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Sets Uniquely Determined by Projections on Axes I. Continuous Case

Abstract

This paper studies sets S in R^n which are uniquely reconstructible from their hyperplane integral projections $P_i(x_i; S) = \int\!\!\int \ldots \int X_S (\{x_1, \ldots, x_i, \ldots, x_n) \, dx_1 \ldots dx_{i-1} \, dx_{i+1} \ldots dx_n$ onto the n coordinate axes of R^n. It is shown that any additive set $S = \{\mathbf{x} = (x_1, \ldots, x_n) : \Sigma_{i=1}^n f_i(x_i) \geqq 0\}$, where each $f_i(x_i)$ is a bounded measurable function, is uniquely reconstructible. In particular, balls are uniquely reconstructible. It is shown that in R^2 all uniquely reconstructible sets are additive. For $n \geqq 3$, Kemperman has shown that there are uniquely reconstructible sets in R^n of bounded measure that are not additive. It is also noted for $n \geqq 3$ that neither of the properties of being additive and being a set of uniqueness is closed under monotone pointwise limits.

A necessary condition for S to be a set of uniqueness is that S contain no bad configuration. A bad configuration is two finite sets of points T_1 in $\text{Int}(S)$ and T_2 in $\text{Int}(S^c)$, where $S^c = R^n - S$, such that T_1 and T_2 have the same number of points in any hyperplane $x_i = c$ for $1 \leqq i \leqq n$, and all $c \in R^2$. We show that this necessary condition is sufficient for uniqueness for open sets S in R^2.

The results show that prior information about a density f in R^2 to be reconstructed in tomography (namely if f is known to have only values 0 and 1) can sometimes reduce the problem of reconstructing f to knowing only two projections of f. Thus even meager prior information can in principle be of enormous value in tomography.

SETS UNIQUELY DETERMINED BY PROJECTIONS ON AXES I. CONTINUOUS CASE*

P. C. FISHBURN†, J. C. LAGARIAS†, J. A. REEDS†, AND L. A. SHEPP†

Abstract. This paper studies sets S in $\mathbb{R}^n$ which are uniquely reconstructible from their hyperplane integral projections $P_i(x_i; S) = \iint \cdots \int \chi_S(x_1, \cdots, x_i, \cdots, x_n) \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n$ onto the n coordinate axes of $\mathbb{R}^n$. It is shown that any *additive set* $S = \{\mathbf{x} = (x_1, \cdots, x_n): \sum_{i=1}^{n} f_i(x_i) \geqq 0\}$, where each $f_i(x_i)$ is a bounded measurable function, is uniquely reconstructible. In particular, balls are uniquely reconstructible. It is shown that in $\mathbb{R}^2$ all uniquely reconstructible sets are additive. For $n \geqq 3$, Kemperman has shown that there are uniquely reconstructible sets in $\mathbb{R}^n$ of bounded measure that are not additive. It is also noted for $n \geqq 3$ that neither of the properties of being additive and being a set of uniqueness is closed under monotone pointwise limits.

A necessary condition for S to be a set of uniqueness is that S contain no bad configuration. A bad configuration is two finite sets of points T_1 in Int (S) and T_2 in Int (S^c), where $S^c = \mathbb{R}^n - S$, such that T_1 and T_2 have the same number of points in any hyperplane $x_i = c$ for $1 \leqq i \leqq n$, and all $c \in \mathbb{R}$. We show that this necessary condition is sufficient for uniqueness for open sets S in $\mathbb{R}^2$.

The results show that prior information about a density f in $\mathbb{R}^2$ to be reconstructed in tomography (namely if f is known to have only values 0 and 1) can sometimes reduce the problem of reconstructing f to knowing only two projections of f. Thus even meager prior information can in principle be of enormous value in tomography.

Key words. projections, uniqueness, additive sets, probability

AMS(MOS) subject classifications. 05B, 53A, 60B, 62H

1. Introduction. Horn [4] asked in connection with a problem of robot vision whether the unit disk D in $\mathbb{R}^2$ is uniquely determined (up to null sets) by its line integral projections onto the x and y axes. At first it may seem unlikely that only two projections of the density

$$f_D(x, y) = \chi_D(x, y) = \begin{cases} 1 & \text{if } (x, y) \in D \\ 0 & \text{if } (x, y) \notin D \end{cases}$$

could determine $f(x, y)$ uniquely, but in fact Horn's question has an affirmative answer despite such tempting "counterexamples" as Fig. 1.1. Here is a simple proof: Any two sets having the same projections on the x and y axes have the same area and moment of inertia. The disk D has the minimum moment of inertia among all sets S having the same area as D, so that the area and moment of inertia of a disk D characterize it uniquely (up to a null set).

This problem is a special kind of tomographic reconstruction problem in which we attempt to reconstruct a density $f(x, y)$, given two projections and the *prior information that the density $f(x, y)$ only takes the values 0 or 1*. For an arbitrary density $f(x, y)$ we generally need to know infinitely many projections to reconstruct $f(x, y)$ uniquely. However, the disk example shows that the prior information that the density $f(x, y)$ is an indicator function can greatly influence the number of projections needed for tomographic reconstruction.

More generally, we consider as given data for reconstructing a set S in $\mathbb{R}^n$ the hyperplane integral projections

$$P_\mathbf{v}(t; S) = \int_{H_t(\mathbf{v})} \chi_S(\mathbf{x}) \, dH_t, \qquad -\infty < t < \infty$$
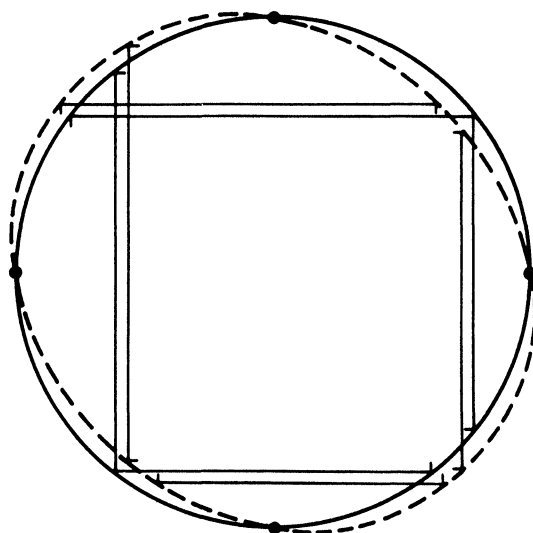
FIG. 1.1. *The solid disk is extended out in quadrants* II *and* IV *and in in quadrants* I *and* III. *The dashed warped disk cannot have the same projections as the original.*

where $\chi_S(\mathbf{x})$ is the characteristic function of S, $H_t(\mathbf{v})$ denotes a hyperplane with normal unit vector $\mathbf{v}$ given by

$$H_t(\mathbf{v}) = \{\mathbf{x}: \langle \mathbf{v}, \mathbf{x} \rangle = t\}$$

and dH_t is Lebesgue $(n-1)$-dimensional measure on H_t. We call the set $\{P_\mathbf{v}(t; S): -\infty < t < \infty\}$ the *projection onto the line* $L_\mathbf{v} = \{t\mathbf{v}: -\infty < t < \infty\}$ and associate $P_\mathbf{v}(t; S)$ to the point $t\mathbf{v}$ on this line.

This paper examines the problem of characterizing sets S of finite volume in $\mathbb{R}^n$ that are reconstructible (up to null sets) from their projections onto the n coordinate axes of $\mathbb{R}^n$, i.e., from

$$(1.1) \qquad P_i(x_i; S) = \int \cdots \int \chi_S(x_1, \cdots, x_i, \cdots, x_n) \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n$$

for $-\infty < x_i < \infty$ and $1 \leq i \leq n$. We call the problem of characterizing sets of uniqueness the *continuous unique reconstruction problem*. We show that such sets S exist. The number n is the minimal number of directions of projection on lines for which there is a set S (not a null set) such that unique reconstruction of S is possible. To see this, note that if projections in $n-1$ or fewer directions are given, then there is a direction orthogonal to all these directions, and we recover no information about the distribution of the mass of S in this direction, so that unique reconstruction is impossible. Furthermore, this argument shows that for unique reconstruction from projections onto n lines to be possible these lines must lie in n linearly independent directions. In this case we may, without loss of generality, reduce the general problem to the special case (1.1) where the lines are the coordinate axes by an invertible linear transformation of coordinates.

There is an analogous discrete version of the problem which we call the *discrete unique reconstruction problem*. The discrete problem is to characterize which finite subsets S of the integer lattice points $\mathbb{Z}^n$ given by their indicator function

$$f_S(x_1, \cdots, x_n) = \begin{cases} 1 & \text{if } (x_1, \cdots, x_n) \in S, \\ 0 & \text{if } (x_1, \cdots, x_n) \notin S, \end{cases}$$

are uniquely reconstructible from the projections $\{P_i : 1 \leq i \leq n\}$, where

$$P_i(x_i) = \sum_{\substack{(y_1, \cdots, y_n) \in \mathbb{Z}^n \\ y_i = x_i}} f_S(y_1, \cdots, y_n).$$

The discrete problem is studied in a companion paper [1].

