5-1976

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Recommended Citation

Brown, L. D. (1976). Closure Theorem for Sequential-Design Processes. *Statistical Decision Theory and Related Topics,* 57-91. http://dx.doi.org/10.1016/B978-0-12-307560-4.50008-9

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Abstract

This chapter focuses on stochastic control and decision processes that occur in a variety of theoretical and applied contexts, such as statistical decision problems, stochastic dynamic programming problems, gambling processes, optimal stopping problems, stochastic adaptive control processes, and so on. It has long been recognized that these are all mathematically closely related. That being the case, all of these decision processes can be viewed as variations on a single theoretical formulation. The chapter presents some general conditions under which optimal policies are guaranteed to exist. The given theoretical formulation is flexible enough to include most variants of the types of processes. In statistical problems, the distribution of the observed variables depends on the true value of the parameter. The parameter space has no topological or other structure here; it is merely a set indexing the possible distributions. Hence, the formulation is not restricted to those problems known in the statistical literature as parametric problems. In nonstatistical contexts, the distribution does not depend on an unknown parameter. All such problems may be included in the formulation by the device of choosing the parameter space to consist of only one point, corresponding to the given distribution.

Disciplines

Statistics and Probability

Reprinted from:
STATISTICAL DECISION THEORY AND RELATED TOPICS, II

ACADEMIC PRESS, INC.
NEW YORK SAN FRANCISCO LONDON

CLOSURE THEOREMS FOR SEQUENTIAL-DESIGN PROCESSES*

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1. *Introduction.* Stochastic control and decision processes occur in a variety of theoretical and applied contexts, for example: "statistical decision problems" (sequential and nonsequential), "stochastic dynamic programming problems", "gambling processes", "optimal stopping problems", "stochastic adaptive control processes", etc. It has long been recognized that these are all mathematically closely related. Since this is the case, all of these decision processes can be viewed as variations on a single theoretical formulation. Such a formulation is the first goal of this paper.

The second goal of the paper is to set forth general conditions under which optimal policies are guaranteed to exist.

The given theoretical formulation is flexible enough to include most variants of the types of processes named above. At the same time we have tried to be parsimonious in our construction and economical in our notation. We hope the reader will find our formulation convenient for many purposes, and particularly for proving theoretical results like the optimal policy theorems mentioned above.

Characteristics of the formulation: In our formulation a countably additive stochastic process is to be observed. At discrete times, or stages, of the process various types of decisions or actions may be taken. (The underlying stochastic process may be either a discrete-time or a continuous-time process.)

At each stage of the process the "statistician" or "controller" chooses from among a set of available actions. This choice may depend on the past history of the process and on the past actions taken. The decision procedure for choosing this action

*Work supported in part by NSF Grant MPS-72-05075-A02.

may be randomized, if desired.

The actions available at any given stage may depend on what actions have been taken in the past. There is no explicit provision in the formulation which allows the set of available actions also to depend on the past observations in the stochastic process, however, such a situation may be included within the formulation by the simple trick described in Section 6. An application to a gambling problem of Dubins and Savage [6] is given in Section 7.

The actions taken at each stage may be of several types. They may involve a decision to stop the process - or to stop and make some sort of terminal decision, as in the classical sequential testing problems. If the action involves a continuation of the observed process it may also involve the stochastic law for the future of the observed process. (This law also may depend on the value of the "parameter" as discussed below.) The word "design" appears in the title of our paper for this reason since a statistician might describe this possibility by saying that the current action includes the experimental design for future observations. As another possibility, the action may concern the time(s) at which future action(s) may be taken.

Finally, the value of any given sequence of actions is measured by either a loss function or a gain function. This loss, or gain, may depend on the actions taken and on the observations in the stochastic process which is observed. Naturally, in problems of a statistical nature the loss, or gain, may depend also on the true state of nature. We have assumed that the loss function is bounded below by a constant. (Equivalently, a gain function must be bounded above.) However this assumption can be weakened as explained in Remark 2.14.

There are certain technical measure theoretic assumptions in addition to the assumptions mentioned above. These assumptions are all of the standard variety, except for Assumption 2.21.

In statistical problems the distribution of the observed variables depends on the true (but unknown) value of the "parameter".

For this purpose our formulation also includes a parameter space. The parameter space has no topological or other structure here; it is merely a set indexing the possible distributions. Hence the formulation is not restricted to those problems known in the statistical literature as parametric problems.

In non-statistical contexts the distribution does not depend on an unknown parameter. All such problems may be included in our formulation by the device of choosing the parameter space to consist of only one point, corresponding to the given distribution.

Existence of optimal decision procedures: Sections 3 and 4 of this paper culminate in Theorems 4.4 and 4.7 which establish the existence of "optimal policies". These theorems involve several extra assumptions. In the usual applications these assumptions require that:

(a) The loss function depends on the actions in a lower semi-continuous way. (Equivalently, the gain is an upper semi-continuous function of the actions taken.)

(b) For all actions whose loss is anywhere finite the possible distributions of the observed process through any finite stage form a dominated family whose densities have a suitable continuity property as a function of the actions taken.

(c) Either the loss function is finitary (Assumption 3.17 (ii)) or sampling without stopping results in an infinite loss.

These consequences are more fully described in Section 4; see especially Discussion 4.6.

The existence of Bayes procedures in the design-decision context is proved in Section 5. The dominatedness assumption of Sections 3 and 4 (referred to in (b), above) is not required.

Our optimality proof is quite different from the usual proofs of such results in "stochastic dynamic programming" settings, including "optimal stopping" problems such as those treated in Chow, Robbins, and Siegmund [5]. (Schäl [13] is an exception.) The "usual proof" proceeds by writing down the basic dynamic programming equation (backward induction equation) for the process, and then demonstrating that this equation has a measurable solution satisfying appropriate boundary conditions. This demonstration proceeds from an examination of the structure of the basic equation. (Generally, one must actually write down a sequence of basic equations for truncated versions of the process, show that each of these truncated processes possesses an optimal policy, and that this sequence of policies converges to an appropriate solution for the original process.) These basic equations do not appear at all in our approach.

The basic equation referred to above is the "pointwise version", in that at each stage of the process and for each possible past sample point one takes an optimal action. There also exists a "global version" of the equations in which at each stage of the process one chooses an optimal procedure - i.e. a procedure which maximizes the expected future gain.

Our optimal policies must, a fortiori, satisfy the "global" version of the basic equations, but not the usual "pointwise" version. However, whenever the pointwise version has a measurable solution then the two versions have the same solution; and the optimal policies whose existence we have established here can be computed by solving the usual basic equations.

[We conjecture that under the technical conditions of our paper it is always the case that the pointwise version really does have a solution. The conjecture appears especially plausible under the conditions of Theorems 3.18 and 4.4 as applied to the dynamic programming setting. See Discussion 6.2.]

In this context our results are closely related to those of Schäl [13], where optimal policies of the type we describe are called weakly optimal policies.

As we have noted, the main theorems in Sections 3 and 4 usually require that the finite-stage distributions of the observed process form a dominated family. Such an assumption is particularly undesirable for certain general applications of the theory. One such application is to the theory of gambling

processes and related dynamic programming processes. Section 7 closes with an optimal policy theorem which does not contain any dominatedness assumptions.

This theorem is proved by using a trick which transfers the problem to become a special case of Theorem 3.18. Presumably, this trick could be used to generate other optimality results which do not contain the dominatedness assumption.

2. *Description of the process.*

2.1. Decision stages: Decisions may be made at discrete stages of the process. The set of stages is labelled T, and T is either $1,\ldots,n$; or $1,2,\ldots$; or $1,2,\ldots,\infty$.

2.2. Action space: The space of available actions is denoted by A. It is assumed that $A \subset K = \underset{t\in T}{\times} K_t$.

Let π_t and $\pi_{(t)}$ denote projection maps with $\pi_t : A \to K_t$, $\pi_{(t)} : A \to \underset{\tau\le t}{\times} K_\tau$. Then $\pi_t(a) = a_t$ describes the action taken (under a) at stage $t\in T$, and $\pi_{(t)}(a) = a_1,\ldots,a_t$ are the actions taken (under a) at stages $1,\ldots,t$.

2.3. Discussion: Later elements of the formulation are arranged so as to allow various types of actions, such as to continue sampling or to stop sampling and make a certain terminal decision. As will be seen, other types of actions are also possible. (In previous literature on the subject, various classes of actions have usually been separated out and given separate notations; and the action space has been divided into explicit subsets - e.g. a "stopping time" and a "terminal decision".)

Clearly, not all actions are available at all stages of the process. In fact, if actions $a_1,\ldots,a_{t-1}$ have been taken at stages $1,\ldots,t-1$ then the only actions which can be taken at stage t are those in $\pi_t((a_1,\ldots,a_{t-1},K_t,K_{t+1},\ldots) \cap A)$. It may be that this set of available actions at stage t consists only of a single value - say a_t. In this case the set of actions now available at stage t is trivial. The statistician is not required to make an actual decision as to which action to follow at this stage of the process. (In the classical sequential testing problem it may be that the particular action a_{t-1} corresponds to the decision to stop and make a certain terminal decision at stage t-1. In that case the set of available actions at stages t, t+1,... will all be trivial - each a_τ, $\tau \geq t$, corresponds to the situation "the process has already stopped" - so that no further actual decisions can be made after stage t-1.)

The general formulation allows an action to be taken at stage 1. This will be before any random variables have been observed. However, in many specific problems the set of actions available at this stage will be trivial. In other problems only two actions may be available, e.g. "Begin!" or "Do not begin!"

2.4. Assumptions: Assume henceforth that the spaces K_t are compact and second countable. Let $\mathcal{A}_t$ denote the Borel field on K_t, and $\mathcal{A}$ the Borel field on K. It is convenient to use the symbol $\mathcal{A}_t$ also to denote the σ-field on A induced by the projection map π_t onto K_t, $\mathcal{A}_t$. In the very few instances where this notation could be ambiguous it will be clarified by writing, for example, $S \varepsilon \mathcal{A}_t$, $S \subset A$.

Assume $A \varepsilon \mathcal{A}$. Let $\bar{A}$ denote the closure of A in K. The symbol $\mathcal{A}$ will also be used to denote the σ-field $\mathcal{A}$ on K as restricted to A. Again, there should be no confusion. Similarly, $\mathcal{A}_{(t)}$ denotes the σ-field induced on $\mathcal{A}$ (or on K, or on $K_{(t)} = \underset{\tau \leq t}{\times} K_\tau$, etc.) by the map $\pi_{(t)}$.

