1995

A New Look at Pricing of the "Russian Option"

Larry A. Shepp
University of Pennsylvania

A. N. Shiryaev

Recommended Citation

Shepp, L. A., & Shiryaev, A. N. (1995). A New Look at Pricing of the "Russian Option". *Theory of Probability & Its Applications, 39* (1), 103-119. http://dx.doi.org/10.1137/1139004

A New Look at Pricing of the "Russian Option"

Abstract
The "Russian option" was introduced and calculated with the help of the solution of the optimal stopping problem for a two-dimensional Markov process in [10]. This paper proposes a new derivation of the general results [10]. The key idea is to introduce the dual martingale measure which permits one to reduce the "two-dimensional" optimal stopping problem to a "one-dimensional" one. This approach simplifies the discussion and explain the simplicity of the answer found in [10].

Keywords
diffusion model of the (B, S)-market, bank account, rational option price, rational expiration time, optimal stopping rules, smooth sewing condition, the Stephan problem, diffusion with reflection

Disciplines
Probability | Statistics and Probability

THEORY PROBAB. APPL.
Vol. 39, No. 1

Translated from Russian Journal

A NEW LOOK AT PRICING OF THE "RUSSIAN OPTION"*

L. A. SHEPP† AND A. N. SHIRYAEV‡

(*Translated by M. V. Khatuntseva*)

Abstract. The "Russian option" was introduced and calculated with the help of the solution of the optimal stopping problem for a two-dimensional Markov process in [10]. This paper proposes a new derivation of the general results [10]. The key idea is to introduce the dual martingale measure which permits one to reduce the "two-dimensional" optimal stopping problem to a "one-dimensional" one. This approach simplifies the discussion and explain the simplicity of the answer found in [10].

Key words. diffusion model of the (B, S)-market, bank account, rational option price, rational expiration time, optimal stopping rules, smooth sewing condition, the Stephan problem, diffusion with reflection

1. Statement of the problem on pricing of the "Russian option".

1. Following Samuelson [9], Black and Scholes [2], and Merton [8], we consider the "diffusion" (B, S)-market consisting of two assets: riskless *bank account* $B = (B_t)_{t \geqq 0}$ and risky stock $S = (S_t)_{t \geqq 0}$.

We assume that the bank account $B = (B_t)_{t \geqq 0}$ is a determinate function

$$B_t = B_0 e^{rt}, \qquad B_0 > 0, \qquad r \geqq 0, \tag{1.1}$$

satisfying the equation

$$dB_t = rB_t dt. \tag{1.2}$$

In order to describe the evolution of the stock price $S = (S_t)_{t \geqq 0}$ as a random process we shall consider following the spirit of the modern "general theory of the random processes" *the canonical filtered Wiener space* $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \geqq 0}, \mathbf{P})$ with components:

Ω a space of continuous functions $\omega = (\omega_t)_{t \geqq 0}$ with $\omega_0 = 0$; $\mathcal{F} = \mathcal{C}$ a Borel σ-algebra generated by cylindric sets; $\mathbf{P}$ a Wiener measure on $(\Omega, \mathcal{F})$; $\mathbf{F} = (\mathcal{F}_t)_{t \geqq 0}$ a filtration, i.e., a flow of σ-algebras $\mathcal{F}_t$, $t \geqq 0$, where $\mathcal{F}_t$ is a σ-algebra $\cap_{s>t} \mathcal{F}_s^0$ with $\mathcal{F}_s^0 = \sigma(\omega_u, u \leqq s)$ completed by the sets of $\mathbf{P}$-null probability from $\mathcal{F}$.

Let $W = (W_t)_{t \geqq 0}$ be a canonical process with $W_t = W_t(\omega)$ such that $W_t(\omega) = \omega_t$. The process W is a standard *Wiener* process (Brownian motion) with respect to the measure $\mathbf{P}$ whose one characterization is the following (Lévy theorem, see [16, Theorem 4.1]): W is a continuous martingale (with respect to $(\mathbf{F}, \mathbf{P})$), i.e., $\mathbf{E}(W_t | \mathcal{F}_s) = W_s$ ($\mathbf{P}$-a.s.) and $\mathbf{E}((W_t - W_s)^2 | \mathcal{F}_s) = t - s$ ($\mathbf{P}$-a.s.), $s \leqq t$.

To describe the evolution of a stock price we shall assume (following [9], [2], [8] and also [18] and [20] in this issue) that the stock $S = (S_t)_{t \geqq 0}$, $S_t = S_t(\omega)$, is a diffusion random process satisfying the stochastic differential equation

$$dS_t = S_t(\mu dt + \sigma dW_t), \tag{1.3}$$

---

*Received by the editors July 5, 1993.

†AT&T Bell Laboratories, Murray Hills, New Jersey, 07974, USA.

‡Steklov Mathematical Institute, RAN, Russia, Vavilov st. 42, 117966 Moscow, Russia. This research was partially supported by Internatinal Science Foundation grant MMK000

with *appreciation rate* $\mu \in \mathbf{R}$, *volatility* $\sigma > 0$, and initial value $S_0 = s > 0$.

Concerning the parameters μ and σ in (1.3) (cf. [20, § 1]) we note that the coefficient σ is defined by a *continuously* observable process S in an arbitrarily small time interval $[0, t_0]$, $t_0 > 0$ (see, for example, [16, Lemma 4.3]). The appreciation rate μ is defined "worse". In this connection the result obtained by Black and Scholes [2] (see also [20]) turned out to be unexpected. It states that the rational price of a standard call option of European type does *not* depend on the parameter $\mu \in \mathbf{R}$, the *a priori* opinion on a value of which may be very fuzzy, and in order to obtain a "good" estimation for μ one need some time of observation.

Section 6 in [20] says that the mentioned above "Black and Scholes effect" of independence of μ is explained by the fact that for the standard call option of European type with expiration time $T > 0$ its payment function $f_T(\omega)$ $(= (S_T(\omega) - K)^+)$ depends on $\omega \in \Omega$ not directly but only via the values $S_T(\omega)$.

By the same reason a similar effect takes place in the case of the "Russian option" considered in [10], for which the system of payment functions $f = (f_t(\omega))_{t \geqq 0}$ is of the following type:

$$f_t = e^{-\lambda t} \max \left[\max_{u \leqq t} S_u, s\psi_0 \right], \tag{1.4}$$

where $\lambda > 0$, $\psi_0 \geqq 1$, $s = S_0 > 0$.

2. Let $f = (f_t(\omega))_{t \geqq 0}$ be some non-negative progressively measurable random process [16, p. 30], [17, p. 11].

Let us consider an option of the American type with a system of payment functions $f = (f_t(\omega))_{t \geqq 0}$ as a contract between a seller and a buyer by which the buyer can exercise the option at any (Markov) time $\tau = \tau(\omega)$ with payment $f_{\tau(\omega)}(\omega)$, $\omega \in \Omega$. (For details see [18], [19], [20].) The following two general problems arise for such options:

(1) What is the *rational* price (i.e., a fair price both from the seller's and buyer's points of view) which the buyer must pay to gain a given contract?

(2) What is the *rational* time in which it is "reasonable" for buyer to exercise the option?

The general theory of option pricing ([4], [5], [7], [1], [18], [19], and [20]) gives the following answers to these questions.

Let $\mathbb{C}_T^*(\mu, f)$ be a rational option cost (see Definition 4 in [19, § 1] and Definition 4 in [20, § 2]) under the assumption that the option (of the American type) can be exercised at any Markov time with values from the interval $[0, T]$. Then

$$\mathbb{C}_T^*(\mu, f) = B_0 \sup_{0 \leqq \tau \leqq T} \mathbf{E}^{\mu - r} \frac{f_\tau}{B_\tau}, \tag{1.5}$$

where $\mathbf{E}^{\mu - r}$ is expectation with respect to the measure $\mathbf{P}^{\mu - r}$ defined below in (2.3).