The problem of characterizing sets of uniqueness was studied in the two-dimensional case by Lorentz [9] in 1948 in an elegant but apparently little known paper. Lorentz gave the following necessary and sufficient condition that a set S be uniquely reconstructible. Let the projections $P_1(x)$, $P_2(y)$ of S onto the x-axis and y-axis in $\mathbb{R}^2$ be

$$P_1(x) = \int \chi_S(x, y) \, dy; \qquad P_2(y) = \int \chi_S(x, y) \, dx$$

and define the *nonincreasing rearrangements* p_i of P_i on $0 \leq u < \infty$ by

$$p_i(u) = \int \chi(x : P_i(x) \geq u) \, dx$$

$$= \text{Leb} \{x : -\infty < x < \infty \text{ and } P_i(x) \geq u\},$$

so that $p_i(u)$ decreases on $0 \leq u < \infty$, where Leb $\{ \cdot \}$ denotes Lebesgue measure.

THEOREM (Lorentz). *A set S in $\mathbb{R}^2$ of finite measure is a set of uniqueness if and only if p_1 and p_2 are inverses of each other on $u \geq 0$, i.e.,*

$$p_1(p_2(u)) \equiv u, \qquad u \geq 0.$$

Lorentz's result can be used to show that the unit disk D is a set of uniqueness. Indeed, it is verified easily that the projections of the unit disk centered at $(0, 0)$ are

$$P_1(x) = P_2(x) = 2\sqrt{(1 - x^2)^+}$$

and that the nonincreasing rearrangements

$$p_1(u) = p_2(u) = \sqrt{(4 - u^2)^+}, \qquad u \geq 0,$$

are inverses of each other for $u \geq 0$.

In this paper we give a *sufficient* condition that S be a set of uniqueness in $\mathbb{R}^n$, namely that S be *additive* (as defined below). We also give a *necessary* condition that S be a set of uniqueness in $\mathbb{R}^n$, which is that S have *no bad configuration* (as defined below).

A set S is said to be *additive* if there are bounded measurable functions f_i for $i = 1, \cdots, n$, where $f_i(\mathbf{x}) = f_i(x_i)$ depends only on the ith coordinate x_i of $\mathbf{x} = (x_1, \cdots, x_n)$, such that

(1.2) $$\mathbf{x} \in S \Leftrightarrow \sum_{i=1}^{n} f_i(x_i) \geq 0.$$

For example, the unit ball is additive as may be seen by taking $f_i(\mathbf{x}) = (1/n) - x_i^2$ for $1 \leq i \leq n$.

THEOREM 1.1. *Any additive set S in $\mathbb{R}^n$ of finite n-dimensional measure is a set of uniqueness.*

Proof. Suppose there were another set T in $\mathbb{R}^n$ having the same projections

$$P_i(x_i) = \int\int \cdots \int \chi_S(\mathbf{x}) \, dx_1 \cdots dx_{i-1} \, dx_{i+1} \cdots dx_n$$

as S. Let $g_i(\mathbf{x}) = g_i(x_i)$ be any integrable (over $S \cup T$) function on $\mathbb{R}^n$ whose values

depend only on the ith coordinate of $\mathbf{x}$. Then

$$\int \cdots_{S} \int g_i(\mathbf{x})\, dx_1 \cdots dx_n = \int g_i(x_i) P_i(x_i)\, dx_i$$

(1.3)

$$= \int \cdots_{T} \int g_i(\mathbf{x})\, dx_1 \cdots dx_n,$$

by integrating over the $n-1$ coordinates other than x_i and using the fact that S and T have the same projections. Clearly, S and T have the same n-dimensional measure. Setting $g_i = f_i$ in (1.3) and subtracting the integral over $S \cap T$ in (1.3) we get

$$(1.4) \qquad \int \cdots_{S-S\cap T} \int f_i(x_i)\, dx_1 \cdots dx_n = \int \cdots_{T-S\cap T} \int f_i(x_i)\, dx_1 \cdots dx_n$$

and

$$(1.5) \qquad \int \cdots_{S-S\cap T} \int dx_1 \cdots dx_n = \int \cdots_{T-S\cap T} \int dx_1 \cdots dx_n.$$

But summing over i in (1.4) we obtain

$$(1.6) \qquad \int \cdots_{S-S\cap T} \int \sum_i f_i(x_i)\, dx_1 \cdots dx_n = \int \cdots_{T-S\cap T} \int \sum_i f_i(x_i)\, dx_1 \cdots dx_n.$$

The integrand on the left in (1.6) is nonnegative by (1.2), and the integrand on the right in (1.6) is strictly negative, also by (1.2). It follows that $\mathrm{Leb}\,(T - S \cap T) = 0$ and then from (1.5) that $\mathrm{Leb}\,(S - S \cap T) = 0$, so S and T differ by null sets. □

Kemperman [7] has subsequently proved that a generalized notion of our version of additivity also guarantees uniqueness.

We can show that various simple sets are uniquely reconstructible by showing that they are additive, as for the unit n-sphere. However, it may be difficult to tell whether a general set S is additive.

There is a simple *necessary* condition for a set S to be a set of uniqueness, which we state next. A set of four points $\mathbf{z}_1 = (x_1, y_1)$, $\mathbf{z}_2 = (x_2, y_2)$, $\mathbf{w}_1 = (x_1, y_2)$, and $\mathbf{w}_2 = (x_2, y_1)$ forming the corners of a rectangle in $\mathbb{R}^2$ are a *2-bad configuration* (or *bad rectangle*) for S if $\mathbf{z}_1$ and $\mathbf{z}_2$ are in the interior $\mathrm{Int}\,(S)$ of S and $\mathbf{w}_1$ and $\mathbf{w}_2$ are in $\mathrm{Int}\,(S^c)$, where $S^c = \mathbb{R}^2 - S$ is the complement of S. More generally, a *k-bad configuration* (or *k-configuration*) for S in $\mathbb{R}^n$ consists of $2k$ vectors $\mathbf{z}_1, \cdots, \mathbf{z}_k$ and $\mathbf{w}_1, \cdots, \mathbf{w}_k$, where all the $\mathbf{z}_j$ are *distinct* points in $\mathrm{Int}\,(S)$ and all the $\mathbf{w}_j$ are *distinct* points in $\mathrm{Int}\,(S^c)$, and every coordinate plane $x_i = c$ contains the same number of $\mathbf{w}_j$'s as $\mathbf{z}_j$'s. We define a *weakly k-bad configuration* for S in $\mathbb{R}^n$ the same way, except that $\mathbf{z}_1, \cdots, \mathbf{z}_k$ need not be distinct points, and $\mathbf{w}_1, \cdots, \mathbf{w}_k$ need not be distinct points. We say that S has *no bad configuration* if S has no k-bad configuration for any $k \geq 2$.

THEOREM 1.2. *A set S of uniqueness in $\mathbb{R}^n$ has no bad configuration.*

Proof. Suppose S has a k-bad configuration consisting of points $\mathbf{z}_1, \cdots, \mathbf{z}_k$ and $\mathbf{w}_1, \cdots, \mathbf{w}_k$ as described above. Then for a small enough positive δ, the set C_1 consisting of the union of k balls of radius δ around the $\mathbf{z}_i$ lies in $\mathrm{Int}\,(S)$, and the set C_2 consisting of the union of k balls of radius δ around the $\mathbf{w}_i$ lies in $\mathrm{Int}\,(S^c)$. Then $T = S - C_1 + C_2$ has the same projections as S so S cannot be a set of uniqueness. □

In the rest of this paper we study the relations between the conditions that a set S is additive, that S is a set of uniqueness, and that S has no bad configuration. We now summarize the results obtained.

In §2 we first show that the concepts "having a bad configuration" and "having a weakly bad congfiuration" coincide for open sets in all dimensions.

THEOREM 2.1. *Let S be an open set of finite measure in $\mathbb{R}^n$. The following are equivalent.*

(1) *S has no k-bad configuration for any $k \geqq 2$.*

(2) *S has no weakly k-bad configuration for any $k \geqq 2$.*

This theorem highlights an essential difference between the continuous case and the discrete case, for in [1] we show that in the discrete case the concepts "having a bad configuration" and "having a weakly bad configuration" are distinct in all dimensions greater than or equal to 3.

In § 2 we study the two-dimensional case in detail. In the two-dimensional case, additivity and uniqueness coincide.

THEOREM 2.2. *Let S be a set in $\mathbb{R}^2$ of finite measure. The following conditions are equivalent.*

(1) *S is a set of uniqueness.*

(2) *S is additive.*

Kuba and Volčič [8] give other necessary and sufficient conditions for being a set of uniqueness in the two-dimensional case.

If S is suitably restricted then the "set of uniqueness" concept essentially coincides with the other concepts in the two-dimensional case.