2.5. Sample space: The sample space is $X = \underset{t \varepsilon T}{\times} X_t$. Each co-ordinate space X_t is endowed with a σ-field, $\mathcal{B}_t$, and $\mathcal{B}$ denotes the product σ-field on X. $\mathcal{B}_{(t)}$ is the product σ-field $\underset{\tau \leq t}{\times} \mathcal{B}_\tau$. For any particular point $x \varepsilon X$ the value $x_t = \pi_t(x)$ is to be thought of as the value of the random variable observed at stage t.

2.6. Observable events: The events which can be observed at stage t are controlled by the actions taken at stages 1,...,t. Thus, for each $a \varepsilon A$ there is a σ-field $\mathcal{B}_t(a) \subset \mathcal{B}_t$. $\mathcal{B}_t(a)$ depends only on $\pi_{(t)}(a)$ - i.e. if $\pi_{(t)}(\alpha) = \pi_{(t)}(\beta)$ then $\mathcal{B}_t(\alpha) = \mathcal{B}_t(\beta)$. $\mathcal{B}_t(a)$ describes the events observable at stage t of the process given that actions $a_1,\ldots,a_t$ have been taken at stages 1,...,t. (It may be that $\mathcal{B}_t(a)$ is the trivial σ-field, meaning that when actions $a_1,\ldots,a_t$ have been taken there is nothing interesting to be observed at stage t.) For convenience, let $\mathcal{B}_0(a)$ denote the trivial σ-field.

Again, the symbol $\mathcal{B}_t(a)$ will also be used to denote the σ-field on X induced by the projection map π_t onto X_t, $\mathcal{B}_t(a)$.

$\mathcal{B}_{(t)}(a)$ denotes the σ-field on X generated by $\{\mathcal{B}_\tau(a): \tau = 1,\ldots, t\}$. The same symbol will also be used to denote the projection of this σ-field on $\underset{\tau \leq t}{\times} X_\tau$. Let $\mathcal{B}(a) = \mathcal{B}_{(\infty)}(a)$ denote the σ-field on X generated by $\{\mathcal{B}_\tau(a): \tau \varepsilon T\}$.

Finally, if $C \subset A$, $C \varepsilon \mathcal{A}_{(t)}$, let $\mathcal{B}_{(t)}(C)$ denote the σ-field generated by the collection $\{\mathcal{B}_{(t)}(a): a \varepsilon C\}$.

2.7. Parameter space: The parameter space is denoted by Θ. For now, and throughout much of the following development this "space" is really just an index set, having no further structure.

2.8. Possible distributions: The possible distributions on X are denoted by $F_\theta(\cdot|a)$, $\theta \varepsilon \Theta$, $a \varepsilon A$. The following standard assumptions are made:

(i) $F_\theta(\cdot|a)$ is a probability distribution on $\mathcal{B}$ for each $\theta \varepsilon \Theta$.

(ii) $F_\theta(B|\cdot)$ is $\mathcal{A}$ measurable for each $B \varepsilon \mathcal{B}$.

In addition it is required that the distribution through stage t should depend (in a measurable way) only on actions taken through stage t. Thus assume

(iii) $F_\theta(B|\cdot)$ is $\mathcal{A}_{(t)}$ measurable for each $B \varepsilon \mathcal{B}_{(t)}$, $t \varepsilon T$.

2.9. Discussion: The preceding assumptions require that the probabilities of certain non-observable events be defined. Thus, $F_\theta(B|a)$ must be defined for $B \varepsilon \mathcal{B}_{(t)}$ even if $B \notin \mathcal{B}_{(t)}(a)$. This requirement may seem counterintuitive from some points of view. Yet, it can be motivated by thinking of $B \varepsilon \mathcal{B}_{(t)}$ as an event that some other observer (possibly, "Nature") might be looking at, - this observer would be observing the same realization of the stochastic process $X_1, X_2, \ldots$ but would be using a different action for which B was observable.

2.10. Conditional distributions: We have found it necessary to also assume the existence of appropriately measurable versions of the conditional distribution on $\mathcal{B}_t$ given $\mathcal{A}$, $\mathcal{B}_{(t-1)}$. Formally,

(i) $F_\theta^t(\cdot|a,x)$ is a version of the conditional probability distribution on $\mathcal{B}_t$ given $\mathcal{A} \times \mathcal{B}_{(t-1)}$.

(ii) $F_\theta^t(B|\cdot,\cdot)$ is $\mathcal{A}_{(t)} \times \mathcal{B}_{(t-1)}$ measurable for all $B \epsilon \mathcal{B}_t$.

[It appears plausible that 2.10 (i) and (ii) should be implied by 2.8 (i) and (iii), at least when X_t, $t \epsilon T$ are Polish spaces. However, we do not know a general theorem to this effect.]

2.11. Sequential decision procedures: A stagewise conditional decision procedure[1] at stage $t \epsilon T$ is a conditional probability measure δ_t satisfying:

(i) $\delta_t(\cdot|a,x)$ is a probability measure on $\mathcal{A}_t$.

(ii) $\delta_t(C|\cdot,\cdot)$ is $\mathcal{A}_{(t-1)} \times \mathcal{B}_{(t-1)}$ measurable for each $C \epsilon \mathcal{A}_t$.

(iii) $\delta_t(C|a,\cdot)$ is $\mathcal{B}_{(t-1)}(a)$ measurable for each $C \epsilon \mathcal{A}_t$, $a \epsilon A$.

(Note that $\delta_t(C|a,\cdot)$ really depends only on $\pi_{(t-1)}(a)$ because of (ii). It is therefore justifiable to abuse the notation slightly by writing $\delta_t(C|a_1,\ldots,a_{t-1};x_1,\ldots,x_{t-1})$ when this is convenient, and to use this notation for $C \subset K_t$, $C \epsilon \mathcal{A}_t$ as well as for $C \subset A$, $C \epsilon \mathcal{A}_t$.)

A sequence over $t \epsilon T$ of such rules determines a sequential decision procedure. We use the notation δ or $\{\delta_t\}$ for a sequential decision procedure. Let $\mathcal{D} = \{\delta\}$.

2.12. The observed process: The preceding suffices to guarantee the existence of a stochastic process of observed actions and sample points at each stage, when $\theta \epsilon \Theta$ is true and the sequential procedure δ has been used. This process is defined on

1. See the note on terminology at the end of this paper.

$\times_{t \epsilon T} (K_t, X_t)$ with the corresponding product σ-field. The probability of a cylinder set $(C_1,B_1)\times\ldots\times(C_n,B_n)\times(K_{n+1},X_{n+1})\times\ldots$ is given by

$$(1) \quad \Delta_{\theta,\delta}\Big(\mathop{\times}_{t=1}^{n}(C_t,B_t) \mathop{\times}_{t>n}(K_t,X_t)\Big)$$

$$= \int_{C_1}\int_{B_1}\cdots\cdots\int_{C_n}\int_{B_n} F_\theta^n(dx_n|a_1,\ldots,a_n;x_1,\ldots,x_{n-1})$$

$$\delta_n(da_n|a_1,\ldots,a_{n-1};x_1,\ldots,x_{n-1})\cdot\ \cdot\ \cdot F_\theta(dx_1|a_1)\delta_1(da_1).$$

It is straightforward to check that the probability measure, $\Delta_{\theta,\delta}$, is well defined by (1), defines a stochastic process on $K_1 \times X_1 \times K_2 \times X_2 \times \ldots$. with the corresponding σ-fields, and that the decision component of this process is concentrated on $A \subset \times_{t \epsilon T} K_t$.

(If $\infty \epsilon T$ then, of course, n may be ∞ in (1) - and similar expressions to follow - but only a finite number of C_t, B_t should be different from K_t, X_t.)

2.13. Loss function: The loss function is a map $L: \Theta \times A \times X \to [0,\infty]$ such that $L(\theta,\cdot,\cdot)$ is $\mathcal{A} \times \mathcal{B}$ measurable.

2.14. Remark: In traditional statistical settings the loss is independent of the observation $x \epsilon X$. However the general theory developed in this paper applies to many other types of problems. In many of these settings - e.g. stochastic adaptive control processes - the loss naturally depends partly or entirely on $x \epsilon X$. Note also that the loss may depend on unobserved (e.g. future) values of x, as well as observed values.

The fact that L is bounded below by the value zero is inessential to the theory, however it is important that the loss be bounded below for each $\theta \epsilon \Theta$ and the value zero is chosen merely for convenience. In settings in which a "gain function" appears one should, of course, define $L = -G$, and it is then important that G be bounded above for each $\theta \epsilon \Theta$.

We mention an important relaxation of this assumption. The condition $L \geq 0$ can be replaced throughout this paper by the following condition:

$$E_{\Delta_{\theta,\delta}}(\inf_{a\in\mathcal{A}} L^-(\theta,a,x)) > -\infty \quad \text{for all } \delta\varepsilon\mathcal{D}$$

The reader may easily check the validity of this assertion. In the appropriate "optimal stopping" context this condition is precisely the condition, "$E(\sup x_n^+) < \infty$", of Chow, Robbins, Siegmund [5, especially Theorems 4.5 and 4.5'].

2.15. Risk: The risk function $R:\Theta \times \mathcal{D} \to [0,\infty]$ is defined to be the expectation of the loss - $E(L(\theta,a,x))$ - computed under the measure $\Delta_{\theta,\delta}$.

As usual the risk functions serve to define a partial ordering, " $>$ ", on $\mathcal{D}$ by $\delta_2 > \delta_1$ if $R(\theta,\delta_2) \leq R(\theta,\delta_1)$ for all $\theta\varepsilon\Theta$. Maximal elements in this partial ordering are called admissible. When Θ contains just one point the maximal elements are called optimal policies.

2.16. The set of risk functions: Let $\Gamma = \{R(\cdot,\delta):\delta\varepsilon\mathcal{D}\}$. Γ may be thought of as a subset of the compact product space $T = \times_{\theta\varepsilon\Theta} [0,\infty]$. Let

$$\hat{\Gamma} = \{r\varepsilon T: \exists\delta\varepsilon\mathcal{D}\ R(\theta,\delta) \leq r(\theta)\ \forall\ \theta\varepsilon\Theta\}.$$

2.17. Discussion: The primary goal of the remainder of the theoretical part of this paper is to provide conditions under which $\hat{\Gamma}$ is compact. For when $\hat{\Gamma}$ is compact various conclusions follow - The following are a few of the most important such conclusions:

(1) There exists a minimal complete class (in abstract terms - every $r\varepsilon\Gamma$ is dominated by a maximal element under $>$). Thus, if Θ contains just one point, then there exist optimal policies.

(2) There exists a minimax procedure, and a least favorable sequence of prior distributions.

(3) If Θ is finite, then the Bayes procedures form a complete class. (This conclusion holds, more generally, if Θ is compact and all risk functions are continuous and real-valued.)

(4) The Stein-Le Cam necessary and sufficient condition for admissibility is valid. See Stein [14] and Farrell [7].