In the case where the expiration time can take any values from the set $[0, \infty)$ we denote by $\mathbb{C}^*(\mu, f)$ the rational price. In analogy to (1.5),

$$\mathbb{C}^*(\mu, f) = B_0 \sup_{0 \leqq \tau < \infty} \mathbf{E}^{\mu - r} \frac{f_\tau}{B_\tau}. \tag{1.6}$$

Therefore, the rational pricing reduces (in view of (1.1)) to the solution of the following optimal stopping problems:

$$\text{“} \sup_{0 \leqq \tau \leqq T} \mathbf{E}^{\mu - r} e^{-r\tau} f_\tau \text{”} \tag{1.7}$$

and

$$\text{“} \sup_{0 \leqq \tau < \infty} \mathbf{E}^{\mu - r} e^{-r\tau} f_\tau \text{”}. \tag{1.8}$$

It turn out that the optimal time τ_T^* (in the problem (1.7)) is exactly the rational time at which the buyer must exercise the option (see [19] and [20]). Similarly, if τ^* is a finite ($\mathbf{P}^{\mu-r}$-a.s.) Markov time for which sup is attained in (1.8), then exactly this time is the rational expiration time.

From the general theory and the practice of solutions of the optimal stopping problem it is well known that the problem (1.7) on a *finite* interval $[0, T]$ is more difficult than the problem (1.8) on an *infinite* interval $[0, \infty)$. This is also the case in the case of the "Russian option" for the function (1.4) under consideration.

Section 2 shows that the costs $\mathbb{C}^*(\mu, f)$, $\mathbb{C}_T^*(\mu, f)$ do not depend on values of the parameter $\mu \in \mathbf{R}$. For the given function f defined in (1.4), we shall for brevity denote these rational costs by $\mathbb{C}^*$, $\mathbb{C}_T^*$.

Our general goal is to give a new proof of the following results of our paper [10].

THEOREM. *The rational cost* $\mathbb{C}^*$ *of the "Russian option" with payment function* $f = (f_t(\omega))_{t \geqq 0}$ *given in* (1.4) *and* (*Markov*) *expiration times with values from* $[0, \infty)$ *is defined by*

$$\mathbb{C}^* = S_0 \cdot \begin{cases} \dfrac{\widetilde{\psi}}{x_2 - x_1} \left[(x_2 - 1) \left(\dfrac{\psi_0}{\widetilde{\psi}} \right)^{x_1} + (1 - x_1) \left(\dfrac{\psi_0}{\widetilde{\psi}} \right)^{x_2} \right], & 1 \leqq \psi_0 < \widetilde{\psi}, \\ \psi_0, & \psi_0 \geqq \widetilde{\psi}, \end{cases} \tag{1.9}$$

or, which is the same,

$$\mathbb{C}^* = S_0 \cdot \begin{cases} \widetilde{\psi} \cdot \dfrac{x_2 \psi_0^{x_1} - x_1 \psi_0^{x_2}}{x_2 \widetilde{\psi}^{x_1} - x_1 \widetilde{\psi}^{x_2}}, & 1 \leqq \psi_0 < \widetilde{\psi}, \\ \psi_0, & \psi_0 \geqq \widetilde{\psi}, \end{cases} \tag{1.10}$$

where x_1, x_2, *being the roots of the quadratic equation* (4.13), *are defined by* (4.15), (4.16), *and*

$$\widetilde{\psi} = \left| \frac{x_2}{x_1} \cdot \frac{x_1 - 1}{x_2 - 1} \right|^{1/(x_2 - x_1)} \tag{1.11}$$

The rational time is

$$\widetilde{\tau} = \inf\{t \geqq 0 : \psi_t \geqq \widetilde{\psi}\}, \tag{1.12}$$

where

$$\psi_t = \frac{\max\{\max_{u \leqq t} S_u, S_0 \psi_0\}}{S_t}, \qquad t \geqq 0. \tag{1.13}$$

The proof of this result is given in § 3. Section 2 introduces a so-called dual martingale measure (see also [20, § 7]), permitting a new representation for $\mathbb{C}_T^*(\mu, f)$ and $\mathbb{C}^*(\mu, f)$. Section 3 studies the properties of the process $(\psi_t)_{t \geqq 0}$ in the phase space $[1, \infty)$ which turns out to be a diffusion Markov process with reflection at the point $\{1\}$.

Note that the idea of using a dual martingale measure is also applied in the papers [14] and [15] published in this issue.

2. Dual martingale measure.

1. Along with the Wiener process $W = (W_t)_{t \geqq 0}$ we introduce for $\mu \in \mathbf{R}$, $t \geqq 0$, the processes

$$W_t^{\mu-r}(\omega) = W_t(\omega) + \frac{\mu - r}{\sigma} t, \tag{2.1}$$

$$Z_t^{\mu-r}(\omega) = \exp\left\{ -\frac{\mu - r}{\sigma} W_t(\omega) - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t \right\}. \tag{2.2}$$

Let $\mathbf{P}_t = \mathbf{P}|\mathcal{F}_t$ be a restriction of the Wiener measure $\mathbf{P}$ on $\mathcal{F}_t$. For $t \geqq 0$ we introduce new measures $\mathbf{P}_t^{\mu-r}$ assuming

$$\mathbf{P}_t^{\mu-r}(d\omega) = Z_t^{\mu-r}(\omega)\mathbf{P}_t(d\omega). \tag{2.3}$$

Since $\mathbf{E}Z_t^{\mu-r}(\omega) = 1$, the measure $\mathbf{P}_t^{\mu-r}$ is probabilistic and due to the consistency of the family $\{\mathbf{P}_t^{\mu-r}, t \geqq 0\}$ we can establish that a probabilistic measure $\mathbf{P}^{\mu-r}$ on $(\Omega, \mathcal{F})$ exists such that its restriction $\mathbf{P}^{\mu-r}|\mathcal{F}_t$ coincides with $\mathbf{P}_t^{\mu-r}$, $t \geqq 0$ (see [20, § 1]).

With respect to the measure $W^{\mu-r} = (W_t^{\mu-r})_{t \geqq 0}$ the process $\mathbf{P}^{\mu-r}$ is a Wiener process

$$\operatorname{Law}(W^{\mu-r}|\mathbf{P}^{\mu-r}) = \operatorname{Law}(W|\mathbf{P}) \ \ (= \mathbf{P}) \tag{2.4}$$

(Girsanov's theorem, [16], [6]).

In order to underline the dependence of the process $S = (S_t)_{t \geqq 0}$ satisfying equation (1.3) on μ we also write $S(\mu) = (S_t(\mu))_{t \geqq 0}$ and $S(\mu, \omega) = (S_t(\mu, \omega))_{t \geqq 0}$ if one also needs to underline the dependence on $\omega \in \Omega$.

Note that the solution of equation (1.3) can be written in the form

$$S_t(\mu) = S_0 \exp\left\{ \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t \right\}, \tag{2.5}$$

or, equivalently, as

$$S_t(\mu) = S_0 e^{\mu t} \mathcal{E}(\sigma W)_t, \tag{2.6}$$

where

$$\mathcal{E}(\sigma W)_t = \exp\left\{ \sigma W_t - \frac{\sigma^2}{2} t \right\} \tag{2.7}$$

is the stochastic Dolean exponent (see, e.g., [17, Chap. 2, § 4]).