THEOREM 2.3. *Let S be an open set in $\mathbb{R}^2$ of finite measure. The following conditions are equivalent.*

(1) *There is no open set $\tilde{S}$ unequal to S having the same projections on the x-axis and y-axis.*

(2) *S has no k-bad configuration for all $k \geqq 2$.*

(3) *S has no weakly k-bad configuration for all $k \geqq 2$.*

If in addition the boundary ∂S has measure zero, these are also equivalent to the following condition.

(4) *S has no 2-bad configuration.*

It seems likely that the hypothesis that ∂S has zero measure is unnecessary in this theorem.

An interesting question concerns whether or not the concepts of uniqueness and additivity coincide for three or more dimensions. In an earlier version of this paper we conjectured that they coincide, but Kemperman [6] settled that conjecture with a counterexample of a set of uniqueness that is not additive. Kellerer found a similar counterexample and kindly transmitted it to us. Kemperman [7] then proposed a generalized notion of additivity that is necessary and sufficient for certain sets to be sets of uniqueness.

Section 3 studies general properties of additive sets. We first give a criterion for a set *not* to be additive.

THEOREM 3.1. *If S is a bounded measurable subset of $\mathbb{R}^n$ and if there are measures μ and ν on $\mathbb{R}^n$ with the same projections (one-dimensional marginals) on all n axes, and μ is concentrated on S and ν is concentrated on S^c, then S is not additive.*

Section 3 then studies the following subset of the cube in $\mathbb{R}^3$:

$$S_0 = \{(x_1, x_2, x_3): 0 \leqq x_i \leqq 1, i = 1, 2, 3, x_3 \geqq \max\{x_1, x_2\}\}.$$

Note that S_0 is the limit as $n \to \infty$ of the additive sets

$$S_n = \{(x_1, x_2, x_3): 0 \leqq x_i \leqq 1 \text{ for } i = 1, 2, 3, x_3^n \geqq x_1^n + x_2^n\},$$

with $S_1 \subset S_2 \subset S_3 \subset \cdots$. Our study of this set was motivated by our search for a set of uniqueness that is not additive. We prove two interesting facts about S_0.

THEOREM 3.2. *S_0 is not additive.*

THEOREM 3.3. S_0 *is not a set of uniqueness.*

These theorems imply that both the additivity property and the set of uniqueness property are not closed under the operation of taking monotone pointwise limits in dimensions greater than or equal to 3.

Theorem 3.2 follows from the fact that there are a pair of measures μ and ν concentrated on S_0 and S_0^c, respectively, having the same projections onto the coordinate axes. An alternative direct proof of Theorem 3.2 has been shown to us by Kemperman.

We prove Theorem 3.3 by showing there exists a pair of measures μ and ν concentrated on S_0 and S_0^c, respectively, *having densities taking only the values 0 and 1*, which have the same projections on the coordinate axes. We actually construct a pair μ, ν having bounded densities and then use a general result ([2]) which asserts that the existence of a pair (μ, ν) with bounded densities implies the existence of such a pair having densities taking only the values 0 and 1.

The related problem of deciding whether there *exists* a set S having given projections is not considered in the paper. The existence problem is studied in the two-dimensional continuous case in [5], [9], where necessary and sufficient conditions for existence are given.

2. Continuous case: general results. Our first result in the continuous case shows that the concepts of "no bad configuration" and "no weakly bad configuration" coincide for *open sets S*, in all dimensions n. This differs from the discrete case for $n \geq 3$.

THEOREM 2.1. *Let S be an open set in $\mathbb{R}^n$. The following conditions are equivalent.*

(1) *S has no weakly k-bad configuration for all $k \geq 2$.*

(2) *S has no k-bad configuration for all $k \geq 2$.*

Proof. It is clear that $(2) \Rightarrow (1)$, so that it suffices to show $(1) \Rightarrow (2)$. To do this we show that if S has a weakly k-bad configuration then it has a k-bad configuration. Let the weakly k-bad configuration consist of the k points $\mathbf{w}^{(i)} = (w_1^{(i)}, \cdots, w_n^{(i)})$ in Int (S), and the k points $\mathbf{z}^{(i)} = (z_1^{(i)}, \cdots, z_n^{(i)})$ in Int (S^c), for $1 \leq i \leq k$. Since the sets $\{w_j^{(i)}\}$ and $\{z_j^{(i)}\}$ have equal projections (counting multiplicities) for each j from 1 to n, there are permutations π_j of $\{1, 2, \cdots, k\}$ such that $z_j^{(i)} = w_j^{(\pi_j(i))}$. Clearly, there are k points $\hat{\mathbf{w}}^{(i)}$ arbitrarily close to the $\mathbf{w}^{(i)}$ in Int (S) whose coordinates consist of kn distinct real numbers. For any such $\hat{\mathbf{w}}^{(i)}$, define $\hat{\mathbf{z}}^{(i)}$ by $\hat{z}_j^{(i)} = \hat{w}_j^{(\pi_j(i))}$ for all i and j. Then the $\hat{\mathbf{z}}^{(i)}$ and $\hat{\mathbf{w}}^{(i)}$ are all distinct points and the sets $\{\hat{\mathbf{w}}^{(i)}\}$ and $\{\hat{\mathbf{z}}^{(i)}\}$ have the same projections. As $\hat{\mathbf{w}}^{(i)}$ approaches $\mathbf{w}^{(i)}$, $\hat{\mathbf{z}}^{(i)}$ approaches $\mathbf{z}^{(i)}$, so by choosing the $\hat{\mathbf{w}}^{(i)}$ sufficiently close to the $\mathbf{w}^{(i)}$ we may ensure both $\hat{\mathbf{w}}^{(i)} \in$ Int (S) and $\hat{\mathbf{z}}^{(i)} \in$ Int (S^c); hence we conclude that there is a k-bad configuration. $\square$

The remainder of this section establishes results for the two-dimensional case.

THEOREM 2.2. *Let S be a set of finite measure in $\mathbb{R}^2$. The following are equivalent.*

(1) *S is a set of uniqueness.*

(2) *S is additive.*

Proof. The result $(2) \Rightarrow (1)$ is just Theorem 1.1. To prove $(1) \Rightarrow (2)$ we use ideas of Lorentz [9]. We are given a set of uniqueness S. Suppose first that S is contained in $\mathbb{R}_+^2 = \{(x_1, x_2): x_1 \geq 0, x_2 \geq 0\}$ and that the projection

$$P_1(x_1) = \int_{-\infty}^{\infty} \chi_S(x_1, x_2) \, dx_2$$

is nonincreasing for $x_1 \geq 0$, and that the projection

$$P_2(x_2) = \int_{-\infty}^{\infty} \chi_S(x_1, x_2) \, dx_1$$

is nonincreasing for $x_2 \geqq 0$. Note that $P_1(x_1)$ is finite for each $x_1 > 0$ since S has finite measure. Lorentz's condition that S be a set of uniqueness is

$$P_2(P_1(x_1)) \equiv x_1 \quad \text{a.e. [Leb]}$$

for $x_1 \geqq 0$. Using this it is easy to verify that

$$(2.1) \qquad S = \{(x_1, x_2): x_1 \geqq 0 \text{ and } 0 \leqq x_2 \leqq P_1(x_1)\}$$

is identical to S up to a null set, since the right side of (2.1) produces the correct projections. Using the fact that $P_1(x_1)$ is nonincreasing, it is easy to verify that

$$(2.2) \qquad S' = \{(x_1, x_2): f_1(x_1) + f_2(x_2) \geqq 0\}$$

where

$$(2.3) \qquad f_1(x_1) = \begin{cases} P_1(x_1) & \text{if } x_1 \geqq 0 \\ -\infty & \text{if } x_1 < 0 \end{cases}$$

and

$$(2.4) \qquad f_2(x_2) = \begin{cases} -x_2 & \text{if } x_2 \geqq 0 \\ -\infty & \text{if } x_2 < 0. \end{cases}$$

This would show that S is additive if f_1 and f_2 did not take the value $-\infty$. We avoid this difficulty in the general case by a change of variable, as described below.

Consider the general case and recall that Lorentz proves that associated to each projection $P_i(x_i; S)$ there are measurable functions $\sigma_i: \mathbb{R} \to \mathbb{R}^+ = \{x: x > 0\}$ which are one-one (almost everywhere) and onto (almost everywhere) such that the functions $p_i(t)$ defined by

$$p_i(\sigma_i(x_i)) = P_i(x_i) \qquad i = 1, 2$$

are nonincreasing on $0 < t < \infty$. These are *monotone rearrangements* of the $P_i(x_i)$ on the x_i-coordinates. Now define $S^+ \subseteq \mathbb{R}^+ \times \mathbb{R}^+$ by

$$S^+ = \{(\sigma_1(x_1), \sigma_2(x_2)): (x_1, x_2) \in S\}.$$

Then S^+ has projections $p_1(x_1)$ and $p_2(x_2)$, and it is easy to check that S^+ is a set of uniqueness if and only if S is a set of uniqueness. By hypothesis, S is a set of uniqueness, hence so is S^+, and by the preceding argument there are functions f_1 and f_2 for S^+ satisfying (2.2)–(2.4). Then (2.2) implies that S is additive for $f_1^*(x_1)$ and $f_2^*(x_2)$, where

$$f_1^*(x_1) = f_1(\sigma_1(x_1)) \quad \text{and} \quad f_2^*(x_2) = f_2(\sigma_2(x_2)),$$

after noting that $f_1^*(x_1)$ and $f_2^*(x_2)$ are finite everywhere because $\sigma_1(x_1) > 0$ and $\sigma_2(x_2) > 0$ for all $x_1, x_2 \in \mathbb{R}$. This completes the proof.   □

We now observe that for open sets S of finite measure our concepts essentially coincide in the two-dimensional case.