Proofs of the above statements are of a topological character, and are well known. For (1)-(3) see for example, Wald [15] and LeCam [8].

2.18. Formula: A sequential decision procedure, $\delta=\{\delta_t:t\varepsilon T\}$, defines a conditional probability measure on $\mathcal{A}$ given $x\varepsilon X$ through the formula (1), below, which gives the measure of cylinder sets. To avoid overburdening the notation we also use the symbol δ as a general notation for this measure.

(1)
$$\delta(A \cap (C_1 \times \dots \times {}_t \times K_{t+1} \times \dots)|x)$$
$$= \int_{a_1\varepsilon C_1} \int_{a_{t-1}\varepsilon C_{t-1}} \delta_t(C_t|a_1,\dots,a_{t-1};x)$$
$$\delta_{t-1}(da_{t-1}|a_1,\dots,a_{t-2};x)\dots\delta(da_1),$$
$$C_i\ \varepsilon\ \mathcal{A}_i, \quad i = 1,\dots,t\varepsilon T.$$

THEOREM 2.19. Let $\{\delta_t\}$ be a sequential decision procedure. Then the conditional measure δ defined by 2.18 (1) satisfies:

(1) $\delta(\cdot|x)$ is a probability measure on A, $\mathcal{A}$, for each $x\varepsilon X$; and

(2) For any $C\varepsilon\mathcal{A}_{(t-1)}$, $D\varepsilon\mathcal{A}_t$ the function $\delta(C \cap D|\cdot)$ is measurable with respect to $\mathcal{B}_{(t-1)}(C)$.

PROOF. Condition (1) is obvious from 2.18 (1). For (2) note that $\mathcal{B}_{(n)}(\pi_{(n)}(C)) \subset \mathcal{B}_{(t-1)}(C)$ for $n \leq t-1$. Consider $\delta_n(E_n|a_{(n-1)};\cdot)$ for $n \leq t$, $E_n\varepsilon\mathcal{A}_n$, and $a_{(n-1)}\varepsilon\pi_{(n-1)}(C)$. By definition it is $\mathcal{B}_{(n-1)}(\pi_{(n-1)}(C))$ measurable, and thus it is $\mathcal{B}_{(t-1)}(C)$ measurable by the preceding remark. The assertion (2) then follows from standard Fubini-type theorems on iterated integration; as given in Meyer [10] or Neveu [11].

2.20. Decision procedures: A measurable conditional probability distribution satisfying conditions 2.19 (1) and (2) is called a decision procedure (as contrasted to a sequential decision procedure as defined in 2.11).

Theorem 2.19 establishes that every sequential decision procedure determines a corresponding decision procedure. In Sections 3 and 4 we find it convenient to work with decision procedures, rather than with sequential decision procedures. Before we can be justified in doing this, it is necessary that we establish conditions under which every decision procedure corresponds to a sequential decision procedure (up to $\{F_\theta\}$ equivalence). This is done in the remainder of this section.

Actually, the following theorem has two important limitations. One of these is Assumption 2.21, which appears to be essentially necessary in any case. The second is that $\{F_\theta(\cdot|a):\theta\in\Theta,\ a\in A\}$ should form a dominated family - dominated by the measure ν. In this way, "$\{F_\theta\}$-equivalence" becomes merely ν-equivalence for the fixed measure ν, and standard conditional probability results can be invoked in the proof of Theorem 2.22. We do not know how far Theorem 2.22 can be extended without this dominatedness assumption; however the dominatedness assumption is in any event a natural one to make in the important Proposition 3.2, and in the remaining theory of Sections 3 and 4. See Assumptions 3.1 or 3.17.

2.21. <u>Assumption</u>: Throughout the remainder of the paper assume that

$$\mathcal{B}_{(t)}(a) = \cap\{\mathcal{B}_t(C):\pi_{(t)}(a)\in C, C\subset\mathcal{A}_{(t)}, C \text{ is open}\},\ \text{for each } a\in A,\ t\in T.$$

[This assumption is a kind of semi-continuity assumption on the σ-fields $\mathcal{B}_{(t)}(a)$. It can be interpreted as saying that if an event is observable at stage t on any neighborhood of $a_{(t)}$ then it should be observable at $a_{(t)}$ itself.]

THEOREM 2.22. <u>Let ν be a given probability measure on $X,\mathcal{B}$, and let δ be a given decision procedure. Then there exists a sequential decision procedure $\{\delta_t:t\in T\}$ such that</u>

$$(1)\quad \Delta((C\cap D)\times B)=\int_{x\in B}\delta(C\cap D|x)\nu(dx)=\int_{a_{(t-1)}\in\pi_{(t-1)}(C),\,x\in B}\cdot\ \cdot\ \cdot\int\delta_t(D|a_1,\ldots,a_{t-1};x)\delta_{t-1}(da_{t-1}|a_1,\ldots,a_{t-2};x)\ldots\delta_1(da_1)\nu(dx),$$

$$C\in\mathcal{A}_{(t-1)},\ D\in\mathcal{A}_t,\ B\in\mathcal{B}.$$

In other words, δ_t is a determination of the conditional distribution of Δ given $\mathcal{A}_{(t-1)}\times\mathcal{B}$ with $\Delta = \nu \circ \delta$ defined on $\mathcal{A}\times\mathcal{B}$ by (1). To see this equivalence more clearly, rewrite (1), by induction, as

$$(2)\quad \Delta((C\cap D)\times B) = \int_{a_{(t-1)}\in\pi_{(t-1)}(C),\,x\in B}\delta_t(D|a_{(t-1)};x)\Delta(da,dx),\ C,\ D,\ B \text{ as in (1)}.$$

PROOF. Fix $t\in T$. We will construct a σ-field $\mathcal{B}'\subset\mathcal{A}_{(t-1)}\times\mathcal{B}$ such that:

(3) $g:A\times X\to R$ is $\mathcal{B}'$ measurable only if $g(a,\cdot)$ is $\mathcal{B}_{(t-1)}(a)$ measurable for all $a\in A$; and

(4) For each $D\in\mathcal{B}_t$ there is a determination of $E_\Delta(\chi_D|\mathcal{A}_{(t-1)}\times\mathcal{B})$ which is $\mathcal{B}'$ measurable.

Once $\mathcal{B}'$ has been constructed, define δ_t as the measurable conditional probability measure of Δ on $\mathcal{A}_t$ given $\mathcal{B}'$. This conditional probability measure exists since A, $\mathcal{A}_t$ is standard Borel (see, e.g., Neveu [11]), and it satisfies (1) and the defining properties of a sequential decision procedure (2.11) because of (3) and (4). It remains to construct $\mathcal{B}'$ and check (3) and (4).

Let ρ be a metric on K. If $B\subset K$ let $\rho(B)$ denote the diameter of B. Let $\mathcal{B}'_\varepsilon$ be the least σ-field on $\mathcal{A}\times X$ containing all sets of the form $C\times B$ where $C\in\mathcal{A}_{(t-1)}$, $\rho(C)\le\varepsilon$, and $B\in\mathcal{B}_{(t-1)}(C)$. Let $\mathcal{B}' = \bigcap_{\varepsilon>0} B'_\varepsilon$. Then, $\mathcal{B}'$ satisfies (3) because of Assumption 2.21.

Fix $D\in\mathcal{B}_t$. Let $\{C_i:i=1,\ldots\}$ be a basis for the topology on $A_{(t-1)}$; and let $\mathcal{A}^*_k$ denote the σ-field generated by the sets $\{\pi^{-1}_{(t-1)}(C_i):i=1,\ldots,k\}$. $\mathcal{A}^*_k$ is actually an algebra of sets generated by a finite number of atoms, say $\{C^*_{k,j}:j=1,\ldots,J(k)\}$. Furthermore

$$(5)\qquad \lim_{k\to\infty}\sup_j \rho(C^*_{k,j}) = 0.$$

One may write

(6) $$E_\Delta(\chi_D | \mathcal{G}_k^* \times \mathcal{B}) = \sum_{j=1}^{J(k)} \delta(D \cap C_{k,j}^* | x) \chi_{C_{k,j}^*}(a).$$

$$= g_k(a,x), \text{ say.}$$

Define $g = \limsup_{k\to\infty} g_k$. For each $a \in \mathcal{G}$, $g_k(a,\cdot)$ is $\mathcal{B}'_\varepsilon$ measurable where $\varepsilon = \sup_j \rho(C_{k,j}^*)$. It follows from (5) that g is $\mathcal{B}'$ measurable.

Finally, it can be checked that $\{g_k\}$ is a Martingale relative to Δ. By the Martingale convergence theorem g is a version of $E_\Delta(\chi_D | \mathcal{G}_{(t-1)} \times \mathcal{B})$ since $\mathcal{G}_{(t-1)} \times \mathcal{B}$ is the least σ-field containing $\{\mathcal{G}_k^* \times \mathcal{B}: k=1,\ldots\}$. Thus $\mathcal{B}'$ has the desired properties and the theorem is proved.

3. *Basic theorems for a compact action space.* It is proved in this section that under certain assumptions $\hat{\Gamma}$ is compact. The basic assumptions in addition to the compactness of A are that $\{F\}$ or $\{F^{(t)}\}$ be a dominated family of distributions whose densities have a suitable continuity property, and that the loss, $L(\theta,\cdot,x)$, be lower semi-continuous on A. The following assumption is somewhat relaxed in Assumption 3.17.

3.1. Assumption: The family $\{F_\theta(\cdot|a): \theta\in\Theta, a\in A\}$ is a dominated family of distributions. In this case it is always possible to choose the dominating σ-finite measure to be a probability measure which is a linear combination of measures in the family. (See Lehmann [9].) Assume this has been done. Denote the dominating probability measure by ν, and let

$$f_\theta(\cdot|a) = \frac{dF_\theta(\cdot|a)}{d\nu} \quad \theta\in\Theta,\ a\in A.$$

Assume that $f_\theta(\cdot|\cdot)$ is $\mathcal{B} \times \mathcal{G}$ measurable. (This measurability condition follows from a Martingale argument in Meyer [10] if X is a Polish space and $\mathcal{B}$ is its Borel field.)

Since ν is a linear combination from $\{F_\theta(\cdot|a)\}$, the conditional distributions of ν given $\mathcal{B}_{(t)}$ exist.