By the Itô formula [17, Chap. 2, § 3], [6,Chap. 1, § 4f],

$$d\left(\frac{S_t(\mu)}{B_t}\right) = \sigma\left(\frac{S_t(\mu)}{B_t}\right) dW_t^{\mu-r}, \tag{2.8}$$

or in integral form

$$\frac{S_t(\mu)}{B_t} = \frac{S_0}{B_0} + \int_0^t \sigma \frac{S_u(\mu)}{B_u} \, dW_u^{\mu-r}. \tag{2.9}$$

Hence with respect to the measure $\mathbf{P}^{\mu-r}$ the process

$$\frac{S(\mu)}{B} = \left(\frac{S_t(\mu)}{B_t}\right)_{t \geqq 0} \tag{2.10}$$

is a martingale and hence we shall name the measure $\mathbf{P}^{\mu-r}$ a *martingale measure.*

From (2.8) it is easy to see that ($\mathbf{P}^{\mu-r}$-a.s.)

$$\frac{S_t(\mu)}{B_t} = \frac{S_0}{B_0} \exp\left\{\sigma W_t^{\mu-r} - \frac{\sigma^2}{2} t\right\} \left(= \frac{S_0}{B_0} \mathcal{E}(\sigma W^{\mu-r})_t \right), \tag{2.11}$$

or, equivalently,

$$S_t(\mu) = S_0 \exp\left\{\sigma W_t^{\mu-r} + \left(r - \frac{\sigma^2}{2}\right) t\right\}. \tag{2.12}$$

From (2.9) or (2.11) it follows that

$$\mathbf{E}^{\mu-r}\left(\frac{S_t(\mu)}{B_t} \cdot \frac{B_0}{S_0}\right) = 1, \tag{2.13}$$

which permits us to define a new *probabilistic* measure $\widetilde{\mathbf{P}}_t^{\mu-r}$ on $(\Omega, \mathcal{F}_t)$ with

$$\widetilde{\mathbf{P}}_t^{\mu-r}(A) = \mathbf{E}^{\mu-r}\left(\frac{S_t(\mu)}{S_0} \cdot \frac{B_0}{B_t} I(A)\right), \qquad A \in \mathcal{F}_t. \tag{2.14}$$

In particular this implies that

$$\widetilde{\mathbf{P}}_t^{0}(A) = \mathbf{E}\left[\exp\left\{\sigma W_t - \frac{\sigma^2}{2} t\right\} I(A)\right], \qquad A \in \mathcal{F}_t, \tag{2.15}$$

since $\mathbf{P}^0 = \mathbf{P}$.

Let $\widetilde{\mathbf{P}}^{\mu-r}$ be a measure on $(\Omega, \mathcal{F})$ such that its restrictions $\widetilde{\mathbf{P}}^{\mu-r}|\mathcal{F}_t = \widetilde{\mathbf{P}}_t^{\mu-r}$, $t \geqq 0$. (The existence of such a measure can be established in the same way as that of the measure $\mathbf{P}^{\mu-r}$.)

Along with the process $W^{\mu-r}$ defined by (2.1) we introduce a new process $\widetilde{W}^{\mu-r} = (\widetilde{W}_t^{\mu-r})_{t\geqq 0}$ with

$$\widetilde{W}_t^{\mu-r} = W_t^{\mu-r} - \sigma t \quad \left(= W_t + \left(\frac{\mu - r}{\sigma} - \sigma\right) t \right). \tag{2.16}$$

It is easy to convince oneself that the process $\widetilde{\mathbf{P}}^{\mu-r}$ with respect to the measure $\widetilde{W}^{\mu-r} = (\widetilde{W}_t^{\mu-r})_{t\geqq 0}$ is a Wiener process. The equality (2.12) yields

$$S_t(\mu) = S_0 \exp\left\{\sigma \widetilde{W}_t^{\mu-r} + \left(r + \frac{\sigma^2}{2}\right) t\right\}, \tag{2.17}$$

and, by the Itô formula,

$$dS_t(\mu) = S_t(\mu)\left[(r + \sigma^2)dt + \sigma d\widetilde{W}_t^{\mu-r}\right], \tag{2.18}$$

$$d\left(\frac{1}{S_t(\mu)}\right) = -\frac{1}{S_t(\mu)}\left[r dt + \sigma d\widetilde{W}_t^{\mu-r}\right], \tag{2.19}$$

$$d\left(\frac{B_t}{S_t(\mu)}\right) = -\sigma\left(\frac{B_t}{S_t(\mu)}\right) d\widetilde{W}_t^{\mu-r}. \tag{2.20}$$

The last equation implies that the process $\widetilde{\mathbf{P}}^{\mu-r}$ is a martingale with respect to the measure $B/S(\mu) = (B_t/S_t(\mu))_{t\geqq 0}$. This circumstance permits one to call this measure a *dual* (with respect to $\mathbf{P}^{\mu-r}$) martingale measure.

It is useful to note that (2.20) implies

$$\frac{B_t}{S_t(\mu)} = \frac{B_0}{S_0} \exp\left\{ -\sigma \widetilde{W}_t^{\mu-r} - \frac{\sigma^2}{2} t \right\} \left(= \frac{B_0}{S_0} \mathcal{E}(-\sigma \widetilde{W}^{\mu-r})_t \right). \tag{2.21}$$

Comparing this representation with (2.11) we find just as it must be that the product of the right-hand sides of (2.11) and (2.21) equals one (by (2.16)).

2. Let $f = (f_t(\omega))_{t \geqq 0}$ be a progressively measurable random process. Then the value f_τ is $\mathcal{F}_\tau$-measurable [16, Lemma 1.8] for any finite Markov time $\tau = \tau(\omega)$. Since, on the sets $A \in \mathcal{F}_\tau$,

$$\widetilde{\mathbf{P}}^{\mu-r}(A) = \mathbf{E}^{\mu-r}\left(\frac{S_\tau(\mu)}{S_0} \cdot \frac{B_0}{B_\tau} I(A) \right), \tag{2.22}$$

we have

$$B_0 \mathbf{E}^{\mu-r} \frac{f_\tau}{B_\tau} = S_0 \mathbf{E}^{\mu-r}\left[\frac{B_0}{B_\tau} \cdot \frac{S_\tau(\mu)}{S_0} \cdot \frac{f_\tau}{S_\tau(\mu)} \right] = S_0 \widetilde{\mathbf{E}}^{\mu-r} \frac{f_\tau}{S_\tau(\mu)}. \tag{2.23}$$

This and (1.5), (1.6) imply new representations for $\mathbb{C}_T^*(\mu, f)$ and $\mathbb{C}^*(\mu, f)$ using the dual martingale measures

$$\mathbb{C}_T^*(\mu, f) = S_0 \sup_{0 \leqq \tau \leqq T} \widetilde{\mathbf{E}}^{\mu-r} \frac{f_\tau}{S_\tau(\mu)} \tag{2.24}$$

and

$$\mathbb{C}^*(\mu, f) = S_0 \sup_{0 \leqq \tau < \infty} \widetilde{\mathbf{E}}^{\mu-r} \frac{f_\tau}{S_\tau(\mu)}. \tag{2.25}$$

In the case where $f_t = f_t(\omega)$ (as, for example, for the "Russian option") depends on ω via values of $S(\mu, \omega)$ (formally, f_t is a $\mathcal{F}_t^{S(\mu,\omega)}$-measurable function, where $\mathcal{F}_t^{S(\mu,\omega)} = \sigma(S_u(\mu, \omega), u \leqq t)$), the values $\mathbb{C}_T^*(\mu, f)$ and $\mathbb{C}^*(\mu, f)$ do not depend on μ since, by (2.17),

$$\operatorname{Law}\big(S(\mu) \mid \widetilde{\mathbf{P}}^{\mu-r}\big) = \operatorname{Law}\big(S(r) \mid \widetilde{\mathbf{P}}\big),$$

where $\widetilde{\mathbf{P}} = \widetilde{\mathbf{P}}^0$.

Denoting by $\mathbb{C}_T^*$ and $\mathbb{C}^*$ the common (with respect to μ) values $\mathbb{C}_T^*(\mu, f)$ and $\mathbb{C}^*(\mu, f)$ we find that

$$\mathbb{C}_T^* = S_0 \sup_{0 \leqq \tau \leqq T} \widetilde{\mathbf{E}} \frac{f_\tau(S(r))}{S_\tau(r)} \tag{2.26}$$

and

$$\mathbb{C}^* = S_0 \sup_{0 \leqq \tau < \infty} \widetilde{\mathbf{E}} \frac{f_\tau(S(r))}{S_\tau(r)}. \tag{2.27}$$

This result, in particular, means that for the pricing problem on the diffusion (B, S)-market under consideration it is sufficient to deal with the value of a parameter μ which is equal to r, i.e., is equal to the interest rate of a bank account $B = (B_t)_{t \geqq 0}$.