THEOREM 2.3. *Let S be an open set in $\mathbb{R}^2$ of finite measure. The following conditions are equivalent.*

(1) *There is no open set $\tilde{S}$ unequal to S having the same projections on the x-axis and y-axis.*

(2) *S has no k-bad configuration for all $k \geqq 2$.*

(3) *S has no weakly k-bad configuration for all $k \geqq 2$.*

If in addition the boundary ∂S of S has measure zero, then these conditions are also equivalent to the following one.

(4) *S has no 2-bad configuration.*

Proof. Condition (1)$\Rightarrow$(2) by Theorem 1.2 and (2)$\Leftrightarrow$(3) by Theorem 2.1. We prove (2)$\Rightarrow$(1) in the contrapositive form by the following lemma.

LEMMA 2.4. *Suppose that S_1 and S_2 are disjoint open sets in $\mathbb{R}^2$ of finite area having the same projections on the x-axis and y-axis. Then there are finite sets T_1 in S_1 and T_2 in S_2 having the same number of points on each line $x = c$ and on each line $y = c$ for all $c \in \mathbb{R}$.*

Proof. We prove the stronger result that the set T_1 can include any given point $\mathbf{v}_0 = (x_0, y_0)$ in S_1. To do this take a small open square U_0 of side δ centered at $\mathbf{v}_0$ that lies entirely in the open set S_1. Since the projections of S_1 and S_2 agree, we can find points $\mathbf{y}_1 = (x_0, y_1)$ and $\mathbf{y}_2 = (x_1, y_0)$ with $\mathbf{y}_1$ and $\mathbf{y}_2$ in S_2. Since S_2 is open, by decreasing δ if necessary, we can find open squares U_0, U_1^*, U_1 of side δ centered at $\mathbf{v}_0$, $\mathbf{y}_1$, and $\mathbf{y}_2$ with the squares U_1, U_1^* being in S_2. (See Fig. 2.1.) We say that the points in U_1^* *cover* those in U_0 in the "down" direction, by which we mean the projections $P_1(x; U_1^*) = P_1(x; U_0)$ for all $x \in \mathbb{R}$, and those in U_1 *cover* those in U_0 in the "across" direction, by which we mean $P_2(y; U_1) \equiv P_2(y; U_0)$. Note that U_0, U_1, and U_1^* each has area δ^2.
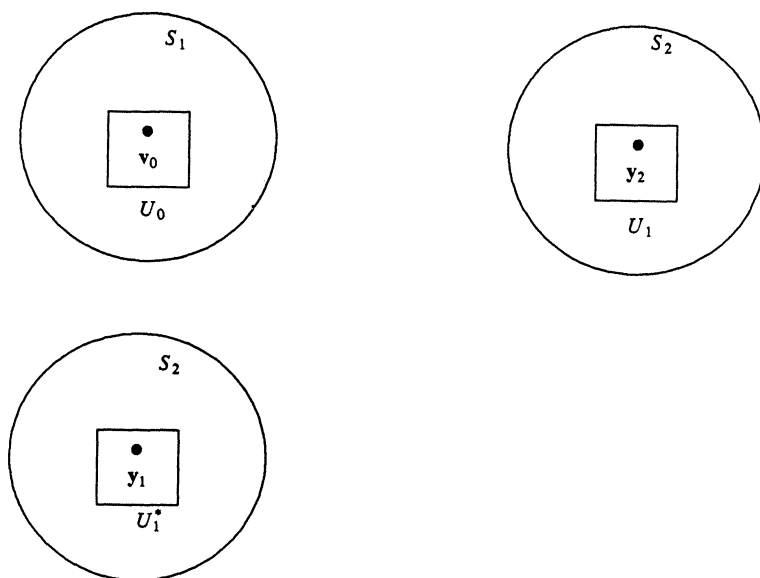

FIG. 2.1. *Initial open squares in S_1 and S_2.*

Now let ε be a positive constant depending on δ that will be picked once and for all. We claim there exists an open set U_2 such that:

    (i) U_2 is in S_1.

    (ii) For each point $\mathbf{w} = (w_1, w_2)$ in U_2 there is some point (w_1, y) in U_1 having the same first coordinate.

    (iii) $P_1(x; U_2) \leqq P_1(x; U_1)$ for all x.

    (iv) $\int_{-\infty}^{\infty} |P_1(x; U_2) - P_1(x; U_1)| \, dx < \varepsilon$, so that Area $(U_2) \geqq \delta^2 - \varepsilon$.

    (v) U_2 is a disjoint union of a finite set of open rectangles.

To see this, we use the fact that since U_1 is in S_2, and S_1 and S_2 have the same projections, and since U_0 has no x-coordinate in common with U_1, then

$$P_1(x; S_1 - \bar{U}_0) \geqq P_1(x; U_1) \quad \text{for all } x \in \mathbb{R},$$

where the bar denotes closure. Now cover the open set $S_1 - \bar{U}_0$ with a mesh of squares of side η and let $\eta \to 0$. Taking η sufficiently small we can approximate the Riemann

integral of $S_1 - \bar{U}_0$ arbitrarily well from below, and choosing an appropriate subcollection of these squares yields a set U_2 such that (i)-(v) hold.

We continue to construct a series of sets U_i such that:

(i) If i is odd, then U_i is in S_2, and is in S_1 otherwise.

(ii) Each point $w = (w_1, w_2)$ in U_i can be reached from a point (w_1, y) in U_{i-1} if i is even and from a point (x, w_2) in U_{i-1} if i is odd.

(iii) $P_1(x; U_i) \leq P_1(x; U_{i-1})$ for all x if i is even. $P_2(y; U_i) \leq P_2(y; U_{i-1})$ for all y if i is odd.

(iv) $\int_{-\infty}^{\infty} |P_j(z; U_i) - P_j(z; U_{i-1})| \, dz \leq i\varepsilon$ where $j = 1$ for even i and $j = 2$ for odd i, so Area $(U_i) \geq \delta^2 - i^2 \varepsilon$.

(v) U_i is a disjoint union of a finite set of open rectangles.

(vi) $U_i \cap U_j = \varnothing$ for $0 \leq j < i$.

This construction is carried out using the fact that (for $i = 2j$)

$$P_1(x; S_1 - \bar{U}_0 - \bar{U}_2 - \cdots - \bar{U}_{2j-2}) \geq P_1(x; U_{2j-1})$$

and (for $i = 2j + 1$)

$$P_2(y; S_2 - \bar{U}_1 - \bar{U}_3 - \cdots - \bar{U}_{2j-1}) \geq P_2(y; U_{2j})$$

provided $U_{2j+1} \cap U_1^* = \varnothing$. Since S_1 has finite area A, if we take ε small enough, say $\varepsilon = 1/100 A^{-2} \delta^4$, then the sets U_i are all of area greater than $\frac{1}{2}\delta^2$ for $1 \leq i \leq 2(A/\delta)$, hence by pairwise disjointness this construction must halt for some $i < 2(A/\delta)$. It follows that there is some $i = 2j$ for which U_{2j} has a point $\mathbf{v}_{2j} = (x_{2j}, y_{2j})$ such that there is a point $\mathbf{v} = (x^*, y_{2j})$ in U_1^*. By condition (ii) we can extend this to a sequence of points $\mathbf{v}_k = (x_k, y_k)$ in U_k such that $x_k = x_{k+1}$ if k is odd and $y_k = y_{k+1}$ if k is even. Now since U_1^* and U_1 are rectangles, $\mathbf{v}_1 = (x_1, y_0) \in U_1$ and $\mathbf{v}_{2j+1} = (x_0, y_{2j}) \in U_1^*$. Then the sets of points $T_1 = \{\mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_4, \cdots, \mathbf{v}_{2j}\}$ in S_1 and $T_2 = \{\mathbf{v}_1, \mathbf{v}_3, \mathbf{v}_5, \cdots, \mathbf{v}_{2j+1}\}$ in S_2 have the required property. $\quad\square$

We now complete the proof of Theorem 2.3. The implication $(2) \Rightarrow (4)$ is always true. We prove $(4) \Rightarrow (2)$ under the assumption that the boundary $\partial S = \bar{S} - S$ of S has Lebesgue measure zero. (Since open sets are Lebesgue measurable, it follows that ∂S is always measurable, but it may have positive measure; cf. [12, p. 59].) We show the contrapositive, which asserts that if S has a k-bad configuration for $k \geq 3$ then S has a 2-bad configuration.

