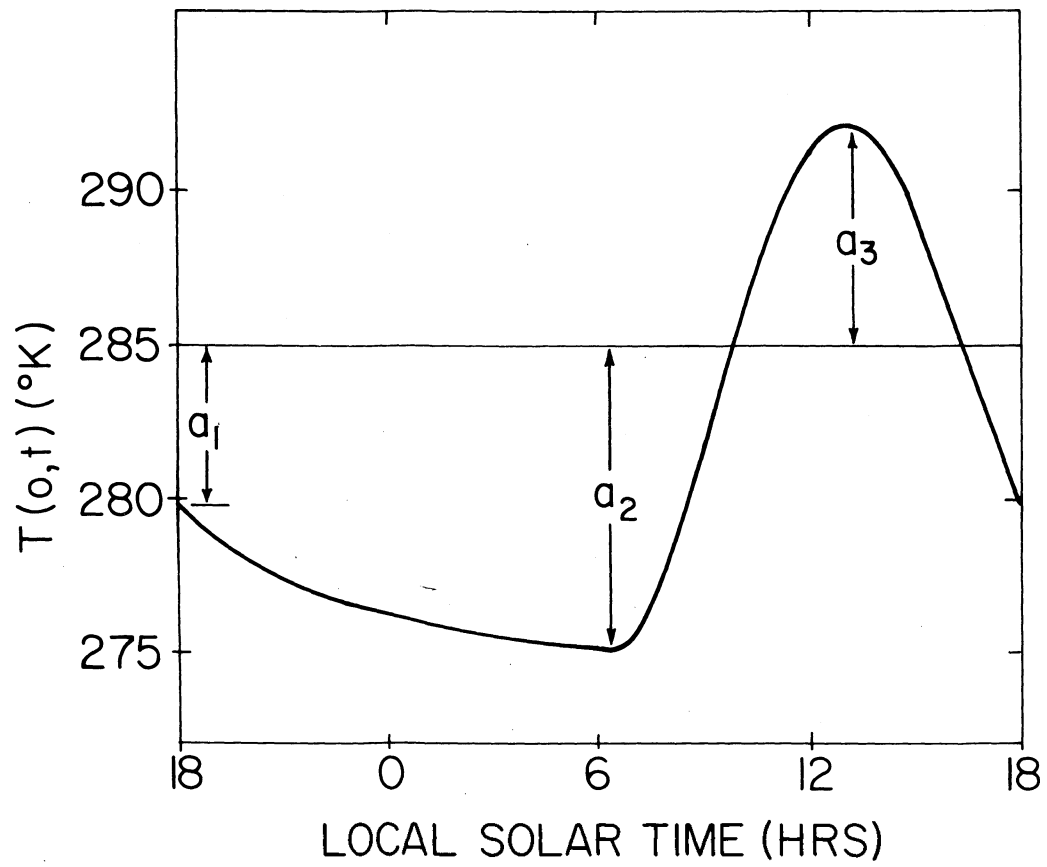


Figure 1. $T(o,t)$ for $5 \text{ days} \leq t \leq 6 \text{ days}$ and with the "standard choice" of constants.

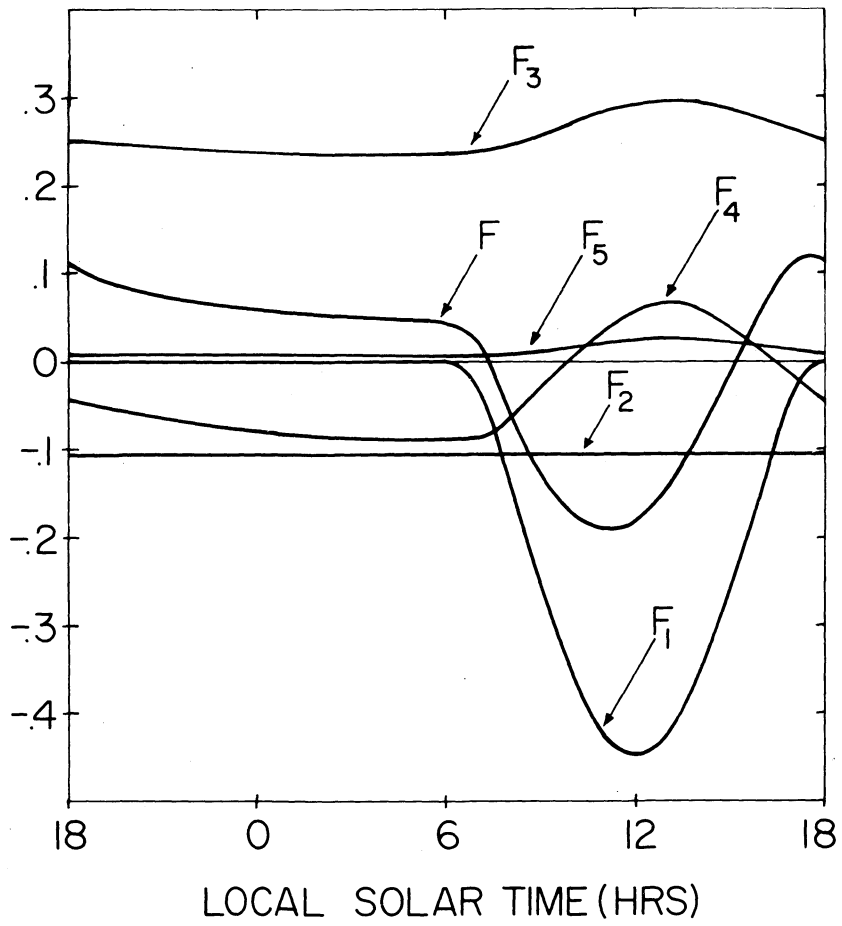


Figure 2. $F_1, F_2, F_3, F_4,$ and F_5 for $5 \text{ days} \leq t \leq 6 \text{ days}$
and with the "standard choice" of constants.