3. In what follows we shall write $S_t = S_t(r)$. Then for the "Russian option" with the function f_t defined in (1.4) we find that (cf. with (1.12))

$$\mathbb{C}^* = S_0 \sup_{0 \leqq \tau < \infty} \widetilde{\mathbf{E}} e^{-\lambda\tau} \psi_\tau, \tag{2.28}$$

where the process $(\psi_t)_{t \geqq 0}$ is defined by (1.12).

Thus, finding, $\mathbb{C}^*$ is reduced to the solution of the optimal stopping problem for the process $(\psi_t)_{t \geqq 0}$. The next section studies the structure of this process.

3. The structure of the process $(\psi_t)_{t \geqq 0}$.

1. First we show that this process is Markov with respect to the measure $\widetilde{\mathbf{P}}$. Since

$$\psi_t = \frac{\max\{\max_{u \leqq t} S_u, S_0 \psi_0\}}{S_t}, \tag{3.1}$$

we have

$$\begin{aligned} \psi_{t+\Delta} &= \max\left\{ \frac{\max_{u \leqq t+\Delta} S_u}{S_{t+\Delta}}, \frac{S_0\psi_0}{S_{t+\Delta}} \right\} \\ &= \max\left\{ \frac{\max_{u \leqq t} S_u}{S_t \cdot S_{t+\Delta}/S_t}, \frac{S_0\psi_0}{S_t \cdot S_{t+\Delta}/S_t}, \frac{\max_{t<u \leqq t+\Delta} S_u/S_t}{S_{t+\Delta}/S_t} \right\} \\ &= \max\left\{ \psi_t \cdot \frac{1}{S_{t+\Delta}/S_t}, \frac{\max_{t<u \leqq t+\Delta}(S_u/S_t)}{S_{t+\Delta}/S_t} \right\}. \end{aligned} \tag{3.2}$$

Note that for the Wiener process $\widetilde{W} = \widetilde{W}^0$ (with respect to the measure $\widetilde{\mathbf{P}}$) we have, for all $t < u \leqq t + \Delta$,

$$\frac{S_u}{S_t} = \exp\left\{ \sigma(\widetilde{W}_u - \widetilde{W}_t) + \left(r + \frac{\sigma^2}{2} \right)(u - t) \right\}, \tag{3.3}$$

by (2.17).

Consequently, (3.2) implies

$$\operatorname{Law}(\psi_{t+\Delta} | \mathcal{F}_t; \widetilde{\mathbf{P}}) = \operatorname{Law}(\psi_{t+\Delta} | \psi_t; \widetilde{\mathbf{P}}), \tag{3.4}$$

which proves the Markov character of the process $(\psi_t, \mathcal{F}_t, \widetilde{\mathbf{P}})_{t \geqq 0}$.

2. Let us study the structure of this process in more detail. To this end we denote

$$M_t = \max\left\{ \max_{u \leqq t} S_u, \ s\psi_0 \right\}, \tag{3.5}$$

where $S_u = S_u(r)$ and $s = S_0$.

$(M_t)_{t \geqq 0}$ is a nondecreasing process of locally bounded variation. Thus, in view of (2.12), (with $\mu = r$, $\widetilde{W} = \widetilde{W}^0$) we find by the Itô formula that

$$d\psi_t = d\left(\frac{M_t}{S_t} \right) = M_t d\left(\frac{1}{S_t} \right) + \frac{1}{S_t} dM_t = -\psi_t[r dt + \sigma d\widetilde{W}_t] + \frac{dM_t}{S_t}, \tag{3.6}$$

or in integral form

$$\psi_t = \psi_0 - r\int_0^t \psi_u\,du - \sigma\int_0^t \psi_u\,d\widetilde{W}_u + \int_0^t \frac{dM_u}{S_u}. \tag{3.7}$$

Similarly, for any function $g = g(\psi)$, $\psi \geqq 1$, of class $\mathbf{C}^2$,

$$dg(\psi_t) = g'(\psi_t)\,d\psi_t + \tfrac{1}{2}g''(\psi_t)\sigma^2\psi_t^2\,dt, \tag{3.8}$$

or

$$\begin{aligned} g(\psi_t) = g(\psi_0) &+ \int_0^t Lg(\psi_u)\,du - \sigma\int_0^t g'(\psi_u)\,\psi_u\,d\widetilde{W}_u \\ &+ \int_0^t g'(\psi_u)\,\frac{dM_u}{S_u}\,I(\psi_u = 1)\,du, \end{aligned} \tag{3.9}$$

where the differential operator is

$$L = -r\psi\frac{\partial}{\partial\psi} + \frac{\sigma^2}{2}\psi^2\frac{\partial^2}{\partial\psi^2}. \tag{3.10}$$

In the last integral we have introduced the indicator $I(\psi_u = 1)$ of the set $\{(\omega, u) : \psi_u(\omega) = 1\}$ since, as can be readily seen, $\psi_u(\omega) > 1$ (in the sense that $\int_0^t I(\psi_u > 1)\,dM_u = 0$, $t > 0$) for $dM_u(\omega) = 0$.

Now we show that, for any $t > 0$,

$$\int_0^t I(\psi_u = 1)\,du = 0 \qquad (\widetilde{\mathbf{P}}\text{-a.s.}). \tag{3.11}$$

Indeed, let $\Psi_\omega = \{t \geqq 0 : \psi_t(\omega) = 1\}$ and let $\Lambda(dt) = dt$ be the Lebesgue measure. Then, by the Fubini theorem,

$$\begin{aligned} \widetilde{\mathbf{E}}\int_0^\infty I(\psi_t(\omega) = 1)\,dt &= \int_0^\infty \widetilde{\mathbf{E}}\,I(\psi_t(\omega) = 1)\,dt \\ &= \int_0^\infty \widetilde{\mathbf{P}}\Big(\frac{\max\{\max_{u\leqq t} S_u,\, s\psi_0\}}{S_t} = 1\Big)\,dt. \end{aligned} \tag{3.12}$$

Taking into account (2.12) (with $\mu = r$) and the properties of the Wiener process $\widetilde{W} = \widetilde{W}^0$ the probability distribution law $\mathrm{Law}(\psi_t|\widetilde{\mathbf{P}})$ has a density, and so the last integral in (3.14) equals zero which proves (3.11).

According to the property (3.11) the process $(\psi_t)_{t\geqq 0}$ spends ($\widetilde{\mathbf{P}}$-a.s.) zero time at the point 1 and hence this point is an *instantly reflecting* or nonsticky *boundary* (see [11, Chap. IV, § 7]).

Denote

$$\varphi_t = \int_0^t I(\psi_u = 1)\,\frac{dM_u}{S_u}, \qquad t \geqq 0. \tag{3.13}$$

It is clear that the non-negative process $(\varphi_t)_{t\geqq 0}$ increases only when the process $(\psi_t)_{t\geqq 0}$ hits at the boundary point $\{1\}$. According to the Definition 7.1 of [11, Chap. IV, § 7], (3.9) and (3.11) imply that the process $(\psi_t)_{t\geqq 0}$ is a *diffusion with instant reflection at the point* $\{1\}$ on a phase (state) set $E = [1, \infty)$. The respective *infinitesimal* *operator* of this process on functions $g \in \mathbf{C}^2$ coincides with the *differential operator* (3.10) and the following condition holds at the boundary point $\{1\}$:

$$g'(1+) = 0, \tag{3.14}$$

where $g'(1+) = \lim_{\psi \downarrow 1} g'(\psi)$.

4. The Stephan problem connected with the optimal stopping of the process $(\psi_t)_{t \geqq 0}$.

1. According to (3.6) and (3.13) the process $(\psi_t)_{t \geq 0}$ satisfies the differential equation with reflection,

$$d\psi_t = -\psi_t\big(r dt + \sigma d\widetilde{W}_t\big) + d\varphi_t, \tag{4.1}$$

with initial condition ψ_0.