PROPOSITION 3.2. Let Assumption 3.1 be satisfied. Let δ be any decision procedure. According to Theorem 1.22, δ corresponds to an essentially unique (a.e.Δ) sequential decision procedure, δ^*, say. Then, $R(\theta,\delta^*)$ (as defined in (1.15)) may be computed as

(1) $$R(\theta,\delta^*) = \iint L(\theta,a,x)f_\theta(x|a)\delta(da|x)\nu(dx).$$

PROOF. This proposition follows directly from Theorem 2.22 and the properties of conditional probability after one writes out explicitly the expressions for $E_{\Delta_{\theta,\delta^*}}(L)$ and $E_\Delta(L)$. These explicit expressions are rather lengthy since they involve repeated expressions like $F_\theta^t(dx_t|a_{(t)},x_{(t-1)}) = c(a_{(t)},x_{(t-1)})f_\theta(x|a)\nu(dx_t|a_{(t)},x_{(t-1)})$. We omit further details.

3.3. Convention: Theorems 2.19, and 2.22 and Proposition 3.2 show that there is a one-one correspondence between equivalence classes of decision procedures and of sequential decision procedures, and this equivalence preserves risk functions.

Hereafter in the context of Assumption 3.1 we will use the symbol $\mathcal{S}$ interchangeably to represent the set of all sequential decision procedures or the set of all decision procedures; and the symbol $R(\theta,\delta)$ will be used when δ is either type of decision procedure.

3.4. Definition: A generalized decision procedure taking decisions on a subset $D \subset K$ is a measurable conditional probability distribution on D given the σ-field $\mathcal{B}$ on X. Such decision procedures may "depend on the future".

Define a topology on the set of such procedures according to the convergence definition: $\delta_\alpha \to \delta$ if

(1) $$\int\delta_\alpha(da|x)f(x)c(a)\nu(dx) \to \int\delta(da|x)f(x)c(a)\nu(dx) \text{ for all}$$

$$f\in L_1(X,\mathcal{B},\nu) = L_1, c\in C(K)$$

where $C(K)$ denotes the continuous functions on K. This topology can equivalently be viewed as the weak topology generated by the maps $\int\delta(da|x)f(x)c(a)\nu(dx)$, $f\in L_1$, $c\in C(K)$.

The preceding topology is not Hausdorff. Hence it is convenient to introduce the equivalence relationship: $\delta_1 \sim \delta_2$ if

$\int \delta_1(da|x)f(x)c(a)\nu(dx) = \int \delta_2(da|x)f(x)c(a)\nu(dx)$ for all $f \varepsilon L_1$, $c \varepsilon C(K)$. Let $\mathcal{D}(K)$ denote the set of equivalence classes of generalized decision procedures on K with the above topology. Let $\mathcal{D}(D)$ denote the subset of $\mathcal{D}(K)$ consisting of the equivalence classes of generalized procedures on D.

Let $\mathcal{D}_p(D)$ be the subset of $\mathcal{D}(D)$ consisting of those equivalence classes of $\mathcal{D}(D)$ such that some member of the class is a decision procedure on D in the sense of Definition 2.20. Thus $\mathcal{D}_p(A)$ is a topological space whose elements are equivalence classes of members of $\mathcal{D}$. When there is no confusion we write the symbol, δ, for either an element of $\mathcal{D}$ or an equivalence class in $\mathcal{D}_p(A)$ (or in $\mathcal{D}(K)$, etc.). The following basic facts can be verified by standard probabilistic arguments.

PROPOSITION 3.5. (i) $\delta_1 \sim \delta_2$ if and only if $\delta_1(C|\cdot)=\delta_2(C|\cdot)$ almost everywhere (ν) for all $C \varepsilon \mathcal{C}$, $C \subset K$; and equivalently if and only if

(1) $$\int \delta_1(da|\cdot)c(a) = \int \delta_2(da|\cdot)c(a) \quad \text{a.e.}(\nu)$$

for all $c \varepsilon C(K)$.

(ii) If $\delta_1(C|\cdot) \varepsilon \mathcal{B}' \subset \mathcal{B}$ for all $C \varepsilon \mathcal{C}$, $C \subset K$, where $\mathcal{B}'$ denotes the intersection of $\mathcal{B}$ with the ν-completion of some given $\mathcal{B}' \subset \mathcal{B}$, then there is a $\delta_2 \sim \delta_1$ such that $\delta_2(C|\cdot) \varepsilon \mathcal{B}'$ for all $C \varepsilon \mathcal{C}$, $C \subset K$.

(iii) If D is a closed subset of K then $\delta \varepsilon \mathcal{D}(D)$ if and only if $\int \delta(da|\cdot)c(a) = 0$ a.e. (ν) for all $c \varepsilon C(K)$ with $\Sigma_c \cap D = \phi$ where

(2) $$\Sigma_c = \{a : c(a) > 0\}$$

(iv) Under Assumption 3.1 (or Assumption 3.17) if $\delta_1, \delta_2 \varepsilon \mathcal{D}$ and $\delta_1 \sim \delta_2$ then $R(\theta,\delta_1) = R(\theta,\delta_2)$ for all $\theta \varepsilon \Theta$.

(v) If $\delta \varepsilon \mathcal{D}(K)$ then $\delta \varepsilon \mathcal{D}_p(K)$ if and only if

(3) $$\int \delta(da|\cdot)c(a)d(a) \ \varepsilon \ L_1(X, \underline{\mathcal{B}}_{(t-1)}(\Sigma_c), \nu)$$

for all $c, d \varepsilon C(K)$ with c being $\mathcal{C}_{(t-1)}$ measurable and d being $\mathcal{C}_t$ measurable.

Proposition 3.5 enables the proof of the following basic

theorem.

THEOREM 3.6. If D is closed in K then $\mathcal{D}_p(D)$ is compact.

PROOF. $\mathcal{D}(K)$ is compact by Le Cam [11, Theorem 2]. By Proposition 3.5 (iii) $\delta \varepsilon \mathcal{D}(D)$ if and only if $\int \delta(da|x)c(a)f(x)\nu(dx) = 0$ for all $f \varepsilon L_1(X,\mathcal{B},\nu)$. Hence $\mathcal{D}(D)$ is closed in $\mathcal{D}(K)$.

If $\delta' \varepsilon \mathcal{D}(D) - \mathcal{D}_p(D)$ then there exists $c, d \varepsilon C(K)$ as in 3.5(3) such that $\int \delta'(da|\cdot)c(a)d(a) \notin L_1(X, \underline{\mathcal{B}}_{(t-1)}(\Sigma_c), \nu)$. There is then an $f' \varepsilon L_1(X,\mathcal{B},\nu)$ such that $\int \delta'(da|x)c(a)d(a)f'(x)\nu(dx) = 1 > 0 = \int \delta(da|x)c(a)d(a)f'(x)\nu(dx)$ for all $\delta \varepsilon \mathcal{D}_p(D)$. Hence there is a neighborhood of δ' which is disjoint from $\mathcal{D}_p(D)$. It follows that $\mathcal{D}_p(D)$ is closed in $\mathcal{D}(D)$; hence compact.

It remains only to discuss some technical methodology which is required for the proof of the first basic statistical result in Theorem 3.14.

3.7. Definition: Let $Q_k = \{q : A \times X \to R : q(a,x) = \sum_{i=1}^{n} c_i(a)h_i(x)$ with $c_i \varepsilon C(K)$, $h_i \varepsilon L_k(x,\mathcal{B},\nu)$, $c_i, h_i \geq 0$, $n < \infty\}$. The two cases of interest are $k=1 (L_1)$ and $k = \infty (L_\infty)$. The respective norms in these spaces will be denoted by $||\cdot||_1$ and $||\cdot||_\infty$.

Here is the second major assumption which is required.

3.8. Assumption: For each $\theta \varepsilon \Theta$ the map of $A \to L_1(x,\mathcal{B},\nu)$ defined by $a \to f_\theta(\cdot|a)$ is continuous. (Thus $||f_\theta(\cdot|a_i) - f_\theta(\cdot|a)||_1 \to 0$ as $a_i \to a$.)

The importance of this assumption is that it implies the conclusion of the following fundamental lemma.

LEMMA 3.9. Suppose Assumption 3.8 is satisfied and A is compact. Then for each $\theta \varepsilon \Theta$ there is a sequence $\langle q_i \rangle \varepsilon Q_1$ such that $q_i(a,\cdot) \to f_\theta(\cdot|a)$ in $L_1(X,\mathcal{B},\nu)$, uniformly in $a \varepsilon A$. (That is, for every $\varepsilon > 0$ there is a $q \varepsilon Q_1$ such that $||q_i(a,\cdot) - f_\theta(\cdot|a)||_1 < \varepsilon$ for all $a \varepsilon A$.)

PROOF. Since A is compact, Assumption 3.8 implies that the set $\{f_\theta(\cdot|a) : a \varepsilon A\}$ is compact. Furthermore, for an $h \varepsilon L_1$ $\{a : ||f_\theta(\cdot|a) - h(\cdot)||_1 < \varepsilon\}$ is open in A. It follows that for any $\varepsilon > 0$ there is a finite collection $h_1, \ldots, h_n$ and a "partition of unity" $c_1,\dots,c_n$ of continuous functions in C(K) with $\Sigma_{c_i} \subset \{a: ||f_\theta(\cdot|a)-h_i(\cdot)||_1 < \varepsilon\}$ such that $||f_\theta(\cdot,a) - \Sigma c_i(a)h_i(\cdot)||_1<\varepsilon$ for all $a\varepsilon A$.

The following important result provides the second basic "product representation".

THEOREM 3.10. Suppose D is closed in K and $\ell: D \times X \to [0,\infty]$ is $\mathcal{A} \times \mathcal{B}$ measurable with $\ell(\cdot,x)$ lower semi continuous on D for each $x\varepsilon X$. Then there is a non-decreasing sequence $<q_i> \varepsilon Q_\infty$ such that $q_i(\cdot,x) \uparrow \ell(\cdot,x)$ for a.e.(ν) $x\varepsilon X$.

PROOF. Let $Q^*_\infty = \{q\varepsilon Q_\infty : q(a,x) = \sum_{i=1}^{n} c_i(a)\chi_{S_i}(x)$, and $S_i \cap S_j = \phi$ for $i \neq j\}$. Clearly $Q^*_\infty \subset Q_\infty$. Let $q^{(1)}, q^{(2)}\ \varepsilon\ Q^*_\infty$. Then

$$\max(q^{(1)},q^{(2)}) = \sum_{i=1}^{n^{(1)}} \sum_{j=1}^{n^{(2)}} \max(c_i^{(1)}(a),c_j^{(2)}(a))\chi_{S_i \cap S_j}(x)\ \varepsilon Q^*_\infty$$

Hence Q^*_∞ is an (increasing) lattice.