We may suppose that the k-bad configuration has the form

$$\mathbf{x}^{(1)} = (x_1^{(1)}, y_1^{(1)}) \in \text{Int} (S), \qquad \mathbf{x}^{(2)} = (x_1^{(1)}, y_1^{(2)}) \in \text{Int} (S^c),$$

$$\mathbf{x}^{(3)} = (x_1^{(2)}, y_1^{(2)}) \in \text{Int} (S), \qquad \mathbf{x}^{(4)} = (x_1^{(2)}, y_1^{(3)}) \in \text{Int} (S^c),$$

$$\mathbf{x}^{(2k-1)} = (x_1^{(k)}, y_1^{(k)}) \in \text{Int} (S), \qquad \mathbf{x}^{(2k)} = (x_1^{(k)}, y_1^{(1)}) \in \text{Int} (S^c).$$

Since these sets are open, for all $\mathbf{y}$ with $\|\mathbf{y}\| < \delta$ and δ small enough, the set

$$S(\mathbf{y}) = \{\mathbf{x}^{(i)} + \mathbf{y} : 1 \leq i \leq 2k\}$$

is a k-bad configuration. Since ∂S has measure 0, the set of $\mathbf{y}$ with $\|\mathbf{y}\| < \delta$ for which some point $\mathbf{x}^{(i)} + \mathbf{y}$ is in ∂S has measure 0, hence we can choose $\mathbf{y}$ so that all k^2 points $(x_1^{(i)}, y_1^{(j)})$ for $1 \leq i, j \leq k$ are in $\text{Int} (S) \cup \text{Int} (S^c)$, i.e., none of them are on ∂S $(= \partial S^c)$. Now the argument of Lemma 1 in [1] in the discrete case can be applied to conclude that there is a subset of four of these k^2 points with $\{(x_1, y_1), (x_2, y_2)\} \subseteq S$ and $\{(x_1, y_2), (x_2, y_1)\} \subseteq S^c$. Since these points are actually in $\text{Int} (S)$ or $\text{Int} (S^c)$, this is a 2-bad configuration. $\quad\square$
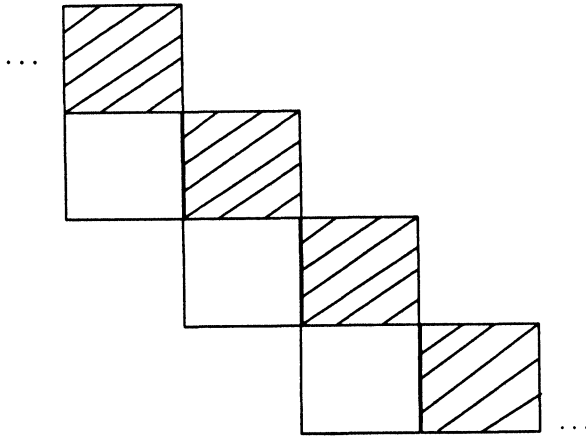
FIG. 2.2. *Staircase example.*

Remarks. (1) Lemma 2.4 is false in two-dimensions if S_1 and S_2 are allowed to have *infinite* area, even if projections are everywhere finite. The infinite checkerboard staircase in Fig. 2.2 provides a counterexample, where shaded open squares are in S_1 and unshaded open squares are in S_2.

(2) The three-dimensional analogue of Lemma 2.4 is false for open sets S_1 and S_2 of finite volume.

LEMMA 2.5. *There exist two disjoint bounded open sets S_1 and S_2 in $\mathbb{R}^3$ such that:*

(1) *S_1 and S_2 have identical projections on all three axes.*

(2) *There exists no finite bad configuration for S_1 or S_2 contained in $S_1 \cup S_2$.*

This result is proved in the Appendix.

3. Properties of additive sets. Recall that S is additive if and only if there are bounded measurable functions $f_i(x_i)$ for $1 \leqq i \leqq n$ such that, up to sets of measure zero,

$$(3.1) \qquad S = \left\{ \mathbf{x} = (x_1, \cdots, x_n): \sum_{i=1}^{n} f_i(x_i) \geqq 0 \right\}.$$

Although it is difficult to decide when a set is additive, there is a simple *necessary* condition for additivity.

THEOREM 3.1. *Suppose S is a bounded measurable subset of $\mathbb{R}^n$ and there are measures μ and ν with the same projections for which μ and ν are concentrated on S and S^c, i.e., $\mu(S) > 0$ and*

$$(3.2) \qquad \mu(S^c) = \nu(S) = 0.$$

Then S is not additive, i.e., (3.1) holds for no f_i, $i = 1, \cdots, n$.

Proof. If f_i exist satisfying (3.1) then for each i

$$(3.3) \qquad \int \cdots \int f_i(x_i) \, d\mu(x) = \int \cdots \int f_i(x_i) \, d\nu(x),$$

since the f_i are bounded measurable and depend only on one coordinate while μ and ν have the same one-dimensional margins. Adding over i in (3.3) gives

$$\int \underset{S}{\cdots} \int \sum_{i=1}^{n} f_i(x_i) \, d\mu(x) = \int \underset{S^c}{\cdots} \int \sum_{i=1}^{n} f_i(x_i) \, d\nu(x).$$

By (3.1), the left integrand is nonnegative while the right integrand is strictly negative.

It follows that ν has measure zero and so μ has measure zero since μ and ν have the same marginals. But $\mu(S) > 0$, so f_i cannot exist.   □

To illustrate some of the difficulties in proving that a set is additive or is one of uniqueness, we study the following particular subset S_0 of the cube in $\mathbb{R}^3$:

$$S_0 = \{(x_1, x_2, x_3): x_3 \geqq \max\{x_1, x_2\}, 0 \leqq x_i \leqq 1 \text{ for } i = 1, 2, 3\}.$$

We will show that S_0 is not additive by exhibiting a pair of measures satisfying Theorem 3.1 for S_0.

THEOREM 3.2. S_0 is not additive, i.e., there are μ and ν satisfying (3.2) with $S = S_0$.

Proof. Since we would naturally believe that $\max\{x_1, x_2\}$ cannot be written as $f_3^{-1}(f_1(x_1) + f_2(x_2))$, the result that S_0 is not additive is perhaps not surprising.

The proof gives a μ and ν not absolutely continuous with respect to Lebesgue measure λ. The idea behind the proof is that if we can find a single probability space on which random variables X_1, X_2, X_1', X_2' are defined, and if X_i and X_i' have the same distribution for $i = 1, 2$, and yet $\max\{X_1, X_2\}$ is everywhere strictly less than $\max\{X_1', X_2'\}$, i.e.,

(3.4) $X_1 \sim X_1', \quad X_2 \sim X_2', \quad \max\{X_1, X_2\} < \max\{X_1', X_2'\},$

then we may assume by change of scale that $0 \leqq X_i \leqq 1$ for $i = 1, 2$ and set

$$X_3 = X_3' = \tfrac{1}{2}(\max\{X_1, X_2\} + \max\{X_1', X_2'\}).$$

Then the measure μ induced by (X_1, X_2, X_3) is concentrated on S_0 since $X_3 \geqq \max\{X_1, X_2\}$ while the measure ν induced by (X_1', X_2', X_3') is concentrated on S_0^c since $X_3' < \max\{X_1', X_2'\}$. The measures μ and ν have the same projections onto the first two coordinate axes since $X_i \sim X_i'$ for $i = 1, 2$; they also have the same projections onto x_3 since $X_3 = X_3'$.

We construct random variables satisfying (3.4). Let $X_1 = X_2 = \beta$ where

$$P(\beta \in du) = \frac{du}{\pi\sqrt{u(1-u)}}, \qquad 0 \leqq u \leqq 1,$$

and let (X_1', X_2') be a uniform point on the union of the line segments from $(0, 1)$ to (β, β) and from (β, β) to $(1, 0)$. Clearly $\max\{X_1, X_2\} = \beta < \max\{X_1', X_2'\}$. It remains only to show that X_1' and X_2' have the distribution of β. We have

$$P(X_1' \in du) = \int_0^1 P(\beta \in d\nu) P(X_1' \in du \mid \beta \in d\nu)$$

$$= \int_0^u \frac{d\nu}{\pi\sqrt{\nu(1-\nu)}} \frac{du}{2(1-\nu)} + \int_u^1 \frac{d\nu}{\pi\sqrt{\nu(1-\nu)}} \frac{du}{2\nu} = \frac{du}{\pi\sqrt{u(1-u)}}.$$

Hence $X_1' \sim \beta$ and therefore $X_2' \sim X_1'$. Of course since μ assigns measure 1 to $X_1 = X_2$, μ is not absolutely continuous.   □

We now show that S_0 is not a set of uniqueness, hence that neither additivity nor being a set of uniqueness is closed under monotone pointwise limits for n greater than or equal to 3. To do this it suffices to show that there are measures μ and ν satisfying the conditions of Theorem 3.1 for S_0 which have Radon–Nikodym derivative 0 and 1 at all points. Indeed, if T_0 is another set with the same projections as S_0 and if $\lambda =$ Lebesgue measure on $\mathbb{R}^n$, $\chi(A)$ is the indicator function of A, and μ_0 and ν_0 are the measures with Radon–Nikodym derivatives

$$\frac{d\mu_0}{d\lambda} = \chi(S_0 - S_0 \cap T_0) \quad \text{and} \quad \frac{d\nu_0}{d\lambda} = \chi(T_0 - S_0 \cap T_0),$$

then μ_0 and ν_0 have the same projections.