Let $\widetilde{\mathbf{P}}_\psi$ denote a probability distribution of the process $(\psi_t)_{t \geqq 0}$ (under the assumption that $\psi_0 = \psi \geqq 1$). Let also

$$\widetilde{V}(\psi) = \sup \widetilde{\mathbf{E}}_\psi e^{-\lambda \tau} \psi_\tau, \tag{4.2}$$

where $\widetilde{\mathbf{E}}_\psi$ is the expectation with respect to the measure $\widetilde{\mathbf{P}}_\psi$ and sup is taken over all finite ($\widetilde{\mathbf{P}}_\psi$-a.s.) Markov times $\tau = \tau(\omega)$. (We assume that $e^{-\lambda\tau(\omega)}\psi_{\tau(\omega)} = 0$ on the set $\{\omega :\ \tau(\omega) = \infty\}$.)

By (2.28),

$$\mathbb{C}^* = S_0 \widetilde{V}(\psi_0), \tag{4.3}$$

where ψ_0 is a constant of (1.4) contained in the definition of the payment function $f = (f_t)_{t \geqq 0}$.

According to the general theory of optimal stopping rules for Markov processes (see, for example, [21, Chap. III]) it is natural to expect that the structure of the optimal stopping time $\widetilde{\tau}$ in the problem (4.2) has the following form:

$$\widetilde{\tau} = \inf\big\{t \geqq 0 :\ \widetilde{V}(\psi_t) = \psi_t\big\}. \tag{4.4}$$

The results of our paper [10] suggests that in fact the structure of $\widetilde{\tau}$ must be quite simple:

$$\widetilde{\tau} = \inf\big\{t \geqq 0 :\ \psi_t \geqq \widetilde{\psi}\big\}, \tag{4.5}$$

where $\widetilde{\psi}$ is some constant. In other words, one can say that $\widetilde{\tau}$ is a first arrival time of the process $(\psi_t)_{t \geq 0}$ into the set of "stopping of observations" $\widetilde{D} = \{\psi :\ \psi \geqq \widetilde{\psi}\}$ (or, simply, of the "stopping region"). It is natural to call $\widetilde{C} = \{\psi :\ 1 \leqq \psi < \widetilde{\psi}\}$ the "region of continuation of observations" (or, simply, the "continuation region").

2. If we assume *a priori* that the function $\widetilde{V}(\psi)$ is sufficiently smooth, then according to the general theory of optimal stopping rules [21, Chap. III] the function $\widetilde{V}(\psi)$ satisfies, for $1 < \psi < \widetilde{\psi}$, the differential equation

$$LV(\psi) = \lambda V(\psi), \tag{4.6}$$

where L is the differential operator (3.10) with the boundary condition

$$V'(1+) = 0 \tag{4.7}$$

(cf. with (3.14)).

Remark. Heuristically, the equation (4.6) may be obtained in the following way.

If $\psi_t = \psi \in \widetilde{C}$, then the observations must not be immediately terminated and must be carried out at least on a "small" time interval $[t, t+\Delta t]$. Then, in view of the Markov property of the process $(\psi_t)_{t\geqq 0}$,

$$\widetilde{V}(\psi) = \widetilde{\mathbf{E}}_\psi\left[e^{-\lambda\Delta t}\widetilde{V}(\psi + \Delta\psi_t)\right] + o(\Delta t), \tag{4.8}$$

where $\Delta\psi_t = \psi_{t+\Delta t} - \psi_t$, $\psi_t = \psi$. Expanding $\widetilde{V}(\psi + \Delta\psi_t)$ and $e^{-\lambda\Delta t}$ into a Fourier series,

$$\begin{aligned}\widetilde{V}(\psi + \Delta\psi_t) &= \widetilde{V}(\psi) + \Delta\psi_t\widetilde{V}'(\psi) + \frac{(\Delta\psi_t)^2}{2}\,\widetilde{V}''(\psi) + o(\Delta t),\\ e^{-\lambda\Delta t} &= 1 - \lambda\Delta t + o(\Delta t),\end{aligned} \tag{4.9}$$

and taking into account that in some probabilistic sense "$(\Delta\psi_t)^2 = \sigma^2\psi_t^2\Delta t + o(\Delta t)$," we obtain from (4.8) and (4.9) letting $\Delta t \to 0$ the following equation:

$$-r\psi\widetilde{V}'(\psi) + \frac{\sigma^2\psi^2}{2}\,\widetilde{V}''(\psi) = \lambda\,\widetilde{V}(\psi), \tag{4.10}$$

which coincides with (4.6).

Similarly, if

$$\widetilde{V}(\psi, t) = \sup_{0\leqq\tau\leqq t} \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}\psi_\tau, \tag{4.11}$$

then this function satisfies in the "continuation region" the equation

$$\lambda\, V(\psi, t) + \frac{\partial V(\psi, t)}{\partial t} = L\, V(\psi, t). \tag{4.12}$$

3. We shall look for the solution of (4.6) in the form $V(\psi) = \psi^x$. Then for x we obtain the quadratic equation

$$x^2 - Ax - B = 0 \tag{4.13}$$

with

$$A = 1 + \frac{2r}{\sigma^2}, \qquad B = \frac{2\lambda}{\sigma^2}. \tag{4.14}$$

Solving equation (4.13) we find its roots

$$x_1 = \frac{A}{2} - \sqrt{\left(\frac{A}{2}\right)^2 + B}, \tag{4.15}$$

$$x_2 = \frac{A}{2} + \sqrt{\left(\frac{A}{2}\right)^2 + B}. \tag{4.16}$$

Note that $x_1 < 0$, $x_2 > 1$.

If equation (4.6) "operates" in the region of ψ with $1 < \psi < \widetilde{\psi}$, then in this region its solution $V(\psi)$ has the following form:

$$V(\psi) = C_1\psi^{x_1} + C_2\psi^{x_2}, \tag{4.17}$$

where C_1 and C_2 are some constants.

Therefore, if we assume that the continuation region $\widetilde{C}$ is of the form $\widetilde{C} = \{\psi : 1 \leqq \psi < \widetilde{\psi}\}$, where $\widetilde{\psi}$ is a constant that must be defined together with C_1 and C_2, then to define these constants one must have *three* additional conditions.

One of these conditions,

$$V(\widetilde{\psi}) = \widetilde{\psi}, \tag{4.18}$$

i.e.,

$$C_1 \widetilde{\psi}^{x_1} + C_2 \widetilde{\psi}^{x_2} = \widetilde{\psi}, \tag{4.19}$$

is quite obvious since it means that the "gain from the continuation of observations" must coincide with the "gain from stopping the observations".

The "smooth pasting" condition

$$V'(\widetilde{\psi}-) = 1, \tag{4.20}$$

is less evident and means that the derivatives of the left- and right-hand sides of (4.18) must coincide ("gains of continuation and stopping of observations must piece together smoothly"; for details see [21, Chap. III, § 8]).

From (4.19) and (4.20) it follows that

$$x_1 C_1 \widetilde{\psi}^{x_1 - 1} + x_2 C_2 \widetilde{\psi}^{x_2 - 1} = 1. \tag{4.21}$$

Finally, the third condition to define $\widetilde{\psi}$, C_1 , and C_2 is the condition (4.7) which, in view of (4.17), leads to the relation

$$C_1 = -\frac{x_2}{x_1} C_2. \tag{4.22}$$

Conditions (4.19) and (4.21) yield

$$C_1 \widetilde{\psi}^{x_1}(x_1 - 1) + C_2 \widetilde{\psi}^{x_2}(x_2 - 1) = 0, \tag{4.23}$$

and by (4.22) we obtain

$$\widetilde{\psi} = \left| \frac{x_2}{x_1} \cdot \frac{x_1 - 1}{x_2 - 1} \right|^{1/(x_2 - x_1)}. \tag{4.24}$$

Remark. The values $\gamma_1 = 1 - x_2$ (< 0) and $\gamma_2 = 1 - x_1$ (> 1) are the roots of the equation

$$\frac{\sigma^2}{2}\gamma^2 + \gamma\left(r - \frac{\sigma^2}{2}\right) - (\lambda + r) = 0. \tag{4.25}$$

If in this equation we let r be μ and $\lambda + r$ be r, then we obtain exactly (2.2) of [10].