Let $\{d_i\}$ be a countable dense subset of D, and let ρ denote a metric on D. Let $M = \{m_i\}$ denote the non-negative rational numbers. Let $c_{ij}(a) = m_i - j\rho(a,d_i)$. Consider the set

$$S'_{ij} = \{x: c_{ij}(a) \leq \ell(a,x) \quad \text{for all } a\varepsilon D\}$$

By standard projection theorems, this set is universally measurable. (See, e.g., Meyer [10].) In particular there exists a set $S_{ij} \subset S'_{ij}$ such that $S_{ij}\ \varepsilon\ \mathcal{B}$ and $\bar\nu(S'_{ij}-S_{ij}) = 0$ where $\bar\nu$ denotes outer ν measure.

It is well known that $\ell(a,x) = \sup c_{ij}(a)\chi_{S'_{ij}}(x)$ since $\ell(\cdot,x)$ is lower semi-continuous. It follows that $\ell(\cdot,x) = \sup c_{ij}(\cdot)\chi_{S_{ij}}(x)$ for almost all (ν) $x\varepsilon X$. The Theorem follows since $\{c_{ij}(a)\chi_{S_{ij}}(x)\} \subset Q^*_\infty \subset Q_\infty$ and Q^*_∞ is a lattice.

[If $X,\mathcal{B}$ is a Polish space with its Borel field then Brown and Purves [4] yields that the S'_{ij} are $\mathcal{B}$ measurable. Hence, in this case, the qualification, "for almost all $(\nu)x\varepsilon X$" may be dropped from the conclusion of the theorem.]

When $\mathcal{S}_p(A)$ is compact then compactness of $\hat\Gamma$ is implied by coordinatewise lower semi-continuity of the maps $\rho_\theta = R(\theta,\cdot): \mathcal{S}_p(A) \to [0,\infty]$. This is the content of the following result.

LEMMA 3.11. Suppose $\mathcal{S}_p(A)$ is compact. Suppose each of the maps $\rho_\theta: \mathcal{S}_p(A) \to [0,\infty]$, $\theta\varepsilon\Theta$, defined by $\rho_\theta(\delta) = R(\theta,\delta)$ is lower semi-continuous. Then $\hat\Gamma$ is compact.

PROOF. Let $s_\alpha(\cdot)$ be any net in $\hat\Gamma$ such that $s_\alpha \to s\ \varepsilon \times_{\theta\varepsilon\Theta} [0,\infty]$. By the definition of $\hat\Gamma$, there is a $\delta_\alpha\ \varepsilon\ \mathcal{S}_p(A)$ such that $R(\theta,\delta_\alpha) \leq s_\alpha(\theta)$. Let $\{\delta_{\alpha'}\}$ denote a convergent subnet, and $\delta_{\alpha'} \to \delta$. Then $R(\theta,\delta) \leq \liminf R(\theta,\delta_{\alpha'}) \leq \liminf s_{\alpha'}(\theta) = s(\theta)$. Hence $s\varepsilon\hat\Gamma$.

The following corollary is required later in this paper.

COROLLARY 3.12. Suppose V is a closed subset of $\mathcal{S}_p(A)$, $\mathcal{S}_p(A)$ is compact and each ρ_θ is lower semi-continuous. Let

$$\hat\Gamma(V) = \{r(\cdot): r(\cdot)\varepsilon \times_{\theta\varepsilon\Theta} [0,\infty],\ \exists\delta\varepsilon V \ni R(\theta,\delta) \leq r(\theta)\ \forall\theta\varepsilon\Theta\}.$$

Then $\hat\Gamma(V)$ is compact.

The remaining assumption required is

3.13. Assumption: $L(\theta,\cdot,x)$ is lower semi-continuous (as a map to $[0,\infty]$) for each $\theta\varepsilon\Theta$, $x\varepsilon X$.

Here is the main result in this case. See Discussion 2.17 for some further consequences.

THEOREM 3.14. Let Assumptions 3.1, 3.8, and 3.13 be satisfied. If A is compact in K then the maps ρ_θ, $\theta\varepsilon\Theta$ are lower semi-continuous, and $\hat\Gamma$ is compact.

PROOF. Fix $\theta\varepsilon\Theta$. By Theorem 3.10 there is a sequence $<q_i>$ in Q_∞ such that $q_i(\cdot,\cdot) \uparrow L(\theta,\cdot,\cdot)$. Then,

$$(1) \qquad \rho_\theta(\delta) = R(\theta,\delta) = \sup_{i\to\infty} \int q_i(a,x) f_\theta(x|a)\delta(da|x)\nu(dx).$$

By Lemma 3.9 there is a sequence $<q'_j>\ \varepsilon\ Q_1$ such that $q'_j(a,\cdot) \to f(\cdot|a)$ in L_1 uniformly in a.

Each expression of the form

$$\int q_i(a,x) q'_j(a,x)\delta(da|x)\nu(dx)$$

describes a continuous map: $\mathcal{D}_p(A) \to [0,\infty)$. Furthermore

$$\begin{aligned} &|\int c_k(a)h_k(x)(f_\theta(x|a) - q_j'(a,x))\delta(da|x)\nu(dx)| \\ &\le \sup_{a\varepsilon A}|c_k(a)| \cdot ||h_k||_\infty \cdot \sup_{a\varepsilon A}||f_\theta(\cdot|a) \\ &- q_j'(a,\cdot)||_1 \to 0 \quad \text{as } j \to \infty. \end{aligned}$$

Hence each expression in the supremum on the right of (1) is a continuous function: $\mathcal{D}_p(A) \to [0,\infty)$.

It follows that $\rho_\theta(\cdot)$ is lower semi-continuous since it is written in (1) as the supremum of continuous functions. The second assertion of the theorem follows from Lemma 3.11.

3.15. Remark: The fact that $\delta\varepsilon\mathcal{D}_p(A)$ rather than merely $\delta\varepsilon\mathcal{D}(A)$ is not used in the semi-continuity part of the proof of the theorem. It is used to connect this assertion to the risk $R(\theta,\cdot)$. Therefore, it is also true that the function defined on $\delta\varepsilon\mathcal{D}(A)$ by $\int L(\theta,a,x)f_\theta(x|a)\delta(da|x)\nu(dx)$ is lower semi-continuous under the other assumptions of the theorem. This function has no natural statistical meaning; but this fact is needed in the proof of Theorem 4.7.

In the more rare circumstance where every loss function is continuous and bounded then a stronger conclusion is possible:

COROLLARY 3.16. *Suppose the assumptions of Theorem* 3.12 *are satisfied and, in addition,* $L(\theta,\cdot,\cdot)$ *is bounded for each* $\theta\varepsilon\Theta$ *and* $L(\theta,\cdot x)$ *is continuous for each* $\theta\varepsilon\Theta$, $x\varepsilon X$. *Then* ρ_θ *is continuous for each* $\theta\varepsilon\Theta$ *and* Γ, *itself, is compact.*

PROOF. Apply the proof of the theorem to L and to $-L(\theta,\cdot,\cdot)+\sup L(\theta,\cdot,\cdot)$ to see that ρ_θ is a continuous map of $\mathcal{D}_p(A) \to [0,\infty)$. The Corollary follows from this.

[This Corollary can also be proved another way. Under the given continuity assumptions it is possible to prove a version of Theorem 3.10 in which the sequence $\{q_j\}$ is uniformly convergent. (This alternate version is actually much easier to establish than is Theorem 3.10, which we needed for the proof of Theorem 3.14.) Substituting this version of Theorem 3.10 in the proof of Theorem 3.14 then proves Corollary 3.16. This proof appears in Portnoy [12].]

If the loss function satisfies another mild structural condition then the above assumption that $\{F_\theta(\cdot|a):\theta\varepsilon\Theta, a\varepsilon\mathcal{A}\}$ be a dominated family (Assumption 3.1) can be significantly relaxed in one respect. The application of the following theorem is further clarified by the Discussions 4.1 and 4.6.

3.17. Assumptions:

(i) Let $F_\theta^{(t)}(\cdot|a)$ denote the marginal distribution of F_θ on $X_{(t)}$. Assume the family $\{F_\theta^{(t)}(\cdot|a):\theta\varepsilon\Theta, a\varepsilon A\}$ satisfies Assumptions 3.1 and 3.8 for each $t\varepsilon T$.

(ii) Suppose there exists a non-decreasing sequence of functions $L^{(t)}:\Theta \times A_{(t)} \times X_{(t)} \to [0,\beta]$ such that $L^{(t)}(\theta,b,y) \le \inf\{L(\theta,a,x):\pi_{(t)}(a)=b,\ \pi_{(t)}(x)=y\}$, and $L^{(t)}(\theta,\cdot,y)$ is lower semi-continuous for each θ,y and t, and

$$\lim_{t\to\infty} L^{(t)} = L. \tag{1}$$

[As an example, suppose $L(\theta,a,x) = \sum_{t=1}^\infty \ell_t(\theta,a_{(t)},x_{(t)})$ where each $\ell_t(\theta,\cdot,x_t) \ge 0$ is lower semi-continuous. Then (ii) is satisfied with $L^{(t)} = \sum_{\tau=1}^t \ell_\tau$. Note that (ii) implies the lower semi-continuity assertion of Assumption 3.13.]

THEOREM 3.18. *Let Assumption* 3.17 *be satisfied. Then the maps* ρ_θ, $\theta\varepsilon\Theta$, *are lower semi-continuous, and* $\hat\Gamma$ *is compact.*

PROOF. Let $\mathcal{P}^{(t)}$ denote the original problem truncated to the index set $T^{(t)} = 1,\ldots,t$, and with loss function $L^{(t)}$. Let $\rho_\theta^{(t)}$ denote the corresponding maps in this problem. Thanks to Assumption 3.17 these maps are lower semi-continuous by Theorem 3.14. Now, $\rho_\theta^{(t)} \uparrow \rho_\theta$ since $L^{(t)} \uparrow L$. Hence ρ_θ is lower semi-continuous. As before, $\hat\Gamma$ is then compact by Lemma 3.11.

The following is analagous to Corollary 3.16.

COROLLARY 3.19. *Suppose the assumptions of Theorem* 3.18 *are* satisfied and, in addition, $L(\theta,\cdot,\cdot)$ is bounded, the functions $L^{(t)}(\theta,\cdot,x)$ of Assumption 3.16 are continuous for each $\theta\in\Theta$, $x\in X$, and the convergence in 3.16 (1) is uniform in a for each $\theta\in\Theta$, $x\in X$. Then ρ_θ is continuous for each $\theta\in\Theta$ and Γ is compact.

PROOF. The assumptions guarantee that L and -L+b satisfy the assumptions of Theorem 3.18, with $b=b(\theta) = \sup L(\theta,\cdot,\cdot)$. (Note, however that $(-L+b)^{(t)} \neq -L^{(t)}+b$.) The corollary then follows from the theorem.

4. *Basic theorems for general action spaces.*

4.1. Discussion: Because of the combined effect of Assumptions 3.1, 3.8, and the assumption that A be compact, Theorem 3.14 is unsuitable for virtually all unbounded-stage sequential applications. In many problems of all general types A is not compact and so neither Theorem 3.14 or 3.18 can be directly applied.