The minimum and maximum values for $T(0,t)$, 5 days $\leq t \leq 6$ days, are respectively $T(0,t)_{min}$ and $T(0,t)_{max}$. Entries shown in the column headed "constants" in Table 2 show the departures from the "standard choice". Thus $A=.15$ in the second row of Table 2 means that the selection of constants was the "standard choice" except that A was equal to .15 rather than .05 as it is in the "standard choice". The reason that these results are in tabular rather than graphical form is that the various curves for $T(0,t)$ are all very similar in shape but have different values for a_1 , a_2 , and a_3 . In every case $T(0,t)_{min}$ was realized at from 12 to 36 minutes after sunrise. Except for the $v_a=15\text{ft/sec}$ and $a=.5$ cases, $T(0,t)$ realized it's maximum value at from 60 to 84 minutes after 12 hrs local solar time. For the $v_a=15\text{ft/sec}$ and $a=.5$ cases, $T(0,t)$ realized it's maximum value at from 36 to 60 minutes after 12 hrs. local solar time.

For an understanding of the results in Table 2 it is important to understand the behavior of the terms F_i , $i=1,2,3,4,5$, as they depend upon the constants in F . F_1 and F_2 are heating terms. F_3 is a cooling term. F_4 and F_5 can either heat or cool the material. The heating effect of F_1 is decreased for increasing A and, for positive ϕ increased for θ increasing from zero. The heating effect of F_2 increases for increasing ϵ , T_a , and R_a .

F_1 is typically of greater importance than F_2 , but could have less net heating effect than F_2 when ϵ , T_a ,

and R_a are large. The cooling effect of F_3 is increased for increasing ϵ . F_3 is perhaps typically the most important cooling term. The effect of F_4 is to cool the material when $T(0,t) > T_a$ but to heat the material when $T(0,t) < T_a$. Its effect is increased for increasing v_a . It's importance becomes considerable when v_a becomes sufficiently large. F_5 can either cool (evaporation) or heat (condensation) the material. It's effect is increased for increasing v_a and a , and can be of considerable importance when v_a and/or a become sufficiently large. It's effect tends to be that of cooling for small R_a and T_a .

Table 2 is quite incomplete. It contains only single variable departures from the "standard choice" evaluation of the constants. The table may, however, be used for crude approximations for a general selection of the constants via the first order Taylor expansion:

$$a_i(c_1 + \Delta c_1, c_2 + \Delta c_2, \dots, c_9 + \Delta c_9) \approx a_i(c_1, c_2, \dots, c_9) + \sum_{j=1}^9 \Delta c_j \partial a_i(c_1, c_2, \dots, c_9) / \partial c_j$$

$$i=1, 2, 3$$

The "standard choice" evaluation of the constants are c_1, c_2, \dots, c_9 . The partial derivatives $\partial a_i / \partial c_j$ may be approximated from Table 2. Departures of the constants from the standard choice are Δc_j .

It is appropriate to consider the limitations and

expected accuracy of these solutions. The term F_1 assumes clear atmospheric conditions, thus our results apply only in this case. The constants T_a , V_a , and R_a are determined by weather conditions and in reality change considerably with time. It has been assumed that they are constant, however. This assumption clearly introduces error into the solutions. Transpiration from vegetation was not considered so that our results do not apply to vegetation.

The model is clearly too simple to predict with any accuracy surface temperatures that one would actually measure in the field. It should, however, be useful for predicting, with limited accuracy, the surface temperatures of non-vegetated materials under clear atmospheric conditions when T_a , V_a , and R_a are relatively constant.

In conclusion it should perhaps be emphasized that relatively little attention has been given to the rate at which the solutions presented in Table 2 become approximately periodic. Only the data presented in Table 1 deals with this phenomena. Representing $T[0, t+n(24hr)]$, $0 \leq t \leq 24hr$, by $T_n(0, t)$, representative values of $T_5(0, t)$ are given in Table 2. It is expected that $T_n(0, t) \approx T_\infty(0, t)$ for n sufficiently large. This is expected to be valid for relatively small n if the 24hr time average of $T_\infty(0, t)$ and T_0 are not significantly different and η is relatively large. Solutions were carried out for $T(0, t)$, $0 \leq t \leq 12$ days for the "standard choice" of constants except $\theta = .3$ radian and with $\eta = .18$ and $.016$. For $\eta = .18$, a_1 , a_2 ,

and a_3 are respectively -1.099, 8.286, and 15.29 for $n=5$ and -1.253, 8.135, and 15.42 for $n=11$. For $\eta=.016$, a_1 , a_2 , and a_3 are respectively -1.421, 4.400 and 6.421 for $n=5$ and -1.844, 4.004, and 6.784 for $n=11$. The changes in a_1 , a_2 , and a_3 are approximately .4 for $\eta=.016$ but approximately .15 for $\eta=.18$. It would be expected that the changes for a_1 , a_2 , and a_3 would not be greater than .15 for the solutions in Table 2. The value taken by η for most earth crust materials is larger than .18. The model would thus predict that the temperature of a typical earth crust material at a given time is dependent only to a rather small degree upon the temperature of the material more than approximately 6 days prior to that time.

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