THEOREM 3.3. S_0 *is not a set of uniqueness, i.e., there are μ and ν with densities taking only values 0 and 1 satisfying $\mu(S_0^c) = \nu(S_0) = 0$.*

Proof. The idea of the proof is to construct μ and ν with bounded densities satisfying (3.2), and then to use a general result that the existence of μ and ν with bounded densities implies the existence of such μ and ν having densities taking only values 0 and 1.

We now exhibit μ and ν *with bounded densities* that satisfy (3.2). We will construct a probability space and random variables as in (3.4). To this end suppose $q(y) > 0$ on $I = (0, 1)$ and define the measure π on I^2 by

$$(3.5) \qquad \pi(dx, dy) = q(x \vee y)\, dx\, dy, \quad \text{where } x \vee y = \max\{x, y\}.$$

We will define a mapping $(x, y) \to (x', y')$ with $x \vee y < x' \vee y'$ and then set $X_1 = x$, $X_2 = y$, $X_1' = x'$, $X_2' = y'$ so that (3.4) holds. To do this, we choose $\phi(y): I \to I$ and $\psi(y): I \to I$ and set

$$(3.6a) \qquad\qquad y' = \phi(y), \quad x' = \frac{x}{y}\psi(y) \quad \text{for } x < y$$

$$(3.6b) \qquad\qquad x' = \phi(x), \quad y' = \frac{y}{x}\psi(x) \quad \text{for } y < x$$

where ϕ and ψ are to satisfy

$$(3.7) \qquad y < \phi(y), \quad \phi(0) = 0, \quad \phi(1) = 1, \quad \phi \text{ increases}, \quad 0 < \psi(y) < 1.$$

For (3.4) to hold we need $\pi(x < a) \equiv \pi(x' < a)$ so that we want

$$(3.8) \qquad \int_0^a dx \int_0^x dy\, q(x) + \int_0^a dx \int_x^1 dy\, q(y) = \int_0^1 dx \int_0^x dy\, q(x)\chi(\phi(x) < a)$$
$$+ \int_0^1 dx \int_x^1 dy\, q(y)\chi\left(\frac{x}{y}\psi(y) < a\right).$$

We will require ψ and q to satisfy

$$(3.9) \qquad\qquad \psi(y) \equiv \alpha \quad \text{where } 0 < \alpha < 1 \text{ is to be chosen}$$

and

$$(3.10) \qquad\qquad aq(a) > \int_0^a q(y)\, dy, \qquad 0 < a < \alpha$$

$$(3.11) \qquad \alpha \int_0^1 q(y)\, dy = \int_0^1 yq(y)\, dy, \qquad q(a) > 0, \quad 0 < a < 1.$$

Then (3.8) with $b = b(a) = \phi^{-1}(a)$ may be written as

$$(3.12) \quad 2\int_0^a xq(x)\, dx + a\int_a^1 q(y)\, dy = \int_0^b xq(x)\, dx + \min\left\{\frac{a}{\alpha}, 1\right\}\int_0^1 yq(y)\, dy.$$

Clearly $b(0) = 0$, $b(1) = 1$, and differentiating on a we obtain

$$(3.13) \qquad aq(a) + \int_a^1 q(y)\, dy - \frac{1}{\alpha}\int_0^1 yq(y)\, dy = bq(b)b'(a), \qquad a < \alpha$$

$$(3.14) \qquad\qquad aq(a) + \int_a^1 q(y)\, dy = bq(b)b'(a), \qquad a > \alpha.$$

Using (3.10) and (3.11) we see that the left side of (3.13) is $aq(a) - \int_0^a q(y)\, dy > 0$ so that $b = b(a)$, defined by (3.12), has $b'(a) > 0$ (if $q(b) > 0$) for $a < \alpha$. By (3.14), $b'(a) > 0$ for $a > \alpha$. Thus $b(a) = \phi^{-1}(a)$ increases from 0 to 1 so that $\phi(a)$ does also. Furthermore, we must have $b = b(a) < a$ for $0 < a < 1$ because if $b \geqq a$ in (3.12) then the right side is greater than the left since under (3.11) it is easy to see that

$$(3.15) \quad 2 \int_0^a xq(x)\, dx + a \int_a^1 q(y)\, dy < \int_0^a xq(x)\, dx + \min\left\{\frac{a}{\alpha}, 1\right\} \int_0^1 yq(y)\, dy.$$

Thus under (3.9)–(3.11), (3.7) and (3.8) hold. Leaving aside the question of finding q to satisfy (3.10) and (3.11) for some $0 < \alpha < 1$, suppose we then set $X_3' = X_3 = z$ where

$$z = (1 - \theta)(x \vee y) + \theta(x' \vee y')$$

and θ is independent of (x, y) and uniform on $I = (0, 1)$. Thus the measure space on which X_i and X_i' are defined is $I^3 = \{(x, y, \theta)\}$. Let μ be the measure induced by $(X_1, X_2, X_3) = (x, y, z)$ and ν be the measure induced by $(X_1', X_2', X_3') = (x', y', z)$. Clearly μ and ν satisfy (3.2) since $x \vee y < z < x' \vee y'$ and μ and ν have the same projections by (3.8) and the symmetry between (x, x') and (y, y').

We now verify that μ and ν have bounded densities. The density of μ for $x < y$ is from (3.5) and (3.6a) and the fact that $P(\theta(y' - y) \in dz - y) = dz/(y' - y)$,

$$
(3.16) \quad
\begin{aligned}
\frac{\mu(dx, dy, dz)}{dx\, dy\, dz} &= q(y) \frac{P((1 - \theta)y + \theta(x' \vee y') \in dz)}{dz} = \frac{q(y)}{(x' \vee y') - y} \\
&\leqq \frac{q(y)}{\phi(y) - y}.
\end{aligned}
$$

The density of ν for $x' < y'$ is as follows. If (x', y') comes from (x_0, y_0) with $x_0 < y_0$, where $x_0 < y_0$ denotes an event A_0 with indicator χ_0, then from (3.5) and (3.6a),

$$
(3.17) \quad
\begin{aligned}
\frac{\pi(A_0 \cap (dx', dy', dz'))}{dx'\, dy'\, dz'} &= \chi_0 q(y) \frac{dx_0\, dy_0}{dx'\, dy'} \frac{P((1 - \theta)y_0 + \theta y' \in dz')}{dz'} \\
&= \chi_0 q(y_0) \frac{y_0}{\psi(y_0)} \frac{1}{\phi'(y_0)} \frac{1}{y' - y_0}.
\end{aligned}
$$

However, (x', y') may also come from (x_1, y_1) with $x_1 > y_1$, an event A_1 with indicator χ_1. But

$$(3.18) \quad \nu(dx', dy', dz')/dx'\, dy'\, dz' = \pi((A_0 \cup A_1) \cap (dx'\, dy'\, dz'))/dx'\, dy'\, dz'$$

since (x', y') comes from either a unique (x_0, y_0) with $x_0 < y_0$ or from a unique (x_1, y_1) with $x_1 > y_1$, or both. That is to say, if (x', y') also comes from say (x_2, y_2), then either $(x_2, y_2) = (x_0, y_0)$ or $(x_2, y_2) = (x_1, y_1)$ because ϕ is strictly monotonic. By (3.5) and (3.6b)

$$
(3.19) \quad
\begin{aligned}
\frac{\pi(A_1 \cap (dx', dy', dz'))}{dx'\, dy'\, dz'} &= \chi_1 q(x) \frac{dx_1}{dx'} \frac{dy_1}{dy'} \frac{P((1 - \theta)x_1 + \theta y' \in dz')}{dz'} \\
&= \chi_1 q(x_1) \frac{1}{\phi'(x_1)} \frac{x_1}{\psi(x_1)} \frac{1}{y_1' - x_1}.
\end{aligned}
$$

By (3.18) and (3.17) and then (3.19) we have

$$(3.20) \quad \frac{\nu(dx', dy', dz')}{dx'\, dy'\, dz'} \leqq q(y_0) \frac{y_0}{\psi(y_0)} \frac{1}{\phi'(y_0)} \frac{1}{\phi(y_0) - y_0} + q(x_1) \frac{x_1}{\psi(x_1)} \frac{1}{\phi'(x_1)} \frac{1}{y_1' - x_1}.$$

But

$$(3.21) \qquad y_1' - x_1 = \frac{y_1}{x_1} \psi(x_1) - x_1 \geqq \phi(x_1) - x_1$$

since $y_1' = y' = (y_1/x_1)\psi(x_1) > x_1' = x_1 = \phi(x_1)$ and we are in case (3.6b). Thus ν will have a bounded density if and only if there is an M for which

$$(3.22) \qquad q(y) \frac{y}{\psi(y)} \frac{1}{\phi'(y)} \frac{1}{\phi(y) - y} \leqq M, \qquad 0 < y < 1.$$

And, from (3.16), μ will have a bounded density if and only if there is an M for which

$$(3.23) \qquad q(y)/(\phi(y) - y) \leqq M$$

since both terms on the right in (3.20) have the same form after (3.21) is used in the denominator of the last term in (3.20).