In sequential problems the action space often contains a sequence $\{\alpha_n: n=1,2,\ldots\}$ corresponding to the decisions "stop at stage n". The only natural limit for such a sequence would be an action (say α_∞) corresponding to the decision "do not stop". But if Assumption 3.1 is to be satisfied this latter action often cannot possibly be in A because there is no suitable choice for $F_\theta(\cdot|\alpha_\infty)$ which would be dominated by ν. This possibility alone need not prevent the application of Theorem 3.18, however.

Even when the preceding difficulty does not occur, it may be that A is not compact simply because certain "natural" actions are prohibited. For example in many fixed sample size estimation problems $A = (-\infty,\infty)$, and $L(\theta,a,x) = (\theta-a)^2$. In this case the actions $\pm\infty$ are "natural", though prohibited, along with the interpretation $L(\theta,\pm\infty,x) = \infty$. (It is possible, and equally "natural", to use the one-point compactification $(-\infty,\infty) \cup \{i\}$ of A, again with the interpretation $L(\theta,i,x) = \infty$.)

The following additional assumptions are required to adapt Theorem 3.18 to the case where A is not compact. Further conditions are needed to adapt Theorem 3.14, and these are given later.

4.2. Assumptions: There is a well formulated statistical problem with action space $\bar{A}$ which "extends" the original problem as described in (i)-(iv), below. Call this problem $\bar{\mathcal{P}}$, and denote its elements by $\bar{\mathcal{B}}$, $\bar{F}$, etc.

There is a map $m:\bar{A} \to A$ with $m(a) = a$ if $a\in A$ such that

(i) m is measurable as map: $(\bar{A},\bar{\mathcal{G}}_t) \to (A,\mathcal{G}_t)$, $\forall\, t\in T$.

(ii) $\mathcal{B}_t(m(a)) = \bar{\mathcal{B}}_t(a)$ $\forall\, a\in\bar{A}$, $t\in T$.

(iii) $F^t_\theta(\cdot|m(a),\cdot) = \bar{F}^t_\theta(\cdot|a,\cdot)$ $\forall\, a\in\bar{A}$, $t\in T$.

(iv) $L(\theta,m(a),x) \leq \bar{L}(\theta,a,x)$ $\forall\, \theta\in\Theta$, $a\in\bar{A}$, $x\in X$.

Assume also that $\bar{L}$ satisfies the lower semi-continuity condition - Assumption 3.13 - on $\bar{A}$. Let $\bar{\rho}_\theta$ denote the map ρ_θ in problem $\bar{\mathcal{P}}$.

4.3. Remark: It is required above that $\bar{L}(\theta,\cdot,x)$ be lower semi-continuous. It can be shown that the formula

(1) $\bar{L}(\theta,b,x) = \sup\{\inf\{L(\theta,a,x): a\in G \cap A\}: G \subset K,\ G\in\mathcal{G},\ b\in G\}$

defines the largest lower semi-continuous extension of $L(\theta,\cdot,x)$ to $\bar{A}$, where $\mathcal{G}$ denotes a countable basis for the topology on A. Furthermore, if X is standard Borel then this extension is $\mathcal{G} \times \mathcal{B}$ measurable. (Use the projection theorem in Brown and Purves [4].) In this case there would be no loss of generality in assuming that the function $\bar{L}$ in Assumption 4.2 were defined by (1).

In contrast, the existence of the function, m, in Assumption 4.2 is not guaranteed. m must be constructed in each example by examining $\bar{L}$ and L. Its existence is not otherwise guaranteed, and may depend also upon the choice for the compactification of A. (The compactification is in turn determined by the choice for the space K in which A is imbedded.)

The following theorem extends Theorem 3.18. The extension of Theorem 3.14 is given in Theorem 4.7.

THEOREM 4.4. Suppose that Assumption 4.2 is satisfied and that the hypotheses of Theorem 3.18 are satisfied in problem $\bar{\mathcal{P}}$. Then the maps ρ_θ and $\bar{\rho}_\theta$, $\theta\in\Theta$, are lower semi-continuous. Furthermore $\hat{\Gamma} = \hat{\bar{\Gamma}}$ is compact.

[Concerning Assumption 3.17 (ii) for problem $\bar{\mathcal{P}}$ note that if the original loss function L satisfies this condition then $\bar{L}$, defined by 4.3 (1) will also if $\bar{A} = \underset{t\in T}{\times} \bar{A}_t$. (Define $L^{(t)}$ from $L^{(t)}$ by 4.3 (1).)]

PROOF. $\bar{\rho}_\theta$ is lower semi-continuous by Theorem 3.18. Hence $\hat{\hat{\Gamma}}$ is compact.

Now, let $\bar{\delta}\varepsilon\bar{\mathcal{D}}$. Define δ by $\delta(B|x) = \bar{\delta}(m^{-1}(B)|x)$, $B\varepsilon\mathcal{Q}$. It is straightforward to check from 4.2 (i)-(ii) that $\delta\varepsilon\mathcal{D} \subset \bar{\mathcal{D}}$ and from 4.2 (iii)-(iv) that $R(\theta,\delta) \leq \bar{R}(\theta,\bar{\delta})$. Hence $\hat{\Gamma} = \hat{\hat{\Gamma}}$, and so $\hat{\Gamma}$ is also compact. The additional assumption needed for the extension of Theorem 3.14 is

4.5. Assumption:

(i) For each $\theta\varepsilon\Theta$ there is locally compact subset $A' = A'(\theta) \subset \bar{A}$ such that Assumptions 3.1 and 3.8 are satisfied relative to the problem with action space $A'(\theta)$ and parameter space $\{\theta\}$.

(ii) For each $\theta\varepsilon\Theta$ and every $c < \infty$ there is a compact subset $D \subset A'(\theta)$ such that

(1) $\bar{L}(\theta,a,x) > c$ for $a\varepsilon\bar{A}-D$, $x\varepsilon X$.

In particular, (1) and the semi-continuity of $\bar{L}$ imply that

(2) $\bar{L}(\theta,a,x) = \infty$ for $a\varepsilon\bar{A}-A'(\theta)$.

[Such a condition is not required in Theorem 4.4. On the other hand, Theorem 4.7 does not require the special Assumption 3.17 on the structure of L. Assumption 3.17 is not satisfied in the most common sequential statistical problems, and so only Theorem 4.7 can be applied in such problems.]

4.6. Discussion: Many common statistical decision problems involve i.i.d. observations from some distribution indexed by $\theta\varepsilon\Theta$ (and each θ indexes a distinct distribution), and the decision $a\varepsilon\bar{A}$ concerns when to stop sampling, and, possibly, other things such as a terminal decision. One or more points in $\bar{A}$ correspond to the situation where sampling never stops. Let α_∞ denote such a point. Then all the distributions $F_\theta(\cdot|\alpha_\infty)$ are mutually singular. It follows that one may have $\alpha_\infty\varepsilon A'(\theta)$ for at most a countable number of points $\theta\varepsilon\Theta$, since $\{\bar{F}_\theta(\cdot|\alpha_\infty):\alpha_\infty\varepsilon A'(\theta)\}$ must be a dominated family of distributions. If $\alpha_\infty \notin A'(\theta)$ then $\bar{L}(\theta,\alpha_\infty,x) = \infty$ by 4.5 (2). (In fact, 4.5 (1) says that $\bar{L}(\theta,a,x) \to \infty$ uniformly in x as $a \to \alpha_\infty$.) Suppose Θ is uncountable. It then follows from Assumption 4.2 (ii)-(iii) that if $\bar{A}$ contains a point under which sampling never stops, then A must also contain at least one such point. To summarize: in such situations some sort of decision under which sampling never stops must be "available" (i.e. in A). Furthermore, the cost of such a decision must be infinite (uniformly in x) for all but at most a countable number of parameter values.

There are many examples in which the conclusion of Theorem 4.7 fails because A fails to contain an action under which sampling never stops (even though the cost of such an action is infinite). However, we do not know whether the second part of the above summary statement is logically necessary (in the special i.i.d. case) for the conclusion of Theorem 4.7.

Assumption 4.5 is phrased in sufficiently general terms to allow other sorts of applications. For example, Assumption 4.5 would be satisfied in the following simple two stage, design and estimation, problem: $A_1 = (0,k]$, $A_2 = (-\infty,\infty) = \Theta$, $F^1_\theta(\cdot|a_1)$ is the normal distribution with mean θ and variance a_1. The process stops after choosing a_1 observing x_1 and estimating by a_2. (Formally, all other co-ordinates of A and X are trivial.) To be specific set $L(\theta,a,x) = (\theta-a_2)^2 + a_1^{-1}$. Then this problem satisfies Assumption 4.5 with $A'(\theta) \equiv (0,k] \times [-\infty,\infty] \subset \bar{A} = [0,k] \times [-\infty,\infty]$.

[Note that $k < \infty$ is required in the preceding example. If $A_1 = (0,\infty)$ then Assumption 4.5 cannot be satisfied and, in fact, the conclusion of Theorem 4.7 fails: No procedure has $R(0,\delta)=0= \lim R(0,\delta_i)$ where δ_i takes action $(i,0)$ with probability one. (In this problem it is an open question whether a minimal complete class exists even though $\hat{\Gamma}$ is not compact.)]

THEOREM 4.7. Let Assumptions 4.2 and 4.5 be satisfied. Then

<u>for each $\theta\varepsilon\Theta$ the map ρ_θ is lower semi-continuous. Furthermore, $\hat{\Gamma}$ is compact.</u>

PROOF. Fix $\theta\varepsilon\Theta$. Let $G \subset \bar{G} \subset A'$ where G is open in $\bar{A}$ and $G \neq A'$. Let $V_G(k) = \{\delta: \int_G \bar{L}(\theta,a,x)\bar{f}_\theta(x|a)\delta(da|x)\nu(dx) \leq k\}$. Pick $a_0 \varepsilon A'-G$. Let $G_0 = \bar{G} \cup \{a_0\}$. For any $\delta\varepsilon\mathcal{D}(\bar{A})$ define δ^* by $\delta^*(x,C) = \delta(x,C \cap \bar{G}) + (1-\delta(x,C \cap \bar{G}))\chi_C(a_0)$. Then $\delta^*\varepsilon\mathcal{D}(G_0)$. (It need not be true that $\delta^*\varepsilon\mathcal{D}_p(G_0)$.) $\mathcal{D}(G_0)$ is compact by Theorem 3.6. Furthermore

$$\int c(a)h(x)\delta(da|x)\nu(dx) = \int c(a)h(x)\delta^*(da|x)\nu(dx) \tag{1}$$

for all $h\varepsilon L_1$, $c\varepsilon C(K)$ with $\Sigma_c \subset G$.