Denoting $\gamma_1 = 1 - x_2$, $\gamma_2 = 1 - x_1$, we can rewrite (4.24) in the form

$$\widetilde{\psi} = \left(\frac{1 - \gamma_1^{-1}}{1 - \gamma_2^{-1}} \right)^{1/(\gamma_2 - \gamma_1)}, \tag{4.26}$$

which coincides with (2.3) at the critical point α $(= \widetilde{\psi})$ in [10].

From (4.21) and (4.23) it follows that

$$C_1 = \frac{x_2 - 1}{x_2 - x_1} \cdot \frac{1}{\widetilde{\psi}^{x_1 - 1}}, \qquad C_2 = \frac{1 - x_1}{x_2 - x_1} \cdot \frac{1}{\widetilde{\psi}^{x_2 - 1}}.$$

Thus we can formulate the following result.

LEMMA. *In a class of twice continuously differentiable functions $V = V(x)$ the solution $(V, \widetilde{\psi})$ of the "Stephan problem" (or the problem with moving boundary $\widetilde{\psi}$):*

$$\text{(4.27)} \qquad \begin{aligned} &LV(\psi) = \lambda V(\psi), \qquad 1 < \psi < \widetilde{\psi}, \\ &V(\widetilde{\psi}) = \widetilde{\psi}, \qquad V'(\widetilde{\psi}-) = 1, \qquad V'(1+) = 0, \end{aligned}$$

is given by formulae:

$$\text{(4.28)} \qquad V(\psi) = \begin{cases} \dfrac{\widetilde{\psi}}{x_2 - x_1}\left[(x_2 - 1)\left(\dfrac{\psi}{\widetilde{\psi}}\right)^{x_1} + (1 - x_1)\left(\dfrac{\psi}{\widetilde{\psi}}\right)^{x_2}\right], & 1 \leqq \psi < \widetilde{\psi}, \\ \psi, & \psi \geqq \widetilde{\psi}, \end{cases}$$

where

$$\text{(4.29)} \qquad \widetilde{\psi} = \left| \frac{x_2}{x_1} \cdot \frac{x_1 - 1}{x_2 - 1} \right|^{1/(x_2 - x_1)},$$

$$\text{(4.30)} \qquad x_1 = \left(\frac{1}{2} + \frac{r}{\sigma^2}\right) - \sqrt{\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}},$$

$$\text{(4.31)} \qquad x_2 = \left(\frac{1}{2} + \frac{r}{\sigma^2}\right) + \sqrt{\left(\frac{1}{2} + \frac{r}{\sigma^2}\right)^2 + \frac{2\lambda}{\sigma^2}}.$$

Remark. Simple transformations show that, along with (4.28), the following equivalent (more compact) representation holds for $V(\psi)$:

$$\text{(4.32)} \qquad V(\psi) = \begin{cases} \widetilde{\psi}\dfrac{x_1 \psi^{x_2} - x_2 \psi^{x_1}}{x_1 \widetilde{\psi}^{x_2} - x_2 \widetilde{\psi}^{x_1}}, & 1 \leqq \psi < \widetilde{\psi}, \\ \psi, & \psi \geqq \widetilde{\psi}. \end{cases}$$

4. In connection with the Stephan problem (4.27) whose solution gives (as will be shown later) the solution of the optimal stopping problem

$$\text{“}\widetilde{V}(\psi) = \sup_{0 \leqq \tau < \infty} \widetilde{\mathbf{E}}_{\psi} e^{-\lambda \tau} \psi_{\tau}\text{”}$$

it is natural to point out the corresponding Stephan problem for

$$\text{(4.33)} \qquad \text{“}\widetilde{V}(\psi, T) = \sup_{0 \leqq \tau \leqq T} \widetilde{\mathbf{E}}_{\psi} e^{-\lambda \tau} \psi_{\tau}\text{”}.$$

Let T be fixed, $0 \leqq t \leqq T$, and let $(V(\psi, t), \widetilde{\psi}_T(t))_{0 \leqq t \leqq T}$ be the solution of the following Stephan problem:

$$\text{(4.34)} \qquad \lambda V(\psi, t) + \frac{\partial V(\psi, t)}{\partial t} = LV(\psi, t)$$

on the set $\{(\psi,t):\ 1 \leqq \psi \leqq \widetilde{\psi}_T(t),\ 0 \leqq t \leqq T\}$; then

$$V\big(\widetilde{\psi}_T(t),\ t\big) = \widetilde{\psi}_T(t), \qquad \frac{\partial V}{\partial \psi}\big(\widetilde{\psi}_T(t)-,\ t\big) = 1, \qquad \frac{\partial V}{\partial \psi}(1+,t) = 0, \tag{4.35}$$

where $\widetilde{\psi}_T(t)$ is a function of the type $\widetilde{\psi}(T-t)$ and $\widetilde{\psi}(s)$, $s \geqq 0$, is an increasing function.

It is highly probable (although it is not discussed here rigorously) that the solution of the problem (4.33) coincides with the solution of the problem (4.35) and, in particular, $\widetilde{V}(\psi,T) = V(\psi,T)$, and the optimal time $\widetilde{\tau}_T = \min\{0 \leqq t \leqq T:\ \psi_t \geqq \widetilde{\psi}(T-t)\}$.

Here $\widetilde{\psi}(T) \to \widetilde{\psi}$, $T \to \infty$, where $\widetilde{\psi}$ is a solution of the problem (4.27).

5. Proof of the theorem.

1. Comparing (4.3) with (1.9), we see that the statement (1.9) of the theorem consists in the following:

$$\widetilde{V}(\psi) = V(\psi), \tag{5.1}$$

where $V = V(\psi)$ is a solution of the Stephan problem (4.27) defined in the previous lemma.

In order to prove (5.1) it is sufficient to state that

(A1) *for any finite* ($\widetilde{\mathbf{P}}_\psi$*-a.s.*, $\psi \geqq 1$) *time* τ,

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}\psi_\tau \leqq V(\psi), \qquad \psi \geqq 1,$$

(A2) *the time* $\widetilde{\tau} = \inf\{t \geqq 0:\ \psi_t \geqq \widetilde{\psi}\}$ *is* ($\widetilde{\mathbf{P}}_\psi$*-a.s.*, $\psi \geqq 1$) *finite and*

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau}\psi_{\widetilde{\tau}} = V(\psi).$$

If these properties are fulfilled, then they imply the statement (1.11) of the theorem on optimality (and rationality) of the time $\widetilde{\tau}$.

The Itô formula, by which, for any finite Markov time τ,

$$\begin{aligned} e^{-\lambda\tau}V(\psi_\tau) &= V(\psi_0) + \int_0^\tau e^{-\lambda u}\big[(LV)(\psi_u) - \lambda V(\psi_u)\big]\,du \\ &\quad - \int_0^\tau e^{-\lambda u}\sigma\psi_u V'(\psi_u)\,d\widetilde{W}_u + \int_0^\tau e^{-\lambda u}V'(\psi_u)\,d\varphi_u, \end{aligned} \tag{5.2}$$

can be applied to the function $V(\psi)$ defined by (4.28).

Note that the definition of $V(\psi)$ implies that $LV(\psi) - \lambda V(\psi) \leqq 0$ for all $\psi \geqq 1$. Furthermore, the last integral in (5.2) equals zero since the function φ_u increases for $\psi_u = 1$ and $V'(1) = 0$. Thus it follows from (5.2) that the integral

$$I_t = -\int_0^t \sigma e^{-\lambda u}\psi_u V'(\psi_u)\,d\widetilde{W}_u$$

is a local martingale (as a stochastic integral over the Wiener process [17, Chap. 2, § 2]; [6, Chap. I, § 4d]) and is uniformly bounded from below:

$$I_t \geqq e^{-\lambda t}V(\psi_t) - V(\psi_0) \geqq -V(\psi_0). \tag{5.3}$$

Therefore, $(I_t)_{t\geqq 0}$ is a supermartingale and $\widetilde{\mathbf{E}}_\psi I_\tau \leqq \widetilde{\mathbf{E}} I_0 = 0$ (the Doob theorem [16, Chap. 2, § 4]).