It remains to show that there is $0 < \alpha < 1$ and $q(y)$, $0 < y < 1$, for which (3.10), (3.11), (3.22), and (3.23) hold for the ϕ determined by (3.8), as in (3.12)–(3.15). We choose in particular

$$(3.24) \qquad \alpha = 2/5, \qquad q(y) = y(1 - y)^2$$

and directly verify that (3.10) and (3.11) hold. It is clear from (3.13) and (3.14) that the only chance for $\phi(y) - y$ or $\phi'(y)$ to be small is near $y = 0$ or $y = 1$. But near $a = 0$, from (3.12), recalling $b = b(a) = \phi^{-1}(a)$, (3.11), (3.13), and (3.24),

$$\int_0^b x^2(1-x)^2 \, dx = 2 \int_0^a xq(x) \, dx - a \int_0^a q(y) \, dy = \frac{a^3}{6} - \frac{a^4}{6} - \frac{a^5}{20} = \frac{b^3}{3} - \frac{b^4}{2} + \frac{b^5}{5}.$$

Let $\approx$ stand for "bounded above and below by a constant multiple of." It follows that near $a = 0$, $\phi'(a) \approx 1$, $\phi(a) - a \approx q(a)$ and hence (3.22) and (3.23) hold near $a = 0$. Near $a = 1$, from (3.12), subtracting from (3.12) with $a = b = 1$, and setting $1 - a = \bar{a}$ and $1 - b = \bar{b}$,

$$\int_b^1 x^2(1-x)^2 \, dx = 2 \int_a^1 xq(x) \, dx - a \int_a^1 q(y) \, dy$$

$$= 2 \int_0^{\bar{a}} x^2(1-x)^2 \, dx - (1 - \bar{a}) \int_0^{\bar{a}} x^2(1-x) \, dx$$

$$= \frac{\bar{a}^3}{3} + \frac{\bar{a}^4}{12} - \frac{3\bar{a}^5}{20} = \frac{\bar{b}^3}{3} - \frac{\bar{b}^4}{2} + \frac{\bar{b}^5}{5}.$$

It follows that near $a = 1$, $\phi'(a) \approx 1$, $\phi(a) - a \approx (1-a)^2 \approx q(a)$ and so (3.22) and (3.23) hold near $a = 1$. This completes the proof that μ and ν have bounded densities.

The fact that there exist μ and ν with densities taking values only 0 and 1 follows easily from the following simple, powerful, and apparently new result, proved in [2]. This result says that if f is any density in $\mathbb{R}^n$ with $0 \leqq f \leqq 1$, then there is a density g with values in $\{0, 1\}$ that has the same projections as f. This is proved in the case $n = 2$ by observing that the necessary and sufficient condition for the existence of a bounded density with given marginals due to Strassen [11] is the same as the necessary and sufficient condition for the existence of a set with given marginals due to Lorentz [9] for $n = 2$. The proof for general n follows easily from the result for $n = 2$. □

Final remarks. The first part of the proof of Theorem 3.2 shows that there is a *fuzzy* set with the same projections as S_0, where a fuzzy set is a function $f_0 = f_0(x) \in [0, 1]$. Indeed, if μ and ν are as in the proof with densities bounded by 1 (multiply by a small constant), we can set $f_0 = \chi(S_0) - d\mu/d\lambda + d\nu/d\lambda$. Note that this is a much different statement than the trivial one that there is a bounded density $g_0(x)$ supported on the cube with the same projections as $\chi(S_0)$, i.e., as S; merely take

$$g_0(x) = 9 P_1(x_1) P_2(x_2) P_3(x_3),$$

where $P_i(x_i)$ is the projection of S_0 onto x_i,

$$P_i(x_i) = \frac{1 - x_i^2}{2} \quad \text{for } i = 1, 2; \qquad P_3(x_3) = x_3^2.$$

The difference is that g_0 is not a fuzzy set because $g_0(x) > 1$ at some points such as $g_0(0, 0, 1) = 9/4$. It is interesting to note that there are analogous reconstruction problems in which a measure exists satisfying a given set of "moment" conditions, but no set exists. For example, it is impossible to find a set $A \subset [0, 1]$ with Leb $(A \cap I) = \frac{1}{2}$ Leb (I) for every interval I, but there certainly is a measure (or fuzzy set), $\frac{1}{2}\lambda$, with this property. Related results are found in [10] (see also [3]), [5], and [11].

Appendix. Proof of Lemma 2.5. The bounded open sets S_1 and S_2 are constructed recursively: each S_i is an infinite union of open unit cubes of size 2^{-k} for various values of k.

The basis of the construction is a configuration which fits inside a $3 \times 2 \times 2$ block made up of sets T_1 and T_2 each of which consists of three unit cubes. Let the block be $B = \{(x, y, z): 0 \le x \le 3, 0 \le y \le 2, 0 \le z \le 2\}$ and label each cube by its corner having the smallest coordinate values. The configuration is:

$$T_1 = \{(0, 0, 0), (1, 0, 0), (2, 1, 1)\}$$

$$T_2 = \{(0, 1, 0), (2, 0, 0), (1, 0, 1)\}.$$

It is pictured in Fig. A.1, where the cubes in T_1 are labeled with a plus sign, and those in T_2 by a minus sign.

The basic configuration has three key properties:

(1) The sets T_1 and T_2 have the same projections on all three axes.

(2) T_1 contains a $2 \times 1 \times 1$ block in the lower left corner of the lower layer of B and a $1 \times 1 \times 1$ block in the upper right corner of the upper layer of B.

(3) Any bad configuration for T_1 (or T_2) in $T_1 \cup T_2$ must contain a point in every cube of $T_1 \cup T_2$.
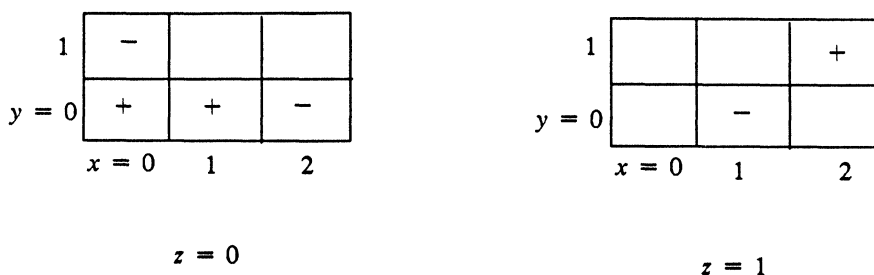

FIG. A.1. *Basic configuration.*

The first two properties are clear. To prove the third property, we form an undirected graph whose vertices are the cubes in $T_1 \cup T_2$ and where an edge represents a condition: "If a bad configuration includes a point in one vertex, then it necessarily includes a point in the other vertex." Such conditions are forced by the requirement that a bad configuration has equal projections for the subsets of T_1 and T_2. For example, if a bad configuration includes a point in the cube $(2, 1, 1)$ in T_1, then it must include a point in the cube $(1, 0, 1)$ in T_2, and vice versa, which is seen by examining its projection in the z-direction. For a point in $(2, 1, 1)$ in T_1 can only be matched in the z-direction by a point in T_2 having $1 < z < 2$, and all such points lie in the cube $(1, 0, 1)$. Proceeding in this way, we obtain the graph pictured in Fig. A.2, where edges are labeled by the direction of projection that verifies their occurrence. Since this graph is connected, it follows that a bad configuration includes every vertex.
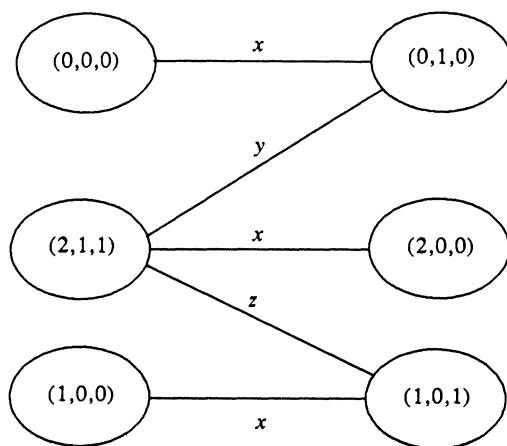