Let $<\delta_\alpha>$ be a convergent net in $V_G(k)$ with $\delta_\alpha \to \delta$. Then $<\delta^*_\alpha>$ is a net in $\mathcal{S}(G_0)$ and so has an accumulation point, say δ^*_0.

It follows from (1) that

$$\int_G \bar{L}(\theta,a,x)\bar{f}_\theta(x|a)\delta_\alpha(da|x)\nu(dx) = \int_G \bar{L}(\theta,a,x)\bar{f}_\theta(x|a)\delta^*_\alpha(da|x)\nu(dx) \tag{2}$$

Hence,

$$\int \bar{L}(\theta,a,x)\bar{f}_\theta(x|a)\delta^*_0(da|x)\nu(dx) \leq k$$

by Remark 3.15 since $\bar{L}(\theta,\cdot,x)\chi_G(\cdot)$ is lower semi-continuous on G_0. It also follows from (1) and the definitions of convergence on $\mathcal{S}(\bar{A})$ and $\mathcal{S}(G_0)$ that (2) holds with δ in place of δ_α and δ^*_0 in place of δ^*_α. (Note: It does not follow that $\delta^* = \delta^*_0$, and this need not be true.) Hence $\delta\varepsilon V_G(k)$. This proves that $V_G(k)$ is closed.

If $H \subset A'$, H closed, define $V'_H(k) = \{\delta: \int_H \bar{f}_\theta(x|a)\delta(da|x)\nu(dx) \geq k\}$. Choose G, H so that $H \subset G \subset \bar{G} \subset A'$, G open. Let $<\delta_\alpha> \subset V'_H(k)$. It follows from (1) that

$$\int_H \bar{f}_\theta(x|a)\delta_\alpha(da|x)\nu(dx) = \int_H \bar{f}_\theta(x|a)\delta^*_\alpha(da|x)\nu(dx) \tag{3}$$

with equality also for $\delta = \lim \delta_\alpha$ and δ^*_0, respectively.

By Remark 3.15, $\int_H \bar{f}_\theta(x|a)\delta^*_\alpha(da|x)\nu(dx)$ is upper semi-continuous on $\mathcal{S}(G_0)$. It follows that $V'_H(k)$ is closed.

Let $G_i \subset \bar{G}_i \subset A'$, $\bigcup_{i=1}^\infty G_i = A'$, G_i open $G_i \neq A'$. Then, for $k < \infty$

$$\{\delta: \bar{\rho}_\theta(\delta) \leq k\} = \left(\bigcap_{i=1}^\infty V_{G_i}(k)\right) \cap \left(\bigcap_{i=1}^\infty V'_{\bar{G}_i}(1-\varepsilon_i)\right)$$

where $2k\varepsilon_i^{-1} = \inf\{\bar{L}(\theta,a,x): a\varepsilon\bar{A}-\bar{G}_i\}$. (To see this, note that the choice of ε_i guarantees that $\bar{\rho}_\theta(\delta) \geq 2k$ for all $\delta \notin V'_{\bar{G}_i}(1-\varepsilon_i)$. Also, 4.5 (1) implies that $\varepsilon_i \to 0$. Thus $\delta \varepsilon \bigcap_{i=1}^\infty V'_{\bar{G}_i}(1-\varepsilon_i)$ only if $\int_{A'} \bar{f}_\theta(x|a)\delta(da|x)\nu(dx) \geq 1$. [Actually, values > 1 are impossible.])

The sets appearing on the right of (4) are closed. Hence $\{\delta: \bar{\rho}_\theta(\delta) \leq k\}$ is closed in $\mathcal{S}(\bar{A})$, which proves the lower semi-continuity of $\bar{\rho}_\theta$ on $\mathcal{S}(\bar{A})$.

As noted, $\bar{R}(\theta,\delta) = \bar{\rho}_\theta(\delta)$. It follows that $\hat{\bar{\Gamma}}$, the dominated-risk set in problem $\bar{\mathcal{P}}$, is compact. Compactness of $\hat{\Gamma}$ follows exactly as in Theorem 4.4.

5. *Bayes procedures*. It is well known that the existence of Bayes procedures may be deduced from dynamic programming results like those described in Discussion 7.2. However we will take an easier path, proceeding directly from Section 4.

5.1. <u>Definitions and Assumptions</u>: Assume Θ is a measure space with σ-field C, say, and L is $C \times \mathcal{G} \times \mathcal{B}$ measurable.

Let P denote a probability measure on Θ, C. δ_P is a <u>Bayes procedure</u> if

$$\int R(\theta,\delta_P)P(d\theta) = \inf_{\delta\varepsilon\mathcal{S}} \int R(\theta,\delta)P(d\theta).$$

In addition to the technical Assumptions 2.8 (ii) and 2.10 (ii) assume that $F_.(B|\cdot)$ is $C \times \mathcal{G}$ measurable for each $B\varepsilon\mathcal{B}$ and that each $F^t_.(B|\cdot,\cdot)$ is $C \times \mathcal{G}_{(t)} \times \mathcal{B}_{(t-1)}$ measurable for each $B\varepsilon\mathcal{B}_t$.

Let $G = P \circ F_\theta$ be the measure which is defined on rectangles by

$$G(D \times B|a) = \int_{\theta \in D} F_\theta(B|a)P(d\theta) \qquad D \in C,\ B \in \mathcal{B}.$$

It can be checked that $G(\cdot|a)$ satisfies Assumption 2.10. Let G^* be the marginal measure:

$$G^*(B|a) = G(\Theta \times B|a).$$

Clearly, it may be the case that $\{G^*(\cdot|a):a\in A\}$ is a dominated family even though $\{F_\theta(\cdot|a):\theta\in\Theta, a\in A\}$ is not. It can be shown that if $\{G^*(\cdot|a):a\in A\}$ is a dominated family then it satisfies Assumption 3.8 if X is a Borel subset of a compact metric space $\bar{X}$ and if $F_\theta(\cdot|\cdot)$ defines a continuous function of A to $P(\bar{X})$ the Borel probability measures on $\bar{X}$ with the weak-$\star$ topology.

THEOREM 5.2. Let P be given. Suppose that Assumption 5.1 is satisfied. Suppose further that the remaining Assumptions of Theorems 4.4 or 4.7 are satisfied for the problem with the family of distributions $\{G^*(\cdot|a):a\in A\}$ for all $\theta\in\Theta$ in place of the family $\{F_\theta(\cdot|a)\}$. Then there is a Bayes procedure, δ_P.

PROOF. Consider a new statistical problem, $\mathcal{P}'$, with a trivial parameter space - $\{\theta'\}$; with $X' = \Theta \times X$, and $\mathcal{B}_t'(a) = \mathcal{J} \times \mathcal{B}_t(a)$ where $\mathcal{J}$ denotes the trivial σ-field on Θ; with $F_{\theta'} = G$ and with $L'(\theta', a, (\theta, x)) = L(\theta, a, x)$. [In other words, make Θ an unobservable part of the sample space, and use the distributions, $\{G(\cdot|a):a\in A\}$.]

$\hat{\Gamma}'$ is closed in $\mathcal{P}'$ by Theorem 4.4 or 4.7. Thus, an optimal policy exists in problem $\mathcal{P}'$. It must not depend on $\theta\in\Theta$, and is thus seen to correspond to a Bayes procedure in the original problem.

5.3. Remark: Note that Theorem 5.2 does not guarantee that a Bayes procedure minimizes the posterior risk (a.e.). Nevertheless in fixed-sample, non-design settings where such a concept is appropriate, it can be shown that this indeed the case. See, for example, Brown and Purves [4] (whose technical conditions can be generalized slightly to include Theorem 5.2 in the fixed-sample non-design case).

6. *Available actions dependent on past observations.* It was pointed out in the introduction that the formulation in Section 2 makes no explicit provision allowing the set of available actions at a given stage to depend on past observations. (The available actions at a given stage may, however, depend on past actions since A need not equal $\times_{t\in T} A_t$.) This section describes a simple trick which allows for such a possibility and also gives an optimal policy theorem in this setting.

6.1. Definitions: Let $\Phi(x)$, $x\in X$, be a non-empty subset of A such that $\pi_{(t)}(\Phi(x))$ depends only on $\pi_{(t-1)}(x)$. The set $\Phi(x)$ is to be interpreted as the set of available actions when x is observed. Thus the K_t-cross-section of $\pi_{(t)}(\Phi(x))$ at $a_1,\ldots,a_{t-1}$ is the set of actions available to the experimenter at stage t given that $\pi_{(t-1)}(x)$ has been observed and actions $a_1,\ldots,a_{t-1}$ have been taken through stage $t-1$. This set will be denoted by $\Phi_t(x;a)$; and is, of course, a function only of $\pi_{(t-1)}(x)$, $\pi_{(t-1)}(a)$.

Let $\mathcal{S}^*(\Phi)$ denote the set of procedures $\delta\in\mathcal{S}$ for which $\delta(\Phi(x)|x) = 1$ for almost all $x\in X$ $(\Lambda_{\theta,\delta})$. In such a problem the experimenter must limit himself to procedures $\delta\in\mathcal{S}^*(\Phi)$.

For convenience, let Φ denote the set $\Phi = \{x,a:a\in\Phi(x)\} \subset X\times A$. The introduction of $\mathcal{S}^*(\Phi)$ is the promised "trick". The reader who is disappointed by this trick may look to the proof of Theorem 6.4 for a better one.

6.2. Remarks: Pathologies can occur. Even though the set $\Phi(x)$ is non-empty for all x it may be that $\mathcal{S}(\Phi)$ is empty for measurability reasons. See Blackwell and Dubins [2]. In order to omit consideration of such situations here we assume directly that $\mathcal{S}^*(\Phi) \neq \phi$ whenever such a condition is desirable.

The following theorem extends the optimality result of Section 4.

THEOREM 6.4. Let the assumptions of Theorem 4.4 or 4.7 be satisfied. Suppose Φ is $\mathcal{B} \times \mathcal{C}$ measurable and $\Phi(x)$ is relatively closed in A for each $x \in X$. Then $\hat{\Gamma}(\mathcal{D}^*(\Phi))$ is compact. (Hence there exist optimal policies in $\mathcal{D}^*(\Phi)$ (so long as $\mathcal{D}^*(\Phi) \neq \phi$), etc.)