Thus, taking the mathematical expectation $\widetilde{\mathbf{E}}_\psi$ in both parts of (5.2), we obtain (the supermartingale property) that

$$\widetilde{\mathbf{E}}_\psi e^{-\lambda\tau} V(\psi_\tau) \leqq V(\psi)$$

for any $\psi \geqq 1$ and finite ($\widetilde{\mathbf{P}}_\psi$-a.s.) Markov time τ. The last inequality together with the evident inequality $\psi \leqq V(\psi)$ prove property (A1).

Now we pass to the proof of property (A2). From (5.2),

$$e^{-\lambda(t\wedge\widetilde{\tau})} V(\psi_{t\wedge\widetilde{\tau}}) = V(\psi_0) + \int_0^{t\wedge\widetilde{\tau}} e^{-\lambda u}\big[(LV)(\psi_u) - \lambda V(\psi_u)\big]\,du + I_{t\wedge\widetilde{\tau}}. \tag{5.4}$$

If $\psi_0 \geqq \widetilde{\psi}$, then $\widetilde{\tau} = 0$ and property (A2) is evidently fulfilled. Let $\psi_0 < \widetilde{\psi}$, then $(LV)(\psi_u) - \lambda V(\psi_u) = 0$ for $u \leqq t \wedge \widetilde{\tau}(\omega)$, $\omega \in \Omega$, and thus

$$V(\widetilde{\psi}) - V(\psi_0) \geqq e^{-\lambda(t\wedge\widetilde{\tau})} V(\psi_{t\wedge\widetilde{\tau}}) - V(\psi_0) = I_{t\wedge\widetilde{\tau}} \geqq -V(\psi_0). \tag{5.5}$$

The process $(I_{t\wedge\widetilde{\tau}})_{t\geqq 0}$ is a local martingale, uniformly bounded (by (5.5)) from below and above.

In what follows we shall show that $\widetilde{\mathbf{P}}_\psi(\widetilde{\tau} < \infty) = 1$ for $\psi \geqq 1$. Thus, since $\widetilde{\mathbf{E}}_\psi I_{\widetilde{\tau}} = \widetilde{\mathbf{E}}_\psi I_0 = 0$ by the Doob theorem, (5.4) implies $\widetilde{\mathbf{E}}_\psi e^{-\lambda\widetilde{\tau}} V(\psi_{\widetilde{\tau}}) = V(\psi)$.

But $\widetilde{\mathbf{P}}_\psi(V(\psi_{\widetilde{\tau}}) = \psi_{\widetilde{\tau}}) = 1$. Hence $\widetilde{\mathbf{E}}_\psi e^{-\lambda\widetilde{\tau}} \psi_{\widetilde{\tau}} = V(\psi)$ which proves property (A2).

2. It only remains to prove the $\widetilde{\mathbf{P}}_\psi$-a.s. finiteness of the time $\widetilde{\tau} = \inf\{t \geqq 0 : \psi_t \geqq \widetilde{\psi}\}$ for any $\psi \geqq 1$.

Note that, for integral $T \geqq 1$,

$$\begin{aligned}
\widetilde{\mathbf{P}}_\psi\Big\{\max_{0\leqq t\leqq T} \psi_t \geqq \widetilde{\psi}\Big\} &\geqq \widetilde{\mathbf{P}}_\psi\Big\{\max_{0\leqq u\leqq t\leqq T} \frac{S_u}{S_t} \geqq \widetilde{\psi}\Big\} \\
&= \widetilde{\mathbf{P}}_\psi\Big\{\max_{0\leqq u\leqq t\leqq T} e^{\sigma(\widetilde{W}_u - \widetilde{W}_t) + (r+\sigma^2/2)(u-t)} \geqq \widetilde{\psi}\Big\} \\
&\geqq \widetilde{\mathbf{P}}_\psi\Big\{e^{(r+\sigma^2/2)} \max\Big[e^{\sigma(\widetilde{W}_1 - \widetilde{W}_0)}, e^{\sigma(\widetilde{W}_2 - \widetilde{W}_1)}, \dots, e^{\sigma(\widetilde{W}_T - \widetilde{W}_{T-1})}\Big] \geqq \widetilde{\psi}\Big\} \\
&= \widetilde{\mathbf{P}}_\psi\Big\{\max\big[\widetilde{W}_1 - \widetilde{W}_0, \dots, \widetilde{W}_T - \widetilde{W}_{T-1}\big] \geqq C\Big\},
\end{aligned} \tag{5.6}$$

where $C = [\log\widetilde{\psi} - (r + \sigma^2/2)]/\sigma$. But it is evident that for *any* real C the probability of the right-hand side of (5.6) tends to 1 as $T \to \infty$. Thus the process $(\psi_t)_{t\geqq 0}$ attains any level with probability 1 and, in particular, the level $\widetilde{\psi}$ which proves the finiteness ($\widetilde{\mathbf{P}}_\psi$-a.s., $\psi \geqq 1$) of the time $\widetilde{\tau}$.

The theorem is proved.

3. In the optimal stopping problem (4.2) under consideration we assume that the sup is taken over *all* Markov times τ. We could also consider a simplified version of this problem supposing

$$\widehat{V}(\psi) = \sup \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau} \psi_\tau, \tag{5.7}$$

where sup is only taken over Markov times τ_a of the type $\tau_a = \inf\{t : \psi_t \geqq a\}$, $a \geqq 1$. In other words, let

$$\widehat{V}(\psi) = \sup_{a\geqq 1} \widehat{V}_a(\psi), \tag{5.8}$$

where

$$\widehat{V}_a(\psi) = \widetilde{\mathbf{E}}_\psi e^{-\lambda\tau_a}\psi_{\tau_a}. \tag{5.9}$$

It is well known [13, Theorem 13.20] that this function is twice continuously differentiable in $C_a = \{\psi:\ 1 < \psi < a\}$ and

$$L\widehat{V}_a(\psi) = \lambda\widehat{V}_a(\psi), \qquad \psi \in C_a, \tag{5.10}$$

$$\widehat{V}_a(a) = a. \tag{5.11}$$

According to (3.14), the condition of instant reflection

$$\frac{\partial \widehat{V}_a}{\partial \psi}(1+) = 0 \tag{5.12}$$

is fulfilled at the point $\psi = 1$.

As in § 4, using on the general solution $\widehat{V}_a(\psi) = C_1\psi^{x_1} + C_2\psi^{x_2}$ of the equation (5.10), we find that

$$\widehat{V}_a(\psi) = \begin{cases} a\dfrac{x_2\psi^{x_1} - x_1\psi^{x_2}}{x_2a^{x_1} - x_1a^{x_2}}, & 1 \leqq \psi \leqq a, \\ \psi, & \psi > a. \end{cases} \tag{5.13}$$

One can immediately prove that $\sup_a \widehat{V}_a(\psi)$ is attained for $a = \widetilde{\psi}$. Needless to say that this fact also follows from our theorem according to which the optimal time $\widetilde{\tau}$ (in the class of *all* finite Markov times) is a time of the *type* τ_a for some a $(= \widetilde{\psi})$.

6. The "Russian option" with dividends. In the context of the outlined results on the Russian option pricing we now consider the case, studied in [3], where a capital *inflow* from the outside takes place in the form of dividends.

In accordance with the notations and concepts (notions) of [20, § 2] also published in this issue, we let $\pi = (\beta, \gamma)$ be a strategy with $\beta = (\beta_t)_{t\geqq 0}$, $\gamma = (\gamma_t)_{t\geqq 0}$, and $X^\pi = (X_t^\pi)_{t\geqq 0}$ a capital corresponding to this strategy. In the "problem with dividends" a change of capital occurs according to (2.8) in [20]:

$$dX_t^\pi(\mu) = \beta_t dB_t + \gamma_t dS_t(\mu) + dD_t, \tag{6.1}$$

where in the case under consideration it is assumed that

$$dD_t = \delta\gamma_t S_t(\mu)\, dt, \tag{6.2}$$

with $\delta < r$.