FIG. A.2. *Graph associated to basic configuration.*

We now begin the recursive construction. Construct a shrunken and rotated copy of the basic configuration by first scaling the first axis by $\frac{1}{2}$ and then permuting the axes cyclically to obtain a configuration in a $2 \times \frac{3}{2} \times 2$ block consisting of $1 \times \frac{1}{2} \times 1$ parallelepipeds. We place this $2 \times \frac{3}{2} \times 2$ block so that its lower left corner in its lowest level exactly overlaps the $1 \times 1 \times 1$ cube $(2, 1, 1)$ in T_1. The $1 \times 1 \times 1$ cube $(2, 1, 1)$ corresponds to the union of two cubes labeled $(0, 0, 0)$ and $(1, 0, 0)$ in the shrunken configuration. Let (T_1', T_2') denote the analogues of T_1 and T_2 in the shrunken configuration. Our second set is:

$$U_1' = T_1 \cup T_2' - \{\text{cube } (2, 1, 1)\}$$

$$U_2' = T_2 \cup T_1' - \{\text{cube } (2, 1, 1)\}.$$

Now we repeat this construction one more time, shrinking and rotating the basic configuration to obtain a $2 \times 1 \times \frac{3}{2}$ block containing the basic configuration as a set of $1 \times \frac{1}{2} \times \frac{1}{2}$ parallelepipeds, and adding it in to obtain:

$$U_1'' = T_1 \cup T_2' \cup T_1'' - \{\text{two overlapping corner cubes}\},$$

$$U_2'' = T_2 \cup T_1' \cup T_2'' - \{\text{two overlapping corner cubes}\}.$$

We call this configuration the *composite configuration.*

Now we describe the composite configuration explicitly. For convenience we scale up the sets U_1'', U_2'' so that all cubes involved have side at least one, by expanding the y and z axes by a factor of 2. When we do so, the resulting configuration U_1, U_2 is embedded in a $5 \times 6 \times 7$ block $B' = \{(x, y, z): 0 \leqq x \leqq 5, 0 \leqq y \leqq 6, 0 \leqq z \leqq 7\}$. U_1 consists of a $2 \times 2 \times 2$ cube A, three $1 \times 1 \times 2$ parallelepipeds B, C, D, and a $1 \times 1 \times 1$ cube E:

A. $(0, 0, 0), (0, 0, 1), (0, 1, 0), (0, 1, 1)$
$\quad$ $(1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)$
B. $(2, 4, 2), (2, 4, 3)$
C. $(2, 3, 4), (2, 3, 5)$
D. $(3, 2, 2), (3, 2, 3)$
E. $(4, 5, 6)$.

The set U_2 consists of three $1 \times 2 \times 2$ parallelepipeds F, G, H and three $1 \times 1 \times 1$ cubes I, J, K, which are:

F. $(0, 2, 0), (0, 3, 0), (0, 2, 1), (0, 3, 1)$
G. $(2, 0, 0), (2, 0, 1), (2, 1, 0), (2, 1, 1)$
H. $(1, 0, 2), (1, 0, 3), (1, 1, 2), (1, 1, 3)$
I. $(3, 5, 4)$
J. $(4, 4, 5)$
K. $(3, 4, 6)$.

The composite configuration (U_1, U_2) has three key properties:

(1) The sets U_1 and U_2 have the same projections on all three axes.
(2) U_1 contains a $2 \times 2 \times 2$ block in the lower left corner of its lowest two layers and a $1 \times 1 \times 1$ block in the upper right corner of its highest layer.
(3) Any bad configuration for U_1 (or U_2) in $U_1 \cup U_2$ must contain a point in its highest cube $(4, 5, 6)$.

Here (1) and (2) are easily checked. Property (3) is proved directly using an argument similar to that given for the basic configuration. We use directed graphs whose vertices consist of several cubes, and whose directed edges indicate: "A bad configuration containing a point in some cube in the entering vertex must contain a point in one of the cubes in the vertex pointed to." The edges in the graph are constructed by arguments similar to those used for the basic configuration. Figure A.3 gives such graphs showing that any bad configuration must include a point in the cube E, thus proving property (3).

To describe the recursive construction of S_1 and S_2, we form an infinite set of copies of the composite configuration, shrunk by factors of $\frac{1}{2}$ at each step, so that the kth composite configuration lies inside a $5/(2^{k-1}) \times 6/(2^{k-1}) \times 7/(2^{k-1})$ block B_k. These blocks are stacked successively so that the corner E of the block B_k exactly overlaps the corner A of block B_{k+1}, as indicated in Fig. A.4.

Let $U_1^{(k)}$, $U_2^{(k)}$ denote the composite configuration in block B_k. The sets S_1, S_2 are given by:

$$S_1 = \left(\bigcup_{k=1}^{\infty} U_1^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} U_2^{(2k)} \right) - \{\text{all overlapped corners}\}$$

$$S_2 = \left(\bigcup_{k=1}^{\infty} U_2^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} U_1^{(2k)} \right) - \{\text{all overlapped corners}\}.$$

If we let $\hat{U}_1$ denote $U_1 - \{A\} - \{E\}$ then

$$S_1 = \{A\} \cup \left(\bigcup_{k=1}^{\infty} \hat{U}_1^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} U_2^{(2k)} \right) \qquad S_2 = \left(\bigcup_{k=1}^{\infty} U_2^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} \hat{U}_1^{(2k)} \right).$$
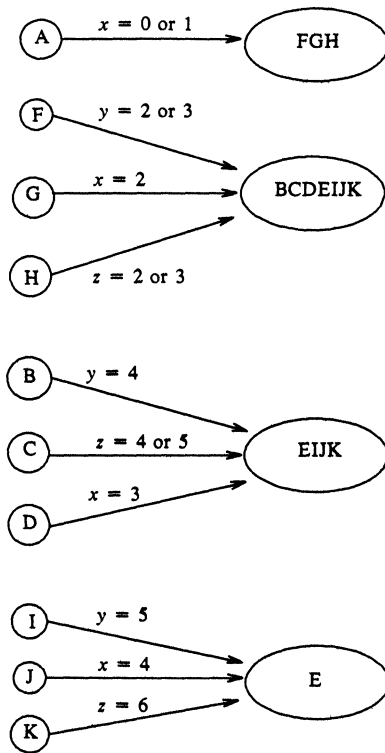
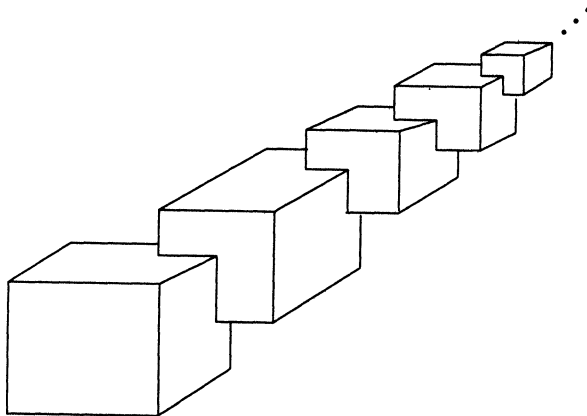FIG. A.3. *Directed graphs associated to composite configuration.*

FIG. A.4. *Recursive construction of blocks B_k.*

It is easy to see that the sets S_1 and S_2 are bounded. Moreover, we show that:

(1) S_1 and S_2 have the same projections on all three axes.

(2) There exists no finite bad configuration for S_1 contained in $S_1 \cup S_2$.

To show that (1) holds, we observe that

$$\hat{S}_1 = \left(\bigcup_{k=1}^{\infty} U_1^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} U_2^{(2k)} \right), \qquad \hat{S}_2 = \left(\bigcup_{k=1}^{\infty} U_2^{(2k-1)} \right) \cup \left(\bigcup_{k=1}^{\infty} U_1^{(2k)} \right)$$

have the same projections on all three axes. Indeed, this holds by property (1) for the composite configuration and the fact that all sets in the union defining $\hat{S}_1$ (respectively, $\hat{S}_2$) are disjoint. The sets $\hat{S}_1$ and $\hat{S}_2$ are not disjoint, since their intersection V is:

$$V = \hat{S}_1 \cap \hat{S}_2 = \bigcup_{k=1}^{\infty} E^{(k)},$$

where $E^{(k)}$ denotes the corner cube E in $U_1^{(k)}$, which is also the corner cube A in $U_1^{(k+1)}$. Since $S_1 = \hat{S}_1 - V$ and $S_2 = \hat{S}_2 - V$, property (1) follows.

To prove property (2), let $\hat{U}^{(k)} = \hat{U}_1^{(k)} \cup U_2^{(k)}$, and call $\hat{U}^{(k)}$ the *level k points in* $S_1 \cup S_2$. We first prove the analogue of property (3) of the composite configuration, which is:

(3′) Any bad configuration for S_1 in $S_1 \cup S_2$ that contains a point in $\hat{U}^{(k)}$ must contain a point in $\hat{U}^{(k+1)}$.

This is proved by exactly the same graph-theoretic argument as in Fig. A.3, except that the vertex E in that graph is interpreted to be "all points in $S_1 \cup S_2$ in all levels $\geq k+1$." The conclusion of property (3) that there must be a point in E is replaced by the conclusion that there must be a point in the bad configuration at a higher level. This point must be on level $k+1$ because points on level k and points on levels greater than or equal to $k+2$ do not have any projections that agree. So (3′) is proved. Finally, property (3′) implies property (2), because repeated applications of property (3′) produce at least one point in the bad configuration at all levels $\geq k$. Hence the bad configuration contains infinitely many points. $\square$

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