PROOF. Let $\bar{\Phi}(x)$ denote the closure of $\Phi(x)$ in $\bar{A}$. $\Phi(x) = \bar{\Phi}(x) \cap A$. $\bar{\Phi} = \{(a,x): a \in \bar{\Phi}(x)\}$ is measurable with respect to $\mathcal{C} \times \hat{\mathcal{B}}$ where $\hat{\mathcal{B}}$ denotes the universal completion of $\mathcal{B}$. (This can be deduced from 4.3 (1).) As a consequence of Corollary 3.11 (applied to $\mathcal{D}(\bar{A})$) and the proof of Theorem 4.5, it is sufficient to show that $\mathcal{D}^*(\Phi)$ is closed in $\mathcal{D}(\bar{A})$ in the topology of Section 3.

Introduce an (artificial) statistical problem with decision space $\bar{A}$; and with X and ν as in the original problem; with $\Theta = \{\theta_0\}$; $f_{\theta_0} \equiv 1$; and where the loss function satisfies $L(\theta_0,a,x) = 0$ if $a \in \bar{\Phi}(x)$, $= \infty$ otherwise for almost every $x(\nu)$. By Theorem 2.14 the map ρ_{θ_0} is lower semi-continuous in this problem. Hence $\{\delta: R(\theta_0,\delta) = 0\} = \mathcal{D}^*(\Phi)$ is closed in $\mathcal{D}(\bar{A})$, as desired. ||

7. *Dynamic programming.*

7.1. Definition (Dynamic programming): Following Blackwell [3], and others, a problem of the type defined in Section 1 is a (stochastic) dynamic programming problem (a la Bellman [1]) when:

(i) Θ is trivial (contains one point).

(ii) X_t, $\mathcal{B}_t(a) \approx X_1$, $\mathcal{B}_1$ for all t, a; and X_1, $\mathcal{B}_1$ is standard Borel (i.e. X_1 is imbeddable as a Borel subset of a Polish space and $\mathcal{B}_1$ is the Borel σ-field).

(iii) A_1 is trivial (contains one point only).

(iv) $F^t(\cdot|x,a) \equiv q(\cdot|x_{t-1},a_t)$, $t \geq 2$, where q is specified and is $\mathcal{B}_{t-1} \times \mathcal{C}_t$ measurable.

[Note that (ii) implies inter alia that the problem has no "design" aspect.]

According to Blackwell [3] any problem of the above form can be modified in an unimportant way to yield an equivalent gambling problem. Here a gambling problem is a problem satisfying (i)-(iv) and the following additional conditions:

(v) $A_t \subset P(X_t)$, the probability distributions on X_t, $\mathcal{B}_t$; and $q(\cdot|x_{t-1},a_t) = a_t$.

(vi) $\delta \in \mathcal{D}^*(\Phi)$ (see Definition 6.1) where (consonant with (i)-(v)) $\pi_t(\Phi(x))$ depends only on x_{t-1} and Φ is $\mathcal{B} \times \mathcal{C}$ measurable.

By (ii) and (vi), $\Phi(x)$ can be rewritten as $\Phi(x) = \{a:(x_{t-1}, a_t) \in \hat{\Phi}(x)$ for $t \geq 2\}$ where $\hat{\Phi}$ is a given $\mathcal{B} \times \mathcal{C}$ measurable set. (As usual, $\hat{\Phi} = \{(x_1,a_2): a_2 \in \hat{\Phi}(x_1)\}$.)

(vii) In all such problems the objective is to maximize the expectation of some given gain function $G = G(a,x)$.

7.2. Discussion: Theorems 3.14, 3.18, 4.4 and 4.7 all yield the existence of optimal policies in dynamic programming problems. Theorem 3.18 appears to be identical in this important limited context to Schäl [13, Theorem 6.5], and the proofs are very similar. (Our statement of the requisite regularity conditions is more explicit - thus better - in that Schäl's paper contains no analog of Theorem 3.10 - one of his regularity conditions is the statement that $L(\theta,\cdot,\cdot)$ satisfy the conclusion of our Theorem 3.10. We should also point out that our dominatedness assumption-Assumption 3.17 (i) - is not explicit in Schäl's paper, but it is a consequence of his other assumptions.)

7.3. Relaxing the dominatedness assumption: Note that the optimality results of Section 3 require that $\{F^{(t)}(\cdot|a): a \in A\}$ be a dominated family for each $t \in T$. For certain types of applications this assumption is both too strong, and unnecessary. For example, in gambling problems $q(\cdot|x_{t-1},a_t)$ represents the distribution of the payoffs at stage t if the fortune at stage t-1 is x_{t-1} and the bet a_t is made for stage t. In many examples such distributions do not form a dominated family.

In order to fill this gap in our results we present the following theorem. A nearly equivalent result was previously given in Schäl [13, Theorem 5.5]. Schäl gives an elegant, independent proof for his theorems; instead we use a trick which reduces it to a corollary of Theorem 3.18.

We give only the corollary result to Theorem 3.18 in the gambling problem setting, because it is easier to state. However, it seems reasonable that other results can be proved from Theorems 4.4 and 4.7 by means of this trick, including results outside the dynamic programming context of this section.

We adopt the following point of view. Since X is a Borel subset of $\bar{X}_t$, the distributions $P(X_t)$ have a natural imbedding in $P(\bar{X}_t)$. Hence we consider A_t as a subset of $P(\bar{X}_t)$ for topological purposes.

7.4. THEOREM: Consider a problem of the form 7.1 (i)-(vii). Let $A_t \subset P(\bar{X}_t)$, $t \geq 2$, have the weak*-topology. Suppose $\hat{\Phi}$ is closed in $X_1 \times A_2$. Suppose $G(\cdot,\cdot)$ is bounded and is a jointly semi-continuous function on $A \times X$. Then there is an optimal policy.

[The boundedness assumption on G may be relaxed by invoking the (gain-function version of the) condition in Remark 2.14.]

PROOF. Since $\bar{X}_1$ is compact and metrizable it is homeomorphic to a closed subset of [0,1]. Therefore, there is no loss of generality in assuming that $X_1 = [0,1]$, and we shall do so.

Consider a (new) statistical problem, $\mathcal{P}'$, in which the sample space is $X' = \underset{t\varepsilon T}{\times} [0,1]_t$ and the observations $(x_1', x_2', \ldots.)$ are all independent with x_1' having the same distribution as did x_1 in the original problem and with $x_2', x_3', \ldots$ each having the uniform distribution on [0,1]; irrespective of the actions taken. The action space in $\mathcal{P}'$ is $a_1 \times (\underset{t\geq 2}{\times} P(X_t))$ (as in the original gampling problem

For each $a \varepsilon P(X_t)$, $x \varepsilon X_t$, define the map

$$C_a(x) = [a([0,x)),\ a([0,x])].$$

(C_a may be called the cumulative distribution map of a.) In problem $\mathcal{P}'$ define the gain function, G, by

(1) $$G'(a,x') = \sup\{G(a,x): x_t' \varepsilon C_{a_t}(x_t),\ t \geq 2\};$$

and define the procedure-restriction via the set function

(2) $$\Phi'(x') = \{a: \exists x \ni x_t' \varepsilon C_{a_t}(x_t),\ t \geq 2, \text{ and } a\varepsilon\Phi(x)\}.$$

From (1) and the joint upper semi-continuity of G follows that $G'(\cdot,x')$ is upper semi-continuous for each $x'\varepsilon X'$.

Consider the set $S_1 = \{(a_1,a_2,\ldots,x_1,x_2,\ldots,x_1',x_2',\ldots): x_t'\varepsilon C_{a_t}(x_t)\}$. This is a closed subset of $A \times X \times X'$. Furthermore, $S_2 = S_1 \cap (\Phi \times X')$ is also closed. Now $\Phi' = \{(x',a): a\varepsilon\Phi'(x')\} = \{(x',a): \exists x \ni (a,x,x')\varepsilon S_2\}$. Thus (except for permutation of co-ordinates) Φ' is the projection of S_2 on the A, X' axes. It follows that Φ' is $\mathcal{B}' \times \mathcal{G}$ measurable and that $\Phi'(x')$ is closed for each $x\varepsilon X$. In fact, Φ' is closed in $X' \times A$; and it follows from this that it contains a Borel graph so that $\mathcal{S}'^*(\Phi') \neq \phi$.

Theorem 6.4 yields the existence of optimal policies in problem $\mathcal{P}'$. Since the original problem, $\mathcal{P}$, and $\mathcal{P}'$ are essentially equivalent via the correspondence $x_t \to C_{a_t}^{-1}(x_t)$ it follows that optimal policies also exist for $\mathcal{P}$.

[The hypotheses of the above theorem are virtually the same as those in an optimal policy theorem of Dubins and Savage [6, Theorem 16.1] (with their improved "upper semi-continuity" condition in place of their condition (e)).]

Note on terminology. A reader has commented that our terminology here may be confusing.

It should be emphasized that the procedure $\delta_t(\cdot|\cdot,\cdot)$ describes the distribution on $\mathcal{G}_t$ "given the observable past". However, it is not obvious in making this definition that $\delta(\cdot|\cdot,\cdot)$ is really the conditional distribution on $\mathcal{G}_t$ of some distribution with respect to some σ-field. It is a consequence of Theorem 1.22 and its proof that if $\{F_\theta(\cdot|a)\} \ll \nu$ then this is in fact the case. To generalize slightly, it follows from Theorem 1.22 that if for given $\theta\varepsilon\Theta$ there is a measure ν_θ such that $\{F_\theta(\cdot|a): a\varepsilon\mathcal{G}\} \ll \nu_\theta$ then $\delta(\cdot|\cdot,\cdot)$ is a version of the conditional distribution on $\mathcal{G}_t$ of $\Delta_{\theta,\delta}$ given $\mathcal{B}'$; where $\mathcal{B}' = \mathcal{B}'(t)$ is defined as in the proof of Theorem 1.22.

On the other hand, the measures $\delta(\cdot|\cdot)$ (defined in 1.18) are not necessarily equal to the conditional distribution on $\mathcal{G}$ of $\Delta_{\theta,\delta}$

given $\mathcal{B}$. For this reason the symbolism, "$\delta(\cdot|\cdot)$" which we have adopted may be misleading.

However, note that for any $C \in \mathcal{G}_{(t-1)}$, $D \in \mathcal{G}_t$, $\delta(C \cap D|x)$ is the conditional probability on $C \cap D$ given $\mathcal{B}_{(t-1)}(C)$. In this sense $\delta(\cdot|\cdot)$ may properly be thought of as describing the conditional distribution of the present given "the observable past" (but not the future or unobservable past). To take a more precise point of view, it can be shown that if $F_\theta(\cdot|a)$ is independent of $a \in \mathcal{G}$ then $\delta(\cdot|\cdot)$ is indeed the conditional distribution on $\mathcal{G}$ given $\mathcal{B}$ of $\Delta_{\theta,\delta}$. However, when $F_\theta(\cdot|a)$ depends on a then $\delta(\cdot|\cdot)$ may not equal this conditional distribution of $\Delta_{\theta,\delta}$.

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