From (6.1) and (6.2) it follows that

$$dX_t^\pi(\mu) = rX_t^\pi(\mu)\, dt + \sigma\gamma_t S_t(\mu)\, dW_t^{\mu-r+\delta}, \tag{6.3}$$

where

$$W_t^{\mu-r+\delta} = \frac{\mu - r + \delta}{\sigma}t + W_t. \tag{6.4}$$

The process $W^{\mu-r+\delta} = (W_t^{\mu-r+\delta})_{t\geqq 0}$ is a Wiener process with respect to the measure $\mathbf{P}^{\mu-r+\delta}$ defined by (2.3) if we replace r by $r - \delta$, and

$$dY_t^\pi(\mu) = \frac{\sigma\gamma_t S_t(\mu)}{B_t}\, dW_t^{\mu-r+\delta}, \tag{6.5}$$

where

$$Y_t^\pi(\mu) = \frac{X_t^\pi(\mu)}{B_t}. \tag{6.6}$$

Let us denote by $\mathbb{C}^*_{\mathrm{div}}(\delta;\mu,\lambda)$ a rational option price of the American type with payment function (1.4). Then, starting from (6.5), using the same method as in the proof of Theorem 1 in [20], we find that

$$\mathbb{C}^*_{\mathrm{div}}(\delta;\mu,\lambda) = \sup_{0\leqq\tau<\infty} \mathbf{E}^{\mu-r+\delta} e^{-r\tau} f_\tau(S(\mu)), \tag{6.7}$$

where

$$f_t(S(\mu)) = e^{-\lambda t}\max\Big[\max_{u\leqq t} S_u(\mu),\ s\psi_0\Big],$$

$\lambda > 0$, $\psi_0 \geqq 1$, $s = S_0 > 0$.

Since

$$\mathrm{Law}\big(S(\mu)|\mathbf{P}^{\mu-r+\delta}\big) = \mathrm{Law}\big(S(r-\delta)|\mathbf{P}\big),$$

(6.7) and (1.4) yield

$$\begin{aligned} \mathbb{C}^*_{\mathrm{div}}(\delta;\mu,\lambda) &= \sup_{0\leqq\tau<\infty} \mathbf{E}e^{-r\tau} f_\tau\big(S(r-\delta)\big) \\ &= \sup_{0\leqq\tau<\infty} \mathbf{E}e^{-(r-\delta+\lambda+\delta)\tau}\max\Big[\max_{u\leqq\tau} S_u(r-\delta),\ s\psi_0\Big]. \end{aligned} \tag{6.8}$$

Denoting by $\mathbb{C}^*(\mu,\lambda)$ the rational price of the initial problem (without dividends, i.e., for $\delta = 0$) we obtain from (6.8) that

$$\mathbb{C}^*_{\mathrm{div}}(\delta;\ \mu,\ \lambda) = \mathbb{C}^*(r-\delta,\ \lambda+\delta).$$

Note now that, as was stated above, $\mathbb{C}^*(\mu,\lambda) = \mathbb{C}^*(r,\lambda)$ for all $\mu\in\mathbf{R}$ and $\mathbb{C}^*(r,\lambda)$ is the function $\mathbb{V}(r,r+\lambda)$ defined by the right-hand side of (1.9) or, equivalently, (1.10).

Then one can see from (6.8) that $\mathbb{C}^*_{\mathrm{div}}(\delta;\mu,\lambda) = \mathbb{V}(r-\delta,(r-\delta)+(\lambda+\delta)) = \mathbb{V}(r-\delta,r+\lambda)$. In on other way this result established in [3] can be stated in the following way: *the formula for the rational price in the problem with dividends is obtained from the formulas for the rational price in the problem without dividends by replacing* r by $r-\delta$, *and* λ *by* $\lambda+\delta$. Then, for all $\mu\in\mathbf{R}$,

$$\mathbb{C}^*(\mu,\lambda) = \mathbb{V}(r,r+\lambda), \qquad \mathbb{C}^*_{\mathrm{div}}(\delta;\ \mu,\ \lambda) = \mathbb{V}(r-\delta,\ r+\lambda) = \mathbb{C}^*(r-\delta,\ \lambda+\delta),$$

where $\mathbb{V}(r,r+\lambda)$ is a function in the right-hand side of (1.9) or (1.10).

Thus, for each $\mu\in\mathbf{R}$,

$$\mathbb{C}^*_{\mathrm{div}}(\delta;\ \mu,\ \lambda) = \mathbb{C}^*(r-\delta,\ \lambda+\delta),$$

i.e., the solution of "the problem with dividends" defined according to (6.2) is obtained from the solution of "the problem without dividends" in which one replaces r by $r-\delta$ and λ by $\lambda+\delta$ (cf. with [3]).

REFERENCES

[1] A. Bensoussan, *On the theory of option pricing*, Acta Appl. Math., 2 (1984), pp. 139–158.

[2] F. Black and M. Scholes, *The pricing of options and corporate liabilities*, J. Political Economy, 81 (1973), pp. 637–657.

[3] J. D. DUFFIC AND J. M. HARRISON, *Arbitrage pricing of a Russian option*, Ann. Appl. Probab., 1993.

[4] J. M. HARRISON AND D. M. KREPS, *Martingales and arbitrage in multiperiod securities market*, J. Economic Theory, 20 (1979), pp. 381–408.

[5] J. M. HARRISON AND S. R. PLISKA, *Martingales and stochastic integrals in the theory of continuous trading*, Stoch. Proc. Appl., 11 (1981), pp. 215–260.

[6] J. JACOD AND A. N. SHIRYAEV, *Limit Theorems for Stochastic Processes*, Springer-Verlag, Berlin–Heidelberg, 1988.

[7] I. KARATZAS AND S. E. SHREVE, *Brownian Motion and Stochastic Calculus*, Springer-Verlag, Berlin–Heidelberg, 1988.

[8] R. C. MERTON, *Theory of rational option pricing*, Bell J. Economics and Management Science, 4 (1973), pp. 141–183.

[9] P. A. SAMUELSON, *Rational theory of warrant pricing*, Industrial Management Rev., 6 (1966), pp. 13–31.

[10] L. A. SHEPP AND A. N. SHIRYAEV, *The Russian option: reduced regret*, Ann. Appl. Probab., 1993.

[11] S. WATANABE AND N. IKEDA, *Stochastic Differential Equations and Diffusion Processes*, North-Holland, Amsterdam, 1981.

[12] C. DELLACHERIE, *Capacités et processus stochastique*, Springer-Verlag, Berlin–Heidelberg, 1972.

[13] E. B. DYNKIN, *Markov Processes*, Vols. 1, 2, Academic Press, New York, 1965.

[14] D. O. KRAMKOV AND E. MORDECKY, *Integral option*, Theory Probab. Appl., 39 (1994) pp. 162–171.

[15] D. O. KRAMKOV AND A. N. SHIRYAEV, *On the rational pricing of the "Russian option" for the symmetrical binomial model of a* (B, S)*-market*, Theory Probab. Appl., 39 (1994), pp. 153–161.

[16] R. SH. LIPTSER AND A. N. SHIRYAEV, *Statistic of Random Processes*, Springer-Verlag, Berlin–Heidelberg, 1977.

[17] ———, *Theory of Martingales*, Moscow, Nauka, 1986.

[18] A. N. SHIRYAEV, *On some basic concepts and some basic stochastic models used in finance*, Theory Probab. Appl., 39 (1994), pp. 1–13.

[19] A. N. SHIRYAEV, YU. M. KABANOV, D. O. KRAMKOV, AND A. V. MEL'NIKOV, *Towards the theory of pricing of options of both European and American types.* I. *Discrete time*, Theory Probab. Appl., 39 (1994), pp. 14–60.

[20] ———, *Towards the theory of pricing of options of both European and American types.* II. *Continuous time*, Theory Probab. Appl., 39 (1994), pp. 61–102.

[21] A. N. SHIRYAEV, *Optimal Stopping Rules*, Springer-Verlag, Berlin–New York, 